Small Ramsey Numbers

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ABSTRACT: We present data which, to the best of our knowledge, includes all known nontrivial values and bounds for specific graph, hypergraph and multicolor Ramsey numbers, where the avoided graphs are complete or complete without one edge. Many results pertaining to other more studied cases are also presented. We give references to all cited bounds and values, as well as to previous similar compilations. We do not attempt complete coverage of asymptotic behavior of Ramsey numbers, but concentrate on their specific values.

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1. Scope and Notation

There is a vast literature on Ramsey type problems starting in 1930 with the original paper of Ramsey [Ram]. Graham, Rothschild and Spencer in their book [GRS] present an exciting development of Ramsey Theory. The subject has grown amazingly, in particular with regard to asymptotic bounds for various types of Ramsey numbers (see the survey paper [GrRö]), but the progress on evaluating the basic numbers themselves has been very unsatisfactory for a long time. In the last decade, however, considerable progress has been obtained in this area, mostly by employing computer algorithms. The few known exact values and several bounds for different numbers are scattered among many technical papers. This compilation is a fast source of references for the best results known for specific numbers. It is not supposed to serve as a source of definitions or theorems, but these can be easily accessed via

* - This paper updates and extends a technical report RIT-TR-93-009 by the author [Ra4].

the references gathered here.

Ramsey Theory studies conditions when a combinatorial object contains necessarily some smaller given objects. The role of Ramsey numbers is to quantify some of the general existential theorems in Ramsey Theory.

Let $G_1, G_2, \ldots, G_m$ be graphs or s-uniform hypergraphs (s is the number of vertices in each edge). $R(G_1, G_2, \ldots, G_m; s)$ denotes the m-color *Ramsey number* for s-uniform graphs/hypergraphs, avoiding G_i in color i for $1 \le i \le m$. It is defined as the least integer n such that, in any coloring with m colors of the s-subsets of a set of n elements, for some i the s-subsets of color i contain a sub-(hyper)graph isomorphic to G_i (not necessarily induced). If $s = 2$ (standard graphs) then s can be omitted. If G_i is a complete graph K_k, then we can write k instead of G_i, and if $G_i = G$ for all i we can use the abbreviation $R_m(G)$ (or $R_m(G; s)$). For $s = 2$, $K_k - e$ denotes a K_k without one edge, and for $s = 3$, $K_k - t$ denotes a K_k without one triangle (hyperedge). P_i is a path on i vertices, C_i is a cycle of length i, and W_i is a wheel with $i-1$ spokes, i.e. a graph formed by some vertex x, connected to all vertices of some cycle C_{i-1}. The book graph $B_i = K_2 + \overline{K}_i$ has $i+2$ vertices, and can be seen as i triangular pages attached to a single edge. $K_{n,m}$ is a complete n by m bipartite graph, in particular $K_{1,n}$ is a star graph. For a graph G, $n(G)$ and $e(G)$ denote the number of vertices and edges, respectively, in G.

Section 2 contains the data for the classical two color Ramsey numbers $R(k, l)$ for complete graphs, and section 3 for the two color case when the avoided graphs are complete or have the form $K_k - e$, but not both are complete. Section 4 lists the most studied two color cases for other graphs. The multicolor and hypergraph cases are gathered in sections 5 and 6, respectively. If some new bound has been not yet published, we also give a reference to the best published previous result, in any. Finally, section 7 gives pointers to cumulative data and to some previous surveys, especially those containing data not subsumed by this compilation.

2. Classical Two Color Ramsey Numbers

We split the data into the table of values and a table with corresponding references. Known exact values appear as centered entries, lower bounds as top entries, and upper bounds as bottom entries.

All the critical graphs for the numbers $R(k, l)$ (graphs on $R(k, l) - 1$ vertices without K_k and without K_l in the complement) are known for $k = 3$ and $l = 3$, 4, 5, 6 [Ka2], 7 [RK3, MZ], and there are 1, 3, 1, 7 and 191 of them, respectively. There exists a unique critical graph for $R(4,4)$ [Ka2]. There are 4 such graphs known for $R(3,8)$ [RK2], 1 for $R(3,9)$ [Ka2] and 350904 for $R(4, 5)$ [MR5], but there might be more of them.

k \ l	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 43	46 51	51 60	59 69	66 78	73 89
4		18	25	35 41	49 62	53 85	69 116	80 151	93 191	97 238	112 291	119 349	123 417
5			43 49	58 87	76 143	95 216	317	445					
6				102 165	300	497	784	1180					
7					205 545	1035	1724	2842					
8						282 1874	3597	6116					
9							565 6680	12795					
10								798 23981					

Table I. Known nontrivial values and bounds for two color Ramsey numbers $R(k,l) = R(k,l;2)$.

k \ l	3	4	5	6	7	8	9	10	11	12	13	14	15
3	GG	GG	GG	Ka2	Ka2 GY	GR MZ	Ka2 GR	Ex2 RK2	Ka2 RK2	Ex8 RK2	Piw1 RK2	Ex7 RK2	WW Ra1
4		GG	Ka1 MR5	Ex8 MR4	Ex3 Spe	Ex3 Spe	RK1 Spe	Piw1 Spe	Piw1 Spe	Piw1 Spe	Piw1 Spe	Piw1 Spe	Wang Spe
5			Ex4 MR4	Ex8 Spe/HZ	RK1 Spe	Piw1 Spe	Spe	Spe					
6				Ka1 Mac	Spe/HZ	Spe/HZ	Spe	Spe					
7					Ma/S1 Mac	Spe/HZ	HZ	Spe					
8						BR Mac	Ea1	HZ					
9							Ma/S1 Mac	Ea1					
10								Ma/S1 Mac					

References for Table I.

Chokbowornpaishand and Longani [ChLo] established $R(3,20) \geq 98$. Wang and Wang in [WW] showed that $R(3,16) \geq 79$, and in [WWY] showed $R(3,17) \geq 92$, $R(3,18) \geq 98$,

$R(3,19) \geq 106$, $R(3,20) \geq 109$, $R(3,21) \geq 122$, $R(3,22) \geq 125$ and $R(3,23) \geq 136$. Most of the lower bounds for $R(4,n)$ presented by Bannani in [Ba], and two other results $R(3,13) \geq 58$ [Ka2] and $R(5,8) \geq 94$ [RK1], were improved by Piwakowski [Piw1].

The best published upper bounds for (k,l) = (4, 5), (4, 6), (5, 5) and (5, 6) of 27, 43, 52 and 94 can be found in [MR2, MR3] and [Wa2], respectively. The bound $R(6,6) \leq 166$ is an immediate consequence of theorem 1 in [Wa1] and $R(4,6) \leq 41$, in this case the best published bound of 169 is due to Giraud [Gi3]. Recently, T. Spencer [Spe], Mackey [Mac], and Huang and Zhang [HZ], using the bounds for minimum and maximum number of edges in (4,5) Ramsey graphs found in [MR2, MR4], were able to establish new upper bounds for several higher Ramsey numbers, improving all the previous longstanding results of Giraud [Gi1, Gi3, Gi4]. Spencer and Mackey keep outdoing each other on these bounds, and when the situation stabilizes we will try to report the outcome of this competition. We have recomputed the bounds marked [HZ] using the method from the paper [HZ], because the bounds in the paper relied on an overly optimistic personal communication from Spencer. The upper bounds implied by $R(k,l) \leq R(k-1,l) + R(k,l-1)$, or by its slight improvement with strict inequality when both terms on the right hand side are even, are marked [Ea1].

For a more in depth study of triangle-free graphs in relation to the case of $R(3,k)$, for which considerable progress has been obtained in recent years, see also [AKS, FL, Fra1, Fra2, Gri, Loc, RK3, RK4, S2, Stat, Yu]. Good asymptotic bounds for $R(k,k)$ can be found, for example, in [Chu3] (lower bound) and [Tho] (upper bound), and for many other asymptotic bounds in the general case of $R(k,l)$ consult [GRS, GrRö].

3. Two Colors - Dropping One Edge from Complete Graph

G \ H	K_3-e	K_4-e	K_5-e	K_6-e	K_7-e	K_8-e	K_9-e	$K_{10}-e$
K_3-e	3	5	7	9	11	13	15	17
K_3	5	7	11	17	21	25	31	36-39
K_4-e	5	10	13	17	28			
K_4	7	11	19					
K_5-e	7	13	22					
K_5	9	16	30-34					
K_6-e	9	17						
K_6	11							

Table II. Two types of Ramsey numbers $R(G,H)$, includes all known nontrivial values.

H / G	K_3-e	K_4-e	K_5-e	K_6-e	K_7-e	K_8-e	K_9-e	$K_{10}-e$
K_3-e	Tr	Tr	Tr	Tr	Tr	Tr	Tr	Tr
K_3	Tr	CH2	Clan	FRS1	GH	Ra2	Ra2	Ea3-Ra2
K_4-e	Tr	CH1	FRS2	McR	McR			
K_4	Tr	CH2	EHM1					
K_5-e	Tr	FRS2	CEHMS					
K_5	Tr	BH	Ex7-Ea1					
K_6-e	Tr	McR						
K_6	Tr							

References for Table II.

For the following numbers it was established that the critical graphs are unique: $R(K_3, K_l - e)$ for $l = 3$ [Tr], 6 and 7 [Ra2], $R(K_4-e, K_4-e)$ [FRS2], $R(K_5-e, K_5-e)$ [Ra3] and $R(K_4-e, K_7-e)$ [McR].

4. General Graph Numbers in Two Colors

This section includes data with respect to general graph results. We tried to include all nontrivial values and identities regarding exact results (or references to them), but only those out of general bounds and other results which, in our opinion, have a direct connection to the evaluation of specific numbers. If some small value cannot be found below, it may be covered by the cumulative data gathered in section 7, or be a special case of a general result listed in this section. Note finally that $B_1 = C_3 = W_3 = K_3$, $B_2 = K_4 - e$, $W_4 = K_4$ and $C_4 = K_{2,2}$ imply other identities not mentioned explicitly.

Paths:

$R(P_n, P_m) = n + \lfloor m/2 \rfloor - 1$ for all $n \geq m \geq 2$ [GeGy]

Cycles:

$R(C_3, C_3) = 6$ [GG]

$R(C_4, C_4) = 6$ [CH1]

Result obtained independently in [Ros] and [FS1]

$$R(C_n, C_m) = \begin{cases} 2n-1 & \text{for } 3 \le m \le n,\ m \text{ odd, } (n,m) \ne (3,3) \\ n-1+m/2 & \text{for } 4 \le m \le n,\ m \text{ and } n \text{ even, } (n,m) \ne (4,4) \\ \max\{n-1+m/2, 2m-1\} & \text{for } 4 \le m < n,\ m \text{ even and } n \text{ odd} \end{cases}$$

$R(nC_3, mC_3) = 3n + 2m$ for $n \ge m \ge 1$, $n \ge 2$ [BES]
Unions of cycles [MS]

Wheels:

$R(W_3, W_5) = 11$ [Clan]
$R(W_3, W_n) = 2n-1$ for all $n \ge 6$ [BE2]
All critical colorings for $R(W_3, W_n)$ for all $n \ge 3$ [RJ]
$R(W_4, W_5) = 17$ [He3]
$R(W_5, W_5) = 15$ [HM2, He2]
$R(W_4, W_6) = 19$, $R(W_5, W_6) = 17$ and $R(W_6, W_6) = 17$,
and all critical colorings (2, 1 and 2) for these numbers [FM]

Books:

$R(B_1, B_n) = 2n + 3$ for all $n > 1$ [RS1]
$R(B_3, B_3) = 14$ [RS1, HM2]
$R(B_2, B_5) = 16$, $R(B_3, B_5) = 17$, $R(B_5, B_5) = 21$,
$R(B_4, B_4) = 18$, $R(B_4, B_6) = 22$, $R(B_6, B_6) = 26$,
in general $R(B_n, B_n) = 4n + 2$ for $4n + 1$ a prime power,
and some other general equalities and bounds for $R(B_n, B_m)$ [RS1].

Complete bipartite graphs:

$R(K_{2,2}, K_{1,15}) = 20$ [La2]
$R(K_{2,3}, K_{1,7}) = 13$ [Par4]
$R(K_{2,3}, K_{2,3}) = 10$ [Bu4]
$R(K_{2,3}, K_{2,4}) = 12$ [ER]
$R(K_{2,3}, K_{3,3}) = 13$ and $R(K_{3,3}, K_{3,3}) = 18$ [HM3]
$R(nK_{1,3}, mK_{1,3}) = 4n + m - 1$ for $n \ge m \ge 1$, $n \ge 2$ [BES]
$R(K_{1,n}, K_{1,m}) = n + m - \varepsilon$, where $\varepsilon = 1$ if both n and m are even and $\varepsilon = 0$ otherwise [Ha1]. It is also a special case of multicolor numbers for stars obtained in [BuRo1].
$R(K_{2,n}, K_{2,n}) \le 4n - 2$ for all $n \ge 2$, exact values 6, 10, 14, 18, 21, 26, 30, 33, 38, 42, 46, 50, 54, 57 and 62 of $R(K_{2,n}, K_{2,n})$ for $2 \le n \le 16$, respectively. The first open case is $65 \le R(K_{2,17}, K_{2,17}) \le 66$ [EHM2].

Special cases:

$R(C_5, W_6) = 13$ [CS]
$R(B_3, K_4) = 14$ [He3]
$R(C_4, K_6) \geq 18$ [Ex9]
Exact values 7, 9, 11, 12, 13, 16 and 16 of $R(C_4, B_n)$ for $2 \leq n \leq 8$, respectively [FRS6]
$26 \leq R(K_{2,2,2}, K_{2,2,2})$ (octahedron) [Ex7]
$17 \leq R(W_5, K_5 - e) \leq 18$ [He2] [G.R.T. Hendry, unpublished]. We believe that this number can be computed by relatively simple methods.

General cases (exact results and bounds):

Paths versus stars [Par2, BEFRS2]
Paths versus books [RS2]
Paths versus cycles [FLPS, BEFRS2]
Paths versus K_n [Par1]
Paths versus $K_{n,m}$ [Häg]
Paths and cycles versus trees [FSS]
Cycles versus stars [La1, Clark, see Par5]
Cycles versus books [FRS5]
Cycles versus K_n [BoEr]
Unicyclic graphs [Gro1, Köh]
C_4 versus some stars [Par3]
C_4 versus books [FRS6]
C_4 versus trees [EFRS3, Bu6]
nC_3 versus mK_4 [LorMu]
$K_{2,n}$ versus some stars [Par4]
$R(nK_4, nK_4) = 7n + 4$ for large n [Bu7]
Stars versus trees [Bu1]
Stars versus stripes, stripes [CL, Lor]
Stars versus books [CRSPS, RS2]
Stars versus $K_{n,m}$ [Stev, Par3]
Stars versus $K_n - tK_2$ [Hua]
Union of two stars [Gro2]
Double stars* [GHK]
Trees [EG, GRS, FSS]
Trees versus K_n [Chv]
Trees versus $K_n + \overline{K}_m$ [RS2, FSR]
Trees versus bipartite graphs [EFRS5]
Trees versus almost complete graphs [GJ2]

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* - A double star is a union of two stars with their centers joined by an edge.

Trees versus small ($n(G) \le 5$) connected G [FRS4]
Linear forests [BuRo2, FS3]
Forests versus K_n [Stahl]
Forests versus almost complete graphs [CGP]
Sparse graphs versus paths and cycles [BEFRS2]
Multipartite complete graphs [BEFRS3, EFRS3, FRS3, Stev]
Multipartite complete graphs versus trees [EFRS6, BEFRSGJ]
Disconnected graphs versus any graph [GJ1]
Graphs with long tails [Bu2, BG]
Brooms[+] [EFRS2]

General results:

[Wa1] $R(k,k) \le 4R(k,k-2)+2$.

[CCD] $R(3,4k+1) \ge 6R(3,k+1)-5$ for all $k \ge 1$.

[Chv] $R(K_n, T_m) = (n-1)(m-1)+1$ for any tree T on m vertices.

[CH2] $R(G,H) \ge (\chi(G)-1)(c(H)-1)+1$, where $\chi(G)$ is the chromatic number of G, and $c(H)$ is the size of the largest connected component of H.

[BE1] $R(G,G) \ge \lfloor (4n(G)-1)/3 \rfloor$ for any connected G.

[BE2] Graphs yielding $R(K_n, G) = (n-1)(n(G)-1)+1$ and related results (see also [EFRS4]).

[BES] Study of Ramsey numbers for multiple copies of graphs (see also [Bu1, LorMu]).

[Zeng] $R(nK_3, nG)$ for all isolate-free graphs G on 4 vertices.

[Bu7, Bu8] Study of Ramsey numbers for large disjoint unions of graphs, in particular $R(nK_k, nK_l) = n(k+l-1)+R(K_{k-1}, K_{l-1})-2$ for n large enough.

[BEFRS1] $R(K_3, G) = 2n(G)-1$ for any connected G on at least 4 vertices and with at most $(17n(G)+1)/15$ edges. In particular it holds for $G = P_i$ and $G = C_i$, for all $i \ge 4$.

[BEFRS4] Graphs H yielding $R(G,H) = (\chi(G)-1)(n(H)-1)+s(G)$, where $s(G)$ is a chromatic surplus of G, defined as the minimum number of vertices in some color class under all vertex colorings in $\chi(G)$ colors (such H's are called G-good). This idea, initiated in [Bu2], is a basis of a number of exact results for $R(G,H)$ for large and sparse graphs H [BG, BEFRS2, Bu5, FS, EFRS3, FRS3, BEFSRGJ]. A survey of this area appeared in [FRS7].

+ - A broom is a star with a path attached to its center.

[BEFS]	Bounds for the difference between consecutive Ramsey numbers.
[Par3, Par4]	Relations between some Ramsey graphs and block designs.
[CSRT]	$R(G,G) \le c_d n(G)$ for all G, where constant c_d depends only on the maximum degree d in G.
[GS]	$R(G,G) \le c_d n$ for all d-arrangeable graphs G on n vertices.
[Al]	$R(G,G) \le 12n$ for all n-vertex graphs G, in which no two vertices of degree at least 3 are adjacent.
[Sid3]	$R(G,K_3) \le 2e(G)+1$ for any graph G without isolated vertices.
[Sid2]	$R(G,K_3) \le n(G)+e(G)$ for all G, a conjecture.
[FSS]	Discussion of the conjecture $R(T_1,T_2) \le n(T_1)+n(T_1)-2$ for any trees T_1 and T_2.
[FM]	Counterexample $G = W_6$ to the Erdös conjecture (see [GRS]) $R(G,G) \ge R(K_{\chi(G)}, K_{\chi(G)})$.
[-]	Special cases of multicolor results listed in section 5.
[-]	See also surveys listed in section 7.

5. Multicolor Graph Numbers

The only known value of a multicolor classical Ramsey number:

$R(3,3,3) = R(3,3,3;2) = 17$	[GG]
2 critical colorings	[KS, LayMa]

Bounds for multicolor classical numbers:

$51 \le R(3,3,3,3) \le 65$	[Chu1] [Fo]
$159 \le R(3,3,3,3,3) \le 322$	[Fre] [Wh, Ea1]
$128 \le R(4,4,4) \le 242$	[HI] [Ea2]
$30 \le R(3,3,4) \le 32$	[Ka2] [Piw2]
$45 \le R(3,3,5) \le 59$	[Ex9, KLR] [Ea2]
$55 \le R(3,4,4) \le 81$	[KLR] [Ea2]
$84 \le R(3,3,3,4) \le 159$	[Ex9] [Ea2]

Multicolor general graphs:

$R(C_4, C_4, C_4) = 11$	[BS, see also Clap]
$R(C_4, C_4, K_3) = 12$	[Schu]
$R(C_4, K_3, K_3) = 17$	[ER]
$R(C_5, C_5, C_5) = 17$	[YR1]
$R(C_6, C_6, C_6) = 12$	[YR3]
$R(K_4 - e, K_4 - e, P_3) = 11$	[Ex6]
$R(C_4, C_4, C_4, T) = 16$ for $T = P_4$ and $T = K_{1,3}$	[ER]
$28 \le R(K_4 - e, K_4 - e, K_4 - e) \le 32$	[Ex6] [Ex6]

General multicolor results:

- General bounds for $R_k(G)$ [CH3].
- Bounds for $R_k(3)$ [Fre, Chu1, Chu2, ChGri, GrRö].
- Formulas for $R_k(G)$ for G being P_3, $2K_2$ and $K_{1,3}$ for all k, and for P_4 if k is not divisible by 3 [Ir]. Wallis [Wal] showed $R_6(P_4) = 13$, which already implied $R_{3t}(P_4) = 6t + 1$, for all $t \ge 2$. Independently, the case $R_k(P_4)$ for $k \ne 3^m$ was completed by Lindström in [Lin], and later Bierbrauer proved $R_{3^m}(P_4) = 2 \cdot 3^m + 1$ for all $m \ge 1$.
- $R_k(C_4) \le k^2 + k + 1$ for all $k \ge 1$, and $R_k(C_4) \ge k^2 - k + 2$ for all $k - 1$ a prime power [Ir, Chu2, ChGra]. For small k some improvements on the latter are known: $R_3(C_4) = 11$ [BS], $R_4(C_4) \ge 18$ and $R_5(C_4) \ge 25$ [Ex9].
- Bounds for the bipartite graphs $R_k(K_{s,t})$, in particular for $K_{2,2} = C_4$ [ChGra].
- Formulas for $R(C_n, C_m, C_k)$ and $R(C_n, C_m, C_k, C_l)$ for n sufficiently large [EFRS1].
- Formulas for $R(P_{n_1}, \cdots, P_{n_k})$, except few cases [FS2].
- Monotone paths and cycles [Lef].
- Formulas for $R(S_1, \cdots, S_k)$, where S_i's are arbitrary stars [BuRo1].
- Formulas for $R(S_1, \cdots, S_k, K_n)$, where S_i's are arbitrary stars [Jac].
- Formulas for $R(S_1, \cdots, S_k, nP_2)$, where S_i's are arbitrary stars [CL2].
- Formulas for $R(pP_3, qP_3, rP_3)$ and $R(pP_4, qP_4, rP_4)$ [Scob].
- Cockayne and Lorimer [CL1] found the exact formula for $R(n_1P_2, \cdots, n_kP_2)$, and later Lorimer [Lor] extended it to a more general case of $R(K_m, n_1P_2, \cdots, n_kP_2)$. Still more general cases of the latter, with multiple copies of the complete graph and forests, were studied in [Stahl, LorSe, LorSo].

- If G is connected and $R(K_k, G) = (k-1)(n(G)-1)+1$, in particular if G is any tree, then $R(K_{n_1}, \cdots, K_{n_k}, G) = (R(K_{n_1}, \cdots, K_{n_k})-1)(n(G)-1)+1$ [BE2].
- Study of $R(S, G_1, \cdots, G_k)$ for large sparse S [EFRS1, Bu3].
- Bounds for trees $R_k(T)$ and forests $R_k(F)$ [EG, GRS, BB, GT].
- See also surveys listed in section 7.

6. Hypergraph Numbers

The only known value of a classical Ramsey number for hypergraphs:

$R(4,4;3) = 13$	[MR1]
more than 200000 critical colorings	

Other hypergraph cases:

$R(K_4-t, K_4-t; 3) = 7$	[Ea4]
$R(K_4-t, K_4; 3) = 8$	[So, Ex1, MR1]
$14 \le R(K_4-t, K_5; 3)$	[Ex1]
$63 \le R(5,5;3)$	[Ea1]
$32 \le R(4,5;3)$	[Ex7]
$33 \le R(5,5;4)$	[Ex10]
$56 \le R(4,4,4;3)$	[Ex7]
$13 \le R(K_4-t, K_4-t, K_4-t; 3) \le 17$	[Ex1] [Ea1]

The computer evaluation of $R(4,4;3)$ in [MR1] consisted of an improvement of the upper bound from 15 to 13, which followed an extensive theoretical study of this number in [Gi2, Is1, Sid1]. Exoo in [Ex1] announced the bounds $R(4,5;3) \ge 30$ and $R(5,5;4) \ge 27$ without presenting the constructions. The best published bound of $R(4,5;3) \ge 24$ was obtained by Isbell [Is2]. Shastri in [Sha] shows a weak bound $R(5,5;4) \ge 19$, nevertheless his lemmas and those in [Ka3, Abb, GRS] can be used to derive other lower bounds for higher numbers. Theoretical results on hypergraph numbers are gathered in [GrRö, GRS].

7. Cumulative Data and Surveys

Cumulative data for two colors:

[CH1] $R(G,G)$ for all graphs G without isolates on at most 4 vertices.

[CH2] $R(G,H)$ for all graphs G and H without isolates on at most 4 vertices.

[Clan] $R(G,H)$ for all graphs G on at most 4 vertices and H on 5 vertices, except five entries.

[He4] All critical colorings for $R(G,H)$, for isolate-free graphs G and H as in [Clan] above.

[Bu4] $R(G,G)$ for all graphs G without isolates and with at most 6 edges.

[He1] $R(G,G)$ for all graphs G without isolates and with at most 7 edges.

[HM2] $R(G,G)$ for all graphs G on 5 vertices and with 7 or 8 edges.

[He2] $R(G,H)$ for all graphs G and H on 5 vertices without isolates, except 7 entries.

[FRS1] $R(K_3,G)$ for all connected graphs G on 6 vertices.

[JR] $R(K_3,G)$ for all connected graphs G on 7 vertices.

[FRS4] $R(G,T)$ for all connected graphs G on at most 5 vertices and all (except some cases) trees T.

Cumulative data for three colors:

[YR2] $R_3(G)$ for all graphs G with at most 4 edges and no isolates.

[YR1] $R_3(G)$ for all graphs G with 5 edges and no isolates, except K_4-e.

[AKM] $R(F,G,H)$ for most triples of isolate-free graphs with at most 4 vertices.

Chvátal and Harary [CH1, CH2] formulated several simple but very useful observations how to discover values of some numbers. All five missing entries in the tables of Clancy [Clan] have been solved. The open entries of [He2] still stand, but may yield soon. The case of $R_3(K_4-e)$ is still also open.

Surveys:

[Bu1] A general survey of results in Ramsey graph theory by S.A. Burr (1974)

[Par5] A general survey of results in Ramsey graph theory by T.D. Parsons (1978)

[Ha2] Summary of progress by Frank Harary (1981)

[ChGri] A general survey of bounds and values by F.R.K. Chung and C.M. Grinstead (1983)

[JGT] Special volume of *the Journal of Graph Theory* (1983)

[Rob] Nice textbook-type review of Ramsey graph theory for newcomers (1984)

[Bu6] What can we hope to accomplish in generalized Ramsey Theory ? (1987)

[GrRö] Survey of asymptotic problems by R.L. Graham and V. Rödl (1987)

[GRS] An excellent book by R.L. Graham, B.L. Rothschild and J.H. Spencer, second edition (1990)

[FRS7] Survey of goodness results of the type $R(G,H) = (\chi(G)-1)(n(H)-1)+s(G)$ (1991)

[Neš] A chapter in *Handbook of Combinatorics*, to appear (1994)

The surveys by S.A. Burr [Bu1] and T.D. Parsons [Par5] contain extensive chapters on general exact results in graph Ramsey theory. F. Harary presented the state of the theory in 1981 in [Ha2], where he also gathered many references including seven to other survey papers. A decade ago, Chung and Grinstead in their survey paper [ChGri] gave less data than in this note, but included a broad discussion of different methods used in Ramsey computations in the classical case. S.A. Burr, one of the most experienced researchers in Ramsey graph theory, formulated in [Bu6] seven conjectures on Ramsey numbers for sufficiently large and sparse graphs, and reviewed the evidence for them found in the literature.

For newer extensive presentations see [GRS, GrRö, FRS7, Neš], though these focus on asymptotic theory not on the numbers themselves. Finally, this compilation could not pretend to be complete without mentioning a special volume of the Journal of Graph Theory [JGT] dedicated entirely to Ramsey theory. Besides a number of research papers, it includes historical notes and presents to us Frank P. Ramsey (1903-1930) as a person.

8. Concluding Remarks

This compilation does not include information on numerous variations of Ramsey numbers, nor related topics, like size Ramsey numbers, zero-sum Ramsey numbers, irredundant Ramsey numbers, local Ramsey numbers, connected Ramsey numbers, chromatic Ramsey numbers, avoiding sets of graphs in some colors, coloring graphs other than complete, or the so called Ramsey multiplicities. Interested reader can find such information in the surveys listed in section 7 here.

The author apologizes for any omissions or other errors in reporting results belonging to the scope of this work. Suggestions for any kind of corrections and/or additions will be greatly appreciated.

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We mark the papers containing results obtained with the help of computer algorithms with stars. We identify two categories of such papers: marked with * involving some use of computers, where the results are easily verifiable with some computations, and those marked with **, where cpu intensive algorithms have to be implemented to replicate or verify the results. The first category contains mostly constructions done by algorithms, while the second mostly nonexistence results or claims of complete enumerations of special kinds of graphs.

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31, *Journal of Graph Theory* 27, *Discrete Mathematics* 26, *Ars Combinatoria* 12, *Utilitas Mathematica* 10, *Journal of Combinatorial Mathematics and Combinatorial Computing* 7, 47 articles in other 25 journals, and 49 references from other sources. The results of 43 references depend on computer algorithms. There were 8 years with exceptional activity in this area, namely 1973-75, 1983, 1987-89 and 1991, where more than ten references correspond to each of these years.