Abstract

A vertex labeling of a graph $G$ is an assignment $f$ of labels to the vertices of $G$ that induces for each edge $xy$ a label depending on the vertex labels $f(x)$ and $f(y)$. The two best known labeling methods are called graceful and harmonious labelings. A function $f$ is called a graceful labeling of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the set $\{0, 1, \ldots, q\}$ such that, when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. A function $f$ is called a harmonious labeling of a graph $G$ with $q$ edges if it is an injection from the vertices of $G$ to the group of integers modulo $q$ such that when each edge $xy$ is assigned the label $f(x) + f(y) \pmod{q}$, the resulting edge labels are distinct. When $G$ is a tree, exactly one label may be used on two vertices. Over the past three decades many variations of graceful and harmonious labelings have evolved and about 300 papers have been written on the subject of graph labeling. In this article we survey what is known about the various methods.

1 Introduction

Most graph labeling methods trace their origin to one introduced by Rosa [Ro1] in 1967, or one given by Graham and Sloane [GS] in 1980. Rosa [Ro1] called a function $f$ a $\beta$-valuation of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the set $\{0, 1, \ldots, q\}$ such that, when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [Go] subsequently called such labelings graceful and this is now the popular term. Rosa introduced $\beta$-valuations as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs.
In particular, $\beta$-valuations originated as a means of attacking the conjecture of Ringel [Ri] that $K_{2n+1}$ can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with $n$ edges. Although an unpublished result of Erdős says that most graphs are not graceful (cf. [GS]), most graphs that have some sort of regularity of structure are graceful. Sheppard [Sh] has shown that there are exactly $q!$ gracefully labeled graphs with $q$ edges. Balakrishnan and Sampathkumar [BS] have shown that every graph is a subgraph of a graceful graph. Rosa [Ro1] has identified essentially three reasons why a graph fails to be graceful: (1) $G$ has “too many vertices” and “not enough edges”, (2) $G$ “has too many edges”, and (3) $G$ “has the wrong parity”. An infinite class of graphs that are not graceful for the second reason is given in [BG]. As an example of the third condition Rosa [Ro1] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles $C_{4n+1}$ and $C_{4n+2}$ are not graceful.

Harmonious graphs naturally arose in the study by Graham and Sloane [GS] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph $G$ with $q$ edges to be harmonious if there is an injection $f$ from the vertices of $G$ to the group of integers modulo $q$ such that when each edge $xy$ is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. When $G$ is a tree, exactly one label may be used on two vertices. Analogous to the “parity” necessity condition for graceful graphs, Graham and Sloane proved that if a harmonious graph has an even number $q$ of edges and the degree of every vertex is divisible by $2^k$ then $q$ is divisible by $2^{k+1}$. Thus, for example, a book with seven pages (i.e., the cartesian product of the complete bipartite graph $K_{1,7}$ and a path of length 1) is not harmonious. Liu and Zhang [LZ2] have generalized this condition as follows: if a harmonious graph with $q$ edges has degree sequence $d_1, d_2, \ldots, d_p$ then gcd($d_1, d_2, \ldots, d_p, q$) divides $q(q - 1)/2$. They have also proved that every graph is a subgraph of a harmonious graph.

Over the past three decades approximately 300 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments. In part because many of the papers have appeared in journals not widely available, frequently the same classes have been done by several authors. In this article, we survey what is known about numerous graph labeling methods. The author requests that he be sent preprints and reprints as well as corrections for inclusion in the updated versions of the survey.

Earlier surveys, restricted to one or two methods, include [Be], [Bl], [KRT2], [Ga2], and [Ga4]. The extension of graceful labelings to directed graphs arose in the characterization of finite neofields by Hsu and Keedwell [HK1], [HK2]. The relationship between graceful digraphs and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings, near-complete mappings and neofields is discussed in [BH1], [BH2]. The connection between graceful labelings and perfect systems of difference sets is given in [BKT]. Labeled graphs serve as useful models for a broad range of applications such as: coding theory, x-ray crystallogra-
phy, radar, astronomy, circuit design, communication network addressing and data base management—see [BG1], [BG2] and [Sut] for details. Terms and notation not defined below follow that used in [CL] and [Ga2].

2 Graceful and Harmonious Labelings

2.1 Trees

The Ringel-Kotzig conjecture that all trees are graceful has been the focus of many papers. Kotzig [HKR] has called the effort to prove it a “disease.” Among the trees known to be graceful are: caterpillars [Ro1] (a caterpillar is a tree with the property that the removal of its endpoints leaves a path); trees with at most 4 end-vertices [HKR], [Zha] and [JMW]; trees with diameter at most 5 [Zha] and [HrHa]; trees with at most 27 vertices [AlM]; symmetrical trees (i.e., a rooted tree in which every level contains vertices of the same degree) [BeS]; and olive trees [PR] and [AB2] (a rooted tree consisting of $k$ branches, where the $i$th branch is a path of length $i$). Stanton and Zarnke [SZ] and Koh, Rogers and Tan [KRT3] gave methods for combining graceful trees to yield larger graceful trees. Burzio and Ferrarese [BurF] have shown that the graph obtained from any graceful tree by subdividing every edge is also graceful. In 1979 Bermond [Be] conjectured that lobsters are graceful (a lobster is a tree with the property that the removal of the endpoints leaves a caterpillar). Special cases of this conjecture have been done by Ng [N1] and by Wang, Jin, Lu and Zhang [WJLZ]. Whether or not lobsters are harmonious seems to have attracted no attention thus far. Chen, Liu and Yeh [CLY] define a firecracker as a graph obtained from the concatenation of stars by linking one leaf from each. They also define a banana tree as a graph obtained by connecting a vertex $v$ to one leaf of each of any number of stars ($v$ is not in any of the stars). They proved that firecrackers are graceful and conjecture that banana trees are graceful. Various kinds of bananas trees have been shown to be graceful by Bhat-Nayak and Deshmukh [BD2], by Murugan and Arumugam [MA2], [MA4] and by Vilfred [Vil]. Despite the efforts of many, the graceful tree conjecture remains open even for trees with maximum degree 3. Aldred and McKay [AlM] used a computer to show that all trees with at most 26 vertices are harmonious. That caterpillars are harmonious has been shown by Graham and Sloane [GS]. More specialized results about trees are contained in [Be], [Bl], [KRT2], [LL], [C4] and [JLLLLZ].

2.2 Cycle-Related Graphs

Cycle-related graphs have been the major focus of attention. Rosa [Ro1] showed that the $n$-cycle $C_n$ is graceful if and only if $n \equiv 0$ or $3 \pmod{4}$ and Graham and Sloane [GS] proved that $C_n$ is harmonious if and only if $n \equiv 1$ or $3 \pmod{4}$. Wheels $W_n = C_n + K_1$ are both graceful and harmonious—[F1], [HK] and [GS]. Notice that a subgraph of a graceful
(harmonious) graph need not be graceful (harmonious). The $n$-cone (also called the $n$-point suspension of $C_m$) $C_m + K_n$ has been shown to be graceful when $m \equiv 0$ or $3 \pmod{12}$ by Bhat-Nayak and Selvam [BNS]. When $n$ is even and $m$ is $2$, $6$ or $10 \pmod{12}$, $C_m + K_n$ violates the parity condition for a graceful graph. Bhat-Nayak and Selvam [BNS] also prove that the following cones are graceful: $C_4 + K_n; C_5 + K_2; C_7 + K_n; C_9 + K_2; C_{11} + K_n$ and $C_{19} + K_n$. The helm $H_n$ is the graph obtained from a wheel by attaching a pendant edge at each vertex of the $n$-cycle. Helms have been shown to be graceful [AF] and harmonious [Gn], [LiuY3], [LiuY4] (see also [LZ2], [SY1], [LiuB2], [DL3] and [RP1]). Koh, et al. [KRTY] define a web graph as one obtained by joining the pendant points of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. They ask whether such graphs are graceful. This was proved by Kang, Liang, Gao and Yang [KLGY]. Yang has extended the notion of a web by iterating the process of adding pendant points and joining them to form a cycle and then adding pendant points to the new cycle. In his notation, $W(2, n)$ is the web graph whereas $W(t, n)$ is the generalized web with $t$ $n$-cycles. Yang has shown that $W(3, n)$ and $W(4, n)$ are graceful (see [KLGY]) and Abhyanker and Bhat-Nayak [AB1] have done $W(5, n)$. Gnanajothi [Gn] has shown that webs with odd cycles are harmonious. Seoud and Youssef [SY1] define a closed helm as the graph obtained from a helm by joining each pendant vertex to form a cycle and a flower as the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm. They prove that closed helms and flowers are harmonious when the cycles are odd. A gear graph is obtained from the wheel $W_n$ by adding a vertex between every pair of adjacent vertices of the $n$-cycle. Ma and Feng [MF2] have proved all gears are graceful. Liu [LiuY3] has shown that if two or more vertices are inserted between every pair of vertices of the $n$-cycle of the wheel $W_n$, the resulting graph is graceful. Liu [LiuY1] has also proved that the graph obtain from a gear graph by attaching one or more pendant points to each vertex between the cycle vertices is graceful.

Delorme, et al. [DMTKT] and Ma and Feng [MF1] showed that any cycle with a chord is graceful. This was first conjectured by Bodendiek, Schumacher and Wegner [BSW2], who proved various special cases. Koh and Yap [KY] generalized this by defining a cycle with a $P_k$-chord to be a cycle with the path $P_k$ joining two nonconsecutive vertices of the cycle. They proved that these graphs are graceful when $k = 3$ and conjectured that all cycles with a $P_k$-chord are graceful. This was proved for $k \geq 4$ by Punnim and Pabhapote in 1987 [PP]. Chen [CheZ2], [CheZ3] obtained the same result except for three cases which were then handled by Gao [Gn2]. Xu [X2] proved that all cycles with a chord are harmonic except for $C_6$ in the case where the distance in $C_6$ between the endpoints of the chord is 2. The gracefulness of cycles with consecutive chords have also been investigated. For $3 \leq p \leq n - r$, let $C_n(p, r)$ denote the $n$-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the $r$ chords $v_1v_p, v_1v_{p+1}, \ldots, v_1v_{p+r-1}$ have been added. Koh and others, [KRTY] and [KP], have handled the cases $r = 2, 3$ and $n - 3$ where $n$ is the length of the cycle. Goh and Lim [GoL] then proved that all remaining cases are graceful. Moreover, Ma [Ma] has shown that $C_n(p, n - p)$ is graceful when $p \equiv 0, 3$
Ma, Liu and Liu [MLL] have proved other special cases of these graphs are graceful. Ma also proved that if one adds to the graph $C_n(3, n - 3)$ any number $k_i$ of paths of length 2 from the vertex $v_1$ to the vertex $v_i$ for $i = 2, \ldots, n$, the resulting graph is graceful. Chen [CheZ1] has shown that apart from four exceptional cases, a graph consisting of three independent paths joining two vertices of a cycle is graceful. Liu [LiuR] has shown that the $n$-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the chords $v_1v_k$ and $v_1v_{k+2}$ ($2 \leq k \leq n - 3$) are adjoined is graceful.

In [DL2] Deb and Limaye use the notation $C(n, k)$ to denote the cycle $C_n$ with $k$ chords sharing a common endpoint. For certain choices of $n$ and $k$ there is a unique $C(n, k)$ graph and for other choices there is more than one graph possible. They call these shell-type graphs and they call the unique graph $C(n, n - 3)$ a shell. Notice that the shell $C(n, n - 3)$ is the same as the fan $F_{n-1}$. Deb and Limaye define a multiple shell to be a collection of edge disjoint shells that have their apex in common. They show that a variety of multiple shells are harmonious and they conjecture that all multiple shells are harmonious.

Sethuraman and Dhavamani [SD1] use $H(n, t)$ to denote the graph obtained from the cycle $C_n$ by adding $t$ consecutive chords incident with a common vertex. If the common vertex is $u$ and $v$ is adjacent to $u$, then for $k \geq 1$, $n \geq 4$ and $1 \leq t \leq n - 3$, Sethuraman and Dhavamani denote by $G(n, t, k)$ the graph obtained by taking the union of $k$ copies of $H(n, k)$ with the edge $uv$ identified. They conjecture that every graph $G(n, t, k)$ is graceful. They prove the conjecture for the case that $t = n - 3$.

Truszczynski [T] studied unicyclic graphs (i.e., graphs with a unique cycle) and proved several classes of such graphs are graceful. Among these are what he calls dragons. A dragon is formed by joining the end point of a path to a cycle (Koh, et al. [KRTY] call these tadpoles). This work led Truszczynski to conjecture that all unicyclic graphs except $C_n$, where $n \equiv 1$ or $2 \pmod{4}$, are graceful. Guo [Gu] has shown that dragons are graceful when the length of the cycle is congruent to 1 or 2 $\pmod{4}$. In his Master’s thesis, Doma [Do] investigates the gracefulness of various unicyclic graphs where the cycle has up to 9 vertices. Because of the immense diversity of unicyclic graphs, a proof of Truszczynski’s conjecture seems out of reach in the near future.

Cycles that share a common edge or a vertex have received some attention. Murugan and Arumugan [MA1] have shown that books with $n$ pentagonal pages (i.e., $nC_5$ with an edge in common) are graceful when $n$ is even and not graceful when $n$ is odd. Let $C_n^{(t)}$ denote the one-point union of $t$ cycles of length $n$. Bermond and others ([BBG] and [BKT]) proved that $C_3^{(t)}$ (that is, the friendship graph or Dutch $t$-windmill) is graceful if and only if $t \equiv 0$ or 1 $\pmod{4}$ while Graham and Sloane [GS] proved $C_3^{(t)}$ is harmonious if and only if $t \equiv 2$ $\pmod{4}$. Koh et al. [KRTL] conjecture that $C_n^{(t)}$ is graceful if and only if $nt \equiv 0$ or 3 $\pmod{4}$. Qian [Q] verifies this conjecture for the case that $t = 2$ and $n$ is even. Figueroa-Centeno, Ichishima and Muntaner-Batle [FIM6] have shown that if $m \equiv 0 \pmod{4}$ then the one-point union of 2, 3 or 4 copies of $C_m$ admits a special kind of graceful labeling called an $\alpha$-valuation (see Section 3.1) and if $m \equiv 2 \pmod{4}$
then the one-point union of 2 or 4 copies of $C_m$ admits an $\alpha$-valuation. Bodendiek, Schumacher and Wegner [BSW1] proved that the one-point union of any two cycles is graceful when the number of edges is congruent to 0 or 3 modulo 4. (The other cases violate the necessary parity condition.) Shee [S2] has proved that $C^t_4$ is graceful for all $t$. Seoud and Youssef [SY8] have shown that the one-point union of a triangle and $C_n$ is harmonious if and only if $n \equiv 1 \pmod{4}$ and that if the one-point union of two cycles is harmonious then the number of edges is divisible by 4. The question of whether this latter condition is sufficient is open. Figueroa-Centeno, Ichishima and Muntaner-Batle [FIM6] have shown that if $G$ is harmonious then the one-point union of an odd number of copies of $G$ using the vertex labeled 0 as the shared point is harmonious. Sethuraman and Selvaraju [SS6] have shown that for a variety of choices of points the one point union of any number of non-isomorphic complete bipartite graphs is graceful. They raise the question of whether this is true for all choices of the common point.

Another class of cycle-related graphs is that of triangular cacti. A triangular cactus is a connected graph all of whose blocks are triangles. A triangular snake is a triangular cactus whose block-cutpoint-graph is a path (a triangular snake is obtained from a path $v_1, v_2, \ldots, v_n$ by joining $v_i$ and $v_{i+1}$ to a new vertex $w_i$ for $i = 1, 2, \ldots, n - 1$). Rosa [Ro2] conjectured that all triangular cacti with $t \equiv 0$ or 1 (mod 4) blocks are graceful (the cases where $t \equiv 2$ or 3 (mod 4) fail to be graceful because of the parity condition.) Moulton [Mo] proved the conjecture for all triangular snakes. A proof of the general case (i.e., all triangular cacti) seems hopelessly difficult. Liu and Zhang [LZ2] gave an incorrect proof that triangular snakes with an odd number of triangles are harmonious while triangular snakes with $n \equiv 2$ (mod 4) triangles are not harmonious. Xu [X2] subsequently proved that triangular snakes are harmonious if and only if the number of triangles is not congruent to 2 (mod 4).

Defining $K_4$-snakes analogous to triangular snakes, Grace [Gr3] showed that these are harmonious. Rosa [Ro2] has also considered analogously defined quadrilateral and pentagonal cacti and examined small cases. Gnanajothi [Gn, pp. 31-34] has shown that quadrilateral snakes are graceful.

Several people have studied cycles with pendant edges attached. Frucht [F1] proved that any cycle with a pendant edge attached at each vertex (i.e., a “crown”) is graceful. Bu, Zhang and He [BZH] and Barrientos [Bar2] have shown that any cycle with a fixed number of pendant edges adjoined to each vertex is graceful. Barrientos [Bar2] proved that the graph obtained from a wheel by attaching one pendant edge to each vertex is graceful. Grace [Gr2] showed that an odd cycle with one or more pendant edges at each vertex is harmonious and conjectured that an even cycle with one pendant edge attached at each vertex is harmonious. This conjecture has been proved by Liu and Zhang [LZ1], Liu [LiuY3] and [LiuY4], Huang [Hua] and Bu [Bu2]. For any $n \geq 3$ and any $t$ with $1 \leq t \leq n$, let $C^t_n$ denote the class of graphs formed by adding a single pendant edge to $t$ vertices of a cycle of length $n$. Ropp [Rop] proved that for every $n$ and $t$ the class $C^t_n$ contains a graceful graph. Gallian and Ropp [Ga2] conjecture that for all $n$ and $t$, all members of $C^t_n$ are graceful. This was proved by Qian [Q] and by Kang, Liang, Gao
and Yang [KLGY]. Of course, this is just a special case of the aforementioned conjecture of Truszczynski that all unicyclic graphs except $C_n$ for $n \equiv 1$ or $2$ (mod 4) are graceful.

2.3 Product Related Graphs.

Graphs that are cartesian products and related graphs have been the subject of many papers. That planar grids, $P_m \times P_n$, are graceful was proved by Acharya and Gill [AG] in 1978 although the much simpler labeling scheme given by Maheo [Mah] in 1980 for $P_m \times P_2$ readily extends to all grids. In 1980 Graham and Sloane [GS] proved ladders, $P_m \times P_2$, are harmonious when $m > 2$ and in 1992 Jungreis and Reid [JR] showed that the grids $P_m \times P_n$ are harmonious when $(m,n) \not= (2,2)$. A few people have looked at graphs obtained from planar grids in various ways. Kathiresan [Kat1] has shown that graphs obtained from ladders by subdividing each step exactly once are graceful and that graphs obtained by appending an edge to each vertex of a ladder are graceful [Kat2]. Acharya [A2] has shown that certain subgraphs of grid graphs are graceful. Lee [L1] defines a Mongolian tent as a graph obtained from $P_m \times P_n$, $n$ odd, by adding one extra vertex above the grid and joining every other vertex of the top row of $P_m \times P_n$ to the new vertex. A Mongolian village is a graph formed by successively amalgamating copies of Mongolian tents with the same number of rows so that adjacent tents share a column. Lee proves that Mongolian tents and villages are graceful. A Young tableau is a subgraph of $P_m \times P_n$ obtained by retaining the first two rows of $P_m \times P_n$ and deleting vertices from the right hand end of other rows in such a way that the lengths of the successive rows form a nonincreasing sequence. Lee and K. C. Ng [LNK] have proved that all Young tableaus are graceful. Lee [L1] has also defined a variation of Mongolian tents by adding an extra vertex above the top row of a Young tableau and joining every other vertex of that row to the extra vertex. He proves these graphs are graceful.

Prisms are graphs of the form $C_m \times P_n$. These can be viewed as grids on cylinders. In 1977 Bodendiek, Schumacher and Wegner [BSW2] proved that $C_m \times P_2$ is graceful when $m \equiv 0$ (mod 4). According to the survey by Bermond [Be], T. Gangopadhyay and S. P. Rao Hebbar did the case that $m$ is even about the same time. In a 1979 paper, Frucht [F1] stated without proof that he had done all $m$. A complete proof of all cases and some related results were given by Frucht and Gallian [FG] in 1988. In 1992 Jungreis and Reid [JR] proved that all $C_m \times P_n$ are graceful when $m$ and $n$ are even or when $m \equiv 0$(mod 4). Yang and Wang [YW1] have shown that the prisms $C_{4m+2} \times P_{4m+3}$ are graceful. Singh [Sin1] proved that $C_3 \times P_n$ is graceful for all $n$. In their 1980 paper Graham and Sloane [GS] proved that $C_m \times P_n$ is harmonious when $n$ is odd and they used a computer to show $C_4 \times P_2$, the cube, is not harmonious. In 1992 Gallian, Prout and Winters [GPW] proved that $C_m \times P_2$ is harmonious when $m \neq 4$. In 1992, Jungreis and Reid [JR] showed that $C_4 \times P_n$ is harmonious when $n \geq 3$. Huang and Skiena [HuS] have shown that $C_m \times P_n$ is graceful for all $n$ when $m$ is even and for all $n$ with $3 \leq n \leq 12$ when $m$ is odd.

Torus grids are graphs of the form $C_m \times C_n$ ($m > 2, n > 2$). Very little success has
been achieved with these graphs. The graceful parity condition is violated for \( C_m \times C_n \) when \( m \) and \( n \) are odd and the harmonious parity condition [GS, Theorem 11] is violated for \( C_m \times C_n \) when \( m \equiv 1, 2, 3 \) (mod 4) and \( n \) is odd. The only result I'm aware of was done in 1992 by Jungreis and Reid [JR] who showed that \( C_m \times C_n \) is graceful when \( m \equiv 0 \) (mod 4) and \( n \) is even. A complete solution to both the graceful and harmonious torus grid problems will most likely involve a large number of cases.

There has been some work done on prism-related graphs. Gallian, Prout and Winters [GPW] proved that all prisms \( C_m \times P_2 \) with a single vertex deleted or single edge deleted are graceful and harmonious. The Möbius ladder \( M_n \) is the graph obtained from the ladder \( P_n \times P_2 \) by joining the opposite end points of the two copies of \( P_n \). In 1989 Gallian [Gal] showed that all Möbius ladders are graceful and all but \( M_3 \) are harmonious. Ropp [Rop] has examined two classes of prisms with pendant edges attached. He proved that all \( C_m \times P_2 \) with a single pendant edge at each vertex are graceful and all \( C_m \times P_2 \) with a single pendant edge at each vertex of one of the cycles are graceful.

Another class of cartesian products that has been studied is that of books and “stacked” books. The book \( B_m \) is the graph \( S_m \times P_2 \) where \( S_m \) is the star with \( m + 1 \) vertices. In 1980 Maheo [Mah] proved that the books of the form \( B_{2m} \) are graceful and conjectured that the books \( B_{4m+1} \) were also graceful. (The books \( B_{4m+3} \) do not satisfy the graceful parity condition.) This conjecture was verified by Delorme [D] in 1980. Maheo [Mah] also proved that \( L_n \times P_2 \) and \( B_{2m} \times P_2 \) are graceful. Both Grace [Gr1] and Reid (see [GJ]) have given harmonious labelings for \( B_{2m} \). The books \( B_{4m+3} \) do not satisfy the harmonious parity condition [GS, Theorem 11]. Gallian and Jungreis [GJ] conjectured that the books \( B_{4m+1} \) are harmonious. Gnanajothi [Gn] has verified this conjecture by showing \( B_{4m+1} \) has an even stronger form of labeling – see Section 4.1. Liang [Li] also proved the conjecture. In their 1988 paper Gallian and Jungreis [GJ] defined a stacked book as a graph of the form \( S_m \times P_n \). They proved that the stacked books of the form \( S_{2m} \times P_n \) are graceful and posed the case \( S_{2m+1} \times P_n \) as an open question. The \( n \)-cube \( K_2 \times K_2 \times \cdots \times K_2 \) (\( n \) copies) was shown to be graceful by Kotzig [K3]—see also [Mah]. In 1986 Reid [Re] found a harmonious labeling for \( K_4 \times P_n \).

The composition \( G_1[G_2] \) is the graph having vertex set \( V(G_1) \times V(G_2) \) and edge set \( \{(x_1, y_1), (x_2, y_2) | x_1 x_2 \in E(G_1) \text{ or } x_1 = x_2 \text{ and } y_1 y_2 \in E(G_2)\} \). The symmetric product \( G_1 \oplus G_2 \) of graphs \( G_1 \) and \( G_2 \) is the graph with vertex set \( V(G_1) \times V(G_2) \) and edge set \( \{(x_1, y_1), (x_2, y_2) | x_1 x_2 \in E(G_1) \text{ or } y_1 y_2 \in E(G_2) \text{ but not both}\} \). Seoud and Youssef [SY4] have proved that \( P_n \oplus K_2 \) is graceful when \( n > 1 \) and \( P_n[K_2] \) is harmonious for all \( n \). They also observe that the graphs \( C_m \oplus C_n \) and \( C_m[C_n] \) violates parity conditions for graceful and harmonious graphs when \( m \) and \( n \) are odd.

### 2.4 Complete Graphs

The questions of the gracefulfulness and harmoniousness of the complete graphs \( K_n \) have been answered. In each case the answer is positive if and only if \( n \leq 4 \) ([Go], [Si], [GS]). Both Rosa [Ro1] and Golomb [Go] proved that the complete bipartite graphs \( K_{m,n} \) are
graceful while Graham and Sloane [GS] showed they are harmonious if and only if \( m \) or \( n = 1 \). Aravamudhan and Murugan [AM] have shown that the complete tripartite graph \( K_{1,m,n} \) is both graceful and harmonious while Gnanajothi [Gn, pp. 25-31] has shown that \( K_{1,1,m,n} \) is both graceful and harmonious and \( K_{2,m,n} \) is graceful. Some of the same results have been obtained by Seoul and Youssef [SY3] who also observed that when \( m, n \) and \( p \) are congruent to \( 2 \mod 4 \), \( K_{m,n,p} \) violates the parity conditions for harmonious graphs.

Define the windmill graphs \( K_n^{(m)}(n > 3) \) to be the family of graphs consisting of \( m \) copies of \( K_n \) with a vertex in common. A necessary condition for \( K_n^{(m)} \) to be graceful is that \( n \leq 5 \)—see [KRTY]. Bermond [Be] has conjectured that \( K_{4}^{(m)} \) is graceful for all \( m \geq 4 \). This is known to be true for \( m \leq 22 \) [HuS]. Bermond, Kotzig and Turgeon [BKT] proved that \( K_n^{(m)} \) is not graceful when \( n = 4 \) and \( m = 2 \) or \( 3 \) and when \( m = 2 \) and \( n = 5 \). In 1982 Hsu [Hs] proved that \( K_4^{(m)} \) is harmonious for all \( m \). Graham and Sloane [GS] conjectured that \( K_n^{(2)} \) is harmonious if and only if \( n = 4 \). They verified this conjecture for the cases that \( n \) is odd or \( n = 6 \). Liu [LiuB2] has shown that \( K_n^{(2)} \) is not harmonious if \( n = 2^sp_1^{a_1} \cdots p_s^{a_s} \) where \( a, a_1, \ldots, a_s \) are positive integers and \( p_1, \ldots, p_s \) are distinct odd primes and there is a \( j \) for which \( p_j \equiv 3 \mod 4 \) and \( a_j \) is odd. He also shows that \( K_n^{(3)} \) is not harmonious when \( n \equiv 0 \mod 4 \) and \( 3n = 4^e(8k + 7) \) or \( n \equiv 5 \mod 8 \). Koh et al. [KRLT] and Rajasingh and Pushpam [RP2] have shown that \( K_{m,n} \), the one-point union of \( t \) copies of \( K_{m,n} \), is graceful. Sethuraman and Selvaraju [SS2] have proved that the one-point union of graphs of the form \( K_{2,m_i} \) for \( i = 1, 2, \ldots, n \) where the union is taken at a vertex from the partite set with \( 2 \) vertices is graceful if at most two of the \( m_i \) are equal. They conjecture that the restriction that at most two of the \( m_i \) are equal is not necessary. Koh et al. [KRTY] introduced the notation \( B(n, r, m) \) for the graph consisting of \( m \) copies of \( K_n \) with a \( K_r \) in common. (Guo [Gu2] has used the notation \( B(n, r, m) \) to denote three independent paths of lengths \( n, r \) and \( m \) joining two vertices.) Bermond [Be] raised the question: “For which \( m, n \) and \( r \) is \( B(n, r, m) \) graceful?” Of course, the case \( r = 1 \) is the same as \( K_n^{(m)} \). For \( r > 1 \), \( B(n, r, m) \) is graceful in the following cases: \( n = 3, r = 2, m \geq 1 \) [KRL]; \( n = 4, r = 2, m \geq 1 \) [D]; \( n = 4, r = 3, m \geq 1 \) (see [Be]), [KRL]. Seoul and Youssef [SY3] have proved \( B(3, 2, m) \) and \( B(4, 3, m) \) are harmonious. Liu [LiuB1] has shown that if there is a prime \( p \) such that \( p \equiv 3 \mod 4 \) and \( p \) divides both \( n \) and \( n - 2 \) and the highest power of \( p \) that divides \( n \) and \( n - 2 \) is odd, then \( B(n, 2, 2) \) is not graceful. More generally, Bermond and Farhi [BF] have considered the class of graphs consisting of \( m \) copies of \( K_n \) having exactly \( k \) copies of \( K_r \) in common. They proved such graphs are not graceful for \( n \) sufficiently large compared to \( r \).

Sethuraman and Elumalai [SE1] have shown that \( K_{1,r,s} \) with a pendent edge attached to each vertex is graceful and \( K_{m,n} \) with a pendent edge attached to each vertex is graceful when \( m \) is even and \( m \leq n \leq 2m + 4 \) and when \( m \) is odd and \( m \leq n \leq 2m - 1 \). In [SK] Sethuraman and Kishore determine the graceful graphs that are the union of \( n \) copies of \( K_4 \) with \( i \) edges deleted for \( 1 \leq i \leq 5 \) with one edge in common. The only cases that are not graceful are those graphs where the members of the union are \( C_4 \) for
n \equiv 3 \mod 4 \text{ and where the members of the union are } P_2. They conjecture that these two cases are the only instances of edge induced subgraphs of the union of } n \text{ copies of } K_4 \text{ with one edge in common that are not graceful. Sethuraman, Selvaraju and Elumalai [SSE] have shown that union of any number of copies of } K_4 \text{ with an edge deleted and one edge in common is harmonious.}

2.5 Disconnected Graphs

There have been many papers dealing with graphs that are not connected. In 1975 Kotzig [K2] considered graphs that are the disjoint union of } r \text{ cycles of length } s, \text{ denoted by } rC_s. \text{ When } rs \equiv 1 \text{ or } 2 \mod 4, \text{ these graphs violate the parity condition and so are not graceful. Kotzig proved that when } r = 3 \text{ and } s = 4k > 4, \text{ then } rC_s \text{ has a stronger form of graceful labeling called } \alpha\text{-labeling (see §3.1) whereas when } r \geq 2 \text{ and } s = 3 \text{ or } 5, \text{ } rC_s \text{ is not graceful. In 1984 Kotzig [K4] once again investigated the gracefulness of } rC_s \text{ as well as graphs that are the disjoint union of odd cycles. For graphs of the latter kind he gives several necessary conditions. His paper concludes with an elaborate table that summarizes what was then known about the gracefulness of } rC_s. \text{ He [He] has shown that graphs of the form } 2C_{2m} \text{ and graphs obtained by connecting two copies of } C_{2m} \text{ with an edge are graceful. Cahit [C7] has shown that } rC_s \text{ is harmonious when } r \text{ and } s \text{ are odd and Seoul, Abdel Maqsoud and Sheen [SAS1] noted that when } r \text{ or } s \text{ is even, } rC_s \text{ is not harmonious. Seoul, Abdel Maqsoud and Sheen [SAS1] proved that } C_n \cup C_{n+1} \text{ is harmonious if and only if } n \geq 4. \text{ They conjecture that } C_3 \cup C_{2n} \text{ is harmonious when } k \geq 3. \text{ In 1978 Kotzig and Turgeon [KT] proved that } mK_n \text{ (i.e., the union of } m \text{ disjoint copies of } K_n) \text{ is graceful if and only if } m = 1 \text{ and } n \leq 4. \text{ Liu and Zhang [LZ2] have shown that } mK_n \text{ is not harmonious for } n \text{ odd and } m \equiv 2 \mod 4 \text{ and is harmonious for } n = 3 \text{ and } m \text{ odd. They conjecture that } mK_3 \text{ is not harmonious when } m \equiv 0 \mod 4. \text{ Bu and Cao [BuC1] give some sufficient conditions for the gracefulness of graphs of the form } K_{m,n} \cup G \text{ and they prove that } K_{m,n} \cup P_t \text{ and the disjoint union of complete bipartite graphs are graceful under some conditions.}

A Skolem sequence of order } n \text{ is a sequence } s_1, s_2, \ldots, s_{2n} \text{ of } 2n \text{ terms such that, for each } k \in \{1, 2, \ldots, n\}, \text{ there exist exactly two subscripts } i(k) \text{ and } j(k) \text{ with } s_{i(k)} = s_{j(k)} = k \text{ and } |i(k) - j(k)| = k. \text{ A Skolem sequence of order } n \text{ exists if and only if } n \equiv 0 \text{ or } 1 \mod 4. \text{ Abraham [Ab2] has proved that any graceful 2-regular graph of order } n \equiv 0 \mod 4 \text{ in which all the component cycles are even or of order } n \equiv 3 \mod 4, \text{ with exactly one component an odd cycle, can be used to construct a Skolem sequence of order } n + 1. \text{ Also, he showed that certain special Skolem sequences of order } n \text{ can be used to generate graceful labelings on certain 2-regular graphs.}

In 1985 Frucht and Salinas [FS] conjectured that } C_s \cup P_n \text{ is graceful if and only if } s + n \geq 7 \text{ and they proved the conjecture for the case that } s = 4. \text{ Frucht [F3] did the case the } s = 3 \text{ and the case that } s = 2n + 1. \text{ Bhat-Nayak and Deshmukh [BD5] also did the case } s = 3 \text{ and they have done the cases of the form } C_{2x+1} \cup P_{x-2\theta} \text{ where } 1 \leq \theta \leq \lfloor (x - 2)/2 \rfloor \text{ [BD1]. Choudum and Kishore [CK2] have done the cases where}
s \geq 5 \text{ and } n \geq (s + 5)/2 \text{ and Kishore [Kis] did the case } s = 5. \text{ Gao and Liang [GaL] have done the following cases: } s > 4, n = 2 \text{ (see also [Gao]); } s = 4k, n = k + 2, n = k + 3, n = 2k + 2; \ s = 4k + 1, n = 2k, n = 3k - 1, n = 4k - 1; s = 4k + 2, n = 3k, n = 3k + 1, n = 4k + 1; s = 4k + 3, n = 2k + 1, n = 3k, n = 4k. \text{ Seoud, Abdel Maqsoud and Sheehan [SAS2] did the case that } s = 2k \ (k \geq 3) \text{ and } n \geq k + 1 \text{ as well as the cases where } s = 6, 8, 10, 12 \text{ and } n \geq 2. \text{ Shimazu [Shi] has handled the cases that } s \geq 5 \text{ and } n = 2, s \geq 4 \text{ and } n = 3 \text{ and } s = 2n + 2 \text{ and } n \geq 2.

Seoud and Youssef [SY2] have shown that } K_5 \cup K_{m,n}, K_{m,n} \cup K_{p,q} (m, n, p, q \geq 2), K_{m,n} \cup K_{p,q} \cup K_{r,s} (m, n, p, q, r, s \geq 2, (p, q) \neq (2, 2)), \text{ and } pK_{m,n} (m, n \geq 2, (m, n) \neq (2, 2)) \text{ are graceful. They also prove that } C_4 \cup K_{1,n} (n \neq 2) \text{ is not graceful whereas Choudum and Kishore [CK4], [Kis] have proved that } C_3 \cup K_{1,n} \text{ is graceful for every } s \geq 7 \text{ and } n \geq 1. \text{ Lee, Quach and Wang [LQW] established the gracefulness of } P_s \cup K_{1,n}.

Seoud and Wilson [SW] have shown that } C_3 \cup K_4, C_3 \cup C_3 \cup K_4 \text{ and certain graphs of the form } C_3 \cup P_n \text{ and } C_3 \cup C_3 \cup P_n \text{ are not graceful. Abrahim and Kotzig [AK4] proved that } C_p \cup C_q \text{ is graceful if and only if } p + q \equiv 0 \text{ or } 3 \text{ (mod } 4). \text{ Zhou [Zho] proved that } K_m \cup K_n \text{ is graceful if and only if } \{m, n\} = \{4, 2\} \text{ or } \{5, 2\}. \text{ Shee [S1] has shown that graphs of the form } P_2 \cup C_{2k+1} (k > 1), P_3 \cup C_{2k+1}, P_n \cup C_3 \text{ and } S_n \cup C_{2k+1} \text{ all satisfy a condition that is a bit weaker than harmonious. Bhat-Nayak and Deshmukh [BD3] have shown that } C_{4t} \cup K_{1,4t-1} \text{ and } C_{4t-3} \cup K_{1,4t+2} \text{ are graceful. Yang and Wang [YW2] proved that } S_m \cup S_n \text{ is graceful if and only if } m \text{ or } n \text{ is odd and that } S_m \cup S_n \cup S_k \text{ is graceful if and only if at least one of } m, n, \text{ or } k \text{ is odd } (m > 1, n > 1, k > 1).

Seoud and Youssef [SY5] investigated the gracefulness of specific families of the form } G \cup K_{m,n}. \text{ They obtained the following results:

- } C_3 \cup K_{m,n} \text{ is graceful if and only if } m \geq 2 \text{ and } n \geq 2;
- } C_4 \cup K_{m,n} \text{ is graceful if and only if } m \geq 2 \text{ and } n \geq 2 \text{ or } \{m, n\} = \{1, 2\};
- } C_7 \cup K_{m,n} \text{ and } C_8 \cup K_{m,n} \text{ are graceful for all } m \text{ and } n;
- } mK_3 \cup nK_{1,r} \text{ is not graceful for all } m, n \text{ and } r;
- } K_i \cup K_{m,n} \text{ is graceful for } i \leq 4 \text{ and } m \geq 2, n \geq 2 \text{ except for } i = 2 \text{ and } (m, n) = (2, 2);
- } K_5 \cup K_{1,n} \text{ is graceful for all } n;
- } K_6 \cup K_{1,n} \text{ is graceful if and only if } n = 1 \text{ or } 3.

### 2.6 Joins of Graphs

A few classes of graphs that are the join of graphs have been shown to be graceful and harmonious. Among these are fans } P_n + K_1 \text{ [GS] and double fans } P_n + 

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and observed that various cases of $K_{m,n} + K_t$ violate the harmonious parity condition in [GS]. Liu and Zhang [LZ2] have proved that $K_2 + K_2 + \cdots + K_2$ is harmonious. Yuan and Zhu [YZ] proved that $K_{m,n} + K_2$ is graceful and harmonious. Gnanajothi [Ga, pp.80-127] obtained the following: $C_n + K_2$ is harmonious when $n$ is odd and not harmonious when $n \equiv 2, 4, 6 \pmod{8}$; $S_n + K_t$ is harmonious; and $P_n + K_t$ is harmonious. Balakrishnan and Kumar [BK2] have proved that the join of $K_n$ and two disjoint copies of $K_2$ is harmonious if and only if $n$ is even. Bu [Bu2] obtained partial results for the gracefulness of $K_n + K_m$. Ramírez-Alfonsín [Ra] has proved that if $G$ is graceful and $|V(G)| = |E(G)| = e$ and either $1$ or $e$ is not a vertex label then $G + K_t$ is graceful for all $t$.

Seoud and Youssef [SY4] have proved: the join of any two stars is graceful and harmonious; the join of any path and any star is graceful; and $C_n + K_t$ are harmonious for every $t$ when $n$ is odd. They also prove that if any edge is added to $K_{m,n}$ the resulting graph is harmonious if $m$ or $n$ is at least 2. Deng [De] has shown certain cases of $C_n + K_t$ are harmonious. Seoud and Youssef [SY7] proved: the graph obtained by appending any number of edges from the two vertices of degree $n \geq 2$ in $K_{2,n}$ is not harmonious; dragons $D_{m,n}$ (i.e., $P_m$ is appended to $C_n$) are not harmonious when $m + n$ is odd; and the disjoint union of any dragon and any number of cycles is not harmonious when the resulting graph has odd order.

Sethuraman and Elumalai [SE3] have proved that for every graph $(p,q)$-graph $G$ the graph $G = K_1 + K_m$ is graceful when $m \geq 2p - p - 1 - q$. As a corollary they deduce that every graph is a vertex induced subgraph of a graceful graph. Balakrishnan and Sampathkumar [BS] ask for which $m$ is even. Liu and Zhang [LZ2] have proved that $K_2 + K_2 + \cdots + K_2$ is harmonious and conjecture that $P_n^k$ is not harmonious when $k > 3$. However, Youssef [Yo] has observed that $P_4^2$ is harmonious. Gnanajothi [Gn, p.50] has shown that the graph that consists of $n$ copies of $C_8$ that have exactly $P_4$ in common is graceful if and only if $n$ is even. For a fixed $n$, let $v_{i1}, v_{i2}, v_{i3}$ and $v_{i4}$ ($1 \leq i \leq n$) be consecutive vertices of $n$ 4-cycles. Gnanajothi [Gn, p.35] also proves that the graph obtained by joining each $v_{i1}$ to $v_{i+1,3}$ is graceful for all $n$ and the generalized Petersen graph $P(n,k)$ is harmonious in all cases (see also [LSS]). $P(n,k)$, where $n \geq 5$ and $1 \leq k \leq n$, has vertex set $\{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$ and edge set $\{a_ia_{i+1} \mid i = 0, 1, \ldots, n-1\} \cup \{a_ib_i \mid i = 0, 1, \ldots, n-1\} \cup \{b_ib_{i+k} \mid i = 0, 1, \ldots, n-1\}$ where all subscripts are taken modulo $n$ [Wat]. The standard Petersen graph is $P(5,2)$. The gracefulness of the generalized Petersen graphs appears to be an open problem.

### 2.7 Miscellaneous Results

It is easy to see that $P_n^2$ is harmonious [Gr2] while a proof that $P_n^2$ is graceful has been given by Kang, Liang, Gao and Yang [KLYG]. ($P_n^k$, the $k$th power of $P_n$, is the graph obtained from $P_n$ by adding edges that join all vertices $u$ and $v$ with $d(u,v) = k$.) This latter result proved a conjecture of Grace [Gr2]. Seoud, Abdel Maqsoud and Sheeham [SAS1] proved that $P_n^3$ is harmonious and conjecture that $P_n^k$ is not harmonious when $k > 3$. However, Youssef [Yo] has observed that $P_4^4$ is harmonious. Gnanajothi [Gn, p.50] has shown that the graph that consists of $n$ copies of $C_8$ that have exactly $P_4$ in common is graceful if and only if $n$ is even. For a fixed $n$, let $v_{i1}, v_{i2}, v_{i3}$ and $v_{i4}$ ($1 \leq i \leq n$) be consecutive vertices of $n$ 4-cycles. Gnanajothi [Gn, p.35] also proves that the graph obtained by joining each $v_{i1}$ to $v_{i+1,3}$ is graceful for all $n$ and the generalized Petersen graph $P(n,k)$ is harmonious in all cases (see also [LSS]). $P(n,k)$, where $n \geq 5$ and $1 \leq k \leq n$, has vertex set $\{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$ and edge set $\{a_ia_{i+1} \mid i = 0, 1, \ldots, n-1\} \cup \{a_ib_i \mid i = 0, 1, \ldots, n-1\} \cup \{b_ib_{i+k} \mid i = 0, 1, \ldots, n-1\}$ where all subscripts are taken modulo $n$ [Wat]. The standard Petersen graph is $P(5,2)$. The gracefulness of the generalized Petersen graphs appears to be an open problem.
Yuan and Zhu [YZ] define a generalization of $P_n^2$ as follows: $P_n(2k)$ is the graph obtained from the path $P_n$ by adding edges that join all vertices $x$ and $y$ with $d(x, y) = 2k$. They proved that $P_n(2k)$ is harmonious when $1 \leq k \leq \frac{n-1}{2}$ and that $P_n(2k)$ has a stronger form of harmonious labeling (see Section 4.1) when $2k-1 \leq n \leq 4k-1$. Cahit [C7] defines a $p$-star as the graph obtained by joining $p$ disjoint paths of a fixed length $k$ to single vertex. They proves all such graphs are harmonious when $p$ is odd and when $k = 2$ and $p$ is even.

Sethuraman and Selvaraju [SS1] define a graph $H$ to be a supersubdivision of a graph $G$, if every edge $uv$ of $G$ is replaced by a copy of $K_{2,m}$ ($m$ may vary for each edge) by identifying $u$ and $v$ with the two vertices in $K_{2,m}$ that form one of the two partite sets. Sethuraman and Selvaraju prove that every supersubdivision of a path is graceful and every cycle has some supersubdivision that is graceful. They conjecture that every supersubdivision of a star is graceful and that paths and stars are the only graphs for which every supersubdivision is graceful. In [SS3] Sethuraman and Selvaraju prove that every connected graph has some supersubdivision that is graceful. They pose the question as to whether this result is valid for disconnected graphs. They also ask if there is any graph other than $K_{2,m}$ that can be used to replace an edge of a connected graph to obtain a supersubdivision that is graceful. In [SS4] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions that have a strong form of graceful labeling called an $\alpha$-labeling (see §3.1).

Kathiresan [Kat4] uses the notation $P_{a,b}$ to denote the graph obtained by identifying the end points of $b$ internally disjoint paths each of length $a$. He conjectures that $P_{a,b}$ is graceful except when $a$ is odd and $b \equiv 2 \pmod{4}$. He proves the conjecture for the case that $a$ is even and $b$ is odd. Kathiresan also shows that the graph obtained by identifying a vertex of $K_n$ with any non-center vertex of the star with $2^n-1 - n(n-1)/2$ edges is graceful.

The total graph $T(P_n)$ has vertex set $V(P_n) \cup E(P_n)$ with two vertices adjacent whenever they are neighbors in $P_n$. Balakrishnan, Selvam and Yegnanarayanan [BSY1] have proved that $T(P_n)$ is harmonious.

For any graph $G$ with vertices $v_1, \ldots, v_n$ and a vector $m = (m_1, \ldots, m_n)$ of positive integers the corresponding replicated graph, $R_m(G)$, of $G$ is defined as follows. For each $v_i$ form a stable set $S_i$ consisting of $m_i$ new vertices $i = 1, 2, \ldots, n$ (recall a stable set $S$ consists of a set of vertices such that there is not an edge $v_i v_j$ for all pairs $v_i, v_j$ in $S$); two stable sets $S_i, S_j, i \neq j$, form a complete bipartite graph if each $v_i v_j$ is an edge in $G$ and otherwise there are edges between $S_i$ and $S_j$. Ramírez-Alfonsín [Ra] has proved that $R_m(P_n)$ is graceful for all $m$ and all $n > 1$ and that $R_{(m,1,1,1)}(C_{4n})$, $R_{(2,1,1,1)}(C_n)$ ($n \geq 8$) and, $R_{(2,2,2,1,1)}(C_{4n})$ ($n \geq 12$) are graceful.

For any permutation $f$ on $1, \ldots, n$, the $f$-permutation graph on a graph $G$, $P(G, f)$, consists of two disjoint copies of $G, G_1$ and $G_2$, each of which has vertices labeled $v_1, v_2, \ldots, v_n$ with $n$ edges obtained by joining each $v_i$ in $G_1$ to $v_{f(i)}$ in $G_2$. In 1983 Lee (see [LWK]) conjectured that for all $n > 1$ and all permutations on $1, 2, \ldots, n$, the permutation graph $P(P_n, f)$ is graceful. Lee, Wang and Kiang [LWK] proved that
\(P(P_{2k}, f)\) is graceful when \(f = (12)(34) \cdots (k, k + 1) \cdots (2k - 1, 2k)\). They conjectured
that if \(G\) is a graceful nonbipartite graph with \(n\) vertices then for any permutation \(f\) on \(1, 2, \ldots, n\), the permutation graph \(P(G, f)\) is graceful. Some families of graceful
permutation graphs are given in \([LLWK]\).

Gnanajothi \([Gn, p.51]\) calls a graph \(G\) bigraceful if both \(G\) and its line graph are
graceful. She shows the following are bigraceful: \(P_m; P_m \times P_n; C_n\) if and only if \(n \equiv 0, 3\)
(mod 4); \(S_n\); \(K_n\) if and only if \(n \leq 3\); and \(B_n\) if and only if \(n \equiv 3\) (mod 4). She also
shows that \(K_{m,n}\) is not bigraceful when \(n \equiv 3\) (mod 4). (Gangopadhyay and Hebbare
\([GH]\) used the term “bigraceful” to mean a bipartite graceful graph.) Murugan and
Arumugan \([MA3]\) have shown that graphs obtained from \(C_4\) by attaching two disjoint
paths of equal length to two adjacent vertices are bigraceful.

Several well-known isolated graphs have been examined. Graceful labelings of the
Petersen graph, the cube, the icosahedron and the dodecahedron can be found in \([Go]\)
and \([Gar]\). On the other hand, Graham and Sloane \([GS]\) showed that all of these except
the cube are harmonious. Winters \([Wi]\) verified that the Grötzsch graph (see \([BoM]\), p.
118), the Heawood graph (see \([BoM]\), p. 236) and the Herschel graph (see \([BoM]\), p. 53)
are graceful. Graham and Sloane \([GS]\) determined all harmonious graphs with at most
five vertices. Seoud and Youssef \([SY8]\) did the same for graphs with six vertices.

### 2.8 Summary

The results and conjectures discussed above are summarized in the tables following. The
letter \(G\) after a class of graphs indicates that the graphs in that class are known to be
graceful; a question mark indicates that the gracefulness of the graphs in the class is an
open problem; we put a “\(G\)” next to a question mark if the graphs have been conjectured
to be graceful. The analogous notation with the letter \(H\) is used to indicate the status
of the graphs with regard to being harmonious. The tables impart at a glimpse what
has been done and what needs to be done to close out a particular class of graphs. Of
course, there is an unlimited number of graphs one could consider. One wishes for some
general results that would handle several broad classes at once but the experience of
many people suggests that this is unlikely to occur soon. The Graceful Tree Conjecture
alone has withstood the efforts of scores of people over the past three decades. Analogous
sweeping conjectures are probably true but appear hopelessly difficult to prove.
Table 1: **Summary of Graceful Results**

<table>
<thead>
<tr>
<th>Graph</th>
<th>Graceful</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trees</td>
<td>G if $\leq 27$ vertices [AlM]</td>
</tr>
<tr>
<td></td>
<td>G if symmetrical [BeS]</td>
</tr>
<tr>
<td></td>
<td>G if at most 4 end-vertices [HKR]</td>
</tr>
<tr>
<td></td>
<td>?G Ringel-Kotzig</td>
</tr>
<tr>
<td>Cycles $C_n$</td>
<td>G iff $n \equiv 0, 3 \pmod{4}$ [Ro1]</td>
</tr>
<tr>
<td>Wheels $W_n$</td>
<td>G [F1], [HK]</td>
</tr>
<tr>
<td>Helms (see §2.2)</td>
<td>G [AF]</td>
</tr>
<tr>
<td>Webs (see §2.2)</td>
<td>G [KLGY]</td>
</tr>
<tr>
<td>Gears (see §2.2)</td>
<td>G [MF2]</td>
</tr>
<tr>
<td>Cycles with $P_k$-chord (see §2.2)</td>
<td>G [DMTKT], [MF1], [KY], [PP]</td>
</tr>
<tr>
<td>$C_n$ with $k$ consecutive chords (see §2.2)</td>
<td>G if $k = 2, 3, n - 3$ [KP], [KRTY]</td>
</tr>
<tr>
<td>Unicyclic graphs</td>
<td>?G iff $G \neq C_n, n \equiv 1, 2 \pmod{4}$ [T]</td>
</tr>
</tbody>
</table>
Table 1: continued

<table>
<thead>
<tr>
<th>Graph</th>
<th>Graceful</th>
</tr>
</thead>
</table>
| $C_n^{(t)}$ (see §2.2) | $n = 3$ G iff $t \equiv 0,1 \pmod{4}$ [BBG], [BKT]  
?G if $nt \equiv 0,3 \pmod{4}$ [KRLT]  
G if $n = 6, t$ even [KRLT]  
G if $n = 4, t > 1$ [S2]  
G if $t = 2, n \not\equiv 1 \pmod{4}$ [Q], [BSW1] |
| Triangular snakes (see §2.2) | G iff number of blocks $\equiv 0,1 \pmod{4}$ [Mo] |
| $K_4$-snakes (see §2.2) | ? |
| Quadilateral snakes (see §2.2) | G [Gn], [Q] |
| Crowns $C_n \odot K_1$ | G [F1] |
| Grids $P_m \times P_n$ | G [AG] |
| Prisms $C_m \times P_n$ | G if $n = 2$ [FG]  
G if $m$ even [HuS], G if $m$ odd,  
$3 \leq n \leq 12$ [HuS]  
G if $m = 3$ [Sin1] |
| Torus grids $C_m \times C_n$ | G if $m \equiv 0 \pmod{4}, n$ even [JR]  
not G if $m, n$ odd (parity condition) |
| Vertex-deleted $C_m \times P_n$ | G if $n = 2$ [GPW] |
| Edge-deleted $C_m \times P_n$ | G if $n = 2$ [GPW] |
| Möbius ladders $M_n$ (see §2.3) | G [Ga1] |
| Stacked books $S_m \times P_n$ (see §2.3) | $n = 2$, G iff $m \not\equiv 3 \pmod{4}$ [Mah], [D], [GJ]  
G if $m$ even [GJ] |
Table 1: continued

<table>
<thead>
<tr>
<th>Graph</th>
<th>Graceful</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)-cube (K_2 \times K_2 \times \cdots \times K_2)</td>
<td>G [K3]</td>
</tr>
<tr>
<td>(K_4 \times P_n)</td>
<td>?</td>
</tr>
<tr>
<td>(K_n)</td>
<td>G iff (n \leq 4) [Go], [Si]</td>
</tr>
<tr>
<td>(K_{m,n})</td>
<td>G [Ro1], [Go]</td>
</tr>
<tr>
<td>(K_{1,m,n})</td>
<td>G [AM]</td>
</tr>
<tr>
<td>(K_{1,1,m,n})</td>
<td>G [Gn]</td>
</tr>
</tbody>
</table>
| Windmills \(K_n^{(m)}(n > 3)\) (see §2.4) | G if \(n = 4, m \leq 22\) [HuS]  
\(?G\) if \(n = 4, m \geq 4\) [Be]  
G if \(n = 4, 4 \leq m \leq 22\) [HuS]  
not G if \(n = 4, m = 2, 3\) [Be]  
not G if \((m, n) = (2, 5)\) [BKT]  
not G if \(n > 5\) [KRTY] |
| \(B(n, r, m)\) \(r > 1\) (see §2.4) | G if \((n, r) = (3, 2), (4, 3)\) [KRL], (4,2) [D] |
| \(mK_n\) (see §2.5) | G iff \(m = 1, n \leq 4\) [KT] |
| \(C_s \cup P_n\) | ? G iff \(s + n \geq 7\) [FS]  
G if \(s = 3\) [F3], \(s = 4\) [FS], \(s = 5\) [Kis]  
G if \(s > 4, n = 2\) [GaL]  
G if \(s = 2n + 1\) [F3]  
G if \(s = 2k, n \geq k + 1\) [SAS2] |
| \(C_p \cup C_q\) | ? G iff \(p + q \equiv 0, 3 \pmod{4}\) [FS]  
G if \(s = 2n + 1\) [F3], \(s \geq 5\)  
and \(n \geq (s + 5)/2\) [CK2] |
Table 1: continued

<table>
<thead>
<tr>
<th>Graph</th>
<th>Graceful</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fans $F_n = P_n + K_1$</td>
<td>$G$ [GS]</td>
</tr>
<tr>
<td>Double fans $P_n + \overline{K_2}$</td>
<td>$G$ [GS]</td>
</tr>
<tr>
<td>$t$-point suspension $P_n + \overline{K_t}$ of $P_n$</td>
<td>$G$ [Gr2]</td>
</tr>
<tr>
<td>$S_m + K_1$</td>
<td>$G$ [H]</td>
</tr>
<tr>
<td>$t$-point suspension of $C_n + \overline{K_t}$</td>
<td>$G$ if $n \equiv 0$ or 3 (mod 12) [BNS]</td>
</tr>
<tr>
<td></td>
<td>not $G$ if $t$ is even and $n \equiv 2, 6$ or 10 (mod 12) [BNS]</td>
</tr>
<tr>
<td></td>
<td>$G$ if $n = 4, 7, 11$ or 19 [BNS]</td>
</tr>
<tr>
<td></td>
<td>$G$ if $n = 5$ or 9 and $t = 2$ [BNS]</td>
</tr>
<tr>
<td>$P_n^2$ (see §2.7)</td>
<td>$G$ [LK]</td>
</tr>
<tr>
<td>Petersen $P(n, k)$ (see §2.7)</td>
<td>?</td>
</tr>
<tr>
<td>Caterpillars</td>
<td>$G$ [Ro1]</td>
</tr>
<tr>
<td>Lobsters</td>
<td>$G$ [Be]</td>
</tr>
</tbody>
</table>
Table 2: Summary of Harmonious Results

<table>
<thead>
<tr>
<th>Graph</th>
<th>Harmonious</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trees</td>
<td>H if ≤ 26 vertices [AlM] (^?)H [GS]</td>
</tr>
<tr>
<td>Cycles (C_n)</td>
<td>H iff (n \equiv 1, 3 \pmod{4}) [GS]</td>
</tr>
<tr>
<td>Wheels (W_n)</td>
<td>H [GS]</td>
</tr>
<tr>
<td>Helms (see §2.2)</td>
<td>H [Gn], [LiuY]</td>
</tr>
<tr>
<td>Webs (see §2.2)</td>
<td>H if cycle is odd</td>
</tr>
<tr>
<td>Gears (see §2.2)</td>
<td>?</td>
</tr>
<tr>
<td>Cycles with (P_k)-chord (see §2.2)</td>
<td>?</td>
</tr>
<tr>
<td>(C_n) with (k) consecutive chords (see §2.2)</td>
<td>?</td>
</tr>
<tr>
<td>Unicyclic graphs</td>
<td>?</td>
</tr>
<tr>
<td>(C_n^{(t)}) (see §2.2)</td>
<td>(n = 3) H iff (t \not\equiv 2 \pmod{4}) [GS] () H if (n = 4, t &gt; 1) [S2]</td>
</tr>
<tr>
<td>Triangular snakes (see §2.2)</td>
<td>H if number of blocks is odd [X2] () not H if number of blocks (\equiv 2 \pmod{4}) [X2]</td>
</tr>
<tr>
<td>(K_4)-snakes (see §2.2)</td>
<td>H [Gr3]</td>
</tr>
<tr>
<td>Quadrilateral snakes (see §2.2)</td>
<td>?</td>
</tr>
<tr>
<td>Crowns (C_n \odot K_1)</td>
<td>H [Gr2], [LZ1]</td>
</tr>
<tr>
<td>Grids (P_m \times P_n)</td>
<td>H iff ((m, n) \neq (2, 2)) [JR]</td>
</tr>
<tr>
<td>Prisms (C_m \times P_n)</td>
<td>H if (n = 2, m \neq 4) [GPW] () H if (n) odd [GS] () H if (m = 4) and (n \geq 3) [JR]</td>
</tr>
</tbody>
</table>
Table 2: continued

<table>
<thead>
<tr>
<th>Graph</th>
<th>Harmonious</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torus grids $C_m \times C_n$,</td>
<td>H if $m = 4$, $n &gt; 1$ [JR]</td>
</tr>
<tr>
<td></td>
<td>not H if $m \not\equiv 0 \pmod{4}$ and $n$ odd [JR]</td>
</tr>
<tr>
<td>Vertex-deleted $C_m \times P_n$</td>
<td>H if $n = 2$ [GPW]</td>
</tr>
<tr>
<td>Edge-deleted $C_m \times P_n$</td>
<td>H if $n = 2$ [GPW]</td>
</tr>
<tr>
<td>Möbius ladders $M_n$ (see §2.3)</td>
<td>H iff $n \not\equiv 3$ [Ga]</td>
</tr>
<tr>
<td>Stacked books $S_m \times P_n$ (see §2.3)</td>
<td>$n = 2$, H if $m$ even [Gr1], [Re]</td>
</tr>
<tr>
<td></td>
<td>not H if $m \equiv 3 \pmod{4}$, $n = 2$, (parity condition)</td>
</tr>
<tr>
<td></td>
<td>H if $m \equiv 1 \pmod{4}$, $n = 2$ [Gn]</td>
</tr>
<tr>
<td>$n$-cube $K_2 \times K_2 \times \cdots \times K_2$</td>
<td>not H if $n = 2, 3$ [GS]</td>
</tr>
<tr>
<td>$K_4 \times P_n$</td>
<td>H [Re]</td>
</tr>
<tr>
<td>$K_n$</td>
<td>H iff $n \leq 4$ [GS]</td>
</tr>
<tr>
<td>$K_{m,n}$</td>
<td>H iff $m$ or $n = 1$ [GS]</td>
</tr>
<tr>
<td>$K_{1,m,n}$</td>
<td>H [AM]</td>
</tr>
<tr>
<td>$K_{1,1,m,n}$</td>
<td>H [Gn]</td>
</tr>
<tr>
<td>Windmills $K_n^{(m)}$ $(n &gt; 3)$ (see §2.4)</td>
<td>H if $n = 4$ [Hs]</td>
</tr>
<tr>
<td></td>
<td>$m = 2$, H iff $n = 4$ [GS]</td>
</tr>
<tr>
<td></td>
<td>not H if $m = 2$, $n$ odd or 6 [GS]</td>
</tr>
<tr>
<td></td>
<td>not H for some cases $m = 3$ [LiuB2]</td>
</tr>
<tr>
<td>$B(n, r, m)$ $r &gt; 1$ (see §2.4)</td>
<td>$(n, r) = (3, 2), (4, 3)$ [SY3]</td>
</tr>
<tr>
<td>$mK_n$ (see §2.5)</td>
<td>H $n = 3$, $m$ odd [LZ2]</td>
</tr>
<tr>
<td></td>
<td>not H for $n$ odd, $m \equiv 2 \pmod{4}$ [LZ2]</td>
</tr>
</tbody>
</table>
Table 2: continued

<table>
<thead>
<tr>
<th>Graph</th>
<th>Harmonious</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_s \cup P_n$</td>
<td>?</td>
</tr>
<tr>
<td>Fans $F_n = P_n + K_1$</td>
<td>H [GS]</td>
</tr>
<tr>
<td>Double fans $P_n + \overline{K}_2$</td>
<td>H [GS]</td>
</tr>
<tr>
<td>$t$-point suspension $P_n + \overline{K}_t$ of $P_n$</td>
<td>H [Re]</td>
</tr>
<tr>
<td>$S_m + K_1$</td>
<td>H [Gn], [CHR]</td>
</tr>
<tr>
<td>$t$-point suspension $C_n + \overline{K}_t$ of $C_n$</td>
<td>H if $n$ odd and $t = 2$ [Re], [Gn] not H if $n \equiv 2, 4, 6 \pmod{8}$ and $t = 2$ [Gn]</td>
</tr>
<tr>
<td>$P_n^2$ (see §2.7)</td>
<td>H [Gr2], [LZ1]</td>
</tr>
<tr>
<td>Petersen $P(n, k)$ (see §2.7)</td>
<td>H [Gn], [LSS]</td>
</tr>
<tr>
<td>Caterpillars</td>
<td>H [GS]</td>
</tr>
<tr>
<td>Lobsters</td>
<td>?</td>
</tr>
</tbody>
</table>

3 Variations of Graceful Labelings

3.1 $\alpha$-labelings

In [Ro1] Rosa defined an $\alpha$-labeling to be a graceful labeling with the additional property that there exists an integer $k$ so that for each edge $xy$ either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. (Other names for such labelings are balanced and interlaced.) It follows that such a $k$ must be the smaller of the two vertex labels that yield the edge labeled 1. Also, a graph with an $\alpha$-labeling is necessarily bipartite and therefore cannot contain a cycle of odd length. Wu [Wu1] has shown that a necessary condition for a bipartite graph with $n$ edges and degree sequence $d_1, d_2, \ldots, d_p$ to have an $\alpha$-labeling is that the gcd($d_1, d_2, \ldots, d_p, n$) divides $n(n - 1)/2$.

A common theme in graph labeling papers is to build up graphs that have desired labelings from pieces with particular properties. In these situations, starting with a graph that possesses an $\alpha$-labeling is a typical approach. (See [CHR], [Gr2], [CLY] and [JR].) Moreover, Jungreis and Reid [JR] showed how sequential labelings of graphs (see
Graphs with \( \alpha \)-labelings have proved useful in the development of the theory of graph decompositions. Rosa [Ro1], for instance, has shown that if \( G \) is a graph with \( q \) edges and has an \( \alpha \)-labeling, then for every natural number \( p \), the complete graph \( K_{2pq+1} \) can be decomposed into copies of \( G \) in such a way that the automorphism group of the decomposition itself contains the cyclic group of order \( n \). In the same vein El-Zanati and Vanden Eynden [EV1] proved that if \( G \) has \( q \) edges and admits an \( \alpha \)-labeling then \( K_{qm,qn} \) can be partitioned into subgraphs isomorphic to \( G \) for all positive integers \( m \) and \( n \). Although a proof of Ringel’s conjecture that every tree has a graceful labeling has withstood many attempts, examples of trees that do not have \( \alpha \)-labelings are easy to construct (see [Ro1]).

As to which graphs have \( \alpha \)-labelings, Rosa [Ro1] observed that the \( n \)-cycle has an \( \alpha \)-labeling if and only if \( n \equiv 0 \pmod{4} \) while \( P_n \) always has an \( \alpha \)-labeling. Other familiar graphs that have \( \alpha \)-labelings include caterpillars [Ro1], the \( n \)-cube [K1], \( B_{4n+1} \) (i.e., books with \( 4n+1 \) pages) [GJ], \( C_{2m} \cup C_{2m} \), and \( C_{4m} \cup C_{4m} \cup C_{4m} \) for all \( m > 1 \) [K2], \( P_n \times Q_n \) [Mah], \( K_{1,2k} \times Q_n \) [Mah], \( C_{4m} \cup C_{4m} \cup C_{4m} \cup C_{4m} \) [IV], \( C_{4m} \cup C_{4m+2} \cup C_{4m} \cup C_{4m} \cup C_{4r} \) when \( m+n \leq r \) [AK4], \( C_{4m} \cup C_{4m} \cup C_{4r} \cup C_{4s} \) when \( m \geq n+r+s \) [ACE], \( C_{4m} \cup C_{4m} \cup C_{4r+2} \cup C_{4s+2} \) when \( m \geq n+r+s+1 \) [ACE], \(((m+1)^2+1)C_4 \) for all \( m \) [Zhi], \( k^2C_4 \) for all \( k \) [Zhi], and \((k^2+k)C_4 \) for all \( k \) [Zhi]. Abrham and Kotzig [AK2] have shown that \( kC_4 \) has an \( \alpha \)-labeling for \( 1 \leq k \leq 10 \) and that if \( kC_4 \) has an \( \alpha \)-labeling then so does \((4k+1)C_4, (5k+1)C_4 \) and \((9k+1)C_4 \).

Figueroa-Centeno, Ichishima and Muntaner-Batle [FIM6] have shown that if \( m \equiv 0 \pmod{4} \) then the one-point union of 2, 3 or 4 copies of \( C_m \) admits an \( \alpha \)-valuation and if \( m \equiv 2 \pmod{4} \) then the one-point union of 2 or 4 copies of \( C_m \) admits an \( \alpha \)-valuation. They conjecture that the one-point union of \( n \) copies of \( C_m \) admits an \( \alpha \)-valuation if and only if \( mn \equiv 0 \pmod{4} \).

Zhile [Zhi] uses \( C_m(n) \) to denote the connected graph all of whose blocks are \( C_m \) and whose block-cutpoint-graph is a path. He proves that for all positive integers \( m \) and \( n \), \( C_{4m}(n) \) has an \( \alpha \)-labeling but \( C_m(n) \) does not have an \( \alpha \)-labeling when \( m \) is odd.

Abrham and Kotzig [AK4] have proved that \( C_m \cup C_n \) has an \( \alpha \)-labeling if and only if both \( m \) and \( n \) are even and \( m+n \equiv 0 \pmod{4} \). Kotzig [K2] has also shown that \( C_4 \cup C_4 \cup C_4 \) does not have an \( \alpha \)-labeling. He asked if \( n = 3 \) is the only integer such that the disjoint union of \( n \) copies of \( C_4 \) does not have an \( \alpha \)-labeling. This was confirmed by Abrham and Kotzig in [AK3]. Eshghi [Es] proved that every 2-regular bipartite graph with \( 3 \) components has an \( \alpha \)-labeling if and only if the number of edges is a multiple of \( 4 \) except for \( C_4 \cup C_4 \cup C_4 \).

Jungreis and Reid [JR] investigated the existence of \( \alpha \)-labelings for graphs of the form \( P_m \times P_n, C_m \times P_n, \) and \( C_m \times C_n \) (see also [Ga4]). Of course, the cases involving \( C_m \) with \( m \) odd are not bipartite, so there is no \( \alpha \)-labeling. The only unresolved cases among these three families are \( C_{4m+2} \times P_{2n+1} \) and \( C_{4m+2} \times C_{4n+2} \). All other cases result in \( \alpha \)-labelings. Balakrishman [Ba] uses the notation \( Q_n(G) \) to denote the graph \( P_2 \times P_2 \times \cdots \times P_2 \times G \) where \( P_2 \) occurs \( n-1 \) times. Snevily [Sn1] has shown that
the graphs $Q_n(C_{4m})$ and the cycles $C_{4m}$ with the path $P_n$ adjoined at each vertex have $\alpha$-labelings. He also has shown [Sn2] that compositions of the form $G[\overline{K_n}]$ have an $\alpha$-labeling whenever $G$ does (see §2.3 for the definition of composition). Balakrishnan and Kumar [BK1] have shown that all graphs of the form $Q_n(G)$ where $G$ is $K_{3,3}, K_{4,4}$, or $P_n$ have an $\alpha$-labeling. Balakrishnan [Ba] poses the following two problems. For which graphs $G$ does $Q_n(G)$ have an $\alpha$-labeling? For which graphs $G$ does $Q_n(G)$ have a graceful labeling? Rosa [Ro1] has shown that $K_{m,n}$ has an $\alpha$-labeling (see also [Bar3]). Barrientos [Bar3] has shown that for $n$ even the graph obtained from the wheel $W_n$ by attaching a pendant edge at each vertex has an $\alpha$-labeling. Qian [Q] has proved that quadrilateral snakes have $\alpha$-labelings. Fu and Wu [FW] showed that if $T$ is a tree that has an $\alpha$-labeling with partite sets $V_1$ and $V_2$ then the graph obtained from $T$ by joining new vertices $w_1, w_2, \ldots, w_k$ to every vertex of $V_1$ has an $\alpha$-labeling. Similarly, they prove that the graph obtained from $T$ by joining new vertices $w_1, w_2, \ldots, w_k$ to the vertices of $V_1$ and new vertices $u_1, u_2, \ldots, u_k$ to every vertex of $V_2$ has an $\alpha$-labeling. They also prove that if one of the new vertices of either of these two graphs is replaced by a star and every vertex of the star is joined to the vertices of $V_1$ or the vertices of both $V_1$ and $V_2$, the resulting graphs have $\alpha$-labelings. Fu and Wu [FW] further show that if $T$ is a tree with an $\alpha$-labeling and the sizes of the two partite sets of $T$ differ by at most 1, then $T \times P_m$ has an $\alpha$-labeling.

Barrientos [Bar4] defines a chain graph as one with blocks $B_1, B_2, \ldots, B_m$ such that for every $i$, $B_i$ and $B_{i+1}$ have a common vertex in such a way that the block-cutpoint graph is a path. He shows that if $B_1, B_2, \ldots, B_m$ are blocks that have $\alpha$-labelings then there exists a chain graph $G$ with blocks $B_1, B_2, \ldots, B_m$ that has an $\alpha$-labeling. He also shows that if $B_1, B_2, \ldots, B_m$ are complete bipartite graphs, then any chain graph $G$ obtained by concatenation of these blocks has an $\alpha$-labeling.

Wu ([Wu2] and [Wu3]) has given a number of methods for constructing larger graceful graphs from graceful graphs. Let $G_1, G_2, \ldots, G_p$ be disjoint connected graphs. Let $w_i$ be in $G_i$ for $1 \leq i \leq p$. Let $w$ be a new vertex not in any $G_i$. Form a new graph $\oplus_w(G_1, G_2, \ldots, G_p)$ by adjoining to the graph $G_1 \cup G_2 \cup \cdots \cup G_p$ the edges $ww_1, ww_2, \ldots, ww_p$. In the case where each of $G_1, G_2, \ldots, G_p$ is isomorphic to a graph $G$ which has an $\alpha$-labeling and each $w_i$ is the isomorphic image of the same vertex in $G_i,$ Wu shows that the resulting graph is graceful. If $f$ is an $\alpha$-labeling of a graph, the integer $k$ with the property that for any edge $uv$ either $f(u) \leq k < f(v)$ or $f(v) \leq k < f(u)$ is called the boundary value or critical number of $f$. Wu [Wu2] has also shown that if $G_1, G_2, \ldots, G_p$ are graphs of the same order and have $\alpha$-labelings where the labelings for each pair of graphs $G_i$ and $G_{p-i+1}$ have the same boundary value for $1 \leq i \leq n/2,$ then $\oplus_w(G_1, G_2, \ldots, G_p)$ is graceful.

Snevily [Sn2] says that a graph $G$ eventually has an $\alpha$-labeling provided that there is a graph $H$, called a host of $G$, which has an $\alpha$-labeling and that the edge set of $H$ can be partitioned into subgraphs isomorphic to $G$. He defines the $\alpha$-labeling number of $G$ to be $G_\alpha = \min\{t : \text{there is a host } H \text{ of } G \text{ with } |E(H)| = t|G|\}$. Snevily proved that even cycles have $\alpha$-labeling number at most 2 and he conjectured that every bipartite
graph has an $\alpha$-labeling number. This conjecture was proved by El-Zanati, Fu and Shiue [EFS]. There are no known examples of a graph $G$ with $G_\alpha > 2$.

Given two bipartite graphs $G_1$ and $G_2$ with partite sets $H_i$ and $L_i$ and $H_2$ and $L_2$, respectively, Snevily [Sn1] defines their weak tensor product $G_1 \boxtimes G_2$ as the bipartite graph with vertex set $(H_1 \times H_2, L_1 \times L_2)$ and with edge $(h_1, h_2)(l_1, l_2)$ if $h_1l_1 \in E(G_1)$ and $h_2l_2 \in E(G_2)$. He proves that if $G_1$ and $G_2$ have $\alpha$-labelings then so does $G_1 \boxtimes G_2$. This result considerably enlarges the class of graphs known to have $\alpha$-labelings.

The sequential join of graphs $G_1, G_2, \ldots, G_n$ is formed from $G_1 \cup G_2 \cup \cdots \cup G_n$ by adding edges joining each vertex of $G_i$ with each vertex of $G_{i+1}$ for $1 \leq i \leq n-1$. Lee and Wang [LWa] have shown that for all $n \geq 2$ and any positive integers $a_1, a_2, \ldots, a_n$ the sequential join of the graphs $K_{a_1}, K_{a_2}, \ldots, K_{a_n}$ has an $\alpha$-labeling.

In [Ga2] Gallian and Ropp conjectured that every graph obtained by adding a single pendant edge to one or more vertices of a cycle is graceful. Qian [Q] has proved this conjecture and in the case that the cycle is even he shows the graphs have an $\alpha$-labeling. He further proves that for $n$ even any graph obtained from an $n$-cycle by adding one or more pendant edges at some vertices has an $\alpha$-labeling as long as at least one vertex has degree 3 and one vertex has degree 2.

For any tree $T(V, E)$ whose vertices are properly 2-colored Rosa and Širáň [RS] define a bipartite labeling of $T$ as a bijection $f : V \rightarrow \{0, 1, 2, \ldots, |E|\}$ for which there is a $k$ such that whenever $f(u) \leq k \leq f(v)$, then $u$ and $v$ have different colors. They define the $\alpha$-size of a tree $T$ as the maximum number of distinct values of the induced edge labels $|f(u) - f(v)|$, $uv \in E$, taken over all bipartite labelings $f$ of $T$. They prove that the $\alpha$-size of any tree with $n$ edges is at least $5(n+1)/7$ and that there exist trees whose $\alpha$-size is at most $(5n + 9)/6$. They conjectured that minimum of the $\alpha$-sizes over all trees with $n$ edges is asymptotically $5n/6$. This conjecture has been proved for trees of maximum degree 3 by Bonnington and Širáň [BoS]. Heinrich and Hell [HeH] defined the grace size of a graph $G$ with $n$ vertices as the maximum, over all bijections $f : V(G) \rightarrow \{1, 2, \ldots, n\}$, of the number of distinct values $|f(u) - f(v)|$ over all edges $uv$ of $G$. So, from Rosa and Širáň’s result, the grace size of any tree with $n$ edges is at least $5(n+1)/7$.

In [GPW] Gallian weakened the condition for an $\alpha$-labeling somewhat by defining a weakly $\alpha$-labeling as a graceful labeling for which there is an integer $k$ so that for each edge $xy$ either $f(x) \leq k \leq f(y)$ or $f(y) \leq k \leq f(x)$. Unlike $\alpha$-labelings, this condition allows the graph to have an odd cycle, but still places a severe restriction on the structure of the graph; namely, that the vertex with the label $k$ must be on every odd cycle. Gallian, Prout and Winters [GPW] showed that the prisms $C_n \times P_2$ with a vertex deleted have $\alpha$-labelings. The same paper reveals that $C_n \times P_2$ with an edge deleted from a cycle has an $\alpha$-labeling when $n$ is even and a weakly $\alpha$-labeling when $n > 3$.

A special case of $\alpha$-labeling called strongly graceful was introduced by Maheo [Mah] in 1980. A graceful labeling $f$ of a graph $G$ is called strongly graceful if $G$ is bipartite with two partite sets $A$ and $B$ of the same order $s$, the number of edges is $2t + s$, there is an integer $k$ with $t - s \leq k \leq t + s - 1$ such that if $a \in A, f(a) \leq k$, and if
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Also note that

that every tree has a near tensor product with respect to the corresponding vertex partitions. They conjecture

if the critical number

same degree, then

a graph with partite sets of odd order such that in each partite set each vertex has the

that they show are bigraceful are: lobsters, trees of diameter at most 5, stars $S_{k,m}$ with

$K_{m,2} \times K_2$ has a strong $\alpha$-labeling and that $K_{m,2} \times P_n$ has

also prove that if $G$ is a connected bipartite graph with partite sets of odd order such that in each partite set each vertex has the

As a corollary they have that $K_{m,n} \times K_2$ does not have a strong $\alpha$-valuation. An $\alpha$-labeling $f$ of a graph $G$ is called free by El-Zanati and Vanden Eynden in [EV3] if the critical number $k$ (in the definition of $\alpha$-labeling) is greater than 2 and if neither 1 nor $k - 1$ is used in the labeling. Their main result is that the union of graphs with free $\alpha$-labelings has an $\alpha$-labeling. In particular, they show that $K_{m,n}$, $m > 1$, $n > 2$, has a free $\alpha$-labeling. They also show that $Q_n$, $n \geq 3$, and $K_{m,2} \times Q_n$, $m > 1$, $n \geq 1$, have free $\alpha$-labelings. El-Zanati [personal communication] has shown that the Heawood graph has a free $\alpha$-labeling.

El-Zanati, Kenig and Vanden Eynden [EKV] call a graceful labeling $\gamma$ a near $\alpha$-labeling if the vertex set can be partitioned into two sets $V_1$ and $V_2$ with the property that every edge has the form $v_1v_2$ where $v_1 \in V_1$, $v_2 \in V_2$ and $\gamma(v_1) < \gamma(v_2)$. They prove that if $G$ is a graph with $n$ edges that has a near $\alpha$-labeling then there exists a cyclic $G$-decomposition of $K_{2n+x+1}$ for all positive integers $x$ and a cyclic $G$-decomposition of $K_{n,n}$. They further prove that if $G$ and $H$ have near $\alpha$-labelings, then so does their weak tensor product with respect to the corresponding vertex partitions. They conjecture that every tree has a near $\alpha$-labeling.

Another kind of labelings for trees was introduced by Ringel, Llado and Serra [RLS] in an approach to proving their conjecture $K_{n,n}$ is edge-decomposable into $n$ copies of any given tree with $n$ edges. If $T$ is a tree with $n$ edges and partite sets $A$ and $B$, they define a labeling $f$ from the set of vertices to $\{1, 2, \ldots, n\}$ to be a bigraceful labeling of $T$ if $f$ restricted to $A$ is injective, $f$ restricted to $B$ is injective, and the edge labels given by $f(y) - f(x)$ where $yx$ is an edge with $y$ in $B$ and $x$ in $A$ is the set $\{0, 1, 2, \ldots, n - 1\}$. (Notice that this terminology conflicts with that given in Section 2.7 In particular, the Ringel, Llado and Serra bigraceful does not imply the usual graceful.) Among the graphs that they show are bigraceful are: lobsters, trees of diameter at most 5, stars $S_{k,m}$ with $k$ spokes of paths of length $m$, and complete $d$-ary trees for $d$ odd. They also prove that
if $T$ is a tree then there is a vertex $v$ and a nonnegative integer $m$ such that the addition of $m$ leaves to $v$ results in a bigraceful tree. They conjecture that all trees are bigraceful.

For connected bipartite graphs Grannell, Griggs and Holroyd [GGH] introduced a labeling that lies between $\alpha$-labelings and graceful labelings. They call a vertex labeling $f$ of a bipartite graph $G$ with $q$ edges and partitive sets $D$ and $U$ gracious if $f$ is a bijection from the vertex set of $G$ to $\{0,1,\ldots,q\}$ such that the set of edge labels induced by $f(u) - f(v)$ for every edge $uv$ with $u \in U$ and $v \in D$ is $\{1,2,\ldots,q\}$. Thus a graceful labeling of $G$ with partite sets $D$ and $U$ is a graceful labeling in which every vertex in $D$ has a label lower than every adjacent vertex. They verified by computer that every tree of size up to 20 has a graceful labeling. This led them to conjecture that every tree has a graceful labeling. For any $k > 1$ and any tree $T$ Grannell et al. say that $T$ has a gracious $k$-labeling if the vertices of $T$ can be partitioned into sets $D$ and $U$ in such a way that there is a function $f$ from the vertices of $G$ to the integers modulo $k$ such that the edge labels induced by $f(u) - f(v)$ where $u \in U$ and $d \in D$ has the following properties: the number of edges labeled with 0 is one less than the number of vertices labeled with 0 and for each nonzero integer $x$ the number of edges labeled with $x$ is the same as the number of vertices labeled with $x$. They prove that every nontrivial tree has a $k$-gracious labeling for $k = 2,3,4,5$ and that caterpillars are $k$-gracious for all $k \geq 2$.

### 3.2 $k$-graceful Labelings

A natural generalization of graceful graphs is the notion of $k$-graceful graphs introduced independently by Slater [Sl2] in 1982 and by Maheo and Thuillier [MT] in 1982. A graph $G$ with $q$ edges is $k$-graceful if there is labeling $f$ from the vertices of $G$ to $\{0,1,2,\ldots,q+k-1\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{k,k+1,\ldots,q+k-1\}$. Obviously, 1-graceful is graceful and it is readily shown that any graph that has an $\alpha$-labeling is $k$-graceful for all $k$. Graphs that are $k$-graceful for all $k$ are sometimes called arbitrarily graceful. Ng [N2] has shown that there are graphs that are $k$-graceful for all $k$ but do not have an $\alpha$-labeling.

Results of Maheo and Thuillier [MT] together with those of Slater [Sl2] show that: $C_n$ is $k$-graceful if and only if either $n \equiv 0$ or 1 (mod 4) with $k$ even and $k \leq (n-1)/2$, or $n \equiv 3$ (mod 4) with $k$ odd and $k \leq (n^2-1)/2$. Maheo and Thuillier [MT] also proved that the wheel $W_{2k+1}$ is $k$-graceful and conjectured that $W_{2k}$ is $k$-graceful when $k \neq 3$ or $k \neq 4$. This conjecture was proved by Liang, Sun and Xu [LSX]. Kang [Ka] proved that $P_n \times C_4n$ is $k$-graceful for all $k$. Lee and Wang [LW] showed that all pyramids, lotuses and diamonds are $k$-graceful and Liang and Liu [LL] have shown that $K_{m,n}$ is $k$-graceful. Bu, Gao and Zhang [BGZ] have proved that $P_n \times P_2$ and $(P_n \times P_2) \cup (P_n \times P_2)$ are $k$-graceful for all $k$. Acharya (see [A2]) has shown that a $k$-graceful Eulerian graph with $q$ edges must satisfy one of the following conditions: $q \equiv 0$ (mod 4), $q \equiv 1$ (mod 4) if $k$ is even, or $q \equiv 3$ (mod 4) if $k$ is odd. Bu, Zhang and He [BZH] have shown that an
even cycle with a fixed number of pendant edges adjoined to each vertex is \( k \)-graceful.

Several authors have investigated the \( k \)-gracefulness of various classes of subgraphs of grid graphs. Acharya \[A1\] proved that all 2-dimensional polyominoes that are convex and Eulerian are \( k \)-graceful for all \( k \); Lee \[L1\] showed that Mongolian tents and Mongolian villages are \( k \)-graceful for all \( k \) (see Section 2.3 for definitions); Lee and K. C. Ng \[LNK\] proved that all Young tableaus are \( k \)-graceful for all \( k \) (see Section 2.3 for definitions). Lee and H. K. Ng \[LNH\] subsequently generalized these results on Young tableaus to a wider class of planar graphs.

Let \( c, m, p_1, p_2, \ldots, p_m \) be positive integers. For \( i = 1, 2, \ldots, m \), let \( S_i \) be a set of \( p_i + 1 \) integers and let \( D_i \) be the set of positive differences of the pairs of elements of \( S_i \). If all these differences are different then the system \( D_1, D_2, \ldots, D_m \) is called a perfect system of difference sets starting at \( c \) if the union of all the sets \( D_i \) is \( c, c+1, \ldots, c-1+\sum_{i=1}^{m} \left( \frac{p_i+1}{2} \right) \). There is a relationship between \( k \)-graceful graphs and perfect systems of difference sets. A perfect system of difference sets starting with \( c \) describes a \( c \)-graceful labeling of a graph that is decomposable into complete subgraphs. A survey of perfect systems of difference sets is given in \[Ab1\].

Acharya and Hegde \[AH2\] generalized \( k \)-graceful to \((k,d)\)-graceful labelings by permitting the vertex labels to belong to \( \{0,1,2,\ldots,k+(q-1)d\} \) and requiring the set of edge labels induced by the absolute value of the difference of labels of adjacent vertices to be \( \{k,k+d,k+2d,\ldots,k+(q-1)d\} \). They also introduce an analog of \( \alpha \)-labelings in the obvious way. Notice that a \((1,1)\)-graceful labeling is a graceful labeling and a \((k,1)\)-graceful labeling is a \( k \)-graceful labeling. Bu and Zhang \[BZ\] have shown that \( K_{m,n} \) is \((k,d)\)-graceful for all \( k \) and \( d \); for \( n \geq 2 \), \( K_n \) is \((k,d)\)-graceful if and only if \( k = d \) and \( n \leq 4 \); if \( m_i, n_i \geq 2 \) and \( \max\{m_i, n_i\} \geq 3 \), then \( K_{m_1,n_1} \cup K_{m_2,n_2} \cup \cdots \cup K_{m_r,n_r} \) is \((k,d)\)-graceful for all \( k, d, \) and \( r \); if \( G \) has an \( \alpha \)-labeling, then \( G \) is \((k,d)\)-graceful for all \( k \) and \( d \); a \( k \)-graceful graph is a \((kd,d)\)-graceful graph; a \((kd,d)\)-graceful connected graph is \( k \)-graceful; and a \((k,d)\)-graceful graph with \( q \) edges that is not bipartite must have \( k \leq (q-2)d \).

Slater \[Sl5\] has extended the definition of \( k \)-graceful graphs to countable infinite graphs in a natural way. He proved that all countably infinite trees, the complete graph with countably many vertices and the countably infinite Dutch windmill is \( k \)-graceful for all \( k \).

More specialized results on \( k \)-graceful labelings can be found in \[L1\], \[LNK\], \[LNH\], \[Sl2\], \[BuF\], \[BuH\], \[BGZ\] and \[ChJ\].

### 3.3 Skolem-Graceful

A number of authors have invented analogues of graceful graphs by modifying the permissible vertex labels. For instance, Lee (see \[LSh\]) calls a graph \( G \) with \( p \) vertices and \( q \) edges Skolem-graceful if there is an injection from the set of vertices of \( G \) to \( \{1,2,\ldots,p\} \) such that the edge labels induced by \( |f(x) - f(y)| \) for each edge \( xy \) are
1, 2, \ldots, q. A necessary condition for a graph to be Skolem-graceful is that \( p \geq q + 1 \). Lee and Wui [LWu] have shown that a connected graph is Skolem-graceful if and only if it is a graceful tree. Although the disjoint union of trees can not be graceful, they can be Skolem-graceful. Lee and Wui [LWu] prove that the disjoint union of 2 or 3 stars is Skolem-graceful if and only if at least one star has even size. In [CK3] Choudum and Kishore show that the disjoint union of \( k \) copies of the star \( K_{1,2p} \) is Skolem graceful if \( k \leq 4p+1 \) and the disjoint union of any number of copies of \( K_{1,2} \) is Skolem graceful. For \( k \geq 2 \), let \( St(n_1, n_2, \ldots, n_k) \) denote the disjoint union of \( k \) stars with \( n_1, n_2, \ldots, n_k \) edges. Lee, Wang and Wui [LWW] showed that the 4-star \( St(n_1, n_2, n_3, n_4) \) is Skolem-graceful for some special cases and conjectured that all 4-stars are Skolem-graceful. Denham, Leu and Liu [DLL] proved this conjecture. Kishore [CK1] proved that all 5-stars are Skolem graceful. Lee, Wang and Wui [LWW] showed that the 4-star is Skolem-graceful when \( n \) is even; the disjoint union of copies of \( P_n \) if and only if \( n \) is even or \( k \equiv 0 \) or \( 1 \) (mod 4). He conjectures that each one of these conditions is sufficient. Choudum and Kishore [CK1] proved that all 5-stars are Skolem graceful. Lee, Quach and Wang [LQW] showed that the disjoint union of the path \( P_n \) and the star of size \( m \) is Skolem-graceful if and only if \( n = 2 \) and \( m \) is even or \( n \geq 3 \) and \( m \geq 1 \). It follows from the work of Skolem [Sk] that \( nP_2 \), the disjoint union of \( n \) copies of \( P_3 \), is Skolem-graceful if and only if \( n \equiv 0 \) or \( 1 \) (mod 4). Harary and Hsu [HaHs] studied Skolem-graceful graphs under the name node-graceful. Frucht [F3] has shown that \( P_m \cup P_n \) is Skolem-graceful when \( m + n \geq 5 \). Bhat-Nayak and Deshmukh [BD4] have shown that \( P_{n_1} \cup P_{n_2} \cup P_{n_3} \) is Skolem-graceful when \( n_1 < n_2 \leq n_3 \), \( n_2 = t(n_1 + 2) + 1 \) and \( n_1 \) is even and when \( n_1 < n_2 \leq n_3; n_2 = t(n_1 + 3) + 1 \) and \( n_1 \) is odd. They also prove that the graphs of the form \( P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_i} \) where \( i \geq 4 \) are Skolem-graceful under certain conditions.

### 3.4 Odd Graceful Labelings

Gnanajothi [Gn, p. 182] defined a graph \( G \) with \( q \) edges to be odd graceful if there is an injection \( f \) from \( V(G) \) to \( \{0, 1, 2, \ldots, 2q - 1\} \) such that, when each edge \( xy \) is assigned the label \( |f(x) - f(y)| \), the resulting edge labels are \( \{1, 3, 5, \ldots, 2q - 1\} \). She proved that the class of odd graceful graphs lies between the class of graphs with \( \alpha \)-labelings and the class of bipartite graphs by showing that every graph with an \( \alpha \)-labeling has an odd graceful labeling and every graph with an odd cycle is not odd graceful. She also proved the following graphs are odd graceful: \( P_n; C_n \) if and only if \( n \) is even; \( K_{m,n} \); combs \( P_n \odot K_1 \) (graphs obtained by joining a single pendant edge to each vertex of \( P_n \)); books; crowns \( C_n \odot K_1 \) (graphs obtained by joining a single pendant edge to each vertex of \( C_n \)) if and only if \( n \) is even; the disjoint union of copies of \( C_4 \); the one-point union of copies of \( C_4 \); \( C_n \times K_2 \) if and only if \( n \) is even; caterpillars; rooted trees of height 2; the graphs obtained from \( P_n \) \((n \geq 3)\) by adding exactly two leaves at each vertex of degree 2 of \( P_n \); the graphs consisting of vertices \( a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n \) with edges \( a_i a_{i+1}, b_i b_{i+1} \) for \( i = 0, \ldots, n - 1 \) and \( a_0 b_1, a_1 b_2, \ldots, a_{n-1} b_n \); the graphs obtained from a star by adjoining to each end vertex the path \( P_3 \) or by adjoining to each end vertex the path \( P_4 \). She conjectures that all trees are odd graceful and proves the conjecture for all trees.
with order up to 10. Eldergill [E] generalized Gnanajothi’s result on stars by showing that the graphs obtained by joining one end point from each of any odd number of paths of equal length is odd graceful. He also proved that the one-point union of any number of copies of $C_6$ is odd graceful. Kathiresan [Kat3] has shown that ladders and graphs obtained from them by subdividing each step exactly once are odd graceful.

Seoud, Diab and Elsakhawi [SDE] have shown that a connected $r$-partite graph is odd graceful if and only if $r = 2$ and that the join of any two connected graphs is not odd graceful.

### 3.5 Graceful-like Labelings

As a means of attacking graph decomposition problems, Rosa [Ro1] invented another analogue of graceful labelings by permitting the vertices of a graph with $q$ edges to assume labels from the set $\{0, 1, \ldots, q + 1\}$, while the edge labels induced by the absolute value of the difference of the vertex labels are $\{1, 2, \ldots, q - 1, q\}$ or $\{1, 2, \ldots, q - 1, q + 1\}$. He calls these $\rho$-labelings. Frucht [F3] used the term nearly graceful labeling instead of $\rho$-labelings. Frucht [F3] has shown that the following graphs have nearly graceful labelings with edge labels from $\{1, 2, \ldots, q - 1, q + 1\}$: $P_m \cup P_n$; $S_m \cup S_n$; $S_m \cup P_n$; $G \cup K_2$ where $G$ is graceful; and $C_3 \cup K_2 \cup S_m$ where $m$ is even or $m \equiv 3 \pmod{14}$. Seoud and Elsakhawi [SE1] have shown that all cycles are nearly graceful. Barrientos [Bar2] proved that $C_n$ is nearly graceful with edge labels $1, 2, \ldots, n - 1, n + 1$ if and only if $n \equiv 1$ or 2 (mod 4). Rosa [Ro2] conjectured that triangular snakes with $t \equiv 0$ or 1 (mod 4) blocks are graceful and those with $t \equiv 2$ or 3 (mod 4) blocks are nearly graceful (a parity condition ensures that the graphs in the latter case cannot be graceful). Moulton [Mo] proved Rosa’s conjecture while introducing the slightly stronger concept of almost graceful by permitting the vertex labels to come from $\{0, 1, 2, \ldots, q - 1, q + 1\}$ while the edge labels are $\{1, 2, \ldots, q - 1, q\}$, or $\{1, 2, \ldots, q - 1, q + 1\}$. Seoud and Elsakhawi [SE1] have shown that the following graphs are almost graceful: $C_n; P_n + K_m; P_n + K_{1,m} + K_{1,m,n}; K_{1,m} + K_{1,m,n}; K_{2,2,m}; K_{1,1,m,n};$ ladders; and $P_n \times P_3 \ (n \geq 3)$.

Barrientos [Bar2] calls a graph a $kC_n$-snake if it is a connected graph with $k$ blocks whose block-cutpoint graph is path and each of the $k$ blocks is isomorphic to $C_n$. (When $n > 3$ and $k > 3$ there is more than one $kC_n$-snake.) If a $kC_n$-snake where the path of minimum length that contains all the cut-vertices of the graph has the property that the distance between any two consecutive cut-vertices is $\lceil n/2 \rceil$ it is called linear. Barrientos proves that $kC_4$-snakes are graceful and that the linear $kC_6$-snakes are graceful when $k$ is even. When $k$ is odd he proves that the linear $kC_6$-snake is nearly graceful. Barrientos further proves that $kC_8$-snakes and $kC_{12}$-snakes are graceful in the cases where the distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph are all even and that certain cases of $kC_{4n}$-snakes and $kC_{5n}$-snakes are graceful (depending on the distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph).

Yet another kind of labeling introduced by Rosa in his 1967 paper [Ro1] is a $\rho$-
valuation. A \(\rho\)-valuation of a graph is an injection from the vertices of the graph with \(q\) edges to the set \(\{0, 1, \ldots, 2q\}\), where if the edge labels induced by the absolute value of the difference of the vertex labels are \(a_1, a_2, \ldots, a_q\), then \(a_i = i\) or \(a_i = 2q + 1 - i\). Rosa [Ro1] proved that a cyclic decomposition of the edge set of the complete graph \(K_{2q+1}\) into subgraphs isomorphic to a given graph \(G\) with \(q\) edges exists if and only if \(G\) has a \(\rho\)-valuation. (A decomposition of \(K_n\) into copies of \(G\) is called cyclic if the automorphism group of the decomposition itself contains the cyclic group of order \(n\).) It is known that every graph with at most 11 edges has a \(\rho\)-labeling and that all lobsters have a \(\rho\)-labeling (see [CRS]). Caro, Roditty and Schönheim [CRS] provide a construction for the adjacency matrix for every graph that has a \(\rho\)-labeling. They ask the following question: If \(H\) is a connected graph having a \(\rho\)-labeling and \(q\) edges and \(G\) is a new graph with \(q\) edges constructed by breaking \(H\) up into disconnected parts does \(G\) also have a \(\rho\)-labeling?

Dufour [Duf] and Eldergill [E] have some results on the decomposition of the complete graph using labeling methods. Balakrishnan and Sampathkumar [BS] showed that for each positive integer \(n\) the graph \(K_n + 2K_2\) admits a \(\rho\)-valuation. Balakrishnan [Ba] asks if it is true that \(K_n + mK_2\) admits a \(\rho\)-valuation for all \(n\) and \(m\).

For graphs with the property \(p = q + 1\), Frucht [F3] has introduced a stronger version of almost graceful graphs by permitting as vertex labels \(\{0, 1, \ldots, q - 1, q + 1\}\) and as edge labels \(\{1, 2, \ldots, q\}\). He calls such a labeling pseudograceful. Frucht proved that \(P_n\) (\(n \geq 3\)), combs, sparklers (i.e., graphs obtained by joining an end vertex of a path to the center of a star), \(C_3 \cup P_n\) (\(n \neq 3\)), and \(C_4 \cup P_n\) (\(n \neq 1\)) are pseudograceful while \(K_{1,n}\) (\(n \geq 3\)) is not. Kishore [Kis] proved that \(C_s \cup P_n\) is pseudograceful when \(s \geq 5\) and \(n \geq (s + 7)/2\) and that \(C_s \cup S_n\) is pseudograceful when \(s = 3, 4,\) and \(s \geq 7\). Seoud and Youssef [SY2] and [SY5] extended the definition of pseudograceful to all graphs with \(p \leq q + 1\). They proved that \(K_m\) is pseudograceful if and only if \(m = 1, 3\) or \(4\) [SY5]; \(K_{m,n}\) is pseudograceful when \(n \geq 2\) and \(P_m + K_n\) (\(m \geq 2\)) [SY2] is pseudograceful. They also proved that if \(G\) is pseudograceful, then \(G \cup K_{m,n}\) is graceful for \(m \geq 2\) and \(n \geq 2\) and \(G \cup K_{m,n}\) is pseudograceful for \(m \geq 2, n \geq 2\) and \((m, n) \neq (2, 2)\) [SY5]. They ask if \(G \cup K_{2,2}\) is pseudograceful whenever \(G\) is.

McTavish [Mc] has investigated labelings where the vertex and edge labels are from \(\{0, \ldots, q, q + 1\}\). She calls these \(\bar{\rho}\)-labelings. Graphs that have \(\bar{\rho}\)-labelings include cycles and the disjoint union of \(P_n\) or \(S_n\) with any graceful graph.

Frucht [F3] has made an observation about graceful labelings that yields nearly graceful analogs of \(\alpha\)-labelings and weakly \(\alpha\)-labelings in a natural way. Suppose \(G(V, E)\) is a graceful graph with the vertex labeling \(f\). For each edge \(xy\) in \(E\), let \([f(x), f(y)]\) (where \(f(x) \leq f(y)\)) denote the interval of real numbers \(r\) with \(f(x) \leq r \leq f(y)\). Then the intersection \(\cap [f(x), f(y)]\) over all edges \(xy \in E\) is a unit interval, a single point, or empty. Indeed, if \(f\) is an \(\alpha\)-labeling of \(G\) then the intersection is a unit interval; if \(f\) is a weakly \(\alpha\)-labeling, but not an \(\alpha\)-labeling, then the intersection is a point; and, if \(f\) is graceful but not a weakly \(\alpha\)-labeling, then the intersection is empty. For nearly graceful labelings, the intersection also gives three distinct classes.
3.6 Cordial Labelings

Cahit [C2] has introduced a variation of both graceful and harmonious labelings. Let $f$ be a function from the vertices of $G$ to $\{0,1\}$ and for each edge $xy$ assign the label $|f(x) - f(y)|$. Call $f$ a cordial labeling of $G$ if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Cahit [C3] proved the following: every tree is cordial; $K_n$ is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all $m$ and $n$; the friendship graph $C_3^{(t)}$ (i.e., the one-point union of $t$ 3-cycles) is cordial if and only if $t \not\equiv 2 \pmod{4}$; all fans are cordial; the wheel $W_n$ is cordial if and only if $n \not\equiv 3 \pmod{4}$ (see also [Du]); maximal outerplanar graphs are cordial; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Kuo, Chang and Kwong [KCK] determine all $m$ and $n$ for which $mK_n$ is cordial.

A $k$-angular cactus is a connected graph all of whose blocks are cycles with $k$ vertices. In [C3] Cahit proved that a $k$-angular cactus with $t$ cycles is cordial if and only if $kt \not\equiv 2 \pmod{4}$. This was improved by Kirchherr [Ki1] who showed any cactus whose blocks are cycles is cordial if and only if the size of the graph is not congruent to 2 (mod 4). Kirchherr [Ki2] also gave a characterization of cordial graphs in terms of their adjacency matrices. Ho, Lee and Shee [HLS2] proved: $P_n \times C_{4m}$ is cordial for all $m$ and all odd $n$; the composition $G[H]$ is cordial if $G$ is cordial and $H$ is cordial and has odd order and even size (see §2.3 for definition of composition); for $n \geq 4$ the composition $C_n[K_2]$ is cordial if and only if $n \not\equiv 2 \pmod{4}$; the cartesian product of two cordial graphs of even size is cordial. The same authors [HLS1] showed that a unicyclic graph is cordial unless it is $C_{4k+2}$ and that the generalized Petersen graph (see §2.7 for definition) $P(n,k)$ is cordial if and only if $n \not\equiv 2 \pmod{4}$. Du [Du] determines the maximal number of edges in a cordial graph of order $n$ and gives a necessary condition for a $k$-regular graph to be cordial.

Seoud and Abdel Maqusoud [SA2] proved that if $G$ is a graph with $n$ vertices and $m$ edges and every vertex has odd degree then $G$ is not cordial when $m+n \equiv 2 \pmod{4}$. They also prove the following: for $m \geq 2$, $C_n \times P_m$ is cordial except for the case $C_{4k+2} \times P_2$; $P_n^2$ is cordial for all $n$; $P_n^3$ is cordial if and only if $n \not\equiv 4 \pmod{4}$; and $P_n^4$ is cordial if and only if $n \not\equiv 4, 5$ or 6. Seoud, Diab and Elsakawi [SDE] have proved the following graphs are cordial: $P_n + P_m$ for all $m$ and $n$ except $(m,n) = (2,2)$; $C_m + C_n$ if $m \not\equiv 0 \pmod{4}$ and $n \not\equiv 2 \pmod{4}$; $P_n + K_{1,m}$ for all $n$ and odd $m$; $C_n + K_{1,m}$ for $n \not\equiv 3 \pmod{4}$ and odd $m$ except $(n,m) = (3,1)$; $C_n + \overline{K}_m$ when $n$ is odd and when $n$ is even and $m$ is odd; $K_{1,m,n}$; $K_{2,2,m}$; the $n$-cube; books $B_n$ if and only if $n \not\equiv 3 \pmod{4}$; $B(3,2,m)$ for all $m$; $B(4,3,m)$ if and only if $m$ is even; and $B(5,3,m)$ if and only if $m \not\equiv 1 \pmod{4}$ (see §2.4 for the notation $B(n,r,m)$).

Lee, Lee and Chang [LLC] prove the following graphs are cordial: the complete $n$-partite graph if and only if at most three of its partite sets have odd cardinality (see also [Du]); the Cartesian product of an arbitrary number of paths; the Cartesian product of two cycles if and only if at least one of them is even; and the Cartesian product of an
arbitrary number of cycles if at least one of them has length a multiple of 4 or at least
two of them are even.

Shee and Ho [SH1] have investigated the cordiality of the one-point union of \( n \) copies
of various graphs. For \( C_m^{(n)} \), the one-point union of \( n \) copies of \( C_m \), they proved:

(i) If \( m \equiv 0 \pmod{4} \), then \( C_m^{(n)} \) is cordial for all \( n \);
(ii) If \( m \equiv 1 \) or \( 3 \pmod{4} \), then \( C_m^{(n)} \) is cordial if and only if \( n \not\equiv 2 \pmod{4} \);
(iii) If \( m \equiv 2 \pmod{4} \), then \( C_m^{(n)} \) is cordial if and only if \( n \) is even.

For \( K_m^{(n)} \), the one-point union of \( n \) copies of \( K_m \), Shee and Ho [SH1] prove:

(i) If \( m \equiv 0 \pmod{8} \), then \( K_m^{(n)} \) is not cordial for \( n \equiv 3 \pmod{4} \);
(ii) If \( m \equiv 4 \pmod{8} \), then \( K_m^{(n)} \) is not cordial for \( n \equiv 1 \pmod{4} \);
(iii) If \( m \equiv 5 \pmod{8} \), then \( K_m^{(n)} \) is not cordial for all odd \( n \);
(iv) \( K_4^{(n)} \) is cordial if and only if \( n \not\equiv 1 \pmod{4} \);
(v) \( K_5^{(n)} \) is cordial if and only if \( n \) is even;
(vi) \( K_6^{(n)} \) is cordial if and only if \( n > 2 \);
(vii) \( K_7^{(n)} \) is cordial if and only if \( n \not\equiv 2 \pmod{4} \);
(viii) \( K_8^{(2)} \) is cordial if and only if \( n \) has the form \( p^2 \) or \( p^2 + 1 \).

Benson and Lee [BL] have investigated the regular windmill graphs \( K_m^{(n)} \) and determined
precisely which ones are cordial for \( m < 14 \).

For \( W_m^{(n)} \), the one-point union of \( n \) copies of the wheel \( W_m \) with the common vertex
being the center, Shee and Ho [SH1] show:

(i) If \( m \equiv 0 \) or \( 2 \pmod{4} \), then \( W_m^{(n)} \) is cordial for all \( n \);
(ii) If \( m \equiv 3 \pmod{4} \), then \( W_m^{(n)} \) is cordial if \( n \not\equiv 1 \pmod{4} \);
(iii) If \( m \equiv 1 \pmod{4} \), then \( W_m^{(n)} \) is cordial if \( n \not\equiv 3 \pmod{4} \).

For all \( n \) and all \( m > 1 \) Shee and Ho [SH1] prove \( F_m^{(n)} \), the one-point union of \( n \) copies
of the fan \( F_m = P_m + K_1 \) with the common point of the fans being the center, is cordial.
The flag \( Fl_m \) is obtained by joining one vertex of \( C_m \) to an extra vertex called the root.
Shee and Ho [SH1] show all \( Fl_m^{(n)} \), the one-point union of \( n \) copies of \( Fl_m \) with the
common point being the root, are cordial.

Andar, Boxwala and Limaye [ABL1] and [ABL2] have proved the following graphs are
cordial: helms; closed helms; generalized helms obtained by taking a web and attaching
pendent vertices to all the vertices of the outermost cycle in the case that the number
cycles is even; flowers, which are obtained by joining the vertices of degree one of a helm
to the central vertex; sunflower graphs, which are obtained by taking a wheel with the
central vertex \( v_0 \) and the cycle \( v_1, v_2, \ldots, v_n, v_1 \), and additional vertices \( w_1, w_2, \ldots, w_n \)
where \( w_i \) is joined by edges to \( v_i, v_{i+1} \), where \( i + 1 \) is taken modulo \( n \), and multiple shells
(see §2.2).

For graphs \( G_1, G_2, \ldots, G_n \) \((n \geq 2)\) which are all copies of a fixed graph \( G \), Shee and
Ho [SH2] call a graph obtained by adding an edge from \( G_i \) to \( G_{i+1} \) for \( i = 1, \ldots, n-1 \) a
path-union of \( G \) (the resulting graph may depend on how the edges are chosen). Among
their results they show the following graphs are cordial: path-unions of cycles; path-unions of \( n \) copies of \( K_m \) when \( m = 4, 6 \) or \( 7 \); path-unions of three or more copies of \( K_5 \); and path-unions of two copies of \( K_m \) if and only if \( m - 2, m \) or \( m + 2 \) is a perfect square. They also show that there exist cordial path-unions of wheels, fans, unicyclic graphs, Petersen graphs, trees and various compositions.

Lee and Liu [LeL] give the following general construction for the forming of cordial graphs from smaller cordial graphs. Let \( H \) be a graph with an even number of edges and a cordial labeling such that the vertices of \( H \) can be divided into \( t \) parts \( H_1, H_2, \ldots, H_t \) each consisting of an equal number of vertices labeled 0 and vertices labeled 1. Let \( G \) be any graph and \( G_1, G_2, \ldots, G_t \) be any \( t \) subsets of the vertices of \( G \). Let \((G, H)\) be the graph which is the disjoint union of \( G \) and \( H \) augmented by edges joining every vertex in \( G_i \) to every vertex in \( H_i \) for all \( i \). Then \( G \) is cordial if and only if \( (G, H) \) is. From this it follows that: all generalized fans \( F_{m,n} = \overline{K_m} + P_n \) are cordial; the generalized bundle \( B_{m,n} \) is cordial if and only if \( m \) is even or \( n \equiv 2 \) (mod 4) \( (B_{m,n} \) consists of \( 2n \) vertices \( v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n \) with an edge from \( v_i \) to \( u_i \) and \( 2m \) vertices \( x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m \) with \( x_i \) joined to \( v_i \) and \( y_i \) joined to \( u_i \); if \( m \) is odd a generalized wheel \( W_{m,n} = \overline{K_m} + C_n \) is cordial if and only if \( n \not\equiv 3 \) (mod 4). If \( m \) is even, \( W_{m,n} \) is cordial if and only if \( n \not\equiv 2 \) (mod 4); a complete \( k \)-partite graph is cordial if and only if the number of parts with an odd number of vertices is at most 3.

Sethuraman, Selvaraju and Elumalai [SSE] have shown that certain cases of the union of any number of copies of \( K_4 \) with one or more edges deleted and one edge in common are cordial.

Cahit [C8] calls a graph \( H\)-cordial if it is possible to label the edges with the numbers from the set \( \{1, -1\} \) in such a way that, for some \( k \), at each vertex \( v \) the algebraic sum of the labels on the edges incident with \( v \) is either \( k \) or \( -k \) and the inequalities \(|v(k) - v(-k)| \leq 1 \) and \(|e(1) - e(-1)| \leq 1 \) are also satisfied, where \( e(i) \) and \( e(j) \) are, respectively, the number of vertices labeled with \( i \) and the number of edges labeled with \( j \). He proves: \( K_n \) is \( H \)-cordial if and only if \( n \equiv 0 \) (mod 4); \( K_{m,n} \) is \( H \)-cordial if and only if \( n > 2 \) and \( n \) is even; \( K_{m,n}, m \not\equiv n \), is \( H \)-cordial if and only if \( n \equiv 0 \) (mod 4), \( m \) is even and \( m > 2, n > 2 \); \( W_n \) is \( H \)-cordial if and only if \( n \equiv 1 \) (mod 4). By allowing 0 as the possible induced vertex label of an \( H \)-cordial labeling he studies semi-\( H \)-cordiality of trees. He also generalizes \( H \)-cordial labelings.

Cahit and Yilmaz [CY] call a graph \( E_k \)-cordial if it is possible to label the edges with the numbers from the set \( \{0, 1, 2, \ldots, k - 1\} \) in such a way that, at each vertex \( v \), the sum modulo \( k \) of the labels on the edges incident with \( v \) satisfies the inequalities \(|v(i) - v(j)| \leq 1 \) and \(|e(i) - e(j)| \leq 1 \), where \( e(s) \) and \( e(t) \) are, respectively, the number of vertices labeled with \( s \) and the number of edges labeled with \( t \). Obviously, \( E_2 \)-cordial is the same as cordial. Cahit and Yilmaz prove the following graphs are \( E_3 \)-cordial: \( P_n \) (\( n \geq 3 \)), stars \( S_n \) if and only if \( n \not\equiv 1 \) (mod 3), \( K_n \) (\( n \geq 3 \)), \( C_n \) (\( n \geq 3 \)), friendship graphs, and fans \( F_n \) (\( n \geq 3 \)). They also prove that \( S_n \) (\( n \geq 2 \)) is \( E_k \)-cordial if and only if \( n \not\equiv 1 \) mod \( k \) when \( k \) is odd or \( n \not\equiv 1 \) mod \( 2k \) when \( k \) is even and \( k \not\equiv 2 \).

Hovey [Ho] has introduced a simultaneous generalization of harmonious and cordial
labelings. For any Abelian group $A$ (under addition) and graph $G(V, E)$ he defines $G$ to be $A$-cordial if there is a labeling of $V$ with elements of $A$ so that for all $a$ and $b$ in $A$ when the edge $ab$ is labeled with $f(a) + f(b)$, the number of vertices labeled with $a$ and the number of vertices labeled $b$ differ by at most one and the number of edges labeled with $a$ and the number labeled with $b$ differ by at most one. In the case where $A$ is the cyclic group of order $k$, the labeling is called $k$-cordial. With this definition we have: $G(V, E)$ is harmonious if and only if $G$ is $|E|$-cordial; $G$ is cordial if and only if $G$ is 2-cordial.

Hovey has obtained the following: caterpillars are $k$-cordial for all $k$; all trees are $k$-cordial for $k = 3, 4$ and $5$; odd cycles with pendant edges attached are $k$-cordial for all $k$; cycles are $k$-cordial for all odd $k$; for $k$ even, $C_{2mk+j}$ is $k$-cordial when $0 \leq j \leq \frac{k}{2} + 2$ and when $k < j < 2k$; $C_{(2m+1)k}$ is not $k$-cordial; $K_m$ is 3-cordial; and, for $k$ even, $K_{mk}$ is $k$-cordial if and only if $m = 1$.

Hovey advances the following conjectures: all trees are $k$-cordial for all $k$; all connected graphs are 3-cordial; and $C_{2mk+j}$ is $k$-cordial if and only if $j \neq k$, where $k$ and $j$ are even and $0 \leq j < 2k$. The last conjecture was verified by Tao [1a]. This result combined with those of Hovey show that for all positive integers $k$ the $n$-cycle is $k$-cordial with the exception that $k$ is even and $n = 2mk + k$. Tao also proved that the crown with $2mk + j$ vertices is $k$-cordial unless $j = k$ is even and for $4 \leq n \leq k$ and the wheel $W_n$ is $k$-cordial unless $k \equiv 5 \pmod{8}$ and $n = (k + 1)/2$.

In [SS4] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph $G$ and successively form supersubdivisions that are cordial in that case that every edge in $G$ is replaced by $K_{2m}$ where $m$ is even. Sethuraman and Selvaraju [SS5] also prove that the one edge union of $k$ copies of shell graphs $C(n, n - 3)$ is cordial for all $n \geq 4$ and all $k$ and that the one vertex union of any number of copies of $K_{m,n}$ is cordial.

Cairnie and Edwards [CE] have determined the computational complexity of cordial and $k$-cordial labelings. They prove a conjecture of Kirchherr [Ki2] that deciding whether a graph admits a cordial labeling is NP-complete. As a corollary, this result implies that the same problem for $k$-cordial labelings is NP-complete. They remark that even the restricted problem of deciding whether connected graphs of diameter 2 have a cordial labeling is also NP-complete.

### 3.7 $k$-equitable Labelings

In 1990 Cahit [C4] proposed the idea of distributing the vertex and edge labels among $\{0, 1, \ldots, k - 1\}$ as evenly as possible to obtain a generalization of graceful labelings as follows. For any graph $G(V, E)$ and any positive integer $k$, assign vertex labels from $\{0, 1, \ldots, k - 1\}$ so that when the edge labels induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with $i$ and the number of vertices labeled with $j$ differ by at most one and the number of edges labeled with $i$ and the number of edges labeled with $j$ differ by at most one. Cahit has called a
graph with such an assignment of labels \( k \)-equitable. Note that \( G(V, E) \) is graceful if and only if it is \(|E| + 1\)-equitable and \( G(V, E) \) is cordial if and only if it is \( 2\)-equitable. Cahit [C3] has shown the following: \( C_n \) is \( 3\)-equitable if and only if \( n \neq 3 \) (mod 6); a triangular snake with \( n \) blocks is \( 3\)-equitable if and only if \( n \) is even; the friendship graph \( C_4^{(n)} \) is \( 3\)-equitable if and only if \( n \) is even; \( W_n \) is \( 3\)-equitable if and only if \( n \neq 3 \) (mod 6); an Eulerian graph with \( q \equiv 3 \) (mod 6) edges is not \( 3\)-equitable; and all caterpillars are \( 3\)-equitable [C3]. Cahit conjectures [C3] that a triangular cactus with \( n \) blocks is \( 3\)-equitable if and only if \( n \) is even. In [C4] Cahit proves that every tree with fewer than five end vertices has a \( 3\)-equitable labeling. He conjectures that all trees are \( k\)-equitable [C5]. In 1999 Speyer and Szaniszlo [SpS] proved Cahit’s conjecture for \( k = 3 \).

In [SA1] Seoud and Abdel Maqsoud prove: a graph with \( n \) vertices and \( q \) edges in which every vertex has odd degree is not \( 3\)-equitable if \( n \equiv 0 \) (mod 3) and \( q \equiv 3 \) (mod 6); all fans except \( P_2 + K_1 \) are \( 3\)-equitable; all double fans except \( P_4 + K_2 \) are \( 3\)-equitable; \( P_2^n \) is \( 3\)-equitable for all \( n \) except 3; \( K_{1,n} \) is \( 3\)-equitable if and only if \( n \equiv 0 \) or 2 (mod 3); \( K_{1,2,n} \), \( n \geq 2 \), is \( 3\)-equitable if and only if \( n \equiv 0 \) or 2 (mod 3); \( K_{m,n} \) is \( 3\)-equitable if and only if \( (m, n) = (4, 4) \); \( K_{1,m,n} \), \( 3 \leq m \leq n \), is \( 3\)-equitable if and only if \( (m, n) = (3, 4) \).

Szaniszlo [Sz] has proved the following: \( P_n \) is \( k\)-equitable for all \( k \); \( K_n \) is \( 2\)-equitable if and only if \( n = 1, 2 \) or 3; \( K_n \) is not \( k\)-equitable for \( 3 \leq k < n \); \( S_n \) is \( k\)-equitable for all \( k \); \( K_{2,n} \) is \( k\)-equitable if and only if \( n \equiv k - 1 \) (mod \( k \)), or \( n = 0, 1, 2, \ldots, [k/2] - 1 \) (mod \( k \)), or \( n = [k/2] \) and \( k \) is odd. She also proves that \( C_n \) is \( k\)-equitable if and only if \( k \) meets all of the following conditions: \( n \neq k \); if \( k \equiv 0, 3 \) (mod 4), then \( n \neq k - 1 \); if \( k \equiv 2, 3 \) (mod 4) then \( n \neq k \) (mod 2k).

Vickrey [Vi] has determined the \( k\)-equitablility of complete multipartite graphs. He shows that for \( m \geq 3 \) and \( k \geq 3 \), \( K_{m,n} \) is \( k\)-equitable if and only if \( K_{m,n} \) is one of the following graphs: \( K_{4,m} \) for \( k = 3 \); \( K_{3,k-1} \) for all \( k \); or \( K_{m,2} \) for \( k > mn \). He also shows that when \( k \) is less than or equal to the number of edges in the graph and at least 3, the only complete multipartite graphs that are \( k\)-equitable are \( K_{mn,k+1} \) and \( K_{km,k+1} \). Partial results on the \( k\)-equitability of \( K_{m,n} \) were obtained by Krussel [Kr].

As a corollary of the result of Cairnie and Edwards [CE] on the computational complexity of cordially labeling graphs, it follows that the problem of finding \( k\)-equitable labelings of graphs is NP-complete as well.

Seoud and Abdel Maqsoud [SA2] call a graph \( k\)-balanced if the vertex labels can be selected from \( \{0, 1, \ldots, k - 1\} \) so that the number of edges labeled \( i \) and the number of edges labeled \( j \) induced by the absolute value of the differences of the vertex labels differ by at most 1. They prove that \( P_n^2 \) is \( 3\)-balanced if and only if \( n = 2, 3, 4 \) or 6; for \( k \geq 4 \), \( P_n^2 \) is not \( k\)-balanced if \( k \leq n - 2 \) or \( n + 1 \leq k \leq 2n - 3 \); for \( k \geq 4 \), \( P_n^2 \) is \( k\)-balanced if \( k \geq 2n - 2 \); for \( k, m, n \geq 3 \), \( K_{m,n} \) is \( k\)-balanced if and only if \( k \geq mn \); for \( m \leq n \), \( K_{1,m,n} \) is \( k\)-balanced if and only if
(i) \( m = 1, n = 1 \) or 2, and \( k = 3 \);
(ii) \( m = 1 \) and \( k = n + 1 \) or \( n + 2 \); or
(iii) \( k \geq (m + 1)(n + 1) \).
Bloom has used the term $k$-equitable to describe another kind of labeling (see [W1], [W2] and [BR]). He calls a graph $k$-equitable if the edge labels induced by the absolute value of the difference of the vertex labels have the property that every edge label induced occurs exactly $k$ times. A graph of order $n$ is called minimally $k$-equitable if the vertex labels are $1, 2, \ldots, n$ and it is $k$-equitable. Both Bloom and Wojciechowski [W1], [W2] proved that $C_n$ is minimally $k$-equitable if and only if $k$ is a proper divisor of $n$. Barrientos and Hevia [BaH] proved that if $G$ is $k$-equitable of size $q = kw$ (in the sense of Bloom) then $\delta(G) \leq w$ and $\Delta(G) \leq 2w$. Barrientos, Dejter and Hevia [BDH] have shown that forests of even size are 2-equitable. They also prove that for $k = 3$ or $k = 4$ a forest of size $kw$ is $k$-equitable if and only if its maximum degree is at most $2w$ and that if 3 divides the size of the double star $S_{m,n}$ ($1 \leq m \leq n$), then $S_{m,n}$ is 3-equitable if and only if $q/3 \leq m \leq [(q-1)/2]$. ($S_{m,n}$ is $K_2$ with $m$ pendant edges attached at one end and $n$ pendant edges attached at the other end.) They discuss the $k$-equitability of forests for $k \geq 5$ and characterize all caterpillars of diameter 2 that are $k$-equitable for all possible values of $k$. In [Bar1] Barrientos proves that the one-point union of a cycle and a path (dragons) and the disjoint union of a cycle and a path are $k$-equitable for all $k$ that divide the size of the graph. Barrientos and Havia [BaH] have shown the following: $C_n \times K_2$ is 2-equitable when $n$ is even; books $B_n$ ($n \geq 3$) are 2-equitable when $n$ is odd; the vertex union of $k$-equitable graphs is $k$-equitable; wheels $W_n$ are 2-equitable when $n \neq 3$ (mod 4). They conjecture that $W_n$ is 2-equitable when $n \equiv 3$ (mod 4) except when $n = 3$. Barrientos and Hevia also proved that the $n$-cube $(n \geq 2)$ has a complete 2-equitable labeling and if $m$ or $n$ is even, then $K(m,n)$ has a complete 2-equitable labeling. Their 2-equitable labelings of $C_n \times K_2$ and the $n$-cube utilized graceful labelings of those graphs.

Bhat-Nayak and M. Acharya [BA] have proved the following: the graphs $C_{2n} \odot K_1$ are minimally 2-equitable, minimally 2n-equitable, minimally 4-equitable, and minimally $n$-equitable; the graphs $C_{3n} \odot K_1$ are minimally 3-equitable, minimally 3n-equitable, minimally $n$-equitable, and minimally 6-equitable; the graphs $C_{5n} \odot K_1$ are minimally 5-equitable, minimally 5n-equitable, minimally $n$-equitable, and minimally 10-equitable; the graphs $C_{2n+1} \odot K_1$ are minimally $(2n + 1)$-equitable; and that the graphs $P_{kn+1}$ are $k$-equitable.

In [Bar3] Barrientos calls a $k$-equitable labeling optimal if the vertex labels are consecutive integers and complete if the induced edge labels are $1, 2, \ldots, w$ where $w$ is the number of distinct edge labels. Note that a graceful labeling is a complete 1-equitable labeling. Barrientos proves that $C_m \odot nK_1$ (that is, an $m$-cycle with $n$ pendant edges attached at each vertex) is optimal 2-equitable when $m$ is even, $C_3 \odot nK_1$ is complete 2-equitable when $n$ is odd and that $C_3 \odot nK_1$ is complete 3-equitable for all $n$. He also shows that $C_n \odot K_1$ is $k$-equitable for every proper divisor $k$ of the size $2n$. Barrientos and Havia [BaH] have shown that the $n$-cube $(n \geq 2)$ has a complete 2-equitable labeling and that $K_{m,n}$ has a complete 2-equitable labeling when $m$ or $n$ is even. They conjecture that every tree of even size has an optimal 2-equitable labeling.
3.8 Hamming-graceful Labelings

Mollard, Payan and Shixin [MPS] introduced a generalization of graceful graphs called Hamming-graceful. A graph \( G = (V, E) \) is called Hamming-graceful if there exists an injective labeling \( g \) from \( V \) to the set of binary \(|E|\)-tuples such that \( \{d(g(v), g(u)) \mid uv \in E\} = \{1, 2, \ldots, |E|\} \). Shixin and Yu [ShY] have shown that all graceful graphs are Hamming-graceful; all trees are Hamming-graceful; \( C_n \) is Hamming-graceful if and only if \( n \equiv 0 \text{ or } 3 \pmod{4} \); if \( K_n \) is Hamming-graceful, then \( n \) has the form \( k^2 \) or \( k^2 + 2 \); \( K_n \) is Hamming-graceful for \( n = 2, 3, 4, 6, 9, 11, 16, \) and \( 18 \). They conjecture that \( K_n \) is Hamming-graceful for \( n \) of the form \( k^2 \) and \( k^2 + 2 \) for \( k \geq 5 \).

4 Variations of Harmonious Labelings

4.1 Sequential and Strongly \( c \)-harmonious Labelings

Chang, Hsu and Rogers [CHR] and Grace [Gr1, Gr2] have investigated subclasses of harmonious graphs. Chang et al. define an injective labeling \( f \) of a graph \( G \) with \( q \) vertices to be strongly \( c \)-harmonious if the vertex labels are from \( \{0, 1, \ldots, q-1\} \) and the edge labels induced by \( f(x) + f(y) \) for each edge \( xy \) are \( c, \ldots, c+q-1 \). Grace called such a labeling sequential. In the case of a tree, Chang et al. modify the definition to permit exactly one vertex label to be assigned to two vertices while Grace allows the vertex labels to range from 0 to \( q \) with no vertex label used twice. By taking the edge labels of a sequentially labeled graph with \( q \) edges modulo \( q \), we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. Grace [Gr2] proved that caterpillars, caterpillars with a pendant edge, odd cycles with zero or more pendant edges, trees with \( \alpha \)-labelings, wheels \( W_{2n+1} \), and \( P_n^2 \) are sequential. Liu and Zhang [LZ1] finished off the crowns \( C_{2n} \odot K_1 \). (The case \( C_{2n+1} \odot K_1 \) was a special case of Grace’s results. Liu [LiuY3] proved crowns are harmonious.) Bu [Bu2] also proved that crowns are sequential as are all even cycles with \( m \) pendant edges attached at each vertex. Figueroa-Centeno, Ichishima and Muntaner-Batle [FIM7] proved that all cycles with \( m \) pendant edges attached at each vertex are sequential.

Singh has proved the following: \( C_n \odot K_2 \) is sequential for all odd \( n > 1 \) [Sin2]; \( C_n \odot P_3 \) is sequential for all odd \( n \) [Sin4]; \( K_2 \odot C_n \) (each vertex of the cycle is joined by edges to the end points of a copy of \( K_2 \)) is sequential for all odd \( n \) [Sin4]; helms \( H_n \) are sequential when \( n \) is even [Sin4]; and \( K_{1,n} + K_2, K_{1,n} + \overline{K}_2, \) and ladders are sequential [Sin5]. Both Grace [Gr1] and Reid (see [GJ]) have found sequential labelings for the books \( B_{2n} \). Jungreis and Reid [JR] have shown the following graphs are sequential: \( P_m \times P_n \ (m, n) \neq (2, 2); C_4 \times P_n \ (m, n) \neq (1, 2); C_{4m+2} \times P_n; C_{2m+1} \times P_n; \) and, \( C_4 \times C_{2n} \ (n > 1) \). The graphs \( C_{4m+2} \times C_{2n+1} \) and \( C_{2m+1} \times C_{2n+1} \) fail to satisfy a necessary parity condition given by Graham and Sloane [GS]. The remaining cases of \( C_m \times P_n \) and \( C_m \times C_n \) are open. Gallian, Prout and Winters [GPW] proved that all
graphs $C_n \times P_2$ with a vertex or edge deleted are sequential.

Gnanajothi [Gn, pp.68-78] has shown the following graphs are sequential: $K_{1,m,n}$; $mC_n$, the disjoint union of $m$ copies of $C_n$, if and only if $m$ and $n$ are odd; books with triangular pages or pentagonal pages; and books of the form $B_{4n+1}$, thereby answering a question and proving a conjecture of Gallian and Jungreis [GJ]. Sun [Su] has also proved that $B_n$ is sequential if and only if $n \neq 3$ (mod 4).

Yuan and Zhu [YZ] have shown that $mC_n$ is sequential when $m$ and $n$ are odd. Although Graham and Sloane [GS] proved that the Möbius $M_3$ is not harmonious, Gallian [Ga1] established that all other Möbius ladders are sequential (see §2.3 for the definition). Chung, Hsu and Rogers [CHR] have shown that $K_{m,n} + K_1$, which includes $S_m + K_1$, is sequential. Seoud and Yousef [SY7] proved that $G$ is sequential and has the same number of edges as vertices, then $G + \overline{K_n}$ is sequential for all $n$. Hegde [Heg3] proved that every graph can be embedded as an induced subgraph of a sequential graph. In [A1] Acharya conjectured that if $G$ is a connected $k$-sequential graph of order $p$ with $k > \lfloor p/2 \rfloor$, then $k = p - 1$ and $G = K_{1,p-1}$ and that, except for $K_{1,p-1}$, every tree in which all vertices are odd is $k$-sequential for all odd positive integer $k \leq p/2$. Hegde [Heg3] gave counterexamples for both of these conjectures.

Among the strongly 1-harmonious (also called strongly harmonious) are: fans $F_n$ with $n \geq 2$ [CHR]; wheels $W_n$ with $n \neq 2 \pmod{3}$ [CHR]; $K_{m,n} + K_1$ [CHR]; French windmills $K^{(t)}_4$ [HS], [KZ]; the friendship graphs $C^{(n)}_3$ if and only if $n \equiv 0$ or 1 (mod 4) [HS], [KZ]; $C^{(t)}_{4k}$ [SW]; and helms [RP1].

Seoud, Diab and Elsakhawi [SDE] have shown that the following graphs are strongly harmonious: $K_{m,n}$ with an edge joining two vertices in the same partite set; $K_{1,m,n}$; the composition $P_n[P_2]$ (see §2.3 for definition); $B(3, 2, m)$ and $B(4, 3, m)$ for all $m$ (see §2.4 for notation); $P^2_n (n \geq 3)$; and $P^3_n (n \geq 3)$. Seoud et al. [SDE] have also proved: $B_{2n}$ is strongly $2n$-harmonious; $P_n$ is strongly $\lfloor n/2 \rfloor$-harmonious; ladders $L_{2k+1}$ are strongly $(k+1)$-harmonious; and that if $G$ is strongly $c$-harmonious and has an equal number of vertices and edges, then $G + \overline{K_n}$ is also strongly $c$-harmonious.

Sethuraman and Selvaraju [SS7] have proved that the graph obtained by adjoining two complete bipartite graphs at one edge is graceful and strongly harmonious. They ask the question if these results extend to any number of complete bipartite graphs.

Acharya and Hegde [AH2] have generalized sequential labelings as follows. Let $G$ be a graph with $q$ edges and let $k$ and $d$ be positive integers. A labeling $f$ of $G$ is said to be $(k, d)$-arithmetic if the vertex labels are distinct nonnegative integers and the edge labels induced by $f(x) + f(y)$ for each edge $xy$ are $k, k+d, k+2d, \ldots, k+(q-1)d$. They obtained a number of necessary conditions for various kinds of graphs to have a $(k, d)$-arithmetic labeling. The case where $k = 1$ and $d = 1$ was called additively graceful by Hegde [Heg1]. Hegde showed that $K_n$ is additively graceful if and only if $n = 2, 3$ or 4; every additively graceful graph except $K_2$ or $K_{1,2}$ contains a triangle; and a unicyclic graph is additively graceful if and only if it is a 3-cycle or a 3-cycle with a single pendant edge attached. Jinnah and Singh [JS] noted that $P^n_4$ is additively graceful. Bu and Shi [BuS1] proved that $K_n$ is not $(k, d)$-arithmetic for $n \geq 5$ and that $K_{m,n}$ is $(k, d)$-arithmetic when $k$
is not of the form \(id\) for \(1 \leq i \leq n - 1\). Yu [Yu1] proved that a necessary condition for \(C_{4t+1}\) to be \((k,d)\)-arithmetic is that \(k = 2dt + r\) for some \(r \geq 0\) and a necessary condition for \(C_{4t+3}\) to be \((k,d)\)-arithmetic is that \(k = (2t + 1)d + 2r\) for some \(r \geq 0\). These conditions were conjectured by Acharya and Hegde [AH2]. Singh proved that the graph obtained by subdividing every edge of the ladder \(L_n\) is \((5,2)\)-arithmetic [Sin3] and that the ladder \(L_n\) is \((n,1)\)-arithmetic [Sin6]. He also proves that that \(P_m \times C_n\) is \(((n-1)/2,1)\)-arithmetic when \(n\) is odd [Sin6].

A graph is called arithmetic if it is \((k,d)\)-arithmetic for some \(k\) and \(d\). Singh and Vilfred [SV] showed that various classes of trees are arithmetic. Singh [Sin6] has proved that the union of an arithmetic graph and an arithmetic bipartite graph is arithmetic. He conjectures that the union of arithmetic graphs is arithmetic. He provides an example to show that the converse is not true.

Acharya and Hegde [AH2] introduced a stronger form of sequential labeling by calling a \((p,q)\)-graph \((V,E)\) strongly \(k\)-indexable if there is an injective function from \(V\) to \(\{0,1,2,\ldots,p-1\}\) such that the set of edge labels induced by adding the vertex labels is \(\{k,k+1,k+2,\ldots,k+q-1\}\). Strongly 1-indexable graphs are simply called strongly indexable. Notice that for trees and unicyclic graphs the notions of sequential labelings and strongly \(k\)-indexable labelings coincide. Acharya and Hegde prove that the only nontrivial regular graphs that are strongly indexable are \(K_2, K_3\) and \(K_2 \times K_3\) and that every strongly indexable graph has exactly one nontrivial component that is either a star or has a triangle. Acharya and Hegde [AH2] call a graph with \(p\) vertices indexable if there is an injective labeling of the vertices with labels from \(\{0,1,2,\ldots,p-1\}\) such that the edge labels induced by addition of the vertex labels are distinct. They conjecture that all unicyclic graphs are indexable. This conjecture was proved by Arumugam and Germina [ArG] who also proved that all trees are indexable. Bu and Shi [BuS2] also proved that all trees are indexable and that all unicyclic graphs with the cycle \(C_3\) are indexable. Hegde [Heg2] has shown the following: every graph can be embedded as an induced subgraph of an indexable graph; if a connected \((p,q)\)-graph \((q \geq 2)\) is \((k,d)\)-indexable then \(d \leq 2\); \(P_m \times P_n\) is indexable for all \(m\) and \(n\); and if a \((p,q)\)-graph is \((k,d)\)-indexable, then \(q \leq (2p-3-k+d)/d\). As a corollary of the latter, it follows that \(K_n(n \geq 4)\) and wheels are not \((k,d)\)-indexable.

### 4.2 Elegant Labelings

An elegant labeling \(f\) of a graph \(G\) with \(q\) edges is an injective function from the vertices of \(G\) to the set \(\{0,1,\ldots,q\}\) such that when each edge \(xy\) is assigned the label \(f(x) + f(y) \pmod{q+1}\) the resulting edge labels are distinct and nonzero. This notion was introduced by Chang, Hsu and Rogers in 1981 [CHR]. Note that in contrast to the definition of a harmonious labeling, it is not necessary to make an exception for trees. While the cycle \(C_n\) is harmonious if and only if \(n\) is odd, Chang et al. [CHR] proved that \(C_n\) is elegant when \(n \equiv 0\) or 3 \((\text{mod } 4)\) and not elegant when \(n \equiv 1\) \((\text{mod } 4)\). Chang et al. further showed that all fans are elegant and the paths \(P_n\) are elegant for \(n \not\equiv 0\) \((\text{mod } 4)\). Cahit
then showed that $P_3$ is the only path that is not elegant. Balakrishnan, Selvam and Yegnanarayanan [BSY2] have proved numerous graphs are elegant. Among them are $K_{m,n}$ and the $m$th-subdivision graph of $K_{1,2n}$. They prove that the bistar $B_{n,n}$ ($K_2$ with $n$ pendant edges at each endpoint) is elegant if and only if $n$ is even. They also prove that every simple graph is a subgraph of an elegant graph and that several families of graphs are not elegant. Deb and Limaye [DL1] have shown that triangular snakes are elegant if and only if the number of triangles is not equal to 3 (mod 4). In the case where the number of triangles is 3 (mod 4) they show the triangular snakes satisfy a weaker condition they call semi-elegant whereby the edge label 0 is permitted. In [DL3] Deb and Limaye define a graph $G$ with $q$ edges to be near-elegant if there is an injective function $f$ from the vertices of $G$ to the set $\{0, 1, \ldots, q\}$ such that when each edge $xy$ is assigned the label $f(x) + f(y) \pmod{(q+1)}$ the resulting edge labels are distinct and not equal to $q$. Thus, in a near-elegant labeling, instead of 0 being the missing value in the edge labels, $q$ is the missing value. Deb and Limaye show that triangular snakes where the number of triangles is 3 (mod 4) are near-elegant. For any positive integers $\alpha \leq \beta \leq \gamma$ where $\beta$ is at least 2 the theta graph $\theta_{\alpha, \beta, \gamma}$ consists of three edge disjoint paths of lengths $\alpha, \beta$ and $\gamma$ having the same end points. Deb and Limaye [DL3] provide elegant and near-elegant labelings for some theta graphs where $\alpha = 1, 2$ or 3. Seoud and Elsakhawi [SE1] have proved that the following graphs are elegant: $K_{1,m,n}; K_{1,1,m,n}; K_2 + \overline{K_m}; K_3 + \overline{K_m}$; and $K_{m,n}$ with an edge joining two vertices of the same partite set.

Sethuraman and Elumalai [SE3] have proved that for every graph $(p, q)$-graph $G$ the graph $G = K_1 + \overline{K_m}$ is graceful when $m \geq 2^p - p - q$. As a corollary they deduce that every graph is a vertex induced subgraph of a elegant graph. In [SS4] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions that have a strong form of elegant labeling.

Sethuraman and Elumalai [SE2] define a graph $H$ to be a $K_{1,m}$-star extension of a $(p, q)$-graph $G$ at a vertex $v$ of $G$ where $m > p - 1 - \deg(v)$ if $H$ is obtained from $G$ by merging the center of the star $K_{1,m}$ with $v$ and merging the $n - 1 - \deg(v)$ pendant vertices of $K_{1,m}$ with the $n - 1 - \deg(v)$ nonadjacent vertices of $v$ in $G$. They prove that for every $(p, q)$-graph $G$ and every vertex $v$ of $G$ and every $m \geq 2^{p-1} - 1 - q$ there is a $K_{1,m}$-star extension of $G$ that is both graceful and harmonious. In the case where $m \geq 2^{p-1} - q$ they show that $G$ has a $K_{1,m}$-star extension that is elegant.

Sethuraman, Selvaraju and Elumalai [SSE] have shown that certain cases of the union of any number of copies of $K_4$ with one or more edges deleted and one edge in common are elegant.

Gallian extended the notion of harmoniousness to arbitrary finite Abelian groups as follows. Let $G$ be a graph with $q$ edges and $H$ a finite Abelian group (under addition) of order $q$. Define $G$ to be $H$-harmonious if there is an injection $f$ from the vertices of $G$ to $H$ such that when each edge $xy$ is assigned the label $f(x) + f(y)$ the resulting edge labels are distinct. When $G$ is a tree, one label may be used on exactly two vertices. Beals, Gallian, Headley and Jungreis [BGHJ] have shown that if $H$ is a finite Abelian
group of order $n > 1$ then $C_n$ is $H$-harmonious if and only if $H$ has a non-cyclic or trivial Sylow 2-subgroup and $H$ is not of the form $Z_2 \times Z_2 \times \cdots \times Z_2$. Thus, for example, $C_{12}$ is not $Z_{12}$-harmonious but is $(Z_2 \times Z_2 \times Z_2)$-harmonious. Analogously, the notion of an elegant graph can be extended to arbitrary finite Abelian groups. Let $G$ be a graph with $q$ edges and $H$ a finite Abelian group (under addition) with $q + 1$ elements. We say $G$ is $H$-elegant if there is an injection $f$ from the vertices of $G$ to $H$ such that when each edge $xy$ is assigned the label $f(x) + f(y)$ the resulting set of edge labels is the non-identity elements of $H$. Beals et al. [BGHJ] proved that if $H$ is a finite Abelian group of order $n$ with $n \neq 1$ and $n \neq 3$, then $C_{n-1}$ is $H$-elegant using only the non-identity elements of $H$ as vertex labels if and only if $H$ has either a non-cyclic or trivial Sylow 2-subgroup. This result completed a partial characterization of elegant cycles given by Chang, Hsu and Rogers [CHR] by showing that $C_n$ is elegant when $n \equiv 2 \pmod{4}$. Mollard and Payan [MP] also proved that $C_n$ is elegant when $n \equiv 2 \pmod{4}$ and gave another proof that $P_n$ is elegant when $n \neq 4$.

For a graph $G(V, E)$ and an Abelian group $H$ Valentin [V] defines a polychrome labeling of $G$ by $H$ to be a bijection $f$ from $V$ to $H$ such that the edge labels induced by $f(uv) = f(u) + f(v)$ are distinct. Valentin investigates the existence of polychrome labelings for paths and cycles by various Abelian groups.

### 4.3 Felicitous Labelings

Another generalization of harmonious labelings are felicitous labelings. An injective function $f$ from the vertices of a graph $G$ with $q$ edges to the set $\{0, 1, \ldots, q\}$ is called felicitous if the edge labels induced by $f(x) + f(y) \pmod{q}$ for each edge $xy$ are distinct. This definition first appeared in a paper by Lee, Schmeichel and Shee in [LSS] and is attributed to E. Choo. Balakrishnan and Kumar [BK2] proved the conjecture of Lee, Schmeichel and Shee [LSS] that every graph is a subgraph of a felicitous graph by showing the stronger result that every graph is a subgraph of a sequential graph. Among the graphs known to be felicitous are: $C_n$ except when $n \equiv 2 \pmod{4}$ [LSS]; $K_{m,n}$ when $m, n > 1$ [LSS]; $P_2 \cup C_{2n+1}$ [LSS]; $P_2 \cup C_{2n}$ [Tel]; $P_3 \cup C_{2n+1}$ [LSS]; $S_m \cup C_{2n+1}$ [LSS]; $K_n$ if and only if $n \leq 4$ [SE1]; $P_n + \overline{K}_m$ [SE1]; the friendship graph $C_3^{(n)}$ for $n$ odd [LSS]; $P_n \cup C_3$ [ShL]; $P_n \cup C_{n+3}$ [Tel]; and the one-point union of an odd cycle and a caterpillar [ShL]. Shee [S1] conjectured that $P_m \cup C_n$ is felicitous when $n > 2$ and $m > 3$. Lee, Schmeichel and Shee [LSS] ask for which $m$ and $n$ is the one-point union of $n$ copies of $C_m$ felicitous. They showed that the case where $mn$ is twice an odd integer is not felicitous. In contrast to the situation for felicitous labelings, we remark that $C_{4k}$ and $K_{m,n}$ where $m, n > 1$ are not harmonious and the one-point union of an odd cycle and a caterpillar is not always harmonious. Lee, Schmeichel and Shee [LSS] conjecture that the $n$-cube is felicitous. This conjecture was proved by Figueroa-Centeno and Ichishima in 2001 [FI].

Balakrishnan, Selvam and Yegnanarayanan [BSY1] obtained numerous results on felicitous labelings. The wreath product, $G \ast H$, of graphs $G$ and $H$ has vertex set
V(G) \times V(H) and (g_1, h_1) is adjacent to (g_2, h_2) whenever g_1g_2 \in E(G) or g_1 = g_2 and h_1h_2 \in E(H). They define H_{n,n} as the graph with vertex set \{u_1, \ldots, u_n; v_1, \ldots, v_n\} and edge set \{u_iv_j \mid 1 \leq i \leq j \leq n\}. They let \langle K_{1,n} : m \rangle denote the graph obtained by taking m disjoint copies of K_{1,n}, and joining a new vertex to the roots of the m copies of K_{1,n}. They prove the following are felicitous: H_{n,n}; P_n * K_2; \langle K_{1,m} : m \rangle; \langle K_{1,2} : m \rangle when m \not\equiv 0 \pmod{3} or m \equiv 3 \pmod{6} or m \equiv 6 \pmod{12}; \langle K_{1,2n} : m \rangle for all m and n \geq 2; \langle K_{1,2t+1} : 2n + 1 \rangle when n \geq t; P_n^k when k = n - 1 and n \not\equiv 2 \pmod{4} or k = 2t and n \geq 3 and k < n - 1; the join of a star and \overline{K}_n; and graphs obtained by joining two end vertices or two central vertices of stars with an edge. Yegnanarayanan [Ye1] conjectures that the graphs obtained from an even cycle by attaching n new vertices to each vertex of the cycle is felicitous. This conjecture was verified by Figueroa-Centeno, Ichishima and Muntaner-Batle in [FIM7]. In [SS4] Sethuraman, Selvaraju and Elhumalai [SSE] have shown that certain cases of the union of any number of copies of K_4 with 3 edges deleted and one edge in common are felicitous. Sethuraman and Selvaraju [SS4] present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions that have a felicitious labeling.

Figueroa-Centeno, Ichishima and Muntaner-Batle [FIM6] define a felicitous graph to be strongly felicitous if there exists an integer k so that for every edge uv \min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}. For a graph with p vertices and q edges with q \geq p - 1 they show that G is strongly felicitous if and only if G has an \(\alpha\)-valuation. They also show that for graphs G_1 and G_2 with strongly felicitous labelings \(f_1\) and \(f_2\) the graph obtained from G_1 and G_2 by identifying the vertices u and v such that \(f_1(u) = 0 = f_2(v)\) is strongly felicitous and that the one-point union two copies of C_m with itself where m \geq 4 and m is even is strongly felicitous. As a corollary they have that the one-point union n copies of C_m with itself where m \geq 4, m is even, and n \equiv 2 \pmod{4} is felicitous. They conjecture that the one-point union of n copies of C_m is felicitous if and only if mn \equiv 0, 1 or 3 \pmod{4}.

Chang, Hsu and Rogers [CHR] have given a sequential counterpart to felicitous labelings. They call a graph strongly c-elegant if the vertex labels are from \{0, 1, \ldots, q\} and the edge labels induced by addition are \{c, c+1, \ldots, c+q-1\}. (A strongly 1-elegant labeling has also been called a consecutive labeling.) Notice that every strongly c-elegant graph is felicitous and that strongly c-elegant is the same as (c,1)-arithmetic in the case where the vertex labels are from \{0, 1, \ldots, q\}. Results on strongly c-elegant graphs are meager. Chang et al. [CHR] have shown: \(K_n\) is strongly 1-elegant if and only if \(n = 2, 3, 4\); \(C_n\) is strongly 1-elegant if and only if \(n = 3\); and a bipartite graph is strongly 1-elegant if and only if it is a star. Shee [S2] has proved that \(K_{m,n}\) is strongly c-elegant for a particular value of c and obtained several more specialized results pertaining to graphs formed from complete bipartite graphs.

Seoud and Elsakhawi [SE2] have shown: \(K_{m,n}\) with an edge joining two vertices of the same partite set is strongly c-elegant for \(c = 1, 3, 5, \ldots, \max\{2m+1, 2n+1\} + 1\); \(K_{1,m,n}\) is strongly c-elegant for \(c = 1, 3, 5, \ldots, 2m\) when \(m = n\), and for \(c = 1, 3, 5, \ldots, m+n+1\) when \(m \neq n\); \(K_{1,1,m,n}\) is strongly c-elegant for \(c = 1, 3, 5, \ldots, 2m + 1\); \(P_n + \overline{K}_m\) is
strongly for $c = \lfloor n/2 \rfloor$-elegant; $C_m + K_n$ is strongly $c$-elegant for odd $m$ and all $n$ for $c = (m-1)/2, (m-1)/2 + 2, \ldots, 2m$ when $(m-1)/2$ is even and for $c = (m-1)/2, (m-1)/2 + 2, \ldots, 2m - (m-1)/2$ when $(m-1)/2$ is odd; ladders $L_{2k+1}$ ($k > 1$) are strongly $(k+1)$-elegant; and $B(3,2,m)$ and $B(4,3,m)$ (see §2.4 for notation) are strongly $1$-elegant and strongly $3$-elegant for all $m$; the composition $P_n[P_2]$ (see §2.3) is strongly $c$-elegant for $1, 3, 5, \ldots, 5n - 6$ when $n$ is odd and for $c = 1, 3, 5, \ldots, 5n - 5$ when $n$ is even; $P_n$ is strongly $\lfloor n/2 \rfloor$-elegant; $P_n^2$ is strongly $c$-elegant for $c = 1, 3, 5, \ldots, q$ where $q$ is the number of edges of $P_n^2$; and $P_n^3$ ($n > 3$) is strongly $c$-elegant for $c = 1, 3, 5, \ldots, 6k - 1$ when $n = 4k$, $c = 1, 3, 5, \ldots, 6k + 1$ when $n = 4k + 1$, $c = 1, 3, 5, \ldots, 6k + 3$ when $n = 4k + 2$, $c = 1, 3, 5, \ldots, 6k + 5$ when $n = 4k + 3$.

5 Miscellaneous Labelings

5.1 Magic, Edge-magic, Vertex-magic and Antimagic Labelings

Motivated by the notion of magic squares in number theory, magic labelings were introduced by Sedláček [Se1] in 1963. Responding to a problem raised by Sedláček, Stewart ([St1] and [St2]) studied various ways to label the edges of a graph in the mid 60s. Stewart calls a connected graph semi-magic if there is a labeling of the edges with integers such that for each vertex $v$ the sum of the labels of all edges incident with $v$ is the same for all $v$. A semi-magic labeling where the edges are labeled with distinct positive integers is called a magic labeling. Stewart calls a magic labeling supermagic if the set of edge labels consists of consecutive integers. The classic concept of an $n \times n$ magic square in number theory corresponds to a supermagic labeling of $K_{n,n}$. Stewart [St1] proved the following: $K_n$ is magic for $n = 2$ and all $n \geq 5$; $K_{n,n}$ is magic for all $n \geq 3$; fans $F_n$ are magic if and only if $n$ is odd and $n \geq 3$; wheels $W_n$ are magic for $n \geq 4$; $W_n$ with one spoke deleted is magic for $n = 4$ and for $n \geq 6$. Stewart [St1] also proved that $K_{m,n}$ is semi-magic if and only if $m = n$. In [St2] Stewart proved that $K_n$ is supermagic for $n \geq 5$ if and only if $n > 5$ and $n \not\equiv 0 \mod 4$. Sedláček [Se2] showed that Mőbius ladders $M_n$ are supermagic when $n \geq 3$ and $n$ is odd and that $C_n \times P_2$ is magic, but not supermagic, when $n \geq 4$ and $n$ is even. Shiu, Lam and Lee [SLL1] have proved that the composition of $C_m$ and $K_n$ is supermagic when $m \geq 3$ and $n \geq 2$. Bača, Holländer and Lih [BHL] have found two families of 4-regular supermagic graphs. Shiu, Lam and Cheng [SLC2] prove that for $n \geq 2$, $mK_{n,n}$ is supermagic if and only if $n$ is even or both $m$ and $n$ are odd. Trenkler [Tre1] has proved that a connected magic $(p,q)$-graph other than $P_2$ exists if and only if $5p/4 < q \leq p(p - 1)/2$.

Trenkler [Tre2] extended the definition of supermagic graphs to include hypergraphs and proved that the complete $k$-uniform $n$-partite hypergraph is supermagic if $n \not\equiv 2$ or $6$ and $k \geq 2$. Sedláček [Se2] also proves that graphs obtained from an odd cycle with at least 5 vertices in which every vertex $v$ of the cycle has two chords joining $v$ to the two vertices at greatest distance from $v$ are magic. (He calls these Mőbius ladders.) Trenkler
and Vetchý [TV] have shown that if \( G \) has order at least 5 then \( G^i \) is magic for all \( i \geq 3 \) and \( G^2 \) is magic if and only if \( G \) is not \( P_5 \) and \( G \) does not have a 1-factor whose every edge is incident with an end-vertex of \( G \). Seoud and Abdel Maqsoud [SA1] proved that \( K_{1,m,n} \) is magic for all \( m \) and \( n \) and that \( P_n^2 \) is magic for all \( n \). Characterizations of regular magic graphs were given by Dood [Do] and necessary and sufficient conditions for a graph to be magic were given in [Je] and [JT].

In 1976 Sedláček [Se2] defined a connected graph with at least two edges to be pseudo-magic if there exists a real-valued function on the edges with the property that distinct edges have distinct values and the sum of the values assigned to all the edges incident to any vertex is the same for all vertices. Sedláček proved that when \( n \geq 4 \) and \( n \) is even, the Möbius ladder \( M_n \) is not pseudo-magic and when \( m \geq 3 \) and \( m \) is odd, \( C_m \times P_2 \) is not pseudo-magic.

For any magic graph \( G \), Kong, Lee and Sun [KLS] let \( M(G) \) denote the set of all magic labelings of \( G \). For any \( L \) in \( M(G) \), they let \( s(L) = \max\{L(e) : e \in E_G\} \) and define the magic strength of \( G \) as \( m(G) = \min\{s(L) : L \in M(G)\} \). They use \( DS(k) \) to denote the graph obtained by taking two copies of \( K_{1,k} \) and connecting the \( k \) pairs of corresponding leaves. Kong et al. [KLS] show: for \( k > 1 \), \( m(DS(k)) = k - 1; m(P_k + K_1) = 1 \) for \( k = 1 \) or \( 2 \), \( k \) if \( k \) is odd and \( k/2 \) if \( k \) is even and \( k \) is odd; \( m(P_m \times P_n) = 1 \) if \( m = n = 2, m(P_m \times P_n) = 2 \) if \( m = 2 \) and \( n > 3 \), \( m(P_m \times P_n) = 3 \) if \( m \) or \( n \) is even and greater than 2; \( m(C_3^{(n)}) \) (Dutch windmill – see §2.4) = 1 if \( n = 1 \) and \( m(C_3^{(n)}) = 2n - 1 \) if \( n > 1 \). They also prove that if \( G \) and \( H \) are magic graphs then \( G \times H \) is magic and \( m(G \times H) = \max\{m(G), m(H)\} \) and that every connected graph is an induced subgraph of a magic graph (see also [ELNR] and [FIM1]). They conjecture that almost all connected graphs are not magic. In [LSS] Lee, Saba and Sun show that the magic strength of \( P_n^k \) is 0 when \( k \) and \( n \) are both odd. Sun and Lee [SuL] show that the Cartesian, conjunctive, normal, lexicographic, and disjunctive products of two magic graphs are magic and the sum of two magic graphs is magic. They also determine the magic strength of the constructed graphs. Avadayappan, Jayanthi and Vasuki [AJV] use the notation \( <K_{1,n} : 2> \) for the tree obtained from the bistar \( B_{n,n} \) by subdividing the edge joining the two stars. They prove: \( m(P_{2n}) = 5n + 1; m(P_{2n+1}) = 5n + 3; m(B_{n,n}) = 5n + 6; m< K_{1,n} : 2 > = 4n + 9; m((2n+1)P_2) = 9n + 6; m(C_{2n+1}) = 5n + 4; m(C_{2n}) = 5n + 2; m(K_{1,n}) = 2n + 4; m(P^2) = 3n; and m(K_{n,m}) \leq (m+2)(n+1) \) where \( n \leq m \).

In 1970 Kotzig and Rosa [KR1] defined a magic labeling of a graph \( G(V,E) \) as a bijection \( f \) from \( V \cup E \) to \( \{1, 2, \ldots, |V \cup E|\} \) such that for all edges \( xy, f(x) + f(y) + f(xy) \) is constant. To distinguish between this usage from that of Stewart we will call this labeling an edge-magic total labeling. Kotzig and Rosa proved: \( K_{m,n} \) has an edge-magic total labeling for all \( m \) and \( n \); \( C_n \) has an edge-magic total labeling for all \( n \geq 3 \) (see also [GoS] and [RB]); and the disjoint union of \( n \) copies of \( P_3 \) has an edge-magic total labeling if and only if \( n \) is odd. They further state that \( K_m \) has an edge-magic total labeling if and only if \( n = 1, 2, 3, 5 \) or 6 (see [KR2], [CT] and [ELNR]) and ask whether all trees have edge-magic total labelings. Wallis et al. [WBMS] enumerate every edge-magic total
labeling of complete graphs. They also prove that the following graphs are edge-magic total: paths, crowns, complete bipartite graphs, and cycles with a single edge attached to one vertex. Enomoto, Llado, Nakamigana and Ringel [ELNR] prove that all complete bipartite graphs and all even cycles are edge-magic total. They also show that wheels $W_n$ are not edge-magic total when $n \equiv 3 \mod 4$ and conjectured that all other wheels are edge-magic total. This conjecture was proved when $n \equiv 0, 1 \mod 4$ by Phillips, Rees and Wallis [PRW1] and when $n \equiv 6 \mod 8$ by Slamin, Baća, Lin, Miller and Simanjuntak [SBLMS]. Fukuchi [Fuk2] verified all cases of the conjecture independently of the work of others. Slamin et al. further show that all fans are edge-magic total. Berkman, Parnas and Roditty [BPR] proved that all cycles are edge-magic total. Berkman, Parnas and Roditty [BPR] proved that all cycles are edge-magic total thereby solving a problem posed in [RL]. Balakrishnan and Kumar [BK2] proved that the join of $K_n$ and two disjoint copies of $K_2$ is edge-magic total if and only if $n = 3$.

Yegnanarayanan [Ye2] has proved the following graphs have edge-magic total labelings: $nP_3$ where $n$ is odd; $P_n + K_1; P_n \times K_3 \ (n \geq 2); (P_n \times K_3) \odot K_{1,n};$ and $C_n \odot K_{1,1}$. He conjectures that for all $n$, $C_n \odot K_{1,n}$ and $nP_3$ have edge-magic total labelings. Yegnanarayanan also introduces several variations of magic and antimagic labelings and provides some results about them.

Craft and Tesar [CT] proved that an $r$-regular graph with $r$ odd and $p \equiv 4 \mod 8$ vertices can not be edge-magic total. Wallis [Wal] proved that if $G$ is an edge-magic total $r$-regular graph $(p, q)$-graph with $r = 2^t s + 1 \ (t > 0)$ and $q$ even then $2^{t+2}$ divides $p$. Figueroa-Centeno, Ichishima and Muntaner-Batle and Ichishima ([FIM1] and [FIM2]) have proved the following graphs are edge-magic total: $P_i \cup nK_2$ for $n$ odd; $P_i \cup nK_2; P_i \cup mK_2; P_i \cup nK_2$ for $n$ odd and $i = 3, 4, 5; 2P_n; P_i \cup P_2 \cup \cdots \cup P_n; mK_{1,n}; K_{1,n} \cup K_{1,n+1}; C_m \odot nK_1; K_1 \odot nK_2$ for $n$ even; $W_{2n}; K_2 \times \overline{K}_n, nK_3$ for $n$ odd; binary trees, generalized Petersen graphs, ladders, books, fans, and odd cycles with pendant edges attached to one vertex. Enomoto et al. [ELNR] conjecture that if $G$ is a graph of order $n + m$ that contains $K_n$ then $G$ is not edge-magic total for $n \gg m$. Wallis [Wal] proves that a cycle with one pendant edge is edge-magic total. In [Wal] Wallis poses a large number of research problems about edge-magic total graphs. Enomoto et al. call an edge-magic labeling total super edge-magic if the set of vertex labels is $\{1, 2, \ldots, |V|\}$ (Wallis [Wal] calls these labelings strongly edge-magic). They prove the following: $C_n$ is super edge-magic if and only if $n$ is odd, caterpillars are super edge-magic, $K_m,n$ is super edge-magic if and only if $m = 1$ or $n = 1$, and $K_n$ is super edge-magic if and only if $n = 1, 2$ or 3. They also prove that if a $(p, q)$-graph is super edge-magic then $q \leq 2p - 3$. Enomoto et al. conjecture that every tree is super edge-magic. Kotzig and Rosa’s ([KR1] and [KR2]) proof that $nK_2$ is edge-magic total when $n$ is odd actually shows that it is super edge-magic. Kotzig and Rosa also prove that every caterpillar is super-edge magic. Figueroa-Centeno, Ichishima and Muntaner-Batle [FIM5] prove the following: if $G$ is a bipartite or tripartite (super) edge-magic graph then $nG$ is (super) edge-magic when $n$
is odd; if $m$ is a multiple of $n + 1$ then $K_{1,m} \cup K_{1,n}$ is super edge-magic; $K_{1,2} \cup K_{1,n}$ is super edge-magic if and only if $n$ is a multiple of 3; $K_{1,m} \cup K_{1,n}$ is edge-magic if and only if $mn$ is even; $K_{1,3} \cup K_{1,n}$ is super edge-magic if and only if $n$ is a multiple of 4; $P_m \cup K_{1,n}$ is super edge-magic when $m \geq 4; 2P_n$ is super edge-magic if and only if $n$ is not 2 or 3; $2P_{4n}$ is super edge-magic for all $n$; and $K_{1,m} \cup 2nK_2$ is super edge-magic for all $m$ and $n$. They conjecture that $K_{1,m} \cup K_{1,n}$ is super edge-magic only when $m$ is a multiple of $n + 1$ and they prove that if $G$ is a super edge-magic $(p,q)$-graph with $p \geq 4$ and $q \geq 2p - 4$, then $G$ contains triangles. Enomoto, Masuda and Nakamigawa [EMN] have proved that every graph can be embedded in a connected super edge-magic graph as an induced subgraph. Slamin et al. [SBLMS] proved that the friendship graph consisting of $n$ triangles is super edge-magic if and only if $n$ is 3, 4, 5 or 7. Fukuchi proved [Fuk1] the generalizied Petersen graph $P(n, 2)$ is super edge-magic if $n$ is odd and at least 3.

Figueroa-Centeno, Muntaner and Ichishima [FMI1] prove that a graph is super edge-magic if and only if it is strongly 1-harmonious and that a super edge-magic graph is cordial. They also prove that $P^2_n$ and $K_2 \times C_{2n+1}$ are super edge-magic. In [FIM2] Figueroa-Centeno et al. show that the following graphs are super edge-magic: $P_3 \cup kP_2$ for all $k$; $kP_n$ when $k$ is odd; and $k(P_2 \cup P_n)$ when $k$ is odd and $n = 3$ or $n = 4$; fans $F_n$ if and only if $n \leq 6$. They conjecture that $kP_2$ is not super edge-magic when $k$ is even. This conjecture has been proved by Z. Chen [Che3] who showed that $kP_2$ is super edge-magic if and only if $k$ is odd. Figueroa-Centeno et al. provide a strong necessary condition for a book to have a super edge-magic labeling and conjecture that for $n \geq 5$ the book $B_n$ is super edge-magic if and only if $n$ is even or $n \equiv 5 \pmod{8}$. They prove that every tree with an $\alpha$-labeling is super edge-magic. Yokomura (see [EIN]) has shown that $P_{2m+1} \times P_2$ and $C_{2m+1} \times P_m$ are super edge-magic (see also [FMI1]). In [FIM7], Figueroa-Centeno et al. proved that if $G$ is a (super) edge-magic 2-regular graph, then $G \circ K_n$ is (super) edge-magic and that $C_m \circ K_n$ is super edge-magic.

In 1970 Kotzig and Rosa [KR1] defined the edge-magic deficiency, $\mu(G)$, of a graph $G$ as the minimum $n$ such that $G \cup nK_1$ is edge-magic total. If no such $n$ exists they define $\mu(G) = \infty$. In 1999 Figueroa-Centeno, Ichishima and Muntaner-Batle [FIM4] extended this notion to super edge-magic deficiency, $\mu_s(G)$, is the analogous way. They prove the following: $\mu_s(nK_2) = \mu(nK_2) = n - 1 \pmod{2}$; $\mu_s(C_n) = 0$ if $n$ is odd; $\mu_s(C_n) = 1$ if $n \equiv 0 \pmod{4}$; $\mu_s(C_n) = \infty$ if $n \equiv 2 \pmod{4}$; $\mu_s(K_n) = \infty$ if and only if $n \geq 5; \mu_s(K_{m,n}) \leq (m-1)(n-1)$; $\mu_s(K_{2,n}) = n - 1$; and $\mu_s(F)$ is finite for all forests $F$. They also prove that if a graph $G$ has $q$ edges with $q/2$ odd and every vertex is even, then $\mu_s(G) = \infty$.

Z. Chin [Che3] has proven: the join of $K_1$ with any subgraph of a star is super edge-magic; the join of two nontrivial graphs is super edge-magic if and only if at least one of them has exactly two vertices and their union has exactly one edge; and if a $k$-regular graph is super edge-magic, then $k \leq 3$. Chin also obtained the following conditions: there is a connected super edge-magic graph with $p$ vertices and $q$ edges if and only if $p - 1 \leq q \leq 2p - 3$; there is a connected 3-regular super edge-magic graph with $p$ vertices if and only if $p \equiv 2 \pmod{4}$; if $G$ is a $k$-regular edge-magic total graph with $p$ vertices
and $q$ edges then $(p + q)(1 + p + q) \equiv 0 \pmod{2d}$ where $d = \gcd(k - 1, q)$. As a corollary of the last result, Chen observes that $nK_2 + nK_2$ is not edge-magic total.

Another labeling that has been called “edge-magic” was introduced by Lee, Seah and Tan in 1992 [LST]. They defined a graph $G = (V, E)$ to be edge-magic if there exists a bijection $f : E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $f^+ : V \to \mathbb{N}$ defined by $f^+(u) = \sum_{(u,v) \in E} f(u, v) \pmod{|V|}$ is a constant map. Lee conjectured that a cubic $(p, q)$-graph is edge-magic if and only if $p \equiv 2 \pmod{4}$. Lee, Pigg and Cox [LPC] verified this conjecture for several classes of cubic graphs. A necessary condition for a $(p, q)$-multigraph to be edge-magic is that $p$ divides $q(q + 1)$ [LST]. Lee, Seah and Tan [LST] proved: $K_{n,n}$ ($n \geq 3$) is edge-magic, $K_n$ is edge-magic for $n \equiv 1, 2 \pmod{4}$ and for $n \equiv 3 \pmod{4}$ ($n \geq 7$). Lee, Seah and Tan also proved that following graphs are not edge-magic: all trees except $P_2$, all unicyclic graphs, and $K_n$ where $n \equiv 0 \pmod{4}$. Schaffer and Lee [ScL] have proved that $C_m \times C_n$ is always edge-magic. Lee, Tong and Seah [LTS] have conjectured that the total graph of a $(p, p)$-graph is edge-magic if and only if $p$ is odd. They prove this conjecture for cycles.

For any graph $G$ and any positive integer $k$ the graph $G[k]$, called the $k$-fold $G$, is the hypergraph obtained from $G$ by replacing each edge of $G$ with $k$ parallel edges. Lee, Seah and Tan [LST] proved that for any $(p, q)$-graph $G$, $G[2p]$ is edge-magic and, if $p$ is odd, $G[p]$ is edge-magic. Shiu, Lam and Lee [SLL3] show that if $G$ is an $(n+1, n)$-multigraph, then $G$ is edge-magic if and only if $n$ is odd and $G$ is isomorphic to the disjoint union of $K_2$ and $(n-1)/2$ copies of $K_2[2]$. They also prove that if $G$ is a $(2m+1, 2m)$-multigraph and $k$ is at least 2, then $G[k]$ is edge-magic if and only if $2m+1$ divides $k(k-1)$. For a $(2m, 2m-1)$-multigraph $G$ and $k$ at least 2, they show that $G[k]$ is edge-magic if $4m$ divides $(2m-1)k((2m-1)k+1)$ or if $4m$ divides $(2m+k-1)k$. In [SLL2] Shiu, Lam and Lee characterize the $(p, p)$-multigraphs that are edge-magic as $mK_2[2]$ or the disjoint union of $mK_2[2]$ and two particular multigraphs or the disjoint union of $K_2, mK_2[2]$ and 4 particular multigraphs. They also show for every $(2m+1, 2m+1)$-multigraph $G$, $G[k]$ is edge-magic for all $k$ at least 2. Lee, Seah and Tan [LST] prove that the multigraph $C_n[k]$ is edge-magic for $k \geq 2$.

MacDougall, Miller, Slamin and Wallis [MMSW] introduced the notion of a vertex-magic total labeling in 1999. For a graph $G(V, E)$ an injective mapping $f$ from $V \cup E$ to the set $\{1, 2, \ldots, |V| + |E|\}$ is a vertex-magic total labeling if there is a constant $k$ so that for every vertex $v$, $f(v) + \sum f(vu) = k$ where the sum is over all vertices $u$ adjacent to $v$. They prove that the following graphs have vertex-magic total labelings: $C_n$ for $n > 2$; $P_n$ for $n > 2$; $K_{m,m}$ for $m > 1$; $K_{m,m} - e$ for $m > 2$; and $K_n$ for $n$ odd. They also prove that when $n > m + 1$, $K_{m,n}$ does not have a vertex-magic total labeling. They conjecture that $K_{m,m+1}$ has a vertex-magic total labeling for all $m$ and that $K_n$ has vertex-magic total labeling for all $n \geq 3$. Lin and Miller [LN] have shown that $K_{m,m}$ is vertex-magic total for all $m > 1$ and that $K_n$ is vertex-magic total for all $n \equiv 2 \pmod{4}, n > 2$. Phillips, Rees and Wallis [PRW2] generalized the Lin and Miller result by proving that $K_{m,n}$ is vertex-magic total if and only if $m$ and $n$ differ by at most 1. Miller, Bača, and MacDougall [MBM] have proved that the generalized Petersen graphs
$P(n, k)$ (see Section 2.7 for the definition) are vertex-magic total when $n$ is even and $k \leq n/2 - 1$. They conjecture that all $P(n, k)$ are vertex-magic total when $k \leq (n-1)/2$ and all prisms $C_n \times P_3$ are vertex-magic total. In [SIM] Slamim and Miller prove the first of these conjectures when $k$ and $n$ are relatively prime and the second of the conjectures for all cases. Bača, Miller and Slamin [BMS] proved that every generalized Petersen graph has a vertex-magic total labeling. MacDougall et al. ([MMSW] and [MMW]) have shown: $W_n$ has a vertex-magic total labeling if and only if $n \leq 11$; fans $F_n$ have a vertex-magic total labelings if and only if $n \leq 10$; friendship graphs have vertex-magic total labelings if and only if $n$ is at most 3; $K_{m,n}(m > 1)$ has a vertex-magic total labeling if and only if $m$ and $n$ differ by at most 1. Wallis [Wal] proved: if $G$ and $H$ have the same order and $G \cup H$ is vertex-magic total then so is $G + H$; if the disjoint union of stars is vertex-magic total then the average size of the stars is less than 3; it a tree has $n$ internal vertices and more than $2n$ leaves then it does not have a vertex-magic total labeling.

Bača, et al. [BBMMSS1] introduced the notion of a $(a, d)$-vertex-antimagic total labeling in 2000. For a graph $G(V, E)$, an injective mapping $f$ from $V \cup E$ to the set $\{1, 2, \ldots, |V| + |E|\}$ is a $(a, d)$-vertex-antimagic total labeling if the set $\{f(v) + \sum f(vu)\}$ where the sum is over all vertices $u$ adjacent to $v$ for all $v$ in $G$ is $\{a, a + d, a + 2d, \ldots, a + |V|d\}$. Among their results are: every super-magic graph has an $(a, 1)$-vertex-antimagic total labeling, every $(a, d)$-antimagic graph $G(V, E)$ is $(a + |E| + 1, d + 1)$-vertex-antimagic total, and, for $d > 1$, every $(a, d)$-antimagic graph $G(V, E)$ is $(a + |V| + |E|, d - 1)$-vertex-antimagic total. They also show that paths and cycles have $(a, d)$-vertex-antimagic total labelings for a wide variety for $a$ and $d$. In [BBMMSS2] Bača et al. use their results in [BBMMSS1] to obtain numerous $(a, d)$-vertex-antimagic total labelings for prisms, antiprisms and generalized Petersen graphs. Lin, Miller, Simanjuntak and Slamin [LMSS] have shown that for $n > 20$, $W_n$ has no $(a, d)$-vertex-antimagic total labeling.

Simanjuntak, Bertault and Miller [SBM] define an $(a, d)$-edge-antimagic vertex labeling for a graph $G(V, E)$ as an injective mapping $f$ from $V$ onto the set $\{1, 2, \ldots, |V|\}$ such that the set $\{f(u) + f(v)|uv \in E\}$ is $\{a, a + d, a + 2d, \ldots, a + (|E| - 1)d\}$. Similarly, they define an $(a, d)$-edge-antimagic total labeling for a graph $G(V, E)$ as an injective mapping $f$ from $V \cup E$ onto the set $\{1, 2, \ldots, |V| + |E|\}$ such that the set $\{f(u) + \sum f(vu)|uv \in E\}$ where $v$ ranges over all of $V$ is $\{a, a + d, a + 2d, \ldots, a + (|E| - 1)d\}$. Among their results are: $C_{2n}$ has no $(a, d)$-edge-antimagic vertex labeling, $C_{2n+1}$ has a $(n + 2, 1)$-edge-antimagic vertex labeling and a $(n + 3, 1)$-edge-antimagic vertex labeling, $P_{2n}$ has a $(n + 2, 1)$-edge-antimagic vertex labeling, $P_n$ has a $(3, 2)$-edge-antimagic vertex labeling, $C_n$ has a $(2n + 2, 1)$- and a $(3n + 2, 1)$-edge-antimagic total labeling, $C_{2n}$ has a $(4n + 2, 2)$- and a $(4n + 3, 2)$-edge-antimagic total labeling, $C_{2n+1}$ has a $(3n + 4, 3)$- and a $(3n + 5, 3)$-antimagic total labeling, $P_{2n+1}$ has a $(3n + 4, 2)$-, a $(3n + 4, 3)$-, a $(2n + 4, 3)$-, a $(5n + 4, 2)$-, a $(3n + 5, 2)$-, and a $(2n + 6, 4)$-edge-antimagic total labeling, $P_{2n}$ has a $(6n, 1)$- and a $(6n + 2, 2)$-edge-antimagic total labeling and several parity conditions for $(a, d)$-edge-antimagic total labelings. They conjecture: paths have no $(a, d)$-edge-antimagic vertex
labelings with \(d > 2\), \(C_{2n}\) has a \((2n+3, 4)\)- or a \((2n+4, 4)\)-edge-antimagic total labeling, \(C_{2n+1}\) has a \((n+4, 5)\)- or a \((n+5, 5)\)-edge-antimagic total labeling, and cycles have no \((a, d)\)-antimagic total labelings with \(d > 5\).

In 2001, Simanjuntak, Rodgers and Miller [SRM] defined a 1-vertex magic vertex labeling of \(G(V, E)\) as a bijection from \(V\) to \(\{1, 2, \ldots, |V| + |E|\}\) with the property that there is a constant \(k\) such that at any vertex \(v\) the sum \(\sum f(u)\) taken over all neighbors of \(v\) is \(k\). Among their results are: \(H \times K_{2k}\) has a 1-vertex-magic vertex labeling for any regular graph \(H\), the symmetric complete multipartite graph with \(p\) parts, each of which contains \(n\) vertices, has a 1-vertex-magic vertex labeling if and only if whenever \(n\) is odd, \(p\) is also odd, and if \(n = 1\), then \(p = 1\), \(P_n\) has a 1-vertex-magic vertex labeling if and only if \(n = 1\) or \(3\), \(C_n\) has a 1-vertex-magic vertex labeling if and only if \(n = 4\), \(K_n\) has a 1-vertex-magic vertex labeling if and only if \(n = 1\), \(W_n\) has a 1-vertex-magic vertex labeling if and only if \(n = 4\), a tree has a 1-vertex-magic vertex labeling if and only if it is \(P_1\) or \(P_3\), and \(r\)-regular graphs with \(r\) odd do not have a 1-vertex-magic vertex labeling.

A magic-type method for labeling the vertices, edges and faces of a planar graph was introduced by Lih [Lih] in 1983. Lih defines a magic labeling of type \((1,1,0)\) of a planar graph \(G(V, E)\) as an injective function from \(\{1, 2, \ldots, |V| + |E|\}\) to \(V \cup E\) with the property that for each interior face the sum of the labels of the vertices and the edges surrounding that face is some fixed value. Similarly, Lih defines a magic labeling of type \((1,1,1)\) of a planar graph \(G(V, E)\) with face set \(F\) as an injective function from \(\{1, 2, \ldots, |V| + |E| + |F|\}\) to \(V \cup E \cup F\) with the property that for each interior face the sum of the labels of the face and the vertices and the edges surrounding that face is some fixed value. Lih calls a labeling involving the faces of a plane graph consecutive if for every integer \(s\) the weights of all \(s\)-sided faces constitute a set of consecutive integers. Lih gave consecutive magic labelings of type \((1,1,0)\) for wheels, friendship graphs, prisms and some members of the Platonic family. In [Bac14] Bača shows that the cylinders \(C_n \times P_m\) have magic labelings of type \((1,1,0)\) when \(m \geq 2\), \(n \geq 3\), \(n \neq 4\). Bača has given magic labelings of type \((1,1,1)\) for fans [Bac1], ladders [Bac1], planar bipyramids (that is, 2-point suspensions of paths) [Bac1], grids [Bac4], hexagonal lattices [Bac3], Möbius ladders [Bac8] and \(P_n \times P_3\) [Bac12]. Bača [Bac2], [Bac10], [Bac11], [Bac12], [Bac13] and Bača and Holländer [BHo2] give magic labelings of type \((1,1,1)\) and type \((1,1,0)\) for certain classes of convex polytopes. Bača [Bac9] also provides consecutive and magic labelings of type \((0,1,1)\) (that is, an injective function from \(\{1, 2, \ldots, |E| + |F|\}\) to \(E \cup F\) with the property that for each interior face the sum of the labels of the face and the edges surrounding that face is some fixed value) and a consecutive labeling of type \((1,1,1)\) for a kind of planar graph with hexagonal faces.

Kathiresan, Muthuvel and Nagasubbu [KaMN] define a Lotus inside a circle as the graph obtained from the cycle with consecutive vertices \(a_1, a_2, \ldots, a_n\) and the star with central vertex \(b_0\) and end vertices \(b_1, b_2, \ldots, b_n\) by joining each \(b_i\) to \(a_i\) and \(a_{i+1} (a_{n+1} = a_1)\). They prove that these graphs \((n \geq 5)\) and subdivisions of ladders have consecutive labelings of type \((1,0,0)\).
Hartsfield and Ringel [HR] introduced antimagic graphs in 1990. A graph with $q$ edges is called antimagic if its edges can be labeled with $1, 2, \ldots, q$ so that the sums of the labels of the edges incident to each vertex are distinct. Among the antimagic graphs are [HR]: $P_n$ ($n \geq 3$), cycles, wheels, and $K_n$ ($n \geq 3$). Hartsfield and Ringel conjecture that every tree except $P_2$ is antimagic and, moreover, every connected graph except $P_2$ is antimagic.

The concept of an $(a, d)$-antimagic labelings was introduced by Wagner and Bodendiek [WB] in 1993. A connected graph $G = (V, E)$ is said to be $(a, d)$-antimagic if there exist positive integers $a$, $d$ and a bijection $f: E \rightarrow \{1, 2, \ldots, |E|\}$ such that the induced mapping $g_f: V \rightarrow N$, defined by $g_f(v) = \sum \{f(u, v): (u, v) \in E(G)\}$, is injective and $g_f(V) = \{a, a + d, \ldots, a + (|V| - 1)d\}$. (In [LMSS] these are called $(a, d)$-vertex-antimagic edge labelings). They prove ([BW1] and [BW3]) the Herschel graph is not $(a, d)$-antimagic and certain cases of graphs called parachutes $P_{g,b}$ are antimagic. ($P_{g,b}$ is the graph obtained from the wheel $W_{g+p}$ by deleting $p$ consecutive spokes.) In [BHo1] Bača and Holländer prove that necessary conditions for $C_n \times P_2$ to be $(a, d)$-antimagic are $d = 1, a = (7n + 4)/2$ or $d = 3, a = (3n + 6)/2$ when $n$ is even and $d = 2, a = (5n + 5)/2$ or $d = 4, a = (n + 7)/2$ when $n$ is odd. Bodendiek and Walther [BW2] conjectured that $C_n \times P_2$ ($n \geq 3$) is $((7n + 4)/2, 1)$-antimagic when $n$ is even and is $((5n + 5)/2, 2)$-antimagic when $n$ is odd. These conjectures were verified by Bača and Holländer [BHo1] who further proved that $C_n \times P_2$ ($n \geq 3$) is $((3n + 6)/2, 3)$-antimagic when $n$ is even. Bača and Holländer [BHo1] conjecture that $C_n \times P_2$ is $((n + 7)/2, 4)$-antimagic when $n$ is odd and at least 7. Bodendiek and Walther [BW3] also conjectured that $C_n \times P_2$ ($n \geq 7$) is $((n + 7)/2, 4)$-antimagic. Bača and Holländer [BHo3] prove that the generalized Petersen graph $P(n, 2)$ is $((3n + 6)/2, 3)$-antimagic for $n \equiv 0 \pmod{4}, n \geq 8$. Bodendiek and Walther [BW5] proved that the following graphs are not $(a, d)$-antimagic: even cycles; paths of even order; stars; $C_3^{(k)}, C_4^{(k)}$; trees of odd order at least 5 that have a vertex that is adjacent to three or more end vertices; $n$-ary trees with at least two layers when $d = 1$; $K_{3,3}$; the Petersen graph; and $K_4$. They also prove: $P_{2k+1}$ is $(k, 1)$-antimagic; $C_{2k+1}$ is $(k+2, 1)$-antimagic; if a tree of order $2k+1$ ($k > 1$) is $(a, d)$-antimagic, then $d = 1$ and $a = k$; if $K_{4k}$ ($k \geq 2$) is $(a, d)$-antimagic then $d$ is odd and $d \leq 2k(4k-3)+1$; if $K_{4k+2}$ is $(a, d)$-antimagic then $d$ is even and $d \leq (2k+1)(4k-1)+1$; and if $K_{2k+1}$ ($k \geq 2$) is $(a, d)$-antimagic then $d \leq (2k+1)(k-1)$. Lin, Müller, Simanjuntak and Slamin [LMSS] show that no wheel $W_n$ ($n > 3$) has an antimagic labeling.

The antiprism on $2n$ vertices has vertex set $\{x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}\}$ and edge set $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i-1}\}$ (subscripts are taken modulo $n$). For $n \geq 3$ and $n \not\equiv 2 \pmod{4}$ Bača [Bac5] gives $(6n + 3, 2)$-antimagic labelings and $(4n + 4, 4)$-antimagic labelings for the antiprism on $2n$ vertices. He conjectures that the antiprism on $2n$ vertices has a $(2n + 5, 6)$-antimagic labeling for $n \geq 4$, and a $(6n + 3, 2)$-antimagic labeling and a $(4n + 4, 4)$-antimagic labeling for $n \not\equiv 2 \pmod{4}$ with $n \geq 6$.

Bača [Bac6] defines a connected plane graph $G$ with edge set $E$ and face set $F$ to be $(a, d)$-face antimagic if there exist positive integers $a$ and $d$ and a bijection $g: E \rightarrow \{1, 2, \ldots, |E|\}$ such that the induced mapping $\psi_g: F \rightarrow \{a, a + d, \ldots, a + (|F(G)| - 1)d\}$ is
also a bijection where for a face \( f \), \( \psi_g(f) \) is the sum of all \( g(e) \) for all edges \( e \) surrounding \( f \). Bača [Bac6] and Bača and Miller [BaM1] describe \((a,d)\)-face antimagic labelings for a certain classes of convex polytopes. In [BaM2] Bača and Miller define the class \( Q^n_m \) of convex polytopes with vertex set \( \{ y_{j,i} : i = 1,2,\ldots,n ; j = 1,2,\ldots,m+1 \} \) and edge set \( \{ y_{j,i}y_{i+1,j} : i = 1,2,\ldots,n ; j = 1,2,\ldots,m+1 \} \cup \{ y_{j,i}y_{j+1,i} : i = 1,2,\ldots,n ; j = 1,2,\ldots,m, j \text{ odd} \} \cup \{ y_{j,i}y_{j+1,i+1} : i = 1,2,\ldots,n ; j = 1,2,\ldots,m, j \text{ even} \} \) where \( y_{j,n+1} = y_{j,1} \). They prove that for \( m \) odd, \( m \geq 3, n \geq 3, Q^m_n \) is \((7n(m+1)/2 + 2, 1)\)-face antimagic and when \( m \) and \( n \) are even, \( m \geq 4, n \geq 4, Q^m_n \) is \((7n(m+1)/2 + 2, 1)\)-face antimagic. They conjecture that when \( n \) is odd, \( n \geq 3, m \) is even, \( m \geq 2, \) then \( Q^m_n \) is \((5n(m+1)+5)/2, 2\)–face antimagic. In [Bac7] Bača proves that for \( n \) even and at least 4 the prism \( C_n \times P_2 \) is \((6n+3, 2)\)-face antimagic and \((4n+4, 4)\)-face antimagic. He also conjectures that \( C_n \times P_2 \) is \((2n+5, 6)\)-face antimagic. In [BLM] Bača, Lin and Miller investigate \((a,d)\)-face antimagic labelings of the convex polytopes \( P_{m+1} \times C_n \). They show that if these graphs are \((a,d)\)-face antimagic then either \( d = 2 \) and \( a = 3n(m+1)+3 \) or \( d = 4 \) and \( a = 2n(m+1) + 4, \) or \( d = 6 \) and \( a = n(m+1) + 5 \). They also prove that if \( n \) is even, \( n \geq 4 \) and \( m \equiv 1 \text{ (mod 4)}, m \geq 3, \) or if \( n \equiv 2 \text{ (mod 4)}, n \geq 6 \) and \( m \) is even, \( m \geq 4, \) then \( P_{n+1} \times C_n \) has a \((3n(m+1)+3, 2)\)-face antimagic labeling and if \( n \) is even, \( n \geq 4 \) and \( m \) is odd, \( m \geq 3, \) or if \( n \equiv 2 \text{ (mod 4)}, n \geq 6 \) and \( m \) is even, \( m \geq 4, \) then \( P_{m+1} \times C_n \) has a \((3n(m+1)+3, 2)\)-face antimagic labeling and a \((2n(m+1)+4, 4)\)-face antimagic labeling. They conjecture that \( P_{n+1} \times C_n \) has \((3n(m+1)+3, 2) \) and \((2n(m+1)+4, 4)\)-face antimagic labelings when \( m \equiv 0 \text{ (mod 4)}, n \geq 4 \) and \( m \geq 4 \) and that \( P_{m+1} \times C_n \) has a \((n(m+1)+5, 6)\)-face antimagic labeling when \( n \) even, \( n \geq 4, \).

For a plane graph \( G \), Bača and Miller [BaM3] call a bijection \( h \) from \( V(G) \cup E(G) \cup F(G) \) to \( \{1,2,\ldots,|V(G)|+|E(G)|+|F(G)| \} \) a \( d \)-antimagic labeling of type \((1,1,1)\) if for every number \( s \) the sets of \( s \)-sided face weights is \( W_s = \{ a_s, a_s+d, a_s+2d,\ldots,a_s+(f_s-1)d \} \) for some integers \( a_s \) and \( d \), where \( f_s \) is the number of \( s \)-sided faces \( (W_s \) varies with \( s \). They show that the prisms \( C_n \times P_2 \) \( n \geq 3 \) have a \( 1 \)-antimagic labeling of type \((1,1,1)\) \( \) and \( \) that for \( n \equiv 3 \text{ (mod 4)} C_n \times P_2 \) have a \( d \)-antimagic labeling of type \((1,1,1)\) for \( d = 2, 3, 4 \) and 6. They conjecture for all \( n \geq 3, C_n \times P_2 \) has a \( d \)-antimagic labeling of type \((1,1,1)\) for \( d = 2, 3, 4, 5 \) and 6. This conjecture has been prove for the case \( d = 3 \) and \( n \neq 4 \) by Bača, Miller and Ryan [BMR] (the case \( d = 3 \) and \( n = 4 \) is open). They also prove that for \( n \geq 4 \) the antiprism on \( 2n \) vertices has a \( d \)-antimagic labeling of type \((1,1,1)\) for \( d = 1, 2 \) and 4. They conjecture the result holds for \( d = 3, 5 \) and 6 as well.

Bača, Baskoro and Miller [BBM] have proved that a honeycomb graph with an even number of columns have a \( 2 \)-antimagic and \( 4 \)-antimagic labelings of type \((1,1,1)\). They conjecture that these honeycombs also have \( d \)-antimagic of type \((1,1,1)\) for \( d = 1, 3 \) and 5. They pose the odd number of columns case for \( 1 \leq d \leq 5 \) as an open problem.

Sonntag [So] has extended the notion of antimagic labelings to hypergraphs. He
shows that certain classes of cacti, cycle and wheel hypergraphs have antimagic labelings. In [BMMSW] Baća et al. survey results on antimagic, edge-magic total and vertex-magic total labelings.

Figueroa-Centeno, Ichishima and Muntaner-Batle [FIM3] have introduced multiplicative analogs of magic and antimagic labelings. They define a graph $G$ of size $q$ to be product magic if there is a labeling $f$ from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that, at each vertex $v$, the product of the labels on the edges incident with $v$ is the same. They call a graph $G$ of size $q$ product antimagic if there is a labeling $f$ from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that the products of the labels on the edges incident at each vertex $v$ are distinct. They prove that a graph of size $q$ is product magic if and only if $q-1$, $P_n$ $(n \geq 4)$ is product antimagic, every 2-regular graph is product antimagic and if $G$ is product antimagic, then so are $G + K_1$ and $G \odot \overline{K}_n$. They conjecture that a connected graph of size $q$ is product antimagic if and only if $q \geq 3$.

5.2 Prime and Vertex Prime Labelings

The notion of a prime labeling originated with Entringer and was introduced in a paper by Tout, Dabboucy and Howalla (see [LWY]). A graph with vertex set $V$ is said to have a prime labeling if its vertices are labeled with distinct integers $1, 2, \ldots, |V|$ such that for each edge $xy$ the labels assigned to $x$ and $y$ are relatively prime. Around 1980, Entringer conjectured that all trees have a prime labeling. So far, there has been little progress towards proving this conjecture. Among the classes of trees known to have prime labelings are: paths, stars, caterpillars, complete binary trees, spiders (i.e., trees with a one vertex of degree at least 3 and with all other vertices with degree at most 2) and all trees of order less than 35 (see [Pi] and [FH]). Other graphs with prime labelings include all cycles and the disjoint union of $C_2k$ and $C_n$ [DLM]. The complete graph $K_n$ does not have a prime labeling for $n \geq 4$ and $W_n$ is prime if and only if $n$ is even (see [LWY]).

Seoud, Diab and Elsakhawi [SDE] have shown the following graphs are prime: fans; helms; flowers; stars; $K_2, n$; and $K_3, n$ unless $n = 3$ or 7. They also shown that $P_n + \overline{K}_m$ $(m \geq 3)$ is not prime.

For $m$ and $n$ at least 3, Seoud and Youssef [SY6] define $S^{(m)}_n$, the $(m, n)$-gon star, as the graph obtained from the cycle $C_n$ by joining the two end vertices of the path $P_{n-2}$ to every pair of consecutive vertices of the cycle such that each of the end vertices of the path is connected to exactly one vertex of the cycle. Seoud and Youssef [SY6] have proved the following graphs have prime labelings: books, $S^{(m)}_n$, $C_n \odot P_m$, $P_n + \overline{K}_2$.
if and only if \( n = 2 \) or \( n \) is odd, and \( C_n \odot K_1 \) with a complete binary tree of order \( 2^k - 1, k \geq 2 \) attached at each pendant vertex. They also prove that every spanning subgraph of a prime graph is prime and every graph is a subgraph of a prime graph. They conjecture that all unicycle graphs have prime labelings. Seoud and Youssef [SY6] prove the following graphs are not prime: \( C_m + C_n, C_n^2 \) for \( n \geq 4 \), \( P_n^2 \) for \( n = 6 \) and for \( n \geq 8 \), and Möbius ladders \( M_n \) for \( n \) even. They also give an exact formula for the maximum number of edges in a prime graph of order \( n \) and an upper bound for the chromatic number of a prime graph. Salmasian [Sa] has shown that every tree with \( n \) vertices \( (n \geq 50) \) can be labeled with \( n \) integers between 1 and \( 4n \) so that every two adjacent vertices have relatively prime labels.

Given a collection of graphs \( G_1, \ldots, G_n \) and some fixed vertex \( v_i \) from each \( G_i \), Lee, Wui and Yeh [LWY] define \( Amal\{\langle G_i, v_i \rangle \} \), the amalgamation of \( \{\langle G_i, v_i \rangle \} i = 1, \ldots, n \}, as the graph obtained by taking the union of the \( G_i \) and identifying \( v_1, v_2, \ldots, v_n \). Lee et al. [LWY] have shown \( Amal\{\{G_i, v_i\}| \) has a prime labeling when \( G_i \) are paths and when \( G_i \) are cycles. They also showed that the amalgamation of any number of copies of \( W_n \), \( n \) odd, with a common vertex is not prime. They conjecture that for any tree \( T \) and \( v \) from \( T \), the amalgamation of two or more copies of \( T \) with \( v \) in common is prime. They further conjecture that the amalgamation of two or more copies of \( W_n \) that share a common point is prime when \( n \) is even \( (n \neq 4) \).

For any finite collection \( \{G_i, u_iv_i\} \) of graphs \( G_i \), each with a fixed edge \( u_i v_i \), Carlson [Ca] defines the edge amalgamation \( Edgeamal\{\langle G_i, u_i v_i \rangle \} \) as the graph obtained by taking the union of all the \( G_i \) and identifying their fixed edges. The case where all the graphs are cycles she calls generalized books. She proves that all generalized books are prime graphs. Moreover, she shows that graphs obtained by taking the union of cycles and identifying in each cycle the path \( P_n \) are also prime. Carlson also proves that \( C_m \)-snakes are prime.

A dual of prime labelings has been introduced by Deretsky, Lee and Mitchem [DLM]. They say a graph with edge set \( E \) has a vertex prime labeling if its edges can be labeled with distinct integers \( 1, \ldots, |E| \) such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is 1. Deretsky, Lee and Mitchem show the following graphs have vertex prime labelings: forests; all connected graphs; \( C_{2k} \cup C_n; C_{2m} \cup C_{2n} \cup C_{2k+1}; C_{2m} \cup C_{2n} \cup C_{2k} \cup C_{2k+1}; C_{2m} \cup C_{2n} \cup C_{2k+1}; \) and \( 5C_{2m} \). They further prove that a graph with exactly two components, one of which is not an odd cycle, has a vertex prime labeling and a 2-regular graph with at least two odd cycles does not have a vertex prime labeling. They conjecture that a 2-regular graph has a vertex prime labeling if and only if it does not have two odd cycles. Let \( G = \bigcup_{i=1}^t C_{2n_i} \) and \( N = \sum_{i=1}^t n_i \). In [BHH] Borosh, Hensley and Hobbs proved that there is a positive constant \( n_0 \) such that the conjecture of Deretsky et al. is true for the cases that (i) \( G \) is the disjoint union of at most seven cycles, or (ii) \( G \) is a union of cycles all of the same even length \( 2n \) if \( n \leq 150,000 \) or if \( n \geq n_0 \), or (iii) \( n_i \geq (\log N)^{4 \log \log \log n} \) for all \( i = 1, \ldots, t \), or (iv) each \( C_{2n_i} \) is repeated at most \( n_i \) times. They end their paper with a discussion of graphs whose components are all even cycles, and of graphs with some components that are not cycles and some
components that are odd cycles.

5.3 Edge-graceful Labelings

In 1985, Lo [Lo] introduced the notion of edge-graceful graphs. A graph $G(V, E)$ is said to be edge-graceful if there exists a bijection $f$ from $E$ to $\{1, 2, \ldots, |E|\}$ so that the induced mapping $f^+: V \to \{0, 1, \ldots, |V| - 1\}$ given by $f^+(x) = \sum \{f(xy) | xy \in E\} \mod |V|$ is a bijection. Note that an edge-graceful graph is anti-magic. A necessary condition for a $(p, q)$-graph to be edge-graceful is that $q(q+1) \equiv p(p+1)/2 \mod p$. Lee [L3] notes that this necessary condition extends to any $(p, q)$-multigraph. Lee, Lee and Murthy [LLM] proved that $K_n$ is edge-graceful if and only if $n \not\equiv 2 \mod 4$. (An edge-graceful labeling for $K_n$ for $n \equiv 2 \mod 4$ in [Lo] was incorrect.) Lee [L3] notes that a multigraph with $p \equiv 2 \mod 4$ vertices is not edge-graceful and conjectures that this condition is sufficient for the edge-gracefulness of connected graphs. Lo proved that all odd cycles are edge-graceful and Wilson and Riskin [WR] proved the Cartesian product of any number of odd cycles is edge-graceful. Lee [L2] has conjectured that all trees of odd order are edge-graceful. Small [Sma] has proved that spiders (see §5.2 for the definition) of odd degree with the property that the distance from the vertex of degree greater than 2 to each end vertex is the same are edge-graceful. Keene and Simoson [KS] proved that all spiders of odd order with exactly three end vertices are edge-graceful. Cabaniss, Low and Mitchem [CLM] have shown that regular spiders of odd order are edge-graceful. Lee and Seah [LSe2] have shown that $K_{n, n, \ldots, n}$ is edge-graceful if and only if $n$ is odd and the number of partite sets is either odd or a multiple of 4. Lee and Seah [LSe1] have also proved that $C_n^k$ (the $k$th power of $C_n$) is edge-graceful for $k < [n/2]$ if and only if $n$ is odd and $C_n^k$ is edge-graceful for $k \geq [n/2]$ if and only if $n$ is a multiple of 4 or $n$ is odd (see also [CLM]). Lee, Seah and Wang [LSW] gave a complete characterization of edge-graceful $P_n^k$ graphs. Shiu, Lam and Cheng [SLC1] proved that the composition of the path $P_3$ and any null graph of odd order is edge-graceful. Shiu, Lee and Schaffer [SLS] investigated the edge-gracefulness of multigraphs derived from paths, combs and spiders obtained by replacing each edge by $k$ parallel edges. Lee and Seah [LSe3] have also investigated edge-gracefulness of various multigraphs.

Lee and Seah (see [L3]) define a sunflower graph $SF(n)$ as the graph obtained by starting with an $n$-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ and creating new vertices $w_1, w_2, \ldots, w_n$ with $w_i$ connected to $v_i$ and $v_{i+1}$ ($v_{n+1}$ is $v_1$). They prove that $SF(n)$ is edge-graceful if and only if $n$ is even. Lee and Seah [LSe4] prove that for $k \leq n/2$, $C_n^k$ is edge-graceful if and only if $n$ is odd and, for $k \geq n/2$, $C_n^k$ is edge-graceful if and only if $n \not\equiv 2 \mod 4$. Lee, Seah and Lo (see [L3]) have proved that for $n$ odd, $C_{2n} \cup C_{2n+1}, C_n \cup C_{2n+2}$ and $C_n \cup C_{4n}$ are edge-graceful. They also show that for odd $k$ and odd $n$, $kC_n$ is edge-graceful. Lee and Seah (see [L3]) prove that the generalized Petersen graph $P(n, k)$ (see Section 2.7) is edge-graceful if and only if $n$ is even and $k < n/2$. In particular, $P(n, 1) = C_n \times P_2$ is edge-graceful if and only if $n$ is even. Lee and Schaffer [ScL] proved that $C_m \times C_n$ ($m > 2, n > 2$) is edge-graceful if and only if
m and n are odd. They also showed that if G and H are edge-graceful regular graphs of odd order then $G \times H$ is edge-graceful and that if G and H are edge-graceful graphs where G is c-regular of odd order m and H is d-regular of odd order n, then $G \times H$ is edge-magic if $\gcd(c, mn) = \gcd(d, m) = 1$. They further show that if H is odd order, 2d-regular, edge-graceful and $\gcd(d, m) = 1$, then $C_{2m} \times H$ is edge-magic and if G is odd-regular, edge-graceful of even order n which is not divisible by 3, and G can be partitioned into 1-factors, then $G \times C_m$ is edge-graceful.

Kendrick and Lee (see [L3]) proved that there are only finitely many n for which $K_{m,n}$ is edge-graceful and they completely solve the problem for m = 2 and m = 3. Ho, Lee and Seah [HLSe] use $S(n; a_1, a_2, \ldots, a_k)$ where n is odd and $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k < n/2$ to denote the $(n, nk)$-multigraph with vertices $v_0, v_1, \ldots, v_{n-1}$ and edge set $\{v_i v_j\} \ i \neq j, i - j \equiv a_t \pmod{n}$ for $t = 1, 2, \ldots, k$. They prove that all such multigraphs are edge-graceful. Lee and Pritikin (see [L3]) prove that the Möbius ladders of order 4n are edge-graceful. Lee, Tong and Seah [LTS] have conjectured that the total graph of a $(p, p)$-graph is edge-graceful if and only if p is even. They have proved this conjecture for cycles.

Kuang, Lee, Mitchem and Wang [KLMW] have conjectured that unicyclic graphs of odd order are edge-graceful. They have verified this conjecture in the following cases: graphs obtained by identifying the end point of a path $P_n$ with a vertex of $C_n$ when $m + n$ is even; crowns with one pendant edge deleted; graphs obtained from crowns by identifying an endpoint of $P_m$, m odd, with a vertex of degree 1; amalgamations of a cycle and a star obtained by identifying the center of the star with a cycle vertex where the resulting graph has odd order; graphs obtained from $C_n$ by joining a pendant edge to $n - 1$ of the cycle vertices and two pendant edges to the remaining cycle vertex.

In 1997 Yilmaz and Cahit [YC] introduced a weaker version of edge-graceful called $E$-cordial. Let G be a graph with vertex set V and edge set E and let f be a function from E to $\{0, 1\}$. Define f on V by $f(v) = \sum f(uv) | uv \in E \pmod{2}$. The function f is called an $E$-cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph that admits an $E$-cordial labeling is called $E$-cordial. Yilmaz and Cahit prove the following graphs are $E$-cordial: trees with n vertices if and only if n $\not\equiv 2 \pmod{4}$; $K_n$ if and only if n $\not\equiv 2 \pmod{4}$; $K_{m,n}$ if and only if $m + n \not\equiv 2 \pmod{4}$; $C_n$ if and only if n $\not\equiv 2 \pmod{4}$; regular graphs of degree 1 on 2n vertices if and only if n is even; friendship graphs $C_3^{(n)}$ for all n; fans $F_n$ if and only if n $\not\equiv 1 \pmod{4}$; and wheels $W_n$ if and only if n $\not\equiv 1 \pmod{4}$. They observe that graphs with n $\equiv 2 \pmod{4}$ vertices can not be $E$-cordial. They generalize $E$-cordial labelings to $E_k$-cordial ($k > 1$) labelings by replacing $\{0, 1\}$ by $\{0, 1, 2, \ldots, k - 1\}$. Of course, $E_2$-cordial is the same as $E$-cordial.
5.4 Line-graceful Labelings

Gnanajothi [Gn] has defined a concept similar to edge-graceful. She calls a graph with \( n \) vertices line-graceful if it is possible to label its edges with \( 0, 1, 2, \ldots, n \) so that when each vertex is assigned the sum modulo \( n \) of all the edge labels incident with that vertex the resulting vertex labels are \( 0, 1, \ldots, n - 1 \). A necessary condition for the line-gracefulness of a graph is that its order is not congruent to 2 (mod 4). Among line-graceful graphs are (see [Gn, pp. 132-181])

- \( P_n \) if and only if \( n \not\equiv 2 \pmod{4} \);
- \( C_n \) if and only if \( n \not\equiv 2 \pmod{4} \);
- \( K_{1,n} \) if and only if \( n \not\equiv 1 \pmod{4} \);
- \( P_n \odot K_1 \) (combs) if and only if \( n \) is even;
- \( (P_n \odot K_1) \odot K_1 \) if and only if \( n \not\equiv 2 \pmod{4} \); (in general, if \( G \) has order \( n \), \( G \odot H \) is the graph obtained by taking one copy of \( G \) and \( n \) copies of \( H \) and joining the \( i \)th vertex of \( G \) with an edge to every vertex in the \( i \)th copy of \( H \);)
- \( mC_n \) when \( mn \) is odd;
- \( K_{1,n} \cup K_{1,n} \) if and only if \( n \) is odd; odd cycles with a chord; even cycles with a tail; even cycles with a tail of length 1 and a chord; graphs consisting of two triangles having a common vertex and tails of equal length attached to a vertex other than the common one; the complete \( n \)-ary tree when \( n \) is even; trees for which exactly one vertex has even degree. She conjectures that all trees with \( p \not\equiv 2 \pmod{4} \) vertices are line-graceful and proved this conjecture for \( p \leq 9 \).

Gnanajothi [Gn] has investigated the line-gracefulness of several graphs obtained from stars. In particular, the graph obtained from \( K_{1,4} \) by subdividing one spoke to form a path of even order (counting the center of the star) is line-graceful; the graph obtained from a star by inserting one vertex in a single spoke is line-graceful if and only if the star has \( p \not\equiv 2 \pmod{4} \) vertices; the graph obtained from \( K_{1,n} \) by replacing each spoke with a path of length \( m \) (counting the center vertex) is line-graceful in the following cases: \( n = 2; n = 3 \) and \( m \not\equiv 3 \pmod{4} \); \( m \) is even and \( mn + 1 \equiv 0 \pmod{4} \).

Gnanajothi studied graphs obtained by joining disjoint graphs \( G \) and \( H \) with an edge. She proved such graphs are line-graceful in the following circumstances: \( G = H \); \( G = P_n, H = P_m \) and \( m+n \not\equiv 0 \pmod{4} \); and \( G = P_n \odot K_1, H = P_m \odot K_1 \) and \( m+n \not\equiv 0 \pmod{4} \).

5.5 Sum Graphs

In 1990, Harary [Ha1] introduced the notion of a sum graph. A graph \( G(V, E) \) is called a sum graph if there is an bijective labeling \( f \) from \( V \) to a set of positive integers \( S \) such that \( xy \in E \) if and only if \( f(x) + f(y) \in S \). Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain isolated vertices. In 1991 Harary, Hentzel and Jacobs [HHJ] defined a real sum graph in an analogous way by allowing \( S \) to be any finite set of positive real numbers. However, they proved that every real sum graph is a sum graph. Bergstrand et al. [BHJ] defined a product graph analogous to a sum graph except that 1 is not permitted to belong to \( S \). They proved that every product graph is a sum graph and vice versa.

For a connected graph \( G \), let \( \sigma(G) \), the sum number of \( G \), denote the minimum
number of isolated vertices that must be added to \( G \) so that the resulting graph is a sum graph (some authors use \( s(G) \) for the sum number of \( G \)). A labeling that makes \( G \) together with \( \sigma(G) \) isolated points a sum graph is called an optimal sum graph labeling. Ellingham [El] proved the conjecture of Harary [Ha1] that \( \sigma(T) = 1 \) for every tree \( T \neq K_1 \). Smyth [Smy1] proved that there is no graph \( G \) with \( e \) edges and \( \sigma(G) = 1 \) when \( n^2/4 < e \leq n(n-1)/2 \). Smyth [Smy2] conjectures that the disjoint union of graphs with sum number 1 has sum number 1. More generally, Kratochvil, Miller and Nguyen [KrMN] conjecture that \( \sigma(G) \leq \sigma(\bar{G}) + \sigma(H) - 1 \). Hao [Hao] has shown that if \( d_1 \leq d_2 \leq \cdots \leq d_n \) is the degree sequence of a graph \( G \) then \( \sigma(G) = \max(d_i - i) \) where the maximum is taken over all \( i \). Bergstand et al. [BHHJKW] proved that \( \sigma(K_n) = 2n - 3 \). Hartsfield and Smyth [HaSm] claimed to have proved that \( \sigma(K_{m,n}) = [3m + n - 3]/2 \) when \( n \geq m \) but Yan and Liu [YL1] found counterexamples to this assertion when \( m \neq n \). Pyatkin [Py], Liaw, Kuo and Chang [LKC], Wang and Liu [WL] and He et al. [HSWCKY] have shown that for \( 2 \leq m \leq n \), \( \sigma(K_{m,n}) = \zeta(K_{m,n}) = \left\lfloor \frac{n}{p} + \frac{(p+1)(m-1)}{2} \right\rfloor \)

where \( p = \left\lfloor \sqrt{\frac{2n}{m-1} + \frac{1}{4} - \frac{1}{4}} \right\rfloor \) is the unique integer such that \( \frac{(p-1)p(m-1)}{2} < n \leq \frac{(p+1)p(m-1)}{2} \).

Miller et al. [MRSS] proved that \( \sigma(W_n) = \frac{n}{2} + 2 \) for \( n \) even and \( \sigma(W_n) = n \) for \( n \) odd (see also [SuM2]). Miller, Ryan and Smyth [MRSm] prove that the complete \( n \)-partite graph on \( n \) sets of 2 nonadjacent vertices has sum number \( 4n - 5 \) and obtain upper and lower bounds on the complete \( n \)-partite graph on \( n \) sets of \( m \) nonadjacent vertices. Gould and Rödl [GR] investigated bounds on the number of isolated points in a sum graph. A group of six undergraduate students [GBGGGJ] proved that \( \sigma(K_n - \text{edge}) \leq 2n - 4 \). The same group of six students also investigated the difference between the largest and smallest labels in a sum graph, which they called the spum. They proved spum of \( K_n \) is \( 4n - 6 \) and the spum of \( C_n \) is at most \( 4n - 10 \). Kratochvil, Miller and Nguyen [KrMN] have proved that every sum graph on \( n \) vertices has a sum labeling such that every label is at most \( 4^n \).

At a conference in 2000 Miller [Mi] posed the following problems. Given any graph \( G \), does there exist an optimal sum graph labeling that uses the label 1? Find a class of graphs \( G \) that have sum number of the order \( |V(G)|^s \) for \( s > 1 \). (Such graphs were shown to exist for \( s = 2 \) by Gould and Rödl in [GR]).

Chang [Cha] generalized the notion of sum graph by permitting \( x = y \) in the definition of sum graph. He calls graphs that have this kind of labeling strong sum graphs and uses \( i^*(G) \) to denote the minimum positive integer \( m \) such that \( G \cup mK_1 \) is a strong sum graph. Chang proves that \( i^*(K_n) = \sigma(K_n) \) for \( n = 2, 3 \) and 4 and \( i^*(K_n) = \sigma(K_n) \) for \( n \geq 5 \). He further shows that for \( n \geq 5 \), \( 3n \log_2 n > i^*(K_n) \geq n^3/4 \).

In 1994 Harary [Ha2] generalized sum graphs by permitting \( S \) to be any set of integers. He calls these graphs integral sum graphs. Unlike sum graphs, integral sum graphs need not have isolated vertices. Sharary [Sha] has shown that \( C_n \) and \( W_n \) are integral sum graphs for all \( n \neq 4 \). Chen [Che2] proved that trees obtained from a star by extending each edge to a path and trees all of whose vertices of degree not 2 are at least distance 4 apart are integral sum graphs. He conjectures that all trees are integral sum
graphs. Chen also gives methods for constructing new connected integral sum graphs from given integral sum graphs by identification. B. Xu [XuB] has shown that the following are integral sum graphs: the union of any three stars; $T \cup K_{1,n}$ for all trees $T$; $mK_3$ for all $m$; the union of any number of integral sum trees; and all caterpillars, with exactly one leave attached to an end point of the spine. Xu also proved that if $2G$ and $3G$ are integral sum graphs, then so is $mG$ for all $m > 1$. Xu poses the question as to whether all disconnected forests are integral sum graphs. Liaw, Kuo and Chang [LKC] proved that all caterpillars and all cycles except $C_4$ are integral sum graphs and they conjecture that every tree is an integral sum graph.

The concepts of sum number and integral sum number have been extended to hypergraphs. Sonntag and Teichert [SoTe1] prove that every hypertree (i.e., every connected, non-trivial, cycle-free hypergraph) has sum number 1 provided that a certain cardinality condition for the number of edges is fulfilled. In [SoTe2] the same authors prove that for $d \geq 3$ every $d$-uniform hypertree is an integral sum graph and that for $n \geq d+2$ the sum number of the complete $d$-uniform hypergraph on $n$ vertices is $d(n-d)+1$. They also prove that the integral sum number for the complete $d$-uniform hypergraph on $n$ vertices is 0 when $d = n$ or $n-1$ and is between $(d-1)(n-d-1)$ and $d(n-d)+1$ for $d \leq n-2$. They conjecture that for $d \leq n-2$ the sum number and the integral sum number of the complete $d$-uniform hypergraph are equal.

Teichert [Tei2] proves that hypercycles have sum number 1 when each edge has cardinality at least 3 and that hyperwheels have sum number 1 under certain restrictions for the edge cardinalities. Teichert [Tei1] determined an upper bound for the sum number of the $d$-partite complete hypergraph $K_{n_1,\ldots,n_d}^d$. In [Tei3] Teichert defines the strong hypercycle $C_n^d$ to be the $d$-uniform hypergraph with the same vertices as $C_n$ where any $d$ consecutive vertices of $C_n$ form an edge of $C_n^d$. He proves that for $n \geq 2d+1 \geq 5$, $\sigma(C_n^d) = d$ and for $d \geq 2$, $\sigma(C_{d+1}^d) = d$. He also shows that $\sigma(C_3^3) = 3$; $\sigma(C_6^3) = 2$ and he conjectures that $\sigma(C_n^d) < d$ for $d \geq 4$ and $d+2 \leq n \leq 2d$.

The integral sum number, $\zeta(G)$, of $G$, is the minimum number of isolated vertices that must be added to $G$ so that the resulting graph is an integral sum graph. Thus, by definition, $G$ is a integral sum graph if and only if $\zeta(G) = 0$. Harary [Ha2] conjectured that for $n \geq 4$ the integral sum number $\zeta(K_n) = 2n-3$. This conjecture was verified by Chen [Che1], by Sharary [Sha] and by B. Xu [XuB]. Yan and Liu proved: $\zeta(K_n - E(K_r)) = n-1$ when $n \geq 6$, $n \equiv 0 \pmod{3}$ and $r = 2n/3-1$ [YL2]; $\zeta(K_{m,m}) = 2m-1$ for $m \geq 2$ [YL2]; $\zeta(K_n - \text{edge}) = 2n-4$ for $n \geq 4$ [YL2], [XuB]; if $n \geq 5$ and $n-3 \geq r$, then $\zeta(K_n - E(K_r)) \geq n-1$ [YL2]; if $[2n/3] - 1 > r \geq 2$, then $\zeta(K_n - E(K_r)) \geq 2n-r-2$ [YL2]; and if $2 \leq m < n$, and $n = (i+1)(im-i+2)/2$, then $\sigma(K_{m,n}) = \zeta(K_{m,n}) = (m-1)(i+1)+1$ while if $[(i+1)(im-i+2)/2 < n < (i+2)(i+1)m-i+1)/2$, then $\sigma(K_{m,n}) = \zeta(K_{m,n}) = [(m-1)(i+1)(i+2)+2n)/(2i+2)]$ [YL2].

Nagamochi, Miller and Slamin [NMS] have determined upper and lower bounds on the sum number a graph. For most graphs $G(V,E)$ they show that $\sigma(G) = \Omega(|E|)$. He et al. [HYM1] investigated $\zeta(K_n - E(K_r))$ where $n \geq 5$ and $r \geq 2$. They proved that $\zeta(K_n - E(K_r)) = 0$ when $r = n$ or $n-1$; $\zeta(K_n - E(K_r)) = n-1$ when $n-2 \geq r \geq$
[2n/3] - 1; \zeta(K_n - E(K_r)) = 3n - 2r - 4 when [2n/3] - 1 > r \geq n/2; \zeta(K_n - E(K_r)) = 2n - 4 when [2n/3] - 1 > n/2 \geq r. Moreover, they prove that if r \neq n - 1, then 
\sigma(K_n - E(K_r)) = \zeta(K_n - E(K_r)).

Alon and Scheinerman [AS] generalized sum graphs by replacing the condition 
f(x) + f(y) \in S with g(f(x), f(y)) \in S where g is an arbitrary symmetric polynomial.
They called a graph with this property a g-graph and proved that for a given 
symmetric polynomial g not all graphs are g-graphs. On the other hand, for every symmetric 
polynomial g and every graph G there is some vertex labeling so that G together 
with at most |E(G)| isolated vertices is a g-graph.

Boland, Laskar, Turner, and Domke [BLTD] investigated a modular version of sum
graphs. They call a graph G(V,E) a mod sum graph (MSG) if there exists a positive integer n and an injective labeling from V to \{1, 2, \ldots, n - 1\} such that xy \in E if and only if f(x) + f(y) (mod n) = f(z) for some vertex z. Obviously, all sum graphs are mod sum graphs. However, not all mod sum graphs are sum graphs. Boland et al. [BLTD] 
have shown the following graphs are MSG: all trees on 3 or more vertices; all cycles on 
4 or more vertices; and all \(K_{2,n}\). They further proved that \(K_p\) (\(p \geq 2\)) is not MSG (see 
also [GLPF]) and that \(W_4\) is MSG. They conjecture that \(W_p\) is MSG for \(p \geq 4\). This 
conjecture was refuted by Sutton, Miller, Ryan and Slamin [SMRS] who proved that for 
n \neq 4, \(W_n\) is not MSG (the case where n is prime had been proved in 1994 by Ghoshal 
et al. [GLPF]). In the same paper Sutton et al. also showed that for \(n \geq 3, K_{n,n}\) is not MSG. Ghoshal, Laskar, Pillone and Fricke [GLPF] proved that every connected graph 
is an induced subgraph of a connected MSG graph and any graph with \(n\) vertices and 
at least two vertices of degree \(n - 1\) is not MSG. Sutton et al. define the mod sum 
number, \(\rho(G)\), of a connected graph G to be the least integer r such that \(G + \overline{K_r}\) is MSG.
Sutton and Miller [SuM1] define the cocktail party graph \(H_{m,n}\) \(m,n \geq 2\), as the 
graph with a vertex set \(V = \{v_1, v_2, v_3, \ldots, v_{mn}\}\) partitioned into \(n\) independent sets 
\(V = \{I_1, I_2, \ldots, I_n\}\) each of size \(m\) such that \(v_iv_j \in E\) for all \(i, j \in \{1, \ldots, mn\}\) where 
i \in I_p, j \in I_q, \(p \neq q\). The graphs \(H_{m,n}\) can be used to model relational database 
management systems (see [Suti]). Sutton and Miller prove that \(H_{m,n}\) is not MSG for 
n > m \geq 3 and \(\rho(K_n) = n\) for \(n \geq 4\). In [SDM] Sutton, Draganova and Miller prove that 
for \(n\) odd and \(n \geq 5\), \(\rho(W_n) = n\) and when \(n\) is even \(\rho(W_n) = 2\). Draganova [Dr] 
has shown that for \(n \geq 5\) and \(n\) odd, \(\rho(F_n) = n\). She poses as an open problem the 
determination of the mod sum number of the t-point suspension of \(C_n\). Wallace [Wa] has 
proved that \(K_{m,n}\) is MSG when \(n\) is even and \(n \geq 2m\) or when \(n\) is odd and \(n \geq 3m - 3\) 
and that \(\rho(K_{m,n}) = m\) when \(3 \leq m \leq n < 2m\). He also proves that the complete 
\(m\)-partite \(K_{n_1,n_2,\ldots,n_m}\) is not MSG when there exist \(n_i\) and \(n_j\) such that 
\(n_i < n_j < 2n_i\). He poses the following conjectures: \(\rho(K_{m,n}) = n\) when \(3m - 3 > n \geq m \geq 3\); if \(K_{n_1,n_2,\ldots,n_m}\) 
where \(n_1 > n_2 > \cdots > n_m\), is not MSG then \((m - 1)n_m \leq \rho(K_{n_1,n_2,\ldots,n_m}) \leq (m - 1)n_1\); if 
G has \(n\) vertices then \(\rho(G) \leq n\); determining the mod sum number of a graph is 
NP-complete (Sutton has observed that Wallace probably meant to say ‘NP-hard’); 
Miller [Mi] has asked if it is possible for the mod sum number of a graph G be of the 
order \(|V(G)|^2\).
Grimaldi [Gr] has investigated labeling the vertices of a graph \( G(V, E) \) with \( n \) vertices with distinct elements of the ring \( \mathbb{Z}_n \) so that \( xy \in E \) whenever \((x + y)^{-1}\) exists in \( \mathbb{Z}_n \).

In his 2001 Ph.D. thesis Sutton [Sut] introduced two methods of graph labelings with applications to storage and manipulation of relational database links specifically in mind. He calls a graph \( G = (V_p \cup V_i, E) \) a \textit{sum* graph} of \( G_p = (V_p, E_p) \) if there is an injective labeling \( \lambda \) of the vertices of \( G \) with non-negative integers with the property that \( uw \in E_p \) if and only if \( \lambda(u) + \lambda(v) = \lambda(z) \) for some vertex \( z \in G \). The \textit{sum* number}, \( \sigma^*(G_p) \), is the minimum cardinality of a set of new vertices \( V_i \) (members of \( V_i \) are called \textit{incidentals}) such that there exists a sum* graph of \( G_p \) on the set of vertices \( V_p \cup V_i \). A \textit{mod sum* graph} of \( G_p \) is defined in the identical fashion except the sum \( \lambda(u) + \lambda(v) \) is taken modulo \( n \) where the vertex labels of \( G \) are restricted to \( \{0, 1, 2, \ldots, n-1\} \). The \textit{mod sum* number}, \( \rho^*(G_p) \), of a graph \( G_p \) is defined in the analogous way. Sum* graphs are a generalization of sum graphs and mod sum graphs are a generalization of mod sum graphs. Sutton shows that every graph is an induced subgraph of a connected sum* graph. The following table summarizing what is known about sum graphs, mod sum graphs, sum* graphs and mod sum* graphs is reproduced from Sutton’s Ph.D. thesis [Sut]. The results on sum* and mod sum* graphs are found in [Sut]. Sutton [Sut] poses the following conjectures: \( \rho(H_{m,n}) \leq mn \) for \( m, n \geq 2 \), \( \sigma^*(G_p) \leq |V_p| \), \( \rho^*(G_p) \leq |V_p| \).
5.6 Representations of Graphs modulo $n$

In 1989 Erdős and Evans [EE] defined a representation modulo $n$ of a graph $G$ with vertices $\{v_1, v_2, \ldots, v_r\}$ as a set $\{a_1, \ldots, a_r\}$ of distinct, nonnegative integers each less than $n$ satisfying $\gcd(a_i - a_j, n) = 1$ if and only if $v_i$ is adjacent to $v_j$. They proved that every finite graph can be represented modulo some positive integer. The representation number, $\text{Rep}(G)$, is smallest such integer. Obviously, a graph is representable by a prime if and only if the graph is complete. Evans, Isaak and Narayan [EIN] have shown that a graph is representable modulo a product of a pair of distinct primes if and only if the graph does not contain an induced subgraph isomorphic to $K_2 \cup 2K_1$, $K_3 \cup K_1$, or the complement of a chordless cycle of length at least five. Nešetřil and Pultr [NP] showed that every graph can be represented modulo a product of some set of distinct
primes. Evans et. al [EIN] proved that if $G$ is representable modulo $n$ and $p$ is a prime divisor of $n$, then $p \geq \chi(G)$. They also determined representation numbers for specific families as follows (here we use $q_i$ to denote the $i$th prime and for any prime $p_i$ we use $p_{i+1}, p_{i+2}, \ldots, p_{i+k}$ to denote the next $k$ primes larger than $p_i$): $\text{Rep}(P_n) = 2 \cdot 3 \cdot \cdots q_{[\log_2(n-1)]};$ $\text{Rep}(C_4) = 4$ and for $n \geq 3$, $\text{Rep}(C_{2n}) = 2 \cdot 3 \cdot \cdots q_{[\log_2(n-1)]};$ $\text{Rep}(C_5) = 3 \cdot 5 \cdot 7 = 105$ and for $n \geq 4$ and not a power of 2, $\text{Rep}(C_{2n+1}) = 3 \cdot 5 \cdot \cdots q_{[\log_2(n)]};$

if $m \geq n \geq 3$, then $\text{Rep}(K_m - P_n) = p_ip_{i+1}$ where $p_i$ is the smallest prime greater than or equal to $m - n = \lceil n/2 \rceil$; if $m \geq n \geq 4$, and $p_i$ is the smallest prime greater than or equal to $m - n = \lceil n/2 \rceil$ then $\text{Rep}(K_m - C_n) = q_iq_{i+1}$ if $n$ is even and $\text{Rep}(K_m - C_n) = q_iq_{i+1}q_{i+2}$ if $n$ is odd; if $n \leq m - 1$, then $\text{Rep}(K_m - K_{1,n}) = p_sp_{s+1} \cdots p_{s+n-1}$ where $p_s$ is the smallest prime greater than or equal to $m - 1$; $\text{Rep}(K_m)$ is the smallest prime greater than or equal to $m$; $\text{Rep}(nK_2) = 2 \cdot 3 \cdot \cdots q_{[\log_2(n)]};$ if $n, m \geq 2$, then $\text{Rep}(nK_m) = p_ip_{i+1} \cdots p_{i+m-1},$ where $p_i$ is the smallest prime satisfying $p_i \geq m,$ if and only if there exists a set of $n - 1$ mutually orthogonal Latin squares of order $m$; $\text{Rep}(mK_1) = 2m$; if $t \leq (m - 1)!$, then $\text{Rep}(K_m + tK_1) = p_sp_{s+1} \cdots p_{s+m-1}$ where $p_s$ is the smallest prime greater than or equal to $m$.

5.7 Binary Labelings

In 1996 Caccetta and Jia [CJ] introduced binary labelings of graphs. Let $G = (V, E)$ be a graph. A mapping $f : E \mapsto \{0, 1\}^m$ is called an M-coding if the induced mapping $g : V \mapsto \{0, 1\}^m,$ defined as $g(v) = \sum_{u \in V, uv \in E} f(uv)$ is injective, where the summation is mod 2. An M-coding is called positive if the zero vector is not assigned to an edge and a vertex of $G$. Cacetta and Jia show that the minimal $m$ for a positive M-coding equals $k + 1$ if $|V| \in \{2^k, 2^k - 2, 2^k - 3\}$ and $k$ otherwise, where $k = \lceil \log_2 |V| \rceil$.

5.8 Average Labelings

In 1997 Harminc [Har] introduced a new kind of labeling in an effort to characterize forests and graphs without edges. Let $G = (V, E)$ be a graph. A mapping $f : V \mapsto N$ is called average labeling if for any $(v, u), (u, w) \in E$ one has $f(u) = (f(v) + f(w))/2$. A labeling is called nontrivial if any connected component of $G$ (excluding isolated vertices) has at least two differently labeled vertices. Harminc provides three results towards the characterization of hereditary graphs properties in terms of average labelings. In particular, all maximal connected subgraphs of $G$ are exactly paths (i.e., $G$ is a linear forest) if and only if there exists a nontrivial average labeling of $G$. He also characterizes forests and graphs without edges by introducing a bit more complicated average-type labelings. In 2001 Harminc and Soták [HaSo] gave a characterization of all non-complete connected graphs that have a non-trivial average labeling.
5.9 \textit{k}-sequential Labelings

In 1981 Bange, Barkauskas and Slater [BBS1] defined a \textit{k}-sequential labeling \( f \) of a graph \( G(V, E) \) as one for which \( f \) is a bijection from \( V \cup E \) to \( \{ k, k + 1, \ldots, |V \cup E| + k - 1 \} \) such that for each edge \( xy \) in \( E \), \( f(xy) = |f(x) - f(y)| \). This generalized the notion of \textit{simply sequential} where \( k = 1 \) introduced by Slater. Bange, Barkauskas and Slater showed that cycles are \( 1 \)-sequential and if \( G \) is \( 1 \)-sequential then \( G + K_1 \) is graceful. In [Sl1], Slater proved: \( K_n \) is \( 1 \)-sequential if and only if \( n \leq 3 \); for \( n \geq 2 \), \( K_n \) is not \( k \)-sequential for any \( k \geq 2 \); and \( K_{1,n} \) is \( k \)-sequential if and only if \( k \) divides \( n \). Acharya and Hegde [AH1] proved: If \( G \) is \( k \)-sequential then \( k \) is at most the independence number of \( G \); \( P_{2n} \) is \( n \)-sequential for all \( n \) and \( P_{2n+1} \) is both \( n \)-sequential and \((n + 1)\)-sequential for all \( n \); \( K_{m,n} \) is \( k \)-sequential for \( k = 1, m \) and \( n \); \( K_{m,n,1} \) is \( 1 \)-sequential; and the join of any caterpillar and \( K_1 \) is \( 1 \)-sequential. Acharya [A1] showed that if \( G(E,V) \) is an odd graph with \( |E| + |V| \equiv 1 \) or 2 (mod 4) when \( k \) is odd or \( |E| + |V| \equiv 2 \) or 3 (mod 4) when \( k \) is even, then \( G \) is not \( k \)-sequential. Acharya also observed that as a consequence of results of Bermond, Kotzig and Turgeon [BKT] we have: \( mK_4 \) is not \( k \)-sequential for any \( k \) when \( m \) is odd and \( mK_2 \) is not \( k \)-sequential for any odd \( k \) when \( m \equiv 2 \) or 3 (mod 4) or for any even \( k \) when \( m \equiv 1 \) or 2 (mod 4). He further noted that \( K_{m,n} \) is not \( k \)-sequential when \( k \) is even and \( m \) and \( n \) are odd, while \( K_{m,k} \) is \( k \)-sequential for all \( k \). Acharya [A1] points out that the following result of Slater’s [Sl2] for \( k = 1 \) linking \( k \)-graceful graphs and \( k \)-sequential graphs holds in general: A graph is \( k \)-sequential if and only if \( G + v \) has a \( k \)-graceful labeling \( f \) with \( f(v) = 0 \). Slater [Sl1] also proved that a \( k \)-sequential graph with \( p \) vertices and \( q > 0 \) edges must satisfy \( k \leq p - 1 \).

5.10 Sequentially Additive Graphs

Bange, Barkauskas and Slater [BBS2] defined a \textit{k}-sequentially additive labeling \( f \) of a graph \( G(V, E) \) to be a bijection from \( V \cup E \) to \( \{ k, \ldots, k + |V \cup E| - 1 \} \) such that for each edge \( xy, f(xy) = f(x) + f(y) \). They proved: \( K_n \) is \( 1 \)-sequentially additive if and only if \( n \leq 3 \); \( C_{3n+1} \) is not \( k \)-sequentially additive for \( k \equiv 0 \) or 2 (mod 3); \( C_{3n+2} \) is not \( k \)-sequentially additive for \( k \equiv 1 \) or 2 (mod 3); \( C_n \) is \( 1 \)-sequentially additive if and only if \( n \equiv 0 \) or 1 (mod 3); and \( P_n \) is 1-sequentially additive. They conjecture that all trees are 1-sequentially additive.

Acharya and Hegde [AH2] have generalized \( k \)-sequentially additive labelings by allowing the image of the bijection to be \( \{ k, k + d, \ldots, (k + |V \cup E| - 1)d \} \). They call such a labeling \textit{additively \((k,d)\)-sequential}.

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