Another infinite sequence of dense triangle-free graphs

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Abstract

The *core* is the unique homorphically minimal subgraph of a graph. A triangle-free graph with minimum degree $\delta > n/3$ is called *dense.* It was observed by many authors that dense triangle-free graphs share strong structural properties and that the natural way to describe the structure of these graphs is in terms of graph homomorphisms. One infinite sequence of cores of dense maximal triangle-free graphs was known. All graphs in this sequence are 3-colourable. Only two additional cores with chromatic number 4 were known. We show that the additional graphs are the initial terms of a second infinite sequence of cores.

Let G and H be graphs. A *homomorphism* $G \to H$ is a function $\sigma : V(G) \to V(H)$ mapping edges on edges, i.e. $vw \in E(G)$ implies $\sigma(v)\sigma(w) \in E(H)$. Every graph G has a unique minimal subgraph G' with $G \to G'$ which is called the *core* (for an introduction and relevant literature see [6]). Note that the core of G has the same chromatic number as G.

A triangle-free graph is *maximal*, if for every pair of non adjacent vertices u, v the addition of the edge uv creates a triangle. Note that a triangle-free graph of order $n \geq 3$ is maximal, if and only if it has diameter 2. Let u_1, u_2 be two non-adjacent vertices of a maximal triangle-free graph G. Assume that there is a vertex v_1 which is adjacent to u_1 but not to u_2. Since u_2 and v_1 are not adjacent, they must have a common neighbour v_2. Therefore $\sigma(u_1) \neq \sigma(u_2)$ for any homomorphism σ from G to a triangle-free graph. It follows that $\sigma(w_1) = \sigma(w_2)$ implies that the set of neighbours of w_1 is the same as the set of neighbours of w_2. Two vertices sharing the same neighbourhood have been called *twins* (or similar, or symmetric).

The "twin" relation is an equivalence relation on the vertex set of a graph where every equivalence class forms an independent set. The identification of twins in any order eventually leads to the unique maximal twin-free induced subgraph which coincides with the core if the graph is maximal triangle-free. Hence the core of a maximal triangle-free graph G is obtained by successively identifying twins until no twins remain.

The reverse operation to twin identification is *vertex duplication* where at each step a new twin is added to a vertex. The final result depends only on the number of times a vertex is duplicated and not on the order of the duplications. Hence the result is uniquely described if we assign positive integers to the vertices of the original graph, each integer specifying the number of duplicates including the original vertex. In case of a core, the numbers represent the cardinalities of the equivalence classes of the "twin" relation.

A triangle-free graph of order n with minimum degree $\delta > n/3$ will be called *dense*. Chen, Jin and Koh [4] characterized the dense maximal triangle-free 3-colourable graphs in terms of their cores (see Brandt [2] for a much simpler proof).

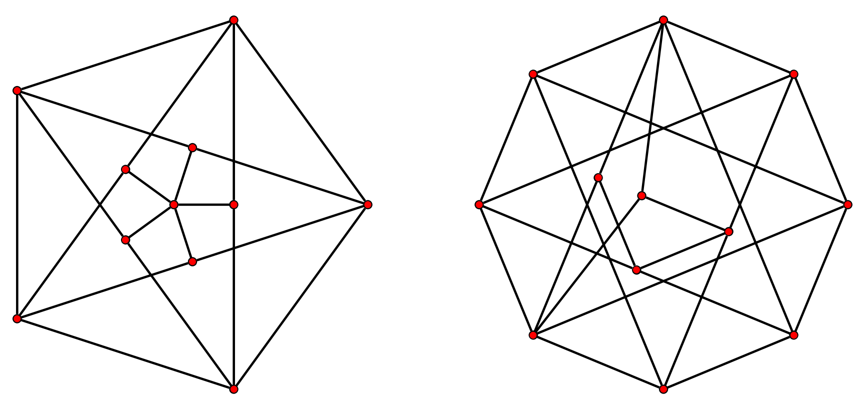

Figure 1: The Grötzsch graph Υ_{11} and the Jin graph Υ_{12}.

Only two 4-chromatic cores of dense maximal triangle-free were known, one being the Grötzsch graph, detected by Häggkvist [5], and the other being the graph, which was found by Jin [8]; see Figure 1.

The objective of this note is to show that these two graphs are the initial terms of an infinite sequence of 4-chromatic cores of dense maximal triangle-free graphs. In fact, we show that for every $p \geq 11$ there exists a 4-chromatic graph Υ_p on p vertices being the core of a dense triangle-free graph.

Let us start with describing an infinite sequence Γ_k, of 3-colourable dense maximal triangle-free cores. Later these graphs will appear as building blocks in the new sequence Υ_p.

For $k \geq 1$ let Γ_k denote the Cayley graph over $\mathbf{Z}_{3k-1}$ with respect to the set of generators $\{i : k \leq i \leq 2k-1\}$. These graphs are complements of cycle powers, defined as follows. Let $\Gamma_1 = K_2$ and, for $k \geq 2$, $\Gamma_k = \overline{C_{3k-1}^{k-1}}$, i.e. Γ_k is the complement of the $(k-1)$st power of the $3k-1$ cycle. As usual, the *kth power* G^k of a graph G is the graph on $V(G)$, where v and w are adjacent if and only if their distance in G is at most k. Clearly, $\Gamma_2 = C_5$, the 5-cycle and Γ_3 is the Möbius ladder on 8 vertices. The graph Γ_k can also be described as the Cayley graph over $\mathbf{Z}_{3k-1}$ with respect to the set of generators $\{3i-2 : 1 \leq i \leq k\}$. We will make use of this representation, by assuming that Γ_k has vertex set $\{w_0, \ldots, w_{3k-2}\}$ where $w_i w_j \in E(\Gamma_k)$ iff $|i-j| \equiv 1 \pmod 3$.

This sequence of graphs was probably first discovered by Andrásfai and Erdős (see [1]) in 1962 and has been rediscovered several times throughout the years. In 1981, Pach [9] observed that the triangle-free graphs where every independent set of vertices is contained in the neighbourhood of a vertex are precisely those which can be obtained from a graph Γ_k by consecutively duplicating vertices. Pach's result was recently rediscovered by Brouwer [3], giving a much shorter proof. In [2], the first author observed that these graphs can be alternatively characterized as those maximal triangle-free graphs which do not contain an induced 6-cycle.

Note that the core is invariant under vertex duplications, since the vertex and its twin can be identified by a homomorphism. Generalizing a result of Jin [7], Chen, Jin and Koh [4] proved that the core of every 3-colourable dense maximal triangle-free graph is a graph Γ_k. So the question arose whether it is possible, to characterize the cores of dense triangle-free graphs with chromatic number at least 4 as well. It was proven by Chen, Jin and Koh [4] that every such graph contains the Grötzsch graph as an induced subgraph (see also [2] for a simple proof of these results). Using the computer program Vega [10], we computed further graphs which are cores of dense maximal triangle-free 4-chromatic graphs, and we observed that they belong to an infinite sequence of such graphs, which we will call, due to their origin, *Vega graphs.*

For every $p \geq 11$ there exists a Vega graph Υ_p with p vertices which is obtained in the following way:

For $k \geq 4$ and $p = 3k+1$ take a 6-cycle $C = (v_1, v_2, \ldots, v_6)$, an edge $u_1 u_2$ and a copy of Γ_{k-2} with vertex set $\{w_1, w_2, \ldots, w_{3k-7}\}$. Join u_1 to v_1, v_3, v_5, u_2 to v_2, v_4, v_6 and w_{3i-j} to v_{3-j} and v_{6-j} for $1 \leq i \leq k-2$ and $0 \leq j \leq 2$. The resulting graph is Υ_{3k+1}; see Figure 2 for the case $k = 5$.

In order to obtain Υ_{3k} delete the vertex w_1 from Υ_{3k+1}, and to obtain Υ_{3k-1} delete the vertex u_1 from Υ_{3k}. Now we can state our main result:

Theorem 1 *For every $p \geq 11$ there is a triangle-free graph with chromatic number 4 of order $p = 3k+\ell$, $-1 \leq \ell \leq 1$, which is the core of a triangle-free $(9k-25+\ell)$-regular graph on $n = 27k-76+3\ell$ vertices.*

Proof. We claim that the Vega graphs Υ_p, $p \geq 11$, have the indicated property. First we show that Υ_p is its own core, which follows from the fact that Υ_p is maximal

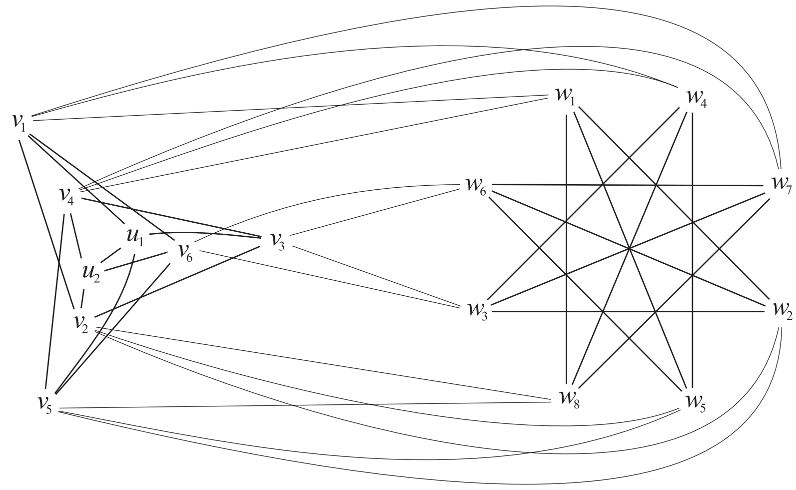

Figure 2: Υ_{16} is composed of two parts. The graph on 8 vertices on the left to which Γ_8 is attached on the right.

triangle-free and the neighbourhood of no vertex is contained in the neighbourhood of another vertex. This can be derived from the fact that the graphs Γ_k do have this property, combined with a simple case analysis. In particular, Υ_p is the core of every graph G which is homomorphic with Υ_p and which contains Υ_p as a subgraph.

It is left to show that for every $p \geq 11$ there is a regular dense maximal triangle-free graph whose core is Υ_p. We do this by duplicating vertices according to an appropriate weight assignment.

Let us start assigning the weights to Υ_{3k+1} and then modify the weights for the cases $p = 3k$ and $p = 3k-1$. For $p = 3k+1$ assign weight 1 to each vertex u_i and to the vertices w_1 and w_{3k-7}, weight 3 to the vertices w_i for $2 \leq i \leq 3k-8$, weight $3k-9$ to v_3 and v_6, and weight $3k-8$ to v_1, v_2, v_4, v_5. The result (by duplicating the vertices the right number of times) is a $(9k-24)$-regular graph of order $27k-73$. For $p = 3k$ (where the vertex w_1 is deleted), leave the labels unchanged except for w_2 and w_{3k-7} which both get weight 2, and v_1 and v_4 which both get weight $3k-9$. The result is a $(9k-25)$-regular graph of order $27k-76$. Finally, the case $p = 3k-1$. Here u_1 is additionally deleted. We leave the labels unchanged, except for u_2 which gets label 2, v_1 and v_3, which get label $3k-10$ and v_5, which gets label $3k-9$. The result is a $(9k-26)$-regular graph of order $27k-79$. $\square$

Finally, we would like to know whether there are further cores of dense maximal triangle-free graphs. If not then the answer to the following problem was affirmative (note that every graph Γ_k is a subgraph of Υ_{3k+7}).

Problem 1 *Is every triangle-free graph with minimum degree $\delta > n/3$ homomorphic with a graph Υ_p?*

As a consequence this would imply that every triangle-free graph with minimum degree $\delta > n/3$ is 4-colourable, in contrast to a conjecture of Jin [8], saying that there are graphs of arbitrarily large chromatic number with this property.

References

[1] B. Andrásfai, Über ein Extremalproblem der Graphentheorie, *Acta Math. Acad. Sci. Hungar.* **13** (1962), 443–455.

[2] S. Brandt, On the structure of dense triangle-free graphs, Preprintreihe Mathematik, No. A 97–16, 1997, to appear in Comb., Prob. and Comp.

[3] A. Brouwer, Finite graphs in which the point neighbourhoods are the maximal independent sets, in: From universal morphisms to megabytes: a Baayen space odyssey (K. Apt, ed.), CWI Amsterdam, 1995, pp. 231–233.

[4] C. C. Chen, G. Jin and K. M. Koh, Triangle-free graphs with large degree, *Comb., Prob. and Comp.* **6** (1997), 381–396.

[5] R. Häggkvist, Odd cycles of specified length in non-bipartite graphs, *Ann. Discr. Math.* **13** (1982), 89–99.

[6] P. Hell and J. Nešetřil, The core of a graph, *Discrete Math.* **109** (1992), 117–126.

[7] G. Jin, Triangle-free graphs with high minimal degrees, *Comb. Prob. and Comp.* **2**, (1993), 479–490.

[8] G. Jin, Triangle-free four-chromatic graphs, *Discrete Math.* **145** (1995), 151–170.

[9] J. Pach, Graphs whose every independent set has a common neighbour, *Discrete Math.* **37** (1981), 217–228.

[10] T. Pisanski (ed.), Vega Version 0.2; Quick Reference Manual and Vega Graph Gallery, IMFM, Ljubljana, 1995. `http://vega.ijp.si/`