Nowhere-zero 3-flows in squares of graphs

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Abstract

It was conjectured by Tutte that every 4-edge-connected graph admits a nowhere-zero 3-flow. In this paper, we give a complete characterization of graphs whose squares admit nowhere-zero 3-flows and thus confirm Tutte's 3-flow conjecture for the family of squares of graphs.

1 Introduction

All graphs considered in this paper are simple. Let $G = (V, E)$ be a graph with vertex set V and edge set E. For any $v \in V(G)$, we use $d_G(v), N_G(v)$ to denote the degree and the neighbor set of v in G, respectively. The minimal degree of a vertex of G is denoted by $\delta(G)$. We use K_m for a complete graph on m vertices, P_t for a path of length t and W_4 for a graph obtained from a 4-circuit by adding a new vertex x and edges joining x to all the vertices on the circuit. We call x the center of this W_4 and each edge with x as one end is called a center edge. Let D be an orientation of G. Then the set of all edges with tails (or heads) at a vertex v is denoted by $E^+(v)$ (or $E^-(v)$). If an edge uv is oriented from u to v under D, then we say $D(uv) = u \to v$. The square of G, denoted by G^2, is the graph obtained from G by adding all the edges that join distance 2 vertices in G. We refer the reader to [1] for terminology not defined in this paper.

Definition 1.1 *Let D be an orientation of G and f be a function: $E(G) \mapsto Z$. Then*

(1). The ordered pair (D, f) is called a **k-flow** *of G if $-k+1 \leq f(e) \leq k-1$ for every edge $e \in E(G)$ and $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$ for every $v \in V(G)$.*

(2). The ordered pair (D, f) is called a **Modular k-flow** *of G if for every $v \in V(G)$, $\sum_{e \in E^+(v)} f(e) \equiv \sum_{e \in E^-(v)} f(e)$ (mod k).*

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The **support** *of a k-flow (Modular k-flow)* (D, f) *of* G *is the set of edges of* G *with* $f(e) \neq 0$ *(*$f(e) \not\equiv 0$ *(mod k)), and is denoted by supp(f). A k-flow* (D, f) *(Modular k-flow) of* G *is* **nowhere-zero** *if* $supp(f) = E(G)$.

For convenience, a nowhere-zero k-flow is abbreviated as a k**-NZF**. The concept of integer-flow was introduced by Tutte([7, 8] also see [9, 4]) as a refinement and generalization of the face-coloring and edge-3-coloring problems. One of the most well known open problems in this subject is the following conjecture due to Tutte:

Conjecture 1.2 (Tutte, unsolved problem 48 in [1]) *Every* 4*-edge-connected graph admits a* 3*-NZF.*

Squares of graphs admitting 3-NZF's are to be characterized in this paper. The following families of graphs are the exceptions in the main theorem.

Definition 1.3 $\mathcal{T}_{1,3} = \{T \mid T \text{ is a tree and } d_T(v) = 1 \text{ or } 3 \text{ for every } v \in V(T)\}$

Definition 1.4 $\bar{\mathcal{T}}_{1,3} = \{T \mid T \in \mathcal{T}_{1,3}$ *or* T *is a* 4*-circuit or* T *can be obtained from some* $T' \in \mathcal{T}_{1,3}$ *by adding some edges each of which joins a pair of distance 2 leaves of* $T'\}$

The following is the main result of this paper.

Theorem 1.5 *Let* G *be a connected simple graph. Then* G^2 *admits a* 3*-NZF if and only if* $G \notin \bar{\mathcal{T}}_{1,3}$.

An immediate corollary of Theorem 1.5 is the following partial result to Tutte's 3-flow conjecture (Conjecture 1.2).

Corollary 1.6 *Let* G *be a graph. If* $\delta(G^2) \geq 4$ *then* G^2 *admits a* 3*-NZF.*

This research is motivated by Conjecture 1.2 and the following open problem:

Conjecture 1.7 (Zhang [11]) *If every edge of a 4-edge-connected graph* G *is contained in a circuit of length at most* 3 *or* 4*, then* G *admits a* 3*-NZF.*

Theorem 1.5 and the following early results are partial results of the open problem above.

Theorem 1.8 (Catlin [2]) *If every edge of a graph* G *is contained in a circuit of length at most* 4*, then* G *admits a* 4*-NZF.*

Theorem 1.9 (Lai [5]) *Every* 2*-edge-connected, locally* 3*-edge-connected graph admits a* 3*-NZF.*

Theorem 1.10 (Imrich and Skrekovski [3]) *Let* G *and* H *be two graphs. Then* $G \times H$ *admits a* 3*-NZF if both* G *and* H *are bipartite.*

2 Splitting operation, flow extension and lemmas

Definition 2.1 *(A special splitting operation) Let* G *be a graph and* $e = xy \in E(G)$*. The graph* G_{*e} *is obtained from* G *by deleting the edge* e *and adding two new vertices* x' *and* y' *and adding two new edges,* e_x *and* e_y*, joining* x *and* y'*,* y *and* x'*, respectively.*

Definition 2.2 *Let* G *be a graph, let* (D, f) *be a 3-flow of* G *and let* $F \subseteq E(G) \setminus supp(f)$*. A 3-flow* (D', f') *of* G *is called an* (F, f)-changer *if* $F \cup supp(f) \subseteq supp(f')$*.*

Lemma 2.3 ([7]) *A graph* G *admits a* k*-flow* (D, f_1) *if and only if* G *admits a Modular* k*-flow* (D, f_2) *such that* $f_1(e) \equiv f_2(e) (\bmod k)$ *for each* $e \in E(G)$*.*

An orientation of a graph G is called a modular 3-orientation if $|E^+(v)| \equiv |E^-(v)| \pmod 3$, for every $v \in V(G)$. The following result appears in [4, 6, 9], but by Lemma 2.3, we can attribute it to Tutte.

Lemma 2.4 ([7]) *Let* G *be a graph. Then* G *admits a* 3*-NZF if and only if* G *has a modular* 3*-orientation.*

A **partial** 3**-orientation** D of G is an orientation of some edges of G satisfying $|E^+(v)| \equiv |E^-(v)| \pmod 3$, for any $v \in V(G)$. The support of D is the set of edges oriented under D and is denoted by $supp(D)$. Clearly the partial orientation obtained by reversing every oriented edge of a partial 3-orientation is also a partial 3-orientation.

Let D be a partial 3-orientation of G and let $C = v_0v_1 \cdots v_{k-1}v_0$ be a circuit of G. A **circuit-operation** along C is defined as following: For $0 \le i \le k-1$, if $D(v_iv_{i+1}) = v_i \to v_{i+1} \pmod k$, then reverse the direction of this edge; if $(v_iv_{i+1}) \pmod k$ is not oriented under D, then orient it as $v_i \to v_{i+1}$; if $D(v_iv_{i+1}) = v_{i+1} \to v_i \pmod k$ then v_iv_{i+1} loses it's orientation.

Lemma 2.5 *Let* G *be a graph,* (D, f) *be a 3-flow of* G *and* H *be a subgraph of* G

(1). If $H \cong W_4$ *and* $e \in E(H) \setminus supp(f)$ *is a center edge, then an* $(\{e\}, f)$*-changer exists.*

(2). If H *is a circuit of length* 3 *with* $E(H) \cap supp(f) = \{e\}$*, then an* $(E(H) \setminus \{e\}, f)$*-changer exists.*

Proof. (1). Since $H \cong W_4$, let x be the center of H and let $u_1u_2u_3u_4u_1$ be the 4-circuit $H \setminus x$. Since G has a 3-flow (D, f), then G has a partial 3-orientation D^* with $supp(D^*) = supp(f)$. We need only to find a partial 3-orientation D' such that $supp(D^*) \cup \{e\} \subseteq supp(D')$. Since e is a center edge, without loss of generality, assume that $e = xu_1$.

First we assume $E(H) \setminus \{e\} \subseteq supp(D^*)$. Without loss of generality, assume $D^*(u_1u_2) = u_1 \to u_2$. Then $D^*(u_2x) = x \to u_2$. Otherwise, we do a circuit-operation along $u_1u_2xu_1$ and then get a needed partial 3-orientation D' of G. For the same reason, u_4 must be the tail (or head) of both u_1u_4 and xu_4. By symmetry, we consider the following two cases.

Case 1. $D^*(u_1u_4) = u_1 \to u_4$ and $D^*(xu_4) = x \to u_4$.

We may assume that u_3 is the tail (or head) of all edges incident with it in H. Otherwise, there exists a directed 2-path xu_3u_i (or u_iu_3x) for some $i \in \{2,4\}$. Then we do circuit-operations along xu_3u_ix (or $u_iu_3xu_i$) and along $u_1u_ixu_1$. Therefore, we get a needed partial 3-orientation of D' of G.

If all edges in H have u_3 as a tail, then we do circuit-operations along xu_1u_4x, along $u_4xu_3u_4$, along xu_3u_2x and along $u_2xu_1u_2$; If all edges in H have u_3 as a head, then we do circuit-operations along $u_1u_2u_3xu_1$ and along $u_3xu_4u_3$. In both cases, we get a needed partial 3-orientation D' of G.

Case 2. $D^*(u_1u_4) = u_4 \to u_1$ and $D^*(xu_4) = u_4 \to x$.

Similar to Case 1, we may assume u_3 be the tail (or head) of all edges incident with it in H. If all edges in H have u_3 as a tail, then we do circuit-operations along xu_1u_4x, along $u_3u_4u_1u_2u_3$ and along $u_3xu_2u_3$; If all edges in H have u_3 as a head, then we do circuit-operations along $u_1xu_2u_1$, along $u_4u_1u_2u_3u_4$ and along $u_4xu_3u_4$. In both cases, we get a needed partial 3-orientation D' of G.

If $supp(D^*)$ misses some other edges of $E(H)$, say $e^* = ab \in E(H) \setminus supp(D^*)$, then we define $D^*(ab) = a \to b$ *or* $b \to a$, by the proof of Case 1 and Case 2, we can find a needed D' of G.

(2). it is trivial. ■

Lemma 2.6 *For each $G \in \bar{\mathcal{T}}_{1,3}$ and each $e_0 \in E(G)$, the graph G^2 admits a 3-flow (D,f) such that $supp(f) = E(G^2) \setminus \{e_0\}$*

Proof. Induction on $|E(G)|$. It is obviously true for graphs G with $G^2 = K_4$ (including $G = C_4$, the circuit of length 4). So, assume that $|V(G)| \geq 5$ and let D be any fixed orientation of G^2.

Let $e = xy$ with $d_G(x) = d_G(y) = 3$. Then G_{*e} consists of two components, say G_1 and G_2. Clearly, $G_1, G_2 \in \bar{\mathcal{T}}_{1,3}$. Without loss of generality, let $e_0 \in E(G_1)$. By induction, G_1^2 admits a 3-flow (D, f_1) such that $supp(f_1) = E(G_1^2) \setminus \{e_0\}$ and G_2^2 admits a 3-flow (D, f_2) that $supp(f_2) = E(G_2^2) \setminus \{e\}$.

Then, identifying the split vertices and edges, back to G, $(D, f_1 + f_2)$ is a 3-flow (D,f) with $supp(f) = E(G^2) \setminus \{e_0\}$. ■

Lemma 2.7 *(1). Let G be a k-path with $k \geq 2$ or an m-circuit with $m = 3$ or $m \geq 5$. Then G^2 admits a 3-NZF.*

(2). Let G be a graph obtained from an r-circuit $x_0x_1 \cdots x_{r-1}x_0$ by attaching an edge x_iv_i at each x_i for $0 \leq i \leq r-1$, where $v_i \neq v_j$ if $i \neq j$. Then G^2 admits a 3-NZF.

(3). Let G be a graph obtained from an m-circuit $x_0x_1 \cdots x_{m-1}x_0$ by attaching an edge $x_{m-1}v$ at x_{m-1} alone, where $m \geq 5$. Then G^2 admits a 3-NZF.

Proof. (1). If G is an m-circuit with $m = 3$ or $m \geq 5$, then G^2 is a cycle (every vertex is of even degree) and G^2 admits 2-NZF. If G is a k-path with $k \geq 2$, by induction on k and using Lemma 2.5-(2), G^2 admits a 3-NZF.

(2). For $r \geq 5$ (or $r = 3$): let D be an orientation such that v_i $(0 \leq i \leq r-1)$ is the tail of every edge of G^2 incident with it and all the other edges are oriented as

$x_i \to x_{i+1},\ x_i \to x_{i+2}$ (mod r) (or $x_i \to x_{i+1}$ (mod 3) only for $r = 3$). Obviously, D is a modular 3-orientation of G^2.

For $r = 4$: let D be the orientation such that v_0 and v_2 be the tail of every edge of G^2 incident with it, v_1 and v_3 be the head of every edge of G^2 incident with it, $x_0x_1x_3x_2x_0$ as a directed circuit and other edges are oriented as $x_3 \to x_0$, $x_1 \to x_2$. Obviously, D is a modular 3-orientation of G^2.

(3). Orient all the edges as $x_i \to x_{i+1},\ x_i \to x_{i+2}$ (mod m) for $0 \le i \le m-1$ and let v be the tail of every edge of G^2 incident with it. Then reverse the direction of the following edges: x_0x_{m-1}, x_0x_{m-2}. Clearly, this orientation is a modular 3-orientation of G^2. ■

3 Proof of the main theorem

Proof. $\Longrightarrow$ By contradiction. Suppose $G \in \bar{\mathcal{T}}_{1,3}$. Let G be a counterexample with $|V(G)| + |E(G)|$ as small as possible. Clearly $|V(G)| \ge 5$ and G contains no circuits. So $G \in \mathcal{T}_{1,3}$. Let $v \in V(G)$ be a degree 3 vertex such that $N_G(v) = \{v_1, v_2, v_3\},\ d_G(v_1) = d_G(v_2) = 1$. Clearly, $G_1 = G \setminus \{v_1, v_2\} \in \mathcal{T}_{1,3}$. Since G^2 has a modular 3-orientation D and both v_1 and v_2 are degree 3 vertices in G^2, then this orientation restricted to the edge set of G_1^2 will generate a modular 3-orientation of G_1^2. Therefore, G_1^2 admits a 3-NZF, a contradiction.

$\Longleftarrow$ Let G be a counterexample to the theorem such that

(i). $|E(G)| - |V(G)|$ is as small as possible,

(ii). subject to (i), $|E(G)|$ is as small as possible.

Note that $|E(G)| - |V(G)| + 1$ is the rank of the cycle space of G.

Claim 1. Let $e_0 = xy \in E(G)$. If $d_G(x) \ge 3$ and $d_G(y) \ge 2$, then xy is not a cut edge of G.

If e_0 is a cut-edge, then at least one component of G_{*e_0} is not in $\bar{\mathcal{T}}_{1,3}$, say, G_1 is not, while G_2 might be. By induction, let (D, f_i) be a 3-flow of G_i^2 for each $i = 1, 2$ such that f_1 is nowhere-zero, f_2 might miss only one edge e_x (that is a copy of e_0). Without loss of generality, assume that $f_1(e_y) + f_2(e_x) \not\equiv 0$ (mod (3)). Then, identifying the split vertices and edges, back to G, $(D, f_1 + f_2)$ is a nowhere-zero Modular 3-flow of G^2. By Lemma 2.3, G^2 admits a 3-NZF, a contradiction.

Claim 2. $d_G(x) \le 3$ for any $x \in V(G)$.

Otherwise, assume that $d_G(x) \ge 4$ for some vertex $x \in V(G)$. Clearly $G \not\cong K_{1,m}$ for $m \ge 4$ since $K_{1,m}$ is not a counterexample. So there exists $e_0 = xy \in E(G)$ with $d_G(y) \ge 2$. By Claim 1, e_0 is not a cut edge of G and $G_1 = G_{*e_0} \notin \bar{\mathcal{T}}_{1,3}$. Then by (i), G_1^2 admits a 3-NZF.

In G_1^2, identify x and x', y and y', and use one edge to replace two parallel edges, by Lemma 2.3, we will get G^2 and a Modular 3-flow (D, f) of G^2 such that $E(G^2) \setminus supp(f) \subseteq \{xv \text{ or } yw \mid v \in N_G(y),\ w \in N_G(x)\}$. Let $C(x) = G^2[N_G(x) \cup \{x\}]$. Then $C(x)$ is a clique of order at least 5. We are to adjust (D, f) so that the resulting Modular 3-flow (D, f')

of G^2 misses only edges of $\{uv \mid u, v \in V(C(x))\}$. For each edge xv which is missed by $supp(f)$ and $xv \notin E(C(x))$, $xyvx$ must be a circuit of G^2, so let (D, f_{xv}) be a 3-flow of G^2 with $supp(f_{xv}) = \{xy, yv, xv\}$ and $f_{xv}(yv) + f(yv) \not\equiv 0 \pmod 3$. Now $(D, f + f_{xv})$ is a Modular 3-flow of G^2 whose support contains xv, yv, but may miss xy. Repeat this adjustment and do the similar adjustment for the edges yw not in the support until we get a Modular 3-flow (D, f') of G^2 such that $E(G^2) \setminus supp(f') \subseteq E(C(x))$. Since each edge in $C(x)$ is contained in some K_5 and thus is a center edge in some W_4, by Lemma 2.3 and Lemma 2.5-(1), G^2 admits a 3-NZF, a contradiction.

Claim 3. No degree 2 *vertex is contained in a* 3*-circuit.*

By contradiction. Assume $xyzx$ is a circuit of G with $d_G(x) = 2$. If $d_G(y) = 2$, then we must have $d_G(z) = 3$. Therefore $G_1 = G \setminus \{xy\} \notin \bar{\mathcal{T}}_{1,3}$ and $G_1^2 = G^2$, contradicting (ii). So $d_G(y) = d_G(z) = 3$.

Let $N_G(y) = \{x, y', z\}$ and $N_G(z) = \{x, y, z'\}$. Let $G_1 = G - \{x\}$. Since $(N_G(y) \cap N_G(z)) \setminus \{x\} = \emptyset$ (otherwise, let $G_2 = G \setminus \{yz\}$, then $G_2^2 = G^2$, $G_2 \notin \bar{\mathcal{T}}_{1,3}$, contradicting (ii)) and $d_{G_1}(y) = 2$, then $G_1 \notin \bar{\mathcal{T}}_{1,3}$. So G_1^2 admits a 3-NZF. Since $E(G^2) \setminus E(G_1^2) = \{xy, xy', xz, xz'\}$, by Lemma 2.5-(2), G^2 admits a 3-NZF, a contradiction.

Claim 4. No degree 2 *vertex of G is contained in a* 4*-circuit.*

Assume $C = xu_1u_2u_3x$ is a 4-circuit of G and $d_G(x) = 2$. By Claim 3, $u_1u_3 \notin E(G)$. Let u_i' be the adjacent vertex of u_i which is not in $V(C)$ if $d_G(u_i) = 3$ for some $i \in \{1, 2, 3\}$. Let $G_1 = G \setminus \{x\}$. We consider the following 3 cases.

Case 1. $d_G(u_1) = d_G(u_3) = 2$.

Then $d_G(u_2) = 3$ and $d_G(u_2') \geq 2$ (if $d_G(u_2') = 1$, it's easy to show G^2 admits a 3-NZF). Clearly, u_2u_2' is a cut edge, contradicting Claim 1.

Case 2. Exactly one of u_1, u_3 has degree 3.

Assume $d_G(u_1) = 3$ and $d_G(u_3) = 2$. Since $d_{G_1}(u_1) = 2$, if $d_{G_1}(u_1') = 2$ then u_1' is not contained in a 3-circuit in G (by Claim 3), and so $G_1 \notin \bar{\mathcal{T}}_{1,3}$. By induction, G_1^2 admits a 3-NZF. Since $E(G^2) \setminus E(G_1^2) = \{xu_1', xu_1, xu_2, xu_3\}$ and $G^2[V(C) \cup \{u_1'\}]$ contains a W_4 with x as its center, by Lemma 2.5-(1), G^2 admits a 3-NZF, a contradiction.

Case 3. $d_G(u_1) = d_G(u_3) = 3$.

If $u_1' = u_3'$, then $u_1'u_1u_2u_3$ is a 3-path, otherwise $u_1'u_1u_2u_3u_3'$ is 4-path. In both cases G_1^2 admits a 3-NZF. Since $E(G^2) \setminus E(G_1^2) = \{xu_1', xu_1, xu_2, xu_3, xu_3'\}$ and each edge xu_i or xu_j' is contained in some W_4 in G^2 as a center edge for $1 \leq i \leq 3$ and $j = 1, 3$, by Lemma 2.5-(1), G^2 admits a 3-NZF. a contradiction.

Claim 5. For any $v \in V(G)$, $d_G(v) \neq 2$.

Otherwise, if there exists $v \in V(G)$ such that $d_G(v) = 2$, then by Claim 3-4, v is not contained in any circuits of length 3 or 4. By Lemma 2.7-(1), G cannot be a k-path with $k \geq 2$ or an m-circuit with $m = 3$ or $m \geq 5$. Let us consider the following cases.

Case 1. There exists a path $P_m = v_1v_2 \cdots v_m$ such that $m \geq 3$, $v = v_t$ for some $2 \leq t \leq m-1$, $d_G(v_k) = 2$ for $2 \leq k \leq m-1$ and $d_G(v_1) \neq 2$, $d_G(v_m) \neq 2$.

Clearly, at least one of v_1, v_m has degree 3. If $d_G(v_i) = 3$ for $i = 1$, or m, let $N_G(v_i) \setminus V(P_m) = \{v_i', v_i''\}$. Clearly, $G_1 = G \setminus \{v_2, v_3, \ldots, v_{m-1}\} \notin \bar{\mathcal{T}}_{1,3}$ (because by

Claim 3, degree 2 vertices are not contained in any 3-circuits of G). By Claim 1, G_1 is connected. So G_1^2 admits a 3-NZF (D, f_1). By Lemma 2.7-(1), P_m^2 admits a 3-NZF (D, f_2). Then G^2 admits a 3-flow (D, f) with $supp(f) = supp(f_1) \cup supp(f_2)$. By Claim 3-4, $E(G^2) \setminus supp(f) = \{v_2v_1', v_2v_1'', v_{m-1}v_m', v_{m-1}v_m''\}$, then by Lemma 2.5-(2), G^2 admits a 3-NZF, a contradiction.

Case 2. There exists a m-circuit $C = v_1v_2 \cdots v_mv_1$ with $m \geq 5$, $d_G(v_i) = 2$ for $1 \leq i \leq m-1$, $d_G(v_m) = 3$ and $v = v_t$ for some $1 \leq t \leq m-1$.

Suppose that $v_0 \in N_G(v_m) \setminus V(C)$. By Claim 1, $d_G(v_0) = 1$. So by Lemma 2.7-(3), G^2 admits a 3-NZF, a contradiction.

Claim 6. Let $e = xy \in E(G)$ with $d_G(x) = d_G(y) = 3$. Then e is contained in a circuit of length 3 or 4.

By contradiction. Let G_1 be the graph obtained from G by deleting the edge e and adding a new vertex y' and a new edge xy'. Since G contains no degree 2 vertices and $d_{G_1}(y) = 2$, then $G_1 \notin \bar{\mathcal{T}}_{1,3}$. By Claim 1, e is not a cut edge of G, then by (i), G_1^2 admits a 3-NZF (D, f_1). Identify y and y', the resulting 3-flow (D, f_2) in G^2 misses only two edges y_1x and y_2x where $N(y) = \{y_1, y_2, x\}$ (since xy is not contained a circuit of length 3 or 4). By Lemma 2.5-(2), G^2 admits a 3-NZF, a contradiction.

Claim 7. For each $x \in V(G)$ with $d_G(x) = 3$, $|N_G(x) \cap V_3| \leq 2$, where V_3 is the set of all the degree 3 vertices of G.

By contradiction. Assume that $U = \{u_1, u_2, u_3\} = N_G(x) \cap V_3$. Let $G_1 = G \setminus \{x\}$. By Claim 1, G_1 is connected. Since G contains no degree 2 vertices, $G_1 \notin \bar{\mathcal{T}}_{1,3}$ and G_1^2 admits a 3-NZF (D, f). By Claim 6, each xu_i $(1 \leq i \leq 3)$ is contained a circuit of length at most 4. We consider the following 3 cases.

Case 1. $G[U]$ contains at least 2 edges.

Suppose that $u_1u_2, u_2u_3 \in E(G)$. Let $u_i' \in N_G(u_i) \setminus U$ for $i = 1, 3$. If $u_1' = u_3'$, then $G^2[U \cup \{u_1', x\}] \cong K_5$, by Lemma 2.5-(1), we can get a 3-NZF of G^2, a contradiction. If $u_1' \neq u_3'$, then $G[u_1'u_1u_2u_3u_3']$ is a 4-path, by Lemma 2.5-(1) (similar to Case 3 of Claim 4), we can get a 3-NZF of G^2, a contradiction.

Case 2. $G[U]$ contains exactly 1 edge.

Assume that $u_1u_2 \in E(G)$. By Claim 6, each edge xu_i $(i = 1, 2, 3)$ is contained in a circuit of length 3 or 4. So we may assume $z \in (N_G(u_2) \cap N_G(u_3)) \setminus \{x\}$. Clearly, $G^* = G^2[U \cup \{x, z\}] \cong K_5$. Let $u_i' \in N_G(u_i) \setminus (U \cup \{z\})$ for $i = 1, 3$. Clearly, $E(G^2) \setminus supp(f) \subseteq E(G^*) \cup \{xu_1', xu_3'\}$. Since $xu_ju_j'x (j = 1, 3)$ is a circuit of G^2, we can get a 3-flow (D, f_1) such that $E(G^2) \setminus supp(f_1) \subseteq E(G^*)$. By Lemma 2.5-(1), we can get a 3-NZF of G^2, a contradiction.

Case 3. $G[U]$ contains no edges.

Assume that $z_1 \in (N_G(u_1) \cap N_G(u_2)) \setminus \{x\}$ and $z_2 \in (N_G(u_1) \cap N_G(u_3)) \setminus \{x\}$. Let $G_2 = G \setminus \{xu_1\}$, then $G_2 \notin \bar{\mathcal{T}}_{1,3}$ and G_2^2 admits a 3-NZF (D, f_1). Clearly, $E(G^2) \setminus supp(f_1) = \{xu_1\}$. Since xu_1 is contained in a W_4 which is contained in the graph induced by $\{u_1, z_1, u_2, u_3, x\}$ in G^2 with x as center, by Lemma 2.5-(1), we can get a 3-NZF of G^2, a contradiction.

Final Step. By Claim 2, Claim 5 and Claim 7, all vertices of G have degree 1 or 3 and each degree 3 vertex is adjacent to at most 2 degree 3 vertices. So $G[V_3]$ is a path or a circuit, hence G must be a graph obtained from an r-circuit $x_0x_1\cdots x_{r-1}x_0$ by attaching an edge x_iv_i at each x_i for $0 \leq i \leq r-1$, where $v_i \neq v_j$ if $i \neq j$, or a path $x_0x_1\cdots x_p$ by attaching an edge v_ix_i $(1 \leq i \leq p-1)$ at each x_i, where $v_i \neq v_j$ if $i \neq j$. Clearly the latter case is a graph in $\bar{\mathcal{T}}_{1,3}$. By Lemma 2.7-(2), G^2 admits a 3-NZF, a contradiction. ■

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