Dynamic Cage Survey

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Submitted: May 22, 2008; Accepted: Sep 15, 2008; Published: Sep 29, 2008
Mathematics Subject Classifications: 05C35,05C25

Abstract

A (k, g)-cage is a k-regular graph of girth g of minimum order. In this survey, we present the results of over 50 years of searches for cages. We present the important theorems, list all the known cages, compile tables of current record holders, and describe in some detail most of the relevant constructions.

Contents

1 Introduction

The cage problem asks for the construction of regular simple graphs with specified degree and girth and minimum order. This problem was first considered by Tutte [87]. A variation of the problem in which the graphs were required to be Hamiltonian was later studied by Kárteszi [58]. At about the same time, the study of Moore graphs, first proposed by E. F. Moore, was developed by A. J. Hoffman and R. R. Singleton [51].

Their study begins with the observation that a regular graph of degree k and diameter d has at most

$$1 + k + k(k-1) + \cdots + k(k-1)^{d-1} \tag{1}$$

vertices, and graphs that achieve this bound must have girth $g = 2d + 1$.

One can turn this around and make a similar observation regarding the order, n, of a regular graph with degree k and girth g. Such a graph is called a (k, g)*-graph*. The precise form of the bound on the order n of a (k, g)-graph depends on the parity of g:

$$n \geq \begin{cases} 1 + k + k(k-1) + ... + k(k-1)^{(g-3)/2}, & g \text{ odd} \\ 2(1 + (k-1) + ... + (k-1)^{(g-2)/2}), & g \text{ even} \end{cases} \tag{2}$$

which is obtained by considering the vertices whose distance from a given vertex (edge) is at most $\lfloor (g-1)/2 \rfloor$.

The bound implied by (2) is called the *Moore bound*, and is denoted by $M(k, g)$. Graphs for which equality holds are called *Moore graphs*. Moore graphs are relatively rare.

Theorem 1 ([10, 30]) *There exists a Moore graph of degree k and girth g if and only if*

(i) $k = 2$ and $g \geq 3$, cycles;

(ii) $g = 3$ and $k \geq 2$, complete graphs;

(iii) $g = 4$ and $k \geq 2$, complete bipartite graphs;

(iv) $g = 5$ and:

$k = 2$, the 5-cycle,

$k = 3$, the Petersen graph,

$k = 7$, the Hoffman-Singleton graph,

and possibly $k = 57$;

(v) $g = 6, 8,$ or 12, and there exists a symmetric generalized n-gon of order $k - 1$ (see 2.2).

Regarding *(v)*, it should be noted that the only known symmetric generalized n-gon have prime power order.

The problem of the existence of Moore graphs is closely related to the *degree/diameter problem* surveyed in [71] (which also contains a further discussion of the history of the above theorem).

As Moore graphs do not exist for all parameters, one is naturally led to consider the more general problem of determining the minimum order of (k, g)-graphs. We denote this minimum value by $n(k, g)$ and refer to a graph that achieves this minimum as a (k, g)*-cage*.

In cases where the order of the (k, g)-cage is not known, we denote the order of the smallest known k-regular graph of girth g by $rec(k, g)$ (the current record holder).

The existence of a (k, g)-cage for any pair of parameters (k, g) is not immediately obvious, and it was first shown by Sachs [81]. Almost immediately thereafter, Sachs' upper bound was improved by Erdős, who used a non-constructive induction method [33] (see also Appendix C) to prove the following theorem.

Theorem 2 ([33]) *For every $k \geq 2$, $g \geq 3$,*

$$n(k, g) \leq 4 \sum_{t=1}^{g-2} (k-1)^t$$

A constructive proof of the existence of (k, g)-cages can also be found in Biggs [14]. In Section A2 we prove the following generalization of his result.

Theorem 3 ([14]) *For every $k \geq 3$, $g \geq 3$, there is k-regular graph G whose girth is at least g.*

The existence of k-regular graphs of girth precisely g, for any $k, g \geq 3$, was first proved by Sachs [81].

Theorem 4 *Let G have the minimum number of vertices for a k-regular graph with girth at least g. Then the girth of G is exactly g.*

This implies that $n(k, g)$ increases monotonically with g. In this form, the result was also proved much later in [41].

Another important early result is the non-constructive *Sauer bound* [82]:

Theorem 5 ([82]) *For every $k \geq 2$, $g \geq 3$,*

$$n(k, g) \leq \begin{cases} 2(k-2)^{g-2}, & g \text{ odd}, \\ 4(k-1)^{g-3}, & g \text{ even}. \end{cases}$$

To avoid trivialities, henceforth we will assume that the degree is at least 3 and the girth is at least 5.

2 Known Cages

Recall that unless there exists a Moore graph, we know that $n(k, g)$ is strictly greater than the Moore bound. Thus, in order to prove that a specific graph is a (k, g)-cage, the non-existence of a smaller (k, g)-graph has to be established. These lower bound proofs are in general very difficult, and consequently, in addition to the Moore graphs, very few cages are known.

In this section, we describe all the known cages. These include three infinite families of geometric graphs, and a finite number of small examples. The latter group includes cages of degree 3 for girths up to 12, cages of girth 5 for degrees up to 7, the $(7, 6)$-cage, and the $(4, 7)$-cage.

2.1 Small Examples

The case of $k = 3$ has received the most attention, and the value of $n(3, g)$ is known for all g up to 12. These values are given in Table 1.

The $(3, 5)$, $(3, 6)$, $(3, 8)$, and $(3, 12)$-cages are Moore graphs. Showing that the remaining cases in the table are indeed cages requires additional arguments:

> There is no Moore graph of girth 7, where the Moore bound is 22, so the lower bound of $n(3, 7) \geq 24$ follows immediately. The proof for girth 10 was computer assisted [74], while the proofs for girth 9 in [23] and girth 11 in [68] involved extensive computer searches.

g	5	6	7	8	9	10	11	12
$n(3, g)$	10	14	24	30	58	70	112	126
number of cages	1	1	1	1	18	3	1	1

Table 1. Known trivalent cages.

The cages for girth five are known for degrees up to 7 and are listed in Table 2. The $(3, 5)$-cage, the Petersen graph, and the $(7, 5)$-cage, the Hoffman-Singleton graph, are Moore graphs. The remaining cases were resolved by a combination of counting arguments and case analysis [79, 72, 91].

k	3	4	5	6	7
$n(k, 5)$	10	19	30	40	50
number of cages	1	1	4	1	1

Table 2. Known Cages of Girth 5.

The case of the $(7, 6)$-cage was settled in [73], and the value of $n(4, 7)$ was recently determined in [37].

Next, we provide brief descriptions of the small cages.

2.1.1 (3,5)-Cage: Petersen Graph

The Petersen graph [77] is the $(3,5)$-cage and has order 10. It can be constructed as the complement of the line graph of K_5, from which it follows that the automorphism group is isomorphic to $Sym(5)$. It is vertex-transitive, edge-transitive, 3-connected, neither planar nor Hamiltonian, and is the subject of an entire book [52]. It is shown in Figure 1.

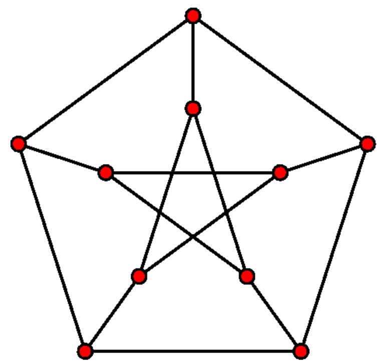

Figure 1: The Petersen Graph

2.1.2 (3,6)-Cage: Heawood Graph

The Heawood graph is the $(3,6)$-cage and has order 14. It is the point-line incidence graph of the projective plane of order 2 (see 2.2.1). It is vertex-transitive, edge-transitive, and the full automorphism group has order 336 (and is isomorphic to $PGL(2,7)$). The usual drawing is shown in Figure 2.

2.1.3 (3,7)-Cage: McGee Graph

The McGee graph is the $(3,7)$-cage and has order 24. It is the first trivalent cage that is not a Moore graph. Its order exceeds the Moore bound by two. It is also the smallest of

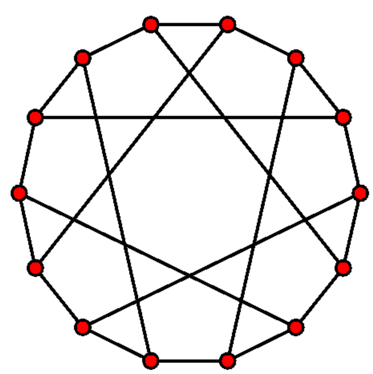

Figure 2: The Heawood Graph

the trivalent cages that is not vertex-transitive. There are two vertex orbits of lengths 8 and 16. The full automorphism group has order 32. The standard drawing is shown in Figure 3. In the figure, the vertices colored red are in one orbit and the vertices in green in the other.

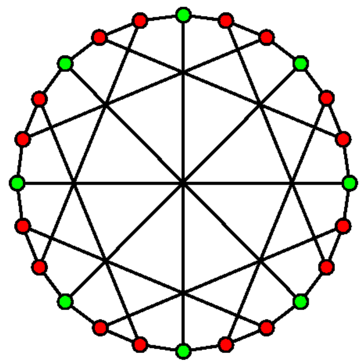

Figure 3: The McGee Graph

2.1.4 (3,8)-Cage: Tutte-Coxeter Graph

The Tutte-Coxeter graph (sometimes called Tutte's cage) is the $(3,8)$-cage and has order 30. It is the point-line incidence graph of the generalized quadrangle of order 2 (see 2.2.2). It is vertex-transitive and 4-arc transitive. The full automorphism group has order 1440. The graph is shown in Figure 4.

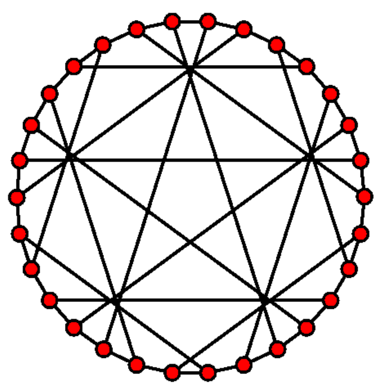

Figure 4: The Tutte-Coxeter Graph

2.1.5 (3,9)-Cages

There are 18 different $(3,9)$-cages, each of order 58. The first of these graphs was discovered by Biggs and Hoare [18]. The list of 18 graphs was shown to be complete by Brinkmann, McKay and Saager in 1995 [23]. The orders of the automorphism groups range from 1 to 24.

2.1.6 (3,10)-Cages

There are three $(3, 10)$-cages of order 70. The first of these was discovered by Balaban [8]. The complete set and the proof of minimality was given by O'Keefe and Wong [74]. None of them are vertex-transitive. The orders of the automorphism groups are 24, 80, and 120.

2.1.7 (3,11)-Cage: Balaban Graph

A $(3, 11)$-graph on 112 vertices was first constructed by Balaban [7] in 1973. It can be obtained from the $(3, 12)$-cage by excision (see B4). The graph was shown to be the unique cage by McKay, Myrvold and Nadon [68]. It is not vertex-transitive, and its automorphism group has order 64.

2.1.8 (3,12)-Cage: Benson Graph

A $(3, 12)$-graph on 126 vertices was first constructed by Benson [11] in 1966. It is the incidence graph of the generalized hexagon of order 2 (see 2.2.3). The graph is vertex-transitive and edge-transitive. Its automorphism group has order 12096 and is a $\mathbb{Z}_2$ extension of $PSU(3, 3)$.

2.1.9 (4,5)-Cage: Robertson Graph

The Robertson graph is the unique $(4, 5)$-cage of order 19 (see [79]). It is not vertex-transitive and the full automorphism group is isomorphic to the dihedral group of order 24. It is shown in Figure 5, wherein the three colored vertices on the right are adjacent to the four vertices on the 12-cycle with the corresponding color.

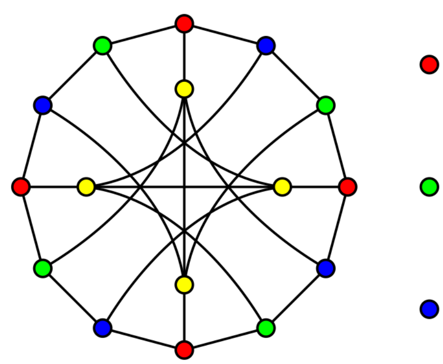

Figure 5: The Robertson Graph

2.1.10 (5,5)-Cages

The four $(5, 5)$-cages have order 30 [58, 25]. Their automorphism groups have orders 20, 30, 96, and 120. The first of these is a subgraph of the Hoffman-Singleton graph (see

2.1.12). The last is known as the Robertson-Wegner graph [91]. It can be constructed as follows.

Begin with a regular dodecahedron, D. The vertices of D determine five (regular) cubes. Furthermore, each of these cubes determines two regular tetrahedra. The vertices of the Robertson-Wegner graph are the 20 vertices of D, plus one vertex for each of the 10 tetrahedra. Each of the tetrahedral vertices is adjacent to its four determining vertices. In addition, two tetrahedral vertices are adjacent if they are contained in the same cube.

2.1.11 (6,5)-Cage

The $(6,5)$-cage is unique and has order 40. It was first presented in [72] and was proved to be minimal in [93]. It can be constructed by removing the vertices of a Petersen graph from the Hoffman-Singleton graph (see 2.1.12). It is vertex-transitive with an automorphism group of order 480.

2.1.12 (7,5)-Cage: Hoffman-Singleton Graph

The Hoffman-Singleton graph is the unique $(7,5)$-cage [51]. The graph was first considered at least as long ago as 1956 by Mesner [70] (further details can be found in [55, 84]). There are several constructions known (see, for example, [46]).

The standard construction, Robertson's *pentagons and pentagrams* [12], begins with five pentagons P_i and five pentagrams Q_j, $0 \leq i,j \leq 4$, obtained by labeling the vertices so that vertex k of P_i is adjacent to vertices $k-1$ and $k+1$ of P_i and vertex k of Q_j is adjacent to vertices $k-2$ and $k+2$ of Q_j (all subscript arithmetic is done modulo 5). The graph is completed by joining vertex k of P_i to vertex $ij+k$ of Q_j (so that each P_i together with each Q_j induce a Petersen graph).

One can obtain the $(6,5)$-cage and one of the $(5,5)$-cages from this construction. To get the $(6,5)$-cage, simply delete a pentagon and the corresponding pentagram. To obtain a $(5,5)$-cage delete two of the pentagons and the corresponding pentagrams.

The graph is vertex-transitive and edge-transitive. Its full automorphism group has order 252000 and is isomorphic to a $\mathbb{Z}_2$ extension of $PSU(3,5)$.

2.1.13 (7,6)-Cage

This case was settled by O'Keefe and Wong [73], who showed that $n(7,6) = 90$. The cage is the incidence graph of an elliptic semiplane discovered some years earlier by Baker [6]. The graph is vertex-transitive and its full automorphism group has order 15120.

2.1.14 (4,7)-Cage

Recently, Exoo, McKay and Myrvold [37] showed that $n(4,7) = 67$. They exhibited one $(4,7)$-cage on 67 vertices whose automorphism group has order 4. It is unknown whether other $(4,7)$-cages exist.

2.2 Geometric Graphs

Geometric graphs are based on generalized polygons whose incidence graphs form three infinite families of cages (girths 6, 8 and 12).

We begin with the definition of a generalized polygon (or n-gon). Let P (the set of points) and B (the set of lines) be disjoint non-empty sets, and let I (the point-line incidence relation) be a subset of $P \times B$. Let $\mathcal{I} = (P, B, I)$, and let $G(\mathcal{I})$ be the associated bipartite *incidence graph* on $P \cup B$ with edges joining the points from P to their incident lines in B ($p \in P$ is adjacent to $\ell \in B$ whenever $(p, \ell) \in I$).

The ordered triple (P, B, I) is said to be a *generalized n-gon* subject to the following four regularity conditions:

GP1: There exist $s \geq 1$ and $t \geq 1$ such that every line is incident to exactly $s + 1$ points and every point is incident to exactly $t + 1$ lines.

GP2: Any two distinct lines intersect in at most one point and there is at most one line through any two distinct points.

GP3: The diameter of the incidence graph $G(\mathcal{I})$ is n.

GP4: The girth of $G(\mathcal{I})$ is $2n$.

While the trivial case $s = t = 1$ leads to two-dimensional polygons, a well-known result of Feit and Higman [40] asserts that if both s and t are integers larger than 1, then n equals $2, 3, 4, 6$ or 8; with the parameters $3, 4, 6$ and 8 corresponding to the *projective planes*, *generalized quadrangles*, *generalized hexagons*, and *generalized octagons*, respectively. Note that the incidence graphs of generalized octagons are not regular, and so they cannot be cages.

2.2.1 The Incidence Graphs of Projective Planes

As mentioned above, finite projective planes are generalized triangles (or 3-gons). In this case, $s = t$, and projective planes are known to exist whenever the order s is a prime power $q = p^k$. If s is not a prime power, $s \equiv 1, 2 \bmod 4$, and s is not the sum of two integer squares, then no plane exists [24]. The first case not covered by the above is the case $n = 10$, for which is has been shown [60] that no plane exists. All remaining cases are unsettled.

A finite projective plane of order q has $q^2 + q + 1$ points and $q^2 + q + 1$ lines, and satisfies the following properties.

PP1: Any two points determine a line.

PP2: Any two lines determine a point.

PP3: Every point is incident with $q + 1$ lines.

PP4: Every line is incident with $q + 1$ points.

The incidence graph of a projective plane of order q is regular of degree $q+1$, has $2(q^2+q+1)$ vertices, diameter 3, and girth 6. Since the Moore bound for degree $q+1$ and girth 6 is equal to the orders of these graphs, the incidence graphs of projective planes are $(q+1, 6)$-cages. For example, the $(3, 6)$-cage Heawood graph (see 2.1.2) is the incidence graph of the plane of order 2. The plane is shown in Figure 6.

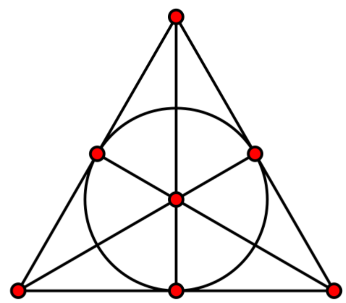

Figure 6: The projective plane of order 2.

2.2.2 The Incidence Graphs of Generalized Quadrangles

A generalized quadrangle is an incidence structure with $s+1$ points on each line, $t+1$ lines through each point. It is said to have order (s, t). A generalized quadrangle of order (s, t) has $(s+1)(st+1)$ points and $(t+1)(st+1)$ lines, and has the following properties.

GQ1: Any two points lie on at most one line.

GQ2: Any two lines intersect in at most one point.

GQ3: Every line is incident with $s+1$ points.

GQ4: Every point is incident with $t+1$ lines.

GQ5: For any point $p \in P$ and line $\ell \in B$, where $(p, \ell) \notin I$, there is exactly one line incident with p and intersecting ℓ.

The incidence graph of a generalized quadrangle of order (q, q) has $2(q+1)(q^2+1)$ vertices and is regular of degree $q+1$, diameter 4, and girth 8. The orders of these graphs match the Moore bound for degree $q+1$ and girth 8, and are therefore cages. Graphs with these parameters are known to exist whenever q is a prime power. For example, the $(3, 8)$-cage Tutte-Coxeter graph (see 2.1.4) is the incidence graph of the generalized quadrangle of order $(2, 2)$ shown in Figure 7. The 15 lines of the quadrangle are represented by the five sides of the pentagon, the five diagonals, and the five partial circles.

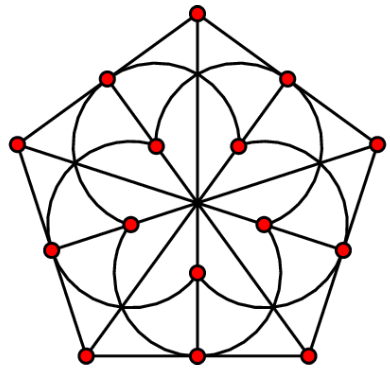

Figure 7: The generalized quadrangle of order (2,2).

2.2.3 The Incidence Graphs of Generalized Hexagons

Generalized hexagons satisfy the following conditions.

GH1: Any two points lie on at most one line.

GH2: Any two lines intersect in at most one point.

GH3: Every line is incident with $s+1$ points.

GH4: Every point is incident with $t+1$ lines.

GH5: For any point $p \in P$ and line $\ell \in B$, where $(p,\ell) \notin I$, there is a unique shortest path from p to ℓ of length 3 or 5.

Once again, the incidence graph of a generalized hexagon of order (q,q) is regular of degree $q+1$, diameter 6, and girth 12. The order of every such graph matches the Moore bound, which in this case is $2(q^3+1)(q^2+q+1)$. Graphs with these parameters are known to exist whenever q is a prime power. The $(3,12)$-cage Benson graph (see 2.1.8) is the incidence graph of the generalized hexagon of order $(2,2)$.

3 Lower Bounds

Outside the cases when Theorem 1 asserts the existence of a Moore graph, the obvious lower bound for the order of a (k,g)-cage is the value of the Moore bound plus one, $M(k,g)+1$, when k is even, and the value of the Moore bound plus two, $M(k,g)+2$, when k is odd.

In [27], Brown showed that $n(k,5)$ is never equal to $M(k,5)+1$. This was further improved by Kovács [59], who showed that $n(k,5)$ is not equal to $M(k,5)+2$ when k is odd and cannot be written in the form $\ell^2+\ell-1$, for ℓ an integer. Eroh and Schwenk [39] also showed that $n(k,5)$ is not equal to $M(k,5)+2$ for $5 \leq k \leq 11$.

For girth 7, Eroh and Schwenk [39] showed the non-existence of k-regular graphs of girth 7 and order $M(k,7)+1$. Note that in this case, the McGee graph 2.1.3 achieves the lower bound $M(3,7)+2$, hence is a cage. The only other known cage for girth 7 is the $(4,7)$-cage [37] of order $67 = M(4,7)+14$.

The cases $n(5,5)$, $n(6,5)$, $n(7,6)$, and $n(3,10)$ were resolved by a combination of counting arguments and case analysis [79, 72, 73, 74, 91].

All the remaining improvements on the lower bounds for cages are based on computer searches (see B). There are three cases where extensive computer searches produced the correct lower bound (and the cages are known). Namely, the cases $n(3,9)$, $n(3,11)$, and $n(4,7)$, [23], [68], and [37], respectively. In the case $n(3,13)$, the lower bound was improved to $202 = M(3,13)+12$ [68], and in the case $n(3,14)$ the lower bound was improved to $258 = M(3,14)+4$ [68].

4 Upper Bounds

In the preceding sections, we have listed and described all the currently known cages; graphs whose orders $n(k,g)$ are provably the smallest, and as such, will permanently stay on the list.

In what follows, we list graphs whose orders, denoted by $rec(k,g)$, are the smallest currently known. Although some of these graphs may actually be cages, the majority will most likely be eventually replaced by smaller graphs.

We adopt a somewhat arbitrary division of the current record holders into two groups:

General constructions – constructions that produce graphs with arbitrarily large values of girth or degree (Section 4.1).

Individual constructions – constructions that work for specific values of girth and degree, and may have been introduced for other purposes, or have been found by the use of computers (Section 4.2).

In the next section, we present the general constructions that have produced the best known asymptotic bounds on $n(k,g)$. These are followed by constructions that are useful for only a limited number of specific values of $n(k,g)$.

4.1 General Constructions

4.1.1 Constructions for Large Girth

In this section we describe constructions for regular graphs with arbitrarily large girth. Included are the constructions of the trivalent sextet, hexagon and triplet graphs [19, 50], as well as the higher degree constructions of Lubotzky, Phillips and Sarnak [63], and Lazebnik, Ustimenko and Woldar [61]. We also discuss a technique of Bray, Parker and Rowley [21].

Biggs observes in [16] that the Moore bounds imply

$$n(k,g) \geq \begin{cases} 1 + \frac{k}{k-2} \times (k-1)^{(g-1)/2}, & g \text{ odd} \\ \frac{2}{k-2} \times (k-1)^{g/2}, & g \text{ even} \end{cases} \tag{3}$$

These bounds imply that minimal k-regular graphs of girth g have approximately $(k-1)^{g/2}$ vertices. Thus, when considering infinite families of k-regular graphs $\{G_i\}$ of increasing girth g_i, we compare their orders v_i to the above bound (3). We say that $\{G_i\}$ is a *family with large girth* if there exists $\gamma > 0$ such that

$$g_i \geq \gamma \log_{k-1}(v_i).$$

It follows from the Moore bound that γ is at most 2, but there are no known families with γ close to 2. Erdős and Sachs [33], and later Sauer [82], using non-constructive methods, showed the existence of infinite families with $\gamma = 1$. The first explicit constructions go back to Margulis [65] who achieved $\gamma = \frac{4}{9}(\approx 0.44)$ for some infinite families with arbitrary large degree and $\gamma \approx 0.83$ for degree 4. These were followed by the results of Imrich [53], who produced infinite families of large degree with $\gamma \approx 0.48$ and a family of trivalent graphs with $\gamma \approx 0.96$. The (trivalent) *sextet graphs* of Biggs and Hoare were shown to satisfy $\gamma \geq 4/3$ by Weiss [92], and the *Ramanujan graphs* of Lubotzky, Phillips and Sarnak [63] were shown to satisfy $\gamma \geq 4/3$ (with arbitrary large degree) by Biggs and Boshier [17]. The (current) best results (for arbitrary large degree) are due to Lazebnik, Ustimenko and Woldar [61] who have constructed infinite families $CD(n,q)$ with $\gamma \geq 4/3 \log_q(q-1)$, q a power of a prime.

Construction I. Sextet Graphs

Sextet graphs are trivalent graphs introduced by Biggs and Hoare in [19]. Let q be an odd prime power. A *duet*, ab, is any unordered pair of elements from $\mathbb{F}_q \cup \{\infty\}$, the points of the projective line $PG(1,q)$. A *quartet* is an unordered pair of duets, ab and cd, satisfying the equality

$$\frac{(a-c)(b-d)}{(a-d)(b-c)} = -1.$$

If one of the vertices is infinity, then $\{\infty, b \mid c, d\}$ is a quartet if

$$\frac{(b-d)}{(b-c)} = -1.$$

A *sextet* is an unordered triple of duets such that every pair of duets from the triple forms a quartet.

Assume that $q \equiv 1 \bmod 8$. Any quartet uniquely determines a sextet. The group $PGL(2,q)$ of projective linear transformations of $PG(1,q)$ preserves and acts transitively on quartets. In addition, given a quartet $\{a, b \mid c, d\}$ there is a unique involution in $PGL(2,q)$ that interchanges a with c and b with d, and whose fixed points constitute a duet ef such that $\{a, b \mid c, d \mid e, f\}$ is a sextet.

Next we define adjacency on the sextets. A sextet $s = \{a, b \mid c, d \mid e, f\}$ is adjacent to three other sextets, each having a different duet in common with s. For example, s

is adjacent to $\{a, b \mid c', d' \mid e', f'\}$ where the duet $c'd'$ is the pair of points fixed by the involution mapping c to e and d to f, and the duet $e'f'$ is the pair of points fixed by the involution mapping c to f and d to e.

It can be shown that there exist exactly $q(q^2 - 1)/24$ sextets. The set of all sextets under the above adjacency relation defines the trivalent graph $\Sigma(q)$. In general, $\Sigma(q)$ is not connected.

If p is an odd prime, the *sextet graph* $S(p)$ is any connected component of $\Sigma(p)$, for $p \equiv 1 \bmod 8$, and its order is $\frac{1}{48}p(p^2 - 1)$.

The graphs $S(73)$ and $S(313)$ are the smallest known trivalent graph of girths 22 and 30, respectively.

Construction II. Hexagons

Hexagon graphs are trivalent graphs introduced by Hoare in [50]. Let p be a prime, $p > 3$, and let $q = p$ if $p \equiv 1 \bmod 4$ and $q = p^2$ otherwise. Consider the complete graph K_{q+1} with vertices labeled by elements of $\mathbb{F}_q \cup \{\infty\}$, the points of the projective line $PG(1, q)$. The concepts of duet, quartet, and sextet are defined as above.

For any 6-cycle C in K_{q+1}, define a *short diagonal* to be a pair of vertices whose distance in C is 2, and a *long diagonal* to be a pair of vertices whose distance in C is 3. A 6-cycle C in K_{q+1} is a *hexagon* if any four of its vertices, v_1, v_2, v_3, v_4, such that v_1, v_2 form a short diagonal and v_3, v_4 form a long diagonal determine a quartet $\{v_1, v_2 \mid v_3, v_4\}$.

This set of hexagons is the vertex set of a trivalent graph denoted $H(q)$, wherein adjacency is defined as follows:

> Let $H = v_1, v_2, v_3, v_4, v_5, v_6$ be a hexagon. Each of its three long diagonals determines one neighbor of H. For example, the long diagonal v_1, v_4 together with the short diagonals v_2, v_6 and v_3, v_5 determines unique sextets $\{v_1, v_4 \mid v_2, v_6 \mid v_7, v_8\}$ and $\{v_1, v_4 \mid v_3, v_5 \mid v_9, v_{10}\}$, which in turn determine the adjacent hexagon $v_1, v_7, v_9, v_4, v_8, v_{10}$.

The graph $H(47)$ is the smallest known trivalent graph of girth 19.

Construction III. Triplets

Let p be an odd prime. The vertex set of the trivalent *triplet graph* $T(p)$ is the set of all 3-subsets of the points of $PG(1, p)$. Two 3-subsets $\{a, b, c\}$ and $\{a, b, d\}$ are adjacent if and only if $\{a, b \mid c, d\}$ is a quartet.

It was shown in [50] that if $p \equiv 1 \bmod 4$ then $T(p)$ has two connected components of size $p(p^2 - 1)/12$, and if $p \equiv 3 \bmod 4$ then $T(p)$ is connected of order $p(p^2 - 1)/6$.

The girth of $T(p)$ is greater than $log_\phi(p)$, where ϕ is the golden ratio $(1 + \sqrt{(}5))/2$. Finally, it may be interesting to note that the connected components of $T(5)$ are Petersen graphs.

Construction IV. Lubotzky, Phillips, Sarnak

The graphs obtained from this construction are Cayley graphs (see A2) of projective linear groups. They belong to the family of *Ramanujan graphs*, which are k-regular graphs whose second largest (in absolute value) eigenvalue λ_2 satisfies the inequality $\lambda_2 \leq 2\sqrt{k-1}$.

Let p and q be distinct primes such that $p, q \equiv 1 \bmod 4$, and let i be an integer satisfying $i^2 \equiv -1 \bmod q$. Then there are $8(p+1)$ solutions $\alpha = (a_0, a_1, a_2, a_3)$ satisfying $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$. Exactly $p+1$ of these 4-tuples α are such that their first coordinate a_0 is positive and odd and the rest of the coordinates a_1, a_2, a_3 are all even. Associate each such α with the matrix $\tilde{\alpha} \in PGL(2, q)$ defined by

$$\tilde{\alpha} = \begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{pmatrix}.$$

Let Λ denote the set of the $p+1$ matrices obtained in this way.

The graphs $X^{p,q}$ are Cayley graphs of degree $p+1$ defined in two different ways depending on the sign of the Legendre symbol $\left(\frac{p}{q}\right)$:

$$X^{p,q} = \begin{cases} C(PSL(2,q), \Lambda) & \text{if } \left(\frac{p}{q}\right) = 1, \\ C(PGL(2,q), \Lambda) & \text{if } \left(\frac{p}{q}\right) = -1. \end{cases}$$

The orders of the graphs are the orders of the linear groups, which are $\frac{q(q^2-1)}{2}$ and $q(q^2-1)$, respectively. The latter graph is bipartite.

In order to state their result precisely, we need their concept of a *good integer*, which is one that cannot be expressed in the form $4^\alpha(8\beta + 7)$ for nonnegative integers α, β.

Theorem 6 ([63])

$$g(X^{p,q}) = \begin{cases} 2\lceil 2\log_p q \rceil & \textit{if } p^{\lceil 2\log_p q \rceil} - q^2 \textit{ is good,} \\ 2\lceil 2\log_p q + \log_p 2 \rceil & \textit{otherwise.} \end{cases}$$

Hence, the girths of the resulting graphs are asymptotically $\frac{4}{3}\log_p(n)$, where n is the order of the graph [63].

Construction V. Lazebnik, Ustimenko, Woldar

Let q be a prime power, and let P (points) and L (lines) be two copies of the set of infinite sequences of elements from the finite field $\mathbb{F}_q$. We adopt the convention that the points in P will be denoted by

$$(p) = \{p_1, p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}, p'_{2,2}, p_{2,3}, \ldots, p_{i,i}, p'_{i,i}, p_{i,i+1}, p_{i+1,i}, \ldots\},$$

and the lines in L will be denoted by

$$[l] = \{l_1, l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2}, l'_{2,2}, l_{2,3}, \ldots, l_{i,i}, l'_{i,i}, l_{i,i+1}, l_{i+1,i}, \ldots\}.$$

A point (p) is incident to the line $[l]$ subject to the following:

$$\begin{array}{lcl} l_{1,1} - p_{1,1} &=& l_1 p_1 \\ l_{1,2} - p_{1,2} &=& l_{1,1} p_1 \\ l_{2,1} - p_{2,1} &=& l_1 p_{1,1} \\ l_{i,i} - p_{i,i} &=& l_1 p_{i-1,i} \\ l'_{i,i} - p'_{i,i} &=& l_{i,i-1} p_1 \\ l_{i,i+1} - p_{i,i+1} &=& l_{i,i} p_1 \\ l_{i+1,i} - p_{i+1,i} &=& l_1 p'_{i,i}, \end{array}$$

with the last four equations defined for $i \geq 2$.

For each positive $n \geq 2$, the first $n-1$ equations define an incidence relation on P_n, L_n, two copies of $(\mathbb{F}_q)^n$, thought of as the projections of the infinite sequences from P and L onto their n first coordinates. The graph $D(n,q)$ is the bipartite incidence graph corresponding to the incidence structure induced on P_n and L_n.

For all $n > 1$, the graphs $D(n,q)$ are q-regular graphs of order $2q^n$ and girth $g \geq n+5$, whose automorphism groups are transitive on points, lines, and edges. Moreover, for $n \geq 6$, the graphs $D(n,q)$ are disconnected with all of their connectivity components $CD(n,q)$ mutually isomorphic. Hence, for $n \geq 6$ and q a prime power, the graphs $CD(n,q)$ are bipartite, connected, q-regular graphs of order $\leq 2q^{n-t+1}$, girth $g \geq n+5$, and have point, line, and edge-transitive automorphism groups [61].

Using this construction, in conjunction with the constructions of Lubotzky, Phillips, and Sarnak [63], and of Füredi, Lazebnik, Seress, Ustimenko, and Woldar [42], one obtains the following bound.

Theorem 7 ([62]) *Let $k \geq 2$ and $g \geq 5$ be integers, and let q denote the smallest odd prime power for which $k \leq q$. Then*

$$n(k,g) \leq 2kq^{\frac{3}{4}g-a}, \tag{2}$$

where $a = 4$, $11/4$, $7/2$, $13/4$ for $g \equiv 0,1,2,3 \bmod 4$, respectively.

Construction VI. Bray, Parker, Rowley

Historically, highly symmetric graphs (see A) repeatedly proved useful in constructions of relatively small (trivalent) graphs of specific girth. A number of the early best constructions were Cayley graphs (see A2). The symmetry of Cayley graphs makes the girth computations more efficient than for asymmetric graphs of the same order. When working with Cayley graphs, it is important to choose groups in which the group operation can be computed quickly. Hence, groups which can be represented as groups of small matrices are a natural choice. Several of the early records constructed in this way can be found in [29].

Improvements on some of the records obtained using Cayley graphs were made by modifying Cayley graphs. Bray, Parker and Rowley [21] constructed a number of current record holders for degree three by factoring out the 3-cycles in trivalent Cayley graphs. Their construction starts with a trivalent Cayley graph, $C(G,X)$, subject to the condition that the generating set X contains an involution, α, and two mutually inverse elements of order 3, δ, δ^{-1}, and that the Cayley graph has no cycles of length 4. The graph $B(G,X)$ is then defined as follows: the vertex set $\mathcal{T}$ of $B(G,X)$ is the set of triangles of $C(G,X)$ with triangle T_i adjacent to triangle T_j in $B(G,X)$ if at least one of the vertices of T_i is adjacent in $C(G,X)$ to at least one of the vertices of T_j via an edge labeled by the involution α.

4.1.2 Constructions for Girth 5

Next we present a series of constructions that produce families of fixed girth 5. We attempt to provide a list that is as complete as possible, including older constructions that may still prove useful, and newer constructions whose efficiency is sometimes hard to compare.

The constructions below are all approximately twice the Moore bound, which is k^2+1, for degree k and girth 5.

Construction VII. Brown

Brown [27] constructed a family of graphs of girth 5 based on his construction for girth 6 [26].

He begins with a set of points $P = \{p_0, \ldots, p_q\}$ and a set of lines $L = \{\ell_0, \ldots, \ell_q\}$, such that p_0 is incident with all the lines in L and ℓ_0 is incident with all the points in P, and removes it from $PG(2,q)$. The incidence graph of the resulting geometry is a q-regular graph of girth 6 and order $2q^2$. He then adds q-cycles to the neighborhoods of the deleted vertices $\{p_1, \ldots, p_q\}$ and $\{\ell_1, \ldots, \ell_q\}$. The resulting graph has degree $q+2$, girth 5, and order $2q^2$.

Construction VIII. Wegner

Wegner [91] constructed a family of graphs of prime degree k, girth 5, and order $2k^2 - 2k$.

Let $p \geq 5$ be a prime. The graph is constructed by connecting the vertices of a p^2-cycle A to an independent set B of size $p(p-2)$. Denote the vertices of A by $a_0, \ldots, a_{p^2-1}$, with each a_i adjacent to a_{i+1}; denote the vertices of B by $b_{s,t}$, for $0 \leq s \leq p-3$ and $0 \leq t \leq p$. The adjacencies between the vertices of the cycle and the independent set are defined as follows. For each a_k, write k as $ip+j$, where $0 \leq i, j < p$, and make a_{ip+j} adjacent to $b_{r,ir+j}$, for $0 \leq r \leq p-3$.

The smallest of Wegner's graphs is of degree $k=5$ and has order 40. Note that the $(5,5)$-cage has order 30.

Construction IX. Parsons

Parsons used finite projective planes to construct infinite families of regular graphs of girth five [76]. The best of these has degree $k = (q+1)/2$ (for prime power q) and has order $2k^2 - 3k + 1$.

Recall that the projective plane $PG(2,q)$ can be constructed from the one and two-dimensional subspaces of a 3-dimensional vector space over the field $\mathbb{F}_q$. Parsons defines a graph $G(q)$ on the points of $PG(2,q)$, with two points adjacent if they are distinct and their dot product (as vectors in $\mathbb{F}_q^3$) is zero. The graphs Parsons constructs are induced subgraphs of $G(q)$.

To specify the graphs, he partitions the vertex set into three subsets, R, S, and T, as follows. Let R be the set of self-orthogonal points (as vectors in $\mathbb{F}_q^3$), S be the set of points adjacent to some point in R, and T be all the remaining points. He proves that if $q \equiv 3 \bmod 4$, the subgraph induced by S has order $q(q+1)/2$ and is regular of degree $(q-1)/2$ and has girth 5, and if $q \equiv 1 \bmod 4$, the subgraph induced by T has order $q(q-1)/2$ and is regular of degree $(q+1)/2$ and has girth 5.

When $q = 5$, Parsons construction produces the Petersen graph, and when $q = 7$, it produces the Coxeter graph.

Construction X. O'Keefe and Wong

A construction based on Latin squares was described by O'Keefe and Wong [75]. It can be viewed as a generalization of Robertson's pentagon-pentagram construction of the Hoffman-Singleton graph 2.1.12. Their construction generates the strongest results when the degree $k = q + 2$, for a prime power q, in which case the order of their graphs is $2k^2 - 8k + 8$. This is the case described below. In the general case, when $3 \leq k \leq q + 1$, the orders of the graphs are $2q(k - 2)$.

Let $q = p^r$ be a prime power. O'Keefe and Wong construct a set of q mutually orthogonal squares with entries from $\mathbb{F}_q$, with the last $q-1$ of these comprising a complete set of mutually orthogonal Latin squares [89]. The Latin squares are constructed as follows. Let a be a primitive element of $\mathbb{F}_q$, and order the field elements as powers of a: $b_0 = 0, b_1 = 1, b_2 = a, \ldots, b_{q-1} = a^{q-2}$. Then the squares L_i, $1 \leq i \leq q - 1$, are defined by setting the (i, j) entry of L_k to $b_k(b_i + b_j)$. Also let L_0 be the square whose (i, j) entry is j. Note that L_0 is not a Latin square, but is orthogonal to each of the Latin squares $L_1, \ldots, L_{q-1}$.

Next define a graph on the two sets of doubly indexed vertices, $X = \{x_{i,j}\}$ and $Y = \{y_{i,j}\}$, $0 \leq i, j \leq q - 1$. Let the $x_{k,j}$ be adjacent to $y_{i,t}$, whenever t is the (i, j) entry of L_k. Since the squares are orthogonal, the resulting graph is a q-regular bipartite graph of girth 6.

To increase the degree to $q + 2$, the edges of a 2-regular graph are added in such a way that no cycles of length 3 or 4 are introduced. Define $X_i = \{x_{i,0}, \ldots, x_{i,q-1}\}$ and $Y_i = \{y_{i,0}, \ldots, y_{i,q-1}\}$. Suppose $q = p^r$ and $p \geq 5$. Add edges between $x_{i,j}$ and $x_{i,j+a}$, and between $y_{i,j}$ and $y_{i,j+a^2}$, for $0 \leq i, j < q$. Note that the 2-regular graph induced on X_i (and Y_i) consists of disjoint p-cycles.

In cases $p = 2$ and $p = 3$ the construction is similar in spirit, but is more intricate, and results in the addition of 8-cycles and 9-cycles, respectively.

When $q = 5$, this construction yields the Hoffman-Singleton graph.

Construction XI. Wang

Wang [90] constructed a family of $(k, 5)$-graphs for $k = 2^s + 1$. This construction uses a complete set of Latin squares of order 2^s, $L_1, \ldots, L_{k-2}$, together with the square $L_0 = [a_{i,j}]$, with $a_{i,j} = i$, for $0 \leq j \leq k - 2$.

The construction is based on a tree obtained by taking an edge uv, and joining $k - 1$ leaves to each of u and v. The resulting tree has diameter 3 and $2k$ vertices.

Begin with 2^{s-1} copies of the above tree, $T_0, \ldots, T_{2^{s-1}-1}$, and label the two sets of leaves in T_i by $\alpha_{2i,j}$ and $\alpha_{2i+1,j}$, for $0 \leq j \leq k - 2$. Next, add $(k - 1)^2$ isolated vertices, $r_{i,j}$, $0 \leq i, j \leq k - 2$, and join $r_{i,j}$ to $\alpha_{h,t}$ if and only if $\alpha_{h,t}$ is the (j, h)-entry of the square L_i. The construction continues by adding edges between each pair of vertices $r_{i,2j}$ and $r_{i,2j+1}$, resulting in $(k, 5)$-graphs of order $2k^2 - 3k + 1$. Finally, the construction is completed by removing all of the vertices $r_{0,j}$ and adding edges between $\alpha_{4h,t}$ and $\alpha_{4h+3,t}$, and between $\alpha_{4h+1,t}$ and $\alpha_{4h+2,t}$.

The resulting $(k,5)$-graph has order $2k^2-4k+2$.

Construction XII. Araujo-Pardo and Montellano-Ballesteros

The authors used finite projective and affine planes to construct an infinite family of regular graphs of girth five and degree k [3]. Their construction gives the strongest results when the degree $k=p+2$, for a prime p, which matches the previous result of O'Keefe and Wong of order $2k^2-8k+8$.

In the general case, where $k-2$ is not a prime, their bound is

$$n(k,5) \le \begin{cases} 4(k-2)^2, & \text{when } 7 \le k \le 3276, \\ 2(k-2)(k-1)(1+\frac{1}{2\ln^2(k-1)}), & \text{when } 3276 < k. \end{cases}$$

We present their general construction, an explicit presentation of the original construction of Brown. Consider the case where the degree k is less than the next prime p, and let A_p be the affine plane of order p. Define a smaller incidence structure $A_{k,p}$ as follows. The points of $A_{k,p}$ are the points of the affine plane whose first coordinates are less than $k-2$. The lines of $A_{k,p}$ are those whose slopes are less than $k-2$. The incidence graph of this structure has order $2kp$, degree $k-2$ and girth 6. Now add edges joining pairs of points whose first coordinates are equal and whose second coordinates differ by 1. Similarly join pairs of lines whose slopes are equal and whose y-intercepts differ by 2.

The resulting graph has girth 5. The precise form of their bound follows by using a new result on the distribution of primes [31].

Construction XIII. Jørgensen

Jørgensen [57] used relative difference sets to construct several infinite families of regular graphs of girth five. The best of these have degree $k=q+3$ (for prime power q) and order $2k^2-12k+16$.

His two general theorems can be summarized as follows.

Theorem 8 ([57]) *Let q be a prime power. Then*

$$n(k,5) \le \begin{cases} 2(k-1)(q-1), & \textit{for } 7 \le q, \quad k \le q+2, \\ 2(k-2)(q-1), & \textit{for } 13 \le q, \quad k \le q+3, \textit{ and } q \textit{ odd}. \end{cases}$$

Jørgensen's constructions are based on the concept of a relative difference set. Let G be a finite group with a normal subgroup $H \triangleleft G$. A subset $D \subset G$ is called a *relative difference set* if any element $g \in G-H$ can be expressed uniquely as a difference using elements from D, $g=x-y$, $x,y \in D$, but no element of H can be expressed in this manner.

Jørgensen's construction starts with a finite group G, a normal subgroup $H \triangleleft G$, a set of coset representatives $T=\{a_1,\dots,a_k\}$ for H in G, and a relative difference set D for H in G. In addition, it also requires two Cayley graphs on H of girth 5, $C(H,S_1)$ and $C(H,S_2)$, such that $S_1 \cap S_2 = \emptyset$.

Once one has all these ingredients, a graph on the vertex set $G \times \{1,2\}$ can be constructed by introducing edges of three types.

Type I.1: $(ha_i, 1)$ *is adjacent to* $(hxa_i, 1)$ *for all* $h \in H$, $x \in S_1$, $a_i \in T$,

Type I.2: $(ha_i, 2)$ *is adjacent to* $(hxa_i, 2)$ *for all* $h \in H$, $x \in S_2$, $a_i \in T$,

Type II: $(g, 1)$ *is adjacent to* $(gy, 2)$ *for all* $g \in G$ *and* $y \in D$.

Note that the first two types create multiple copies of the original Cayley graphs on the cosets of H in G.

To obtain the bounds in Theorem 8, one needs to make specific choices for G, H, T, S_1 and S_2. For the first of the inequalities, Jørgensen took G to be the cyclic group of order $(q+1)(q-1)$, H to be the cyclic subgroup of order $(q-1)$, T to be any complete set of coset representatives for H in G, and D to be a relative difference set of size $q-1$ whose existence is guaranteed by the results of Bose [20] and Elliot and Butson [32]. The two Cayley graphs in this case are edge disjoint cycles on the vertices of H.

For example, using this method with the choices $G = \mathbb{Z}_5 \times \mathbb{Z}_5$, $H = \mathbb{Z}_5 \times \{0\}$, $S_1 = \{(1,0),(4,0)\}$, $S_2 = \{(2,0),(3,0)\}$, $T = \{(0,0),(0,1),(0,2),(0,3),(0,4)\}$, and $D = \{(0,0),(1,1),(4,2),(4,3),(1,4)\}$, results in yet another construction for the Hoffman-Singleton Graph.

For the second inequality in Theorem 8, one proceeds along similar lines using dihedral groups.

4.1.3 Constructions for Girth 6, 8, and 12

When $k-1$ is a prime power and $g = 6, 8$, or 12, the (k, g)-cage is known, and is the incidence graph of the generalized n-gon, for $n = 3, 4$, or 6, respectively (see 2.2). In this section, we discuss constructions for degrees k where no generalized n-gon is known. A number of different constructions for these values are obtained by removing particular sets of vertices from the incidence graphs of the generalized n-gons. This might be compared to the situation for girth 5 where the cages $(3,5)$, $(5,5)$ and $(6,5)$ can be constructed by removing vertices from the Hoffman-Singleton graph (see 2.1.12), which is the $(7,5)$-cage [94].

In [43], Gács and Héger present a unified view of these constructions using the concept of a t-*good structure*. A t-good structure in a generalized n-gon is a pair (P, L) consisting of a set of points P, and a set of lines L, subject to the condition that there are t lines in L through any point not in P, and t points in P on any line not in L. Removing the points and lines of a t-good structure from the incidence graph of a generalized n-gon results in a $(q+1-t)$-regular graph of girth at least $2n$.

Construction XIV. Brown

Brown [26] was the first who explicitly considered the case of girth 6, and most subsequent constructions are directly or indirectly derived from his. His construction can be stated in the language of t-good structures as follows.

He begins with a 1-good structure consisting of a set of points $P = \{p_0, \ldots, p_q\}$ and a set of lines $L = \{\ell_0, \ldots, \ell_q\}$, such that p_0 is incident with all the lines in L and ℓ_0 is incident with all the points in P, and removes it from $PG(2,q)$. The incidence graph of the resulting geometry is a q-regular graph of girth 6 and order $2q^2$.

For $t > 1$, he also removes all the neighbors of $p_1, \ldots, p_{t-1}$ and $\ell_1, \ldots, \ell_{t-1}$ to obtain a geometry whose incidence graph has girth (at least) 6, whose degree is $q-t+1$, and whose order is $2q(q-t+1)$. He then uses Bertrand's postulate that asserts for every $k > 2$ the existence of a prime p such that $k < p < 2k$ [48] to show that $n(k,6) < 4k^2$, for all k, and other number theoretic results to show that for any $\epsilon > 0$ there exists and integer N such that for all $k > N$, $n(k,6) < 2(1+\epsilon)k^2$. Note that $M(k,6) = 2k^2-2k+2$.

Construction XV. Araujo, González, Montellano-Ballesteros, Serra

In [4], Araujo, González, Montellano-Ballesteros, and Serra apply Brown's construction to generalized quadrangles and hexagons, and obtain the following bounds which are valid for $g = 6, 8$, or 12.

$$n(k,g) \leq 2kq^{\frac{g-4}{2}}$$

By using recent results on the distribution of primes they obtain:

$$n(k,g) \leq \begin{cases} 2k(k-1)^{\frac{g-4}{2}}(\frac{7}{6})^{\frac{g-4}{2}}, & 7 \leq k \leq 3275 \\ 2k(k-1)^{\frac{g-4}{2}}(1+\frac{1}{2\ln^2(k)})^{\frac{g-4}{2}}, & 3276 \leq k. \end{cases}$$

Construction XVI. Abreu, Funk, Labbate, Napolitano

In [1], Abreu, Funk, Labbate, and Napolitano construct several families of regular graphs of girth 6. Their constructions make use of the addition and multiplication tables of $\mathbb{F}_q$. We describe two of them using the language of Gács and Héger [43].

In the first construction, let (p_1, ℓ_1) be a non-incident point-line pair. Choose $t-1$ points on ℓ_1: $p_2, \ldots, p_t$; and choose $t-1$ lines through p_1: $\ell_2, \ldots, \ell_t$. The pair (P, L), where P is the set of all points on any of the lines ℓ_i, and Q is the set of all lines through any of the points p_i, is a t-good set. Removing (P, L) from $PG(2,q)$ leaves a geometry whose incidence graph is $q-t+1$-regular, has girth at least 6, and order $2(q^2+(1-t)q+(t-2))$.

The second construction requires the concept of a *Baer subplane* $(\mathcal{P}', \mathcal{L}')$ of a projective plane $(\mathcal{P}, \mathcal{L})$, which is a subplane satisfying the property that any point $p \notin \mathcal{P}'$ is incident with exactly one line in $\mathcal{L}$, and any line $\ell \notin \mathcal{L}'$ is incident with exactly one point in $\mathcal{P}'$. Baer subplanes of $PG(2,q)$ exist whenever q is a square. In fact, it is known that such planes can be partitioned into $q-\sqrt{q}+1$ disjoint Baer subplanes [49]. Note that a Baer subplane is a 1-good structure. To obtain a t-good structure, $1 \leq t \leq q-\sqrt{q}+1$, use t disjoint Baer subplanes. The resulting graph is $q-t+1$-regular, has girth at least 6, and order $2(q^2+(1-t)q-t\sqrt{q}+(1-t))$.

Construction XVII. Bretto, Gillibert

A construction for $(k,6)$ and $(k,8)$-graphs was given by Bretto and Gillibert [22]. They construct graphs $\Phi(G,S,m)$, with G a finite group, S a non-empty subset of G, and m a positive integer. Each $s \in S$ defines a partition $G = \cup_{x \in T_s} \langle s \rangle x$, where T_s is a complete set of right coset representatives of $\langle s \rangle$, the subgroup generated by s.

The vertices of $\Phi(G, S, m)$ are the cosets $\langle s \rangle x$ of the subgroups $\langle s \rangle$ generated by elements of S. Two such cosets are adjacent if the cardinality of their intersection is m.

For example, they choose G to be the Klein group, $\mathbb{Z}_2 \times \mathbb{Z}_2$. The elements of G are $\{1, a, b, ab\}$. Take $S = \{a, b, ab\}$. Then the cosets of a are $\{1, a\}$ and $\{b, ab\}$; the cosets of b are $\{1, b\}$ and $\{a, ab\}$; and the cosets of ab are $\{1, ab\}$ and $\{a, b\}$. Taking $m = 1$, we obtain the octahedral graph.

Using the nonabelian groups of order p^3 in which every nonidentity element has order p, one can construct $(p, 6)$-graphs of order $2p^2$. Note that this order is the same as the order obtained by Brown [26].

Similarly, they employed a semidirect product of Z_p^3 by Z_p to construct p-regular graphs of girth 8 and order $2p^3$ in the cases where k is a prime.

Construction XVIII. Gács, Héger

Applying their concept of a t-good structure to generalized 4-gons and 6-gons, Gács and Héger constructed graphs which establish the following two bounds.

$$\begin{aligned} n(k,8) &\leq 2(k^3 - 2k) \\ n(k,12) &\leq 2(k^5 - k^3) \end{aligned} \tag{4}$$

Construction XIX. Balbuena

This construction [9] covers a case for girth 8 not included in the above upper bound of Gács and Héger, namely the case where the degree is one less than a prime power q:

$$n(q-1, 8) \leq 2(q^3 - q^2 - q). \tag{5}$$

Construction XX. Araujo-Pardo, Balbuena

In this construction, the authors apply the subgraph idea of Brown to girths 6, 8, and 12 [2]. For girth 12, and q a prime power larger than 3, they obtain a result for degree $q-1$ that is not covered by any of the above constructions:

$$n(q-1, 12) \leq 2q^2(q^3 - q^2 - q + 1). \tag{6}$$

4.2 Individual Constructions

4.2.1 Individual Constructions for Degree 3

The orders of the trivalent cages of girth 13 and up are all unsettled. The following table shows the current state of knowledge of degree up to 32. Note that for the sake of completeness we have also included the known cages for girths 5 through 12. In cases where the lower and upper bounds were established independently, the authors responsible for the lower bounds are listed first, and separated by a semi-colon from the authors responsible for the upper bound.

Girth g	Lower Bound	Upper Bound	# of Cages	Due to
5	10	10	1	Petersen
6	14	14	1	Heawood
7	24	24	1	McGee
8	30	30	1	Tutte
9	58	58	18	Brinkmann-McKay-Saager
10	70	70	3	O'Keefe-Wong
11	112	112	1	McKay-Myrvold; Balaban
12	126	126	1	Benson
13	202	272		McKay-Myrvold; Hoare
14	258	384		McKay; Exoo
15	384	620		Biggs
16	512	960		Exoo
17	768	2176		Exoo
18	1024	2560		Exoo
19	1536	4324		Hoare, H(47)
20	2048	5376		Exoo
21	3072	16028		Exoo
22	4096	16206		Biggs-Hoare, S(73)
23	6144	49326		Exoo
24	8192	49608		Bray-Parker-Rowley
25	12288	108906		Exoo
26	16384	109200		Bray-Parker-Rowley
27	24576	285852		Bray-Parker-Rowley
28	32768	415104		Bray-Parker-Rowley
29	49152	1143408		Bray-Parker-Rowley
30	65536	1227666		Biggs-Hoare, S(313)
31	98304	3649794		Bray-Parker-Rowley
32	131072	3650304		Bray-Parker-Rowley

Table 3. Bounds for trivalent cages.

Next, we briefly describe the graphs from the above table that are not cages.

Rec(3,13) = 272

The smallest known trivalent graph of girth 13 has 272 vertices and is a Cayley graph (see A2) of the group of transformations of the affine plane over $\mathbb{F}_{17}$. Biggs reports [14] that the graph was discovered by Hoare. Royle [80] has shown that a smaller $(3,13)$-graph cannot be a Cayley graph.

Rec(3,14) = 384

The smallest known trivalent graph of girth 14 has 384 vertices [34] and is a lift (see B3) of the multigraph shown in Figure 8.

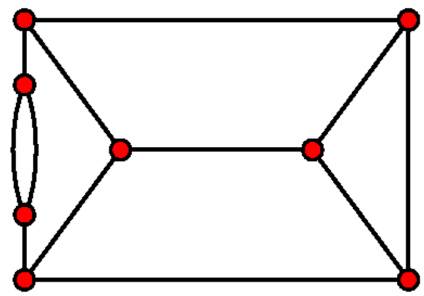

Figure 8: Base Graph for Girth 14

The voltage group used in the construction is a semidirect product of the cyclic group of order 3 by the generalized quaternion group of order 16, and is SmallGroup(48,18) in the Small Group Library in GAP [44]. The automorphism group of this graph has order 96.

Rec(3,15) = 620

The smallest known trivalent graph of girth 15 has 620 vertices and is the sextet graph (see 4.1.1), S(31), discovered by Biggs and Hoare [19]. The automorphism group of this graph has order 14880.

Rec(3,16) = 960

The smallest known trivalent graph of girth 16 has 960 vertices and was discovered by Exoo [35]. The graph is a lift (see B3) of the Petersen graph with voltage assignments in $\mathbb{Z}_2 \times \mathbb{Z}_{48}$. In Figure 9 the voltages assigned to each edge are given. Unlabeled edges are assigned the group identity $(0,0)$.

The automorphism group of this graph has order 96.

Rec(3,17) = 2176

The smallest known trivalent graph of girth 17 has order 2176 and was discovered by Exoo [36]. It is a lift (see B3) of the multigraph shown in Figure 10.

The voltage group is a group of order 272 and is SmallGroup(272,28) in the GAP Small Group Library [44]. Note that the group is not the affine group of the same order that was used by Biggs and Hoare in the construction of the $(3,13)$-graph discussed above. The automorphism group of this graph has order 544.

Rec(3,18) = 2560

The smallest known trivalent graph of girth 18 has 2560 vertices and was discovered by Exoo [36]. It is a lift (see B3) of the multigraph in Figure 11.

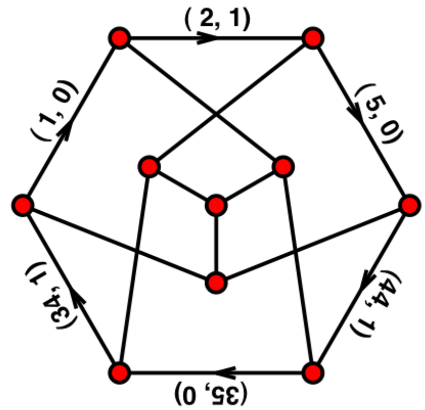

Figure 9: Base Graph for Girth 16

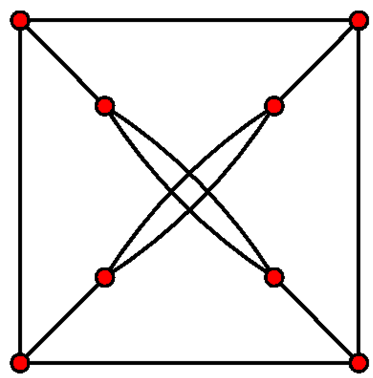

Figure 10: Base Graph for Girth 17

The voltage group has order 320, and is SmallGroup(320,696) in the GAP Small Group Library [44]. The maximum order among group elements is 20. The Sylow 2-subgroups are nonabelian with exponent 8. The Sylow 5-subgroup is normal, and so the full group is a semi-direct product. The automorphism group of the graph has order 640.

Rec(3,19) = 4324

The smallest known trivalent graph of girth 19 has order 4324. It is the Hexagon graph (see 4.1.1), H(47), discovered by Hoare [50]. The automorphism group of the graph has order 51888.

Rec(3,20) = 5376

The smallest known trivalent graph of girth 20 has order 5376 and was discovered by Exoo [36]. It is a lift (see B3) over the multigraph of order two and size three, with the three edges joining the two vertices. The voltage group is a group of order 2688. The automorphism group of the graph has order 5376 and is transitive on the vertices. The graph turns out to be a Cayley graph, and hence, a graphical regular representation [45].

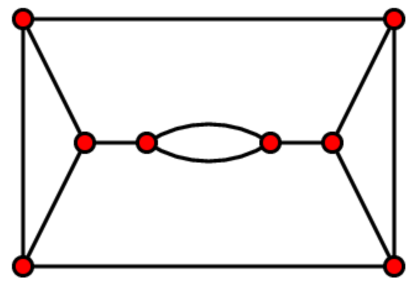

Figure 11: Base Graph for Girth 18

Rec(3,21) = 16028

The smallest known trivalent graph of girth 21 has order 16028 and was obtained by Exoo [38] using excision (see B4) on S(73) [19].

Rec(3,22) = 16206

The smallest known trivalent graph of girth 22 has order 16206 and is the sextet graph (see 4.1.1), S(73), discovered by Biggs and Hoare [19].

Rec(3,23) = 49326

The smallest known trivalent graph of girth 23 was discovered by Exoo [38] and was obtained by excision (see B4) from the graph $B(PSL(2,53) \times \mathbb{Z}_2, \{\alpha, \delta, \delta^{-1}\})$ defined next. It is of order 49326.

Rec(3,24) = 49608

The smallest known trivalent graph of girth 24 was discovered by Bray, Parker and Rowley [21]. It is the graph $B(PSL(2,53) \times \mathbb{Z}_2, \{\alpha, \delta, \delta^{-1}\})$ of order 49608. The generating permutations α and δ are as follows.
α :

```
( 1, 4)( 2, 5)( 3, 6)( 7,23)( 8,16)( 9,14)(10,24)(11,46)
(12,21)(15,32)(17,47)(18,41)(19,49)(20,55)(22,29)(25,48)
(26,36)(27,38)(28,39)(30,40)(31,35)(33,60)(34,51)(37,44)
(42,56)(43,59)(45,54)(50,58)(52,57)
```

δ :

```
( 7,49,12)( 8,52,57)( 9,35,44)(10,34,18)(11,53,48)(13,56,55)
(14,17,21)(15,31,38)(16,25,51)(19,60,40)(20,54,41)(22,29,45)
(23,24,32)(26,50,42)(27,33,30)(28,36,37)(39,43,46)(47,59,58)
```

Rec(3,25) = 108906

The smallest known trivalent graph of girth 25 is of order 108906 and was obtained by Exoo [38], using excision, from $B((PSL(2,25) \times 7:3):2, \{\alpha, \delta, \delta^{-1}\})$ defined next.

Rec(3,26) = 109200

The smallest known trivalent graph of girth 26 is of order 109200. It is the graph $B((PSL(2,25) \times 7:3):2, \{\alpha, \delta, \delta^{-1}\})$ discovered by Bray, Parker and Rowley [21]. The generating permutations α and δ are as follows.
α :

```
( 2, 7)( 3, 6)( 4, 5)( 8,18)( 9,20)(10,17)(11,12)(13,32)
(14,29)(15,31)(16,25)(19,21)(22,26)(23,27)(24,33)(28,30)
```

δ :

```
( 1, 7, 5)( 3, 4, 6)( 8,15,30)( 9,32,11)(10,27,22)(13,24,26)
(14,21,25)(16,29,31)(18,23,20)(19,33,28)
```

Rec(3,27) = 285852

The smallest known trivalent graph of girth 27 is of order 285852. It is the graph $B(PSL(2,83) \times \mathbb{Z}_3, \{\alpha, \delta, \delta^{-1}\})$ discovered by Bray, Parker and Rowley [21]. The generating permutations α and δ are as follows.
α :

```
( 7,14)( 8,73)( 9,24)(10,89)(11,23)(12,60)(13,87)(15,76)
(16,68)(17,25)(18,40)(19,64)(20,26)(21,43)(22,31)(27,67)
(28,74)(29,79)(30,39)(32,66)(33,71)(34,78)(35,41)(36,44)
(37,52)(38,50)(42,81)(45,77)(46,69)(47,54)(48,58)(49,85)
(51,56)(53,72)(55,80)(57,70)(59,62)(61,63)(65,90)(75,88)
(82,84)(83,86)
```

δ :

```
( 1, 3, 5)( 2, 4, 6)( 7,38,65)( 8,30,43)( 9,64,18)(10,27,37)
(11,31,72)(12,32,55)(13,35,84)(14,75,68)(15,51,57)(16,33,90)
(17,49,19)(20,80,42)(21,66,40)(22,87,61)(23,85,47)(24,78,26)
(25,63,34)(28,70,76)(29,59,52)(36,62,89)(39,79,86)(41,48,88)
(44,58,77)(45,73,81)(46,54,82)(50,69,83)(53,56,67)(60,71,74)
```

Rec(3,28) = 415104

The smallest known trivalent graph of girth 28 is of order 415104. It is the graph $B(PGL(2,47) \times Alt(4), \{\alpha, \delta, \delta^{-1}\})$ discovered by Bray, Parker and Rowley [21]. The generating permutations α and δ are as follows.
α :

```
( 1, 4)( 2, 3)( 5,20)( 6,52)( 7,45)( 8,12)( 9,28)(10,32)
(11,33)(13,47)(14,42)(15,34)(16,22)(17,24)(18,41)(19,26)
(21,27)(23,38)(25,50)(29,49)(30,44)(31,35)(36,46)(37,39)
(43,48)
```

δ :

```
( 1, 4, 2)( 5,33,11)( 6,30,17)( 7,19,21)( 8,38,45)( 9,34,39)
(10,37,20)(12,14,26)(13,44,23)(15,35,31)(16,51,27)(18,50,46)
(22,48,28)(24,42,47)(25,36,43)(29,32,41)(40,49,52)
```

Rec(3,29) = 1143408

The smallest known trivalent graph of girth 29 is of order 1143408. It is the graph $B(PGL(2,83) \times Alt(4), \{\alpha, \delta, \delta^{-1}\})$ discovered by Bray, Parker and Rowley [21]. The generating permutations α and δ are as follows.
α :

```
( 1, 3)( 2, 4)( 5, 6)( 7,42)( 8,73)( 9,69)(10,62)(11,43)
(12,36)(13,32)(14,63)(15,16)(17,34)(18,86)(19,87)(20,27)
(21,77)(22,33)(23,44)(24,79)(25,66)(26,81)(28,84)(29,41)
(30,67)(31,65)(35,53)(37,46)(38,75)(39,80)(40,74)(45,48)
(47,55)(49,51)(50,58)(52,70)(54,56)(57,60)(59,68)(61,82)
(64,76)(71,88)(72,83)(78,85)
```

δ :

```
( 2, 4, 3)( 5,10,49)( 6,26,77)( 7,41,24)( 8,53,50)( 9,85,75)
(11,44,45)(12,78,80)(13,28,60)(14,23,81)(15,71,69)(16,19,62)
(17,31,66)(18,46,56)(20,35,72)(21,68,74)(22,42,55)(25,34,51)
(27,58,64)(29,32,70)(30,76,86)(33,63,61)(36,52,54)(37,87,88)
(38,43,83)(39,57,47)(40,82,79)(48,59,67)(65,73,84)
```

Rec(3,30) = 1227666

The smallest known trivalent graph of girth 30 has order 1227666 and is the sextet graph (see 4.1.1), S(313), discovered by Biggs and Hoare [19].

Rec(3,31) = 3649794

The smallest known trivalent graph of girth 31 is of order 3649794 and was obtained by Bray, Parker and Rowley [21], using excision, from the graph $B(PGL(2,97) \times Alt(4), \{\alpha, \delta, \delta^{-1}\})$ defined next.

Rec(3,32) = 3650304

The smallest known trivalent graph of girth 32 is of order 3650304. It is the graph $B(PGL(2,97) \times Alt(4), \{\alpha, \delta, \delta^{-1}\})$ discovered by Bray, Parker and Rowley [21]. The generating permutations α and δ are as follows.
α :

```
( 1,  3)( 2,  4)( 5, 34)( 6, 91)( 7, 22)( 8, 45)( 9, 66)
(10, 41)(11, 95)(12, 88)(13, 67)(14, 80)(15, 19)(16, 17)
(18, 68)(20, 30)(21, 26)(23, 31)(24, 61)(25, 71)(27, 65)
(28, 59)(29, 36)(32, 97)(33, 40)(35, 64)(37, 70)(38, 46)
(39, 49)(42,102)(43, 48)(44, 96)(47, 62)(50, 54)(51, 99)
(52, 53)(55, 87)(56,100)(57, 79)(58, 72)(60,101)(63, 76)
(69, 86)(73, 92)(74, 83)(75, 94)(77, 89)(78, 90)(81, 98)
(82, 85)(84, 93)
```

δ :

```
( 2,  3,  4)( 5, 62, 69)( 6, 87, 20)( 7, 13, 49)( 8, 25, 79)
( 9, 56, 72)(10, 21, 33)(11, 19, 38)(12, 52, 44)(14, 39, 30)
(15, 43, 81)(16, 97, 83)(17, 85, 29)(18, 86, 74)(22, 60, 88)
(23, 28, 27)(24, 78, 95)(26, 67,100)(31, 47, 94)(32, 45, 42)
(34, 41, 98)(35, 53, 46)(36, 77,101)(37,102, 55)(48, 99, 66)
(50, 68, 57)(51, 91, 59)(54, 90, 96)(58, 71, 61)(64, 89, 73)
(65, 84, 92)(70, 82, 93)(75, 80, 76)
```

4.2.2 Individual Constructions for Girth 5

For graphs of degree k and girth 5, the Moore bound is k^2+1. In this case there are Moore graphs for degrees 3, 7, and perhaps 57. The case $k = 57$ is unresolved, and has received a lot of attention. Aschbacher showed that there does not exist a rank 3 permutation group with subdegree 57. This means that a Moore graph of degree 57 is not a rank 3 graph and is not distance transitive. In an unpublished work, G. Higman also showed that such a graph is not vertex-transitive [28]. Further restrictions on the automorphism group of a potential Moore graph of degree 57 were obtained by Makhnev and Paduchikh [64], who showed that if the automorphism group of such a graph contains an involution, then the order of the group must be relatively small.

The best currently known graphs of girth 5 and degree up to 20 are listed in the next table.

Degree k	Lower Bound	Upper Bound	Due to
3	10	10	Petersen
4	19	19	Robertson
5	30	30	Robertson-Wegner-Wong
6	40	40	Wong
7	50	50	Hoffman-Singleton
8	67	80	Royle
9	86	96	Jørgensen
10	103	126	Exoo
11	124	156	Jørgensen
12	147	203	Exoo
13	174	240	Exoo
14	199	288	Jørgensen
15	230	312	Jørgensen
16	259	336	Jørgensen
17	294	448	Schwenk
18	327	480	Schwenk
19	364	512	Schwenk
20	403	576	Jørgensen

Table 4. Bounds for girth 5 cages.

Next, we briefly describe the graphs from the above table that are not cages.

Rec(8,5) = 80

The smallest known $(8,5)$-graph is of order 80. It was discovered by Royle [80], and is a Cayley graph. It can be constructed using either SmallGroup(80,32) or SmallGroup(80,33) in the GAP Small Group Library [44].

Rec(9,5) = 96

The smallest known $(9,5)$-graph, of order 96, was constructed by Jørgensen [57] using a cyclic relative difference set. The construction is from the first part of Theorem 8, for the case $q = 7$. The group used is cyclic of order 48, the subgroup is cyclic of order 6, and the relative difference set has size 6.

Rec(10,5) = 126

The smallest known $(10,5)$-graph has order 126 and was discovered by Exoo [38]. It is a voltage graph constructed from a base graph of order 2 and voltage group of order 63. Its automorphism group has order 3528 and two vertex orbits.

Rec(11,5) = 156

The smallest known $(11,5)$-graph, of order 156, was constructed by Jørgensen [57] using a cyclic relative difference set. The construction is a variation of Jørgensen's method using the group $\mathbb{Z}_{13} \times Sym(3)$, the subgroup $\{0\} \times Sym(3)$, and the relative difference set

$$\{(1, id), (10, id), (11, id), (0, (12)), (5, (12)), (2, (23)), (8, (23)), (7, (13)), (9, (13))\}.$$

Rec(12,5) = 203

The smallest known $(12,5)$-graph has order 203 and was discovered by Exoo [38]. It is a Cayley graph of the semi-direct product $\mathbb{Z}_{29} \rtimes \mathbb{Z}_7$ (of order 203), whose full automorphism group is the same group again, and hence, it is a graphical regular representation [45].

Rec(13,5) = 240

The smallest known $(13,5)$-graph has order 240 and was discovered by Exoo [38]. It is a voltage graph constructed from a base graph of order 10 and voltage group of order 24. Its automorphism group has order 13200 and two vertex orbits.

Rec(14,5) = 288

The smallest known $(14,5)$-graph, of order 288, was constructed by Jørgensen [57] using a dihedral relative difference set. The construction is from the second part of Theorem 8, for the case $q = 13$ and $k = 14$.

Rec(15,5) = 312

The smallest known $(15,5)$-graph, of order 312, was constructed by Jørgensen [57] using a dihedral relative difference set. The construction is from the second part of Theorem 8, for the case $q = 13$ and $k = 15$.

Rec(16,5) = 336

The smallest known $(16,5)$-graph, of order 336, was constructed by Jørgensen [57] using a dihedral relative difference set. The construction is from the second part of Theorem 8, for the case $q = 13$ and $k = 16$.

Rec(17,5) = 448

The smallest known $(17,5)$-graph has order 448, and was constructed by Schwenk [83] from the $(19,5)$-graph by removing two copies of the Möbius-Kantor graph from each of the sets P and Q.

Rec(18,5) = 480

The smallest known $(18,5)$-graph has order 480, and was constructed by Schwenk [83] from the $(19,5)$-graph by removing one copy of the Möbius-Kantor graph from each of the sets P and Q.

Rec(19,5) = 512

The smallest known $(19,5)$-graph has order 512, and was discovered by Schwenk [83]. The graph is constructed from two sets P and Q each consisting of 16 copies, indexed by $\mathbb{F}_{16}$, of the Möbius-Kantor graph (see Figure 12). In addition, the vertices of each copy of the graph are labeled by $\mathbb{F}_{16}$. The remaining edges join vertices from copies in P to vertices from copies in Q according to a rule based on field operations in $\mathbb{F}_{16}$ analogous to the rule used in Robertson's construction of the Hoffmann-Singleton graph (see 2.1.12).

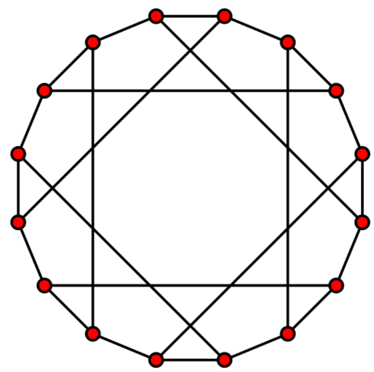

Figure 12: The Möbius-Kantor Graph

Rec(20,5) = 576

The smallest known $(20,5)$-graph, of order 576, was constructed by Jørgensen [57] using a dihedral relative difference set. The construction is from the second part of Theorem 8, for the case $q = 17$ and $k = 20$.

4.2.3 Individual Constructions for Girth 6

When $k = q+1$, for a prime power q, the $(k,6)$-cage is the incidence graph of a projective plane. Outside these cases, there is only one case where the order of the cage has been established, namely $k = 7$ [73]. The remaining cases listed in the following table all come from infinite families discussed in Section 4.1.3.

Degree k	Lower Bound	Upper Bound	Due to
3	14	14	Projective Plane
4	26	26	Projective Plane
5	42	42	Projective Plane
6	62	62	Projective Plane
7	90	90	O'Keefe-Wong
8	114	114	Projective Plane
9	146	146	Projective Plane
10	182	182	Projective Plane
11	224	240	Wong
12	266	266	Projective Plane
13	314	336	Abreu-Funk-Labbate-Napolitano
14	366	366	Projective Plane
15	422	462	Abreu-Funk-Labbate-Napolitano
16	482	504	Abreu-Funk-Labbate-Napolitano
17	546	546	Projective Plane
18	614	614	Projective Plane
19	686	720	Abreu-Funk-Labbate-Napolitano
20	762	762	Projective Plane

Table 5. Bounds on cages of girth 6.

4.2.4 Summary of Individual Constructions

In the table below, we summarize the best known upper bounds for degrees up to 20 and girths up to 16. Recall that entries for degree 3 and girths greater than 16 can be found in Table 3.

Most of the cases where there is an entry in the table that does not appear in any of the previous tables are for girths 8 and 12. For girths 8 and 12, there is a cage known whenever $k-1$ is a prime power. All but two of the other entries for girths 8 and 12 come from Gács and Héger [43]. The remaining entries, $(15, 8)$ and $(15, 12)$, are due to Balbuena [9] and Araujo-Pardo and Balbuena [2], respectively.

The four entries for girth 7 consist of the two known cages (degrees 3 and 4), and two graphs constructed using excision by McKay and Yuanshen [69]. The remaining two entries, that have not appeared previously, are for degree 4 and girths 9 and 10 [38].

k/g	5	6	7	8	9	10	11	12	13	14	15	16
3	10	14	24	30	58	70	112	126	272	384	620	960
4	19	26	67	80	275	384		728				
5	30	42	152	170				2730				
6	40	62	294	312				7812				
7	50	90		686				32928				
8	80	114		800				39216				
9	96	146		1170				74898				
10	126	182		1640				132860				
11	156	240		2618				319440				
12	203	266		2928				354312				
13	240	336		4342				738192				
14	288	366		4760				804468				
15	312	462		7648				1957376				
16	336	504		8092				2088960				
17	448	546		8738				2236962				
18	480	614		10440				3017196				
19	512	720		13642				4938480				
20	576	762		14480				5227320				

Table 6. Summary of Upper Bounds for $n(k, g)$.

Appendix

A Highly Symmetric Graphs

Many of the graphs discussed above exhibit a high level of symmetry. All the known Moore graphs possess an automorphism group that acts transitively on the set of vertices of the graph. Similarly, the majority of the other graphs discussed in this survey have a large automorphism group with very few orbits. This observation suggests that highly symmetric graphs deserve a careful examination.

In the case of small cages the reason behind the usefulness of symmetric graphs may be due to the fact that small regular graphs are more likely to have a relatively large group of automorphisms, while for larger instances of the cage problem, this may be due to the availability of group theoretic packages like GAP [44].

A *graph automorphism* of a graph G is a permutation φ of the vertices of G that preserves the structure of G, i.e., any two vertices, u and v, are adjacent if and only if $\varphi(u)$ is adjacent to $\varphi(v)$. The set of all automorphisms of G forms a group, $Aut(G)$, under the operation of composition.

In the following sections, we review the class of vertex-transitive graphs and its subclass of Cayley graphs.

A1 Vertex-Transitive Graphs

A group G acting on a set V is said to *act transitively* on V if for any $u, v \in V$, there exists an element $g \in G$ that maps u to v. A graph G is *vertex-transitive* if $Aut(G)$, the automorphism group of G, acts transitively on the set of vertices, $V(G)$ ([15]). Thus, vertex-transitive graphs look the same at each vertex, and all the vertices of such graphs lie on the same number of cycles of any particular length. In particular, each vertex lies on a cycle of length g, the girth of the graph. In fact, the number of cycles of any fixed length through any vertex v must satisfy additional arithmetic properties related to the order of the graph, $|V(G)|$, (see, for example, [54]). Looking at the other end of the cycle spectrum, note that all but four of the known non-trivial vertex-transitive graphs are Hamiltonian; the four exceptions being the Petersen and Coxeter graph, and the two graphs obtained from these by replacing their vertices by triangles.

Of all the known constructions of vertex-transitive graphs, let us mention the most direct one. For the purpose of simplification we restrict ourselves to the case of finite groups acting on finite sets.

Let Γ be *a permutation group acting transitively on a set* V (i.e., $\Gamma \leq Sym(V)$, the full symmetric group of all permutations of the set V), and let v^g denote the vertex resulting from the action of $g \in \Gamma$ on $v \in V$. Then Γ has a natural induced action on the Cartesian product $V \times V$ defined by $(u, v)^g = (u^g, v^g)$, for all $(u, v) \in V \times V$ and $g \in \Gamma$, and the Γ orbits in $V \times V$ are called the *orbitals* of Γ. For each orbital Δ of the action of Γ on $V \times V$, there is a *paired orbital* $\Delta^* = \{(v, u) \mid (u, v) \in \Delta\}$.

Let Ω be any set of orbitals of Γ closed under taking paired orbitals, and let E be the set of (unordered) pairs $\{u, v\}$ such that (u, v) belongs to Ω. Then the graph $G = (V, E)$ is called the *orbital graph* of Γ (with respect to Ω). It is easy to see that the orbital graph of any transitive permutation group is a vertex-transitive graph, and moreover, that any vertex-transitive graph $G = (V, E)$ is the orbital graph of any of its vertex-transitive automorphism groups $\Gamma \leq Aut(G)$ with $\Omega = \{(u, v)^g \mid \{u, v\} \in E, g \in \Gamma\}$, i.e., the class of vertex-transitive graphs is the class of orbital graphs of transitive permutations groups (for more details see [78]).

Note in addition, that an abstract group Γ has a transitive permutation representation on a set of size n if and only if Γ has a subgroup Λ of index n; in which case Γ can be thought of as acting on the (right) cosets of Λ in Γ via (right) multiplication ([13]). Hence, the class of (finite) vertex-transitive graphs is the class of orbital graphs of (finite) groups acting via left multiplication on the left cosets of their subgroups, and *every* vertex-transitive graph whose automorphism group contains Γ acting transitively on its vertices can be obtained by choosing appropriate Λ and Ω.

A2 Cayley Graphs

A vertex-transitive graph G is Cayley if there exists an automorphism group Γ of G that *acts regularly* on $V(G)$; i.e., for each $u, v \in V(G)$, there exists *exactly one* automorphism $\varphi \in \Gamma$ such that $\varphi(u) = v$.

Cayley graphs constitute a subclass of the class of vertex-transitive graphs that have proved particularly useful in the construction of cages. The following definition is the one used throughout our survey.

Let Γ be an (abstract) finite group with a generating set X that does not contain the identity of Γ and is closed under taking inverses, $X = X^{-1}$ (no assumptions are made about the minimality of X). The *Cayley graph* $C(\Gamma, X)$ is the regular graph of degree $|X|$ that has Γ for its set of vertices and whose adjacency is defined by making each vertex g of the graph, $g \in \Gamma$, adjacent to all the vertices in the set $g \cdot X = \{g \cdot x \mid x \in X\}$. Alternatively, for any two vertices $g, h \in \Gamma$, h is adjacent to g if and only if $g^{-1}h \in X$. Note that the fact that X is closed under inverses makes the resulting graph undirected.

Besides being useful in cage construction, Cayley graphs have also been useful in proofs. The proof of following result mentioned in the introduction is a generalization of a proof due to Biggs [14].

Theorem 3 *Given any $k, g \geq 3$, there is k-regular graph G whose girth is at least g.*

Proof. The graph we construct is a Cayley graph.

Let $k, g \geq 3$, $r = \lfloor \frac{g}{2} \rfloor$, and $T_{k,r}$ denote the finite tree of radius r with center x in which all the vertices, whose distance from x is less than r, are of degree k; and the vertices at distance r from x are leaves of degree 1. Color the edges of $T_{r,k}$ by the k colors $\{1, 2, 3, \ldots, k\}$ subject to the edge-coloring rule that no two adjacent edges are of the same color. For each color i, let α_i denote the involutory [1] permutation of the vertices of $T_{k,r}$:

$$\alpha_i(u) = v \text{ if and only if the edge } \{u, v\} \text{ is colored by } i.$$

Let $\Gamma = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$ be the finite permutation group generated by the involutions α_i, and take $X = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. We claim that the k-regular graph $C(\Gamma, X)$ has girth at least g. First observe that any cycle of length s in $C(\Gamma, X)$ corresponds to a reduced word $w(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of length s that is equal to 1_Γ, $w(\alpha_1, \alpha_2, \ldots, \alpha_k) = 1_\Gamma$ (a word is reduced if no α_i's is immediately followed by itself). Clearly, $w(\alpha_1, \alpha_2, \ldots, \alpha_k) = 1_\Gamma$ implies $w(\alpha_1, \alpha_2, \ldots, \alpha_k)(x) = x$, for all $x \in X$, in the action of Γ on the vertices of $T_{k,r}$. Consider the effect of $w(\alpha_1, \alpha_2, \ldots, \alpha_k)$ on x, a neighbor of 1_Γ. Initially, each element of the word moves x one step toward the leaves. In order for the image of x to return back to x, an additional $r + 1$ elements are required. Thus, any reduced word representing the identity must have length at least $2r + 1$. Equivalently, the girth of $C(\Gamma, X)$ is at least g.
$\square$

Note that not all vertex-transitive graphs are Cayley. The Petersen graph is the smallest exception.

[1] *Involutory* is a word stolen from Biggs [14]

B Computer Methods

Computers have been used in the study of cages since the 1960's, starting with O'Keefe and Wong who used them to complete detailed case analyses [72, 74, 73, 93]. More intensive use of computers followed the introduction of McKay's *nauty* package [67], which proved to be particularly useful in work on lower bounds. Similarly, progress on upper bounds followed the development of programs such as GAP [44], which facilitated the use of large groups in Cayley and voltage graph constructions.

B1 Lower Bound Proofs and Isomorphism Checking

Computational proofs that have established the correct lower bounds for the orders of some of the cages were made possible by the ability to do fast isomorphism testing. The best example of this type of program is McKay's *nauty* package [67].

This approach has been used to establish the correct lower bound for $n(3,9)$ [23], $n(3,11)$ [68], and $n(4,7)$ [37]. Such proofs are organized by splitting the problem into a large number of subproblems, which can then be handled independently, and the work can be done in parallel on many different computers.

The computations begin by selecting a root vertex and constructing a rooted k-ary tree of radius $\frac{g-1}{2}$. The actual computation proceeds in two phases. First, lists of all non-isomorphic ways to add sets of m edges to the tree are determined (for some experimentally determined value of m). This phase involves extensive isomorphism checking. The second phase is the one that is more easily distributed across a large number of computers. Each of the isomorphism classes found in the first phase becomes an independent starting point for an exhaustive search to determine whether the required graph can be completed. Of all possible edges that could be added to the graph at this point, those that would violate the degree or girth conditions are eliminated. The order in which the remaining edges are considered is then determined by heuristics.

B2 Computer Searches

A number of computational techniques have been employed in finding small (k, g)-graphs.

Several investigators have developed fast methods for searching Cayley graphs, see for example [29, 47].

However, nearly all of the current record holders for degree 3 and girth 14 and up were found by relaxing the Cayley graph symmetry requirement. While Bray, Parker, and Rowley [21] achieved this by collapsing triangles in Cayley graphs, the use of voltage graphs allows one to expand the search space beyond Cayley graphs, and still restrict it via symmetry assumptions. Voltage graphs were used as a search space in [34, 35, 36].

At the other extreme, the graph which helped to established the value $n(4,7) = 67$ was found by a search that made no symmetry assumptions.

B3 Voltage Graphs

Informally, voltage graphs are lifts of base graphs determined by an assignment of group elements to oriented edges of the base graph. A *base graph* is a finite digraph with possible loops and multiple edges. We denote its vertex, edge, and arc (oriented edge) sets by $V(G)$, $E(G)$, and $D(G)$, respectively. Each edge $e \in E(G)$ is represented twice in $D(G)$ (once with each of the two possible orientations) and if $e \in D(G)$, we denote the reverse arc by e^{-1}.

Given a finite group Γ, a *voltage assignment* is a function $\alpha : D(G) \to \Gamma$ satisfying the property $\alpha(e^{-1}) = g^{-1}$ whenever $\alpha(e) = g$. The group Γ is called the *voltage group*.

Given a voltage assignment $\alpha : D(G) \to \Gamma$, the *lift* of G, often called the *derived graph* or the *derived regular cover* of G, denoted by G^α, is the graph whose vertex set is $V(G) \times \Gamma$ with two vertices (u, g) and (v, h) adjacent if and only if $uv \in E(G)$ and $g \cdot \alpha(uv) = h$.

We include two simple examples of the derived graph construction.

First, consider the *dumb-bell* graph depicted in Figure B1. with a voltage assignment chosen from Z_5. The derived graph in this example is the Petersen graph.

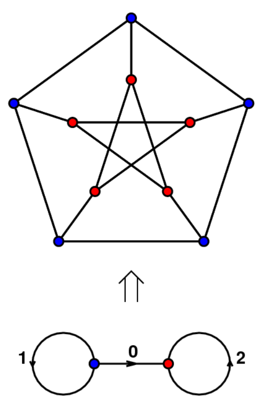

Figure B1: The Petersen Graph as a Lift

Our second example, shown in Figure B2, gives rise to the Heawood graph. The base graph is a dipole with a voltage assignment from Z_7.

B4 Excision

Several of the best known constructions are obtained by removing vertices and edges from graphs with the next largest girth. This process is known as *excision*.

The first example of the use of this technique was the $(3, 11)$-cage, constructed by Balaban [7] from the $(3, 12)$-cage by removing a small subtree and suppressing the resulting vertices of degree two.

The $rec(3, 21)$, $rec(3, 23)$, and $rec(3, 25)$-graphs were also obtained using excision [38], as was the $rec(3, 31)$-graph [21].

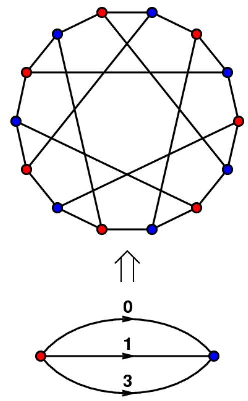

Figure B2: The Heawood Graph as a Lift

C The Upper Bound of Erdős

The following is essentially a verbatim translation of the Erdős' original proof in German [33], subject to some notational changes.

Theorem 2 *For every $k \geq 2$, $l \geq 3$,*

$$n(k,g) \leq 4 \sum_{t=1}^{g-2} (k-1)^t \tag{C1}$$

Note that for $k = 2$ as well as $g = 3$ the bound (C1) is sharp, and so from now on we will assume $k > 2$ and $g > 3$.

In order to prove the upper bound, we prove the following slightly stronger claim:

Let $k \geq 2$, $g \geq 4$, $m \geq 2\sum_{t=1}^{g-2}(k-1)^t$. Then there exists a k-regular graph $G^{(2m)}$ of order $2m$ with the property that all of its cycles are of length at least g.

We prove this theorem for a fixed g using induction on k. For $k = 2$, everything is trivial, $G^{(2m)}$ is simply the cycle of length $2m$ ($2m > g$ is clear). Let us assume now that our theorem holds for $k-1$; we want to prove it for k. ¿From now on, let $G^{(n)}(k,g)$ denote a k-regular graph of girth $\geq g$ with exactly n vertices. By our induction hypothesis, there exists a graph $G^{(2m)}(k-1,g)$. Let $G^{(2m)}$ now be a graph with the following three properties:

(I) $k-1 \leq v(x) \leq k$ for the degree $v(x)$ of all the $2m$ vertices

(II) all cycles have length at least g

(III) $G^{(2m)}$ has the maximal number of edges among all the graphs satisfying (I) and (II).

As $G^{(2m)}(k-1,g)$ satisfies (I) and (II), it is clear that $G^{(2m)}$ exists. We intend to show that the degrees of all the vertices of $G^{(2m)}$ are exactly k, hence $G^{(2m)}$ is k-regular, which will complete the proof.

First, we want to show that $G^{(2m)}$ contains at most one vertex of degree $< k$. Since this proof is not at all easy, let us first show that this result already implies the k-regularity of $G^{(2m)}$. Because of (I), the exceptional vertex must be of degree $k-1$. But that is impossible, due to the well-known fact that the number of vertices of odd degree must be even, and an even k would force the existence of exactly one vertex of odd degree, while an odd k would force the existence of exactly $2m-1$ such vertices; it therefore follows that all the vertices are of degree k and the proof of our theorem is completed.

Hence, it remains to prove that the existence of two vertices x_1 and x_2 in $G^{(2m)}$ of degree $< k$ (i.e., of degree $k-1$) leads to a contradiction. Let $N(x_i, g-2)$ stands for the set of vertices in $G^{(2m)}$ whose distance from x_i is at most $g-2$. Then we claim:

Lemma 1 *The set of all the vertices in $G^{(2m)}$ of degree less than k is contained in $N(x_1, g-2) \cap N(x_2, g-2)$.*

It is enough to show this for $N(x_1, g-2)$. If there existed an $x \notin N(x_1, g-2)$ of degree $< k$, the graph $G^{(2m)} + (x_1, x)$ (i.e., $G^{(2m)}$ with an added edge) would obviously satisfy conditions (I) and (II). Property (I) follows because the degree of both x_1 and x is assumed to be less than k, and property (II) follows because $x \notin N(x_1, g-2)$. This would however contradict the maximality property (III) of $G^{(2m)}$, and so Lemma 1 is proved.

Lemma 2 *Let x be a vertex of $G^{(2m)}$ of degree $< k$. Then*

$$|N(x,r)| \leq \sum_{t=0}^{r} (k-1)^t. \tag{C2}$$

Since the degree of x is smaller than k, (C2) follows for $r=1$. The rest of the proof for $r>1$ follows from a simple induction on r using the fact that the degree of all the vertices is at most k.

Combining Lemmata 1 and 2, we obtain

$$|N(x_1, g-2) \cup N(x_2, g-2)| \leq m, \tag{C3}$$

as it follows obviously from (C2) and Lemma 1 that

$$\begin{aligned} |N(x_1, g-2) \cup N(x_2, g-2)| &= |N(x_1, g-2)| + |N(x_2, g-2)| \\ &\quad - |N(x_1, g-2) \cap N(x_2, g-2)| \\ &\leq 2\sum_{t=0}^{g-2} (k-1)^t - 2 \leq m \end{aligned}$$

(because of Lemma 1 and because the degrees of x_1 and x_2 are smaller than k, $|N(x_1, g-2) \cap N(x_2, g-2)| \geq 2$). This proves (C3).

Now, let $x_1, \ldots, x_p$ be the vertices contained in $N(x_1, g-2) \cup N(x_2, g-2)$, and $y_1, \ldots, y_{2m-p}$ be the remaining vertices of $G^{(2m)}$. It follows from (C3) that

$$2m - p \geq p \tag{C4}$$

We want to show now that at least two of the y_j's are joined by an edge. First, because of Lemma 1, the degree of all the vertices y_j is k. If no two of the vertices y_j were connected through an edge, there would have to be $k(2m-p)$ edges connecting the y_j's to the x_i's. However, due to (C4) and the fact that the degrees of all the $(2m-p)$ vertices y_j is k, this would force the degrees of x_i's to be equal to k as well; a contradiction.

We may assume without loss of generality that $G^{(2m)}$ contains the edge y_1y_2. Let us consider now the graph $(G^{(2m)} - y_1y_2 + x_1y_1 + x_2y_2) = \overline{G}^{(2m)}$ (the edges x_1y_1 and x_2y_2 do not belong to $G^{(2m)}$). We claim that $\overline{G}^{(2m)}$ satisfies (I) and (II). Clearly, (I) holds true. If $\overline{G}^{(2m)}$ contained a cycle of length less than g, this cycle would have to contain one (or both) of the edges x_1y_1, x_2y_2. Because of the way the vertices y_j were defined

$$\begin{aligned} e(G^{(2m)}; x_1, y_1) &\geq g-1 \\ e(G^{(2m)}; x_2, y_2) &\geq g-1 \\ e(G^{(2m)} - (y_1, y_2); y_1, y_2) &\geq g-1 \end{aligned}$$

where $e(G^{(2m)}; x_i, y_i)$ stands for the distance between x_i and y_i in $G^{(2m)}$, and the last inequality follows from the fact that $G^{(2m)}$ does not contain cycles of length less than g.

It follows easily from these inequalities that $\overline{G}^{(2m)}$ satisfies (II). That, however, contradicts (III), as $\overline{G}^{(2m)}$ has more edges than $G^{(2m)}$.

Hence, $G^{(2m)}$ can contain at most one vertex of degree $< k$, and that completes the proof of Theorem 2.

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