A Dynamic Survey of Graph Labeling

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Abstract

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labelings were first introduced in the late 1960s. In the intervening years dozens of graph labelings techniques have been studied in over 600 papers. Finding out what has been done for any particular kind of labeling and keeping up with new discoveries is difficult because of the sheer number of papers and because many of the papers have appeared in journals that are not widely available. In this survey I have collected everything I could find on graph labeling. For the convenience of the reader the survey includes a detailed table of contents and index.

Contents

1 Introduction

Most graph labeling methods trace their origin to one introduced by Rosa [452] in 1967, or one given by Graham and Sloane [249] in 1980. Rosa [452] called a function f a *β-valuation* of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb [243] subsequently called such labelings *graceful* and this is now the popular term. Rosa introduced β-valuations as well as a number of other labelings as tools for decomposing the complete graph into isomorphic subgraphs. In particular, β-valuations originated as a means of attacking the conjecture of Ringel [447] that K_{2n+1} can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with n edges. Although an unpublished result of Erdős says that most graphs are not graceful (cf. [249]), most graphs that have some sort of regularity of structure are graceful. Sheppard [502] has shown that there are exactly $q!$ gracefully labeled graphs with q edges. Balakrishnan and Sampathkumar [68] have shown that every graph is a subgraph of a graceful graph. Rosa [452] has identified essentially three reasons why a graph fails to be graceful: (1) G has "too many vertices" and "not enough edges", (2) G "has too many edges", and (3) G "has the wrong parity". An infinite class of graphs that are not graceful for the second reason is given in [103]. As an example of the third condition Rosa [452] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles C_{4n+1} and C_{4n+2} are not graceful.

Harmonious graphs naturally arose in the study by Graham and Sloane [249] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G with q edges to be *harmonious* if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices. Analogous to the "parity" necessity condition for graceful graphs, Graham and Sloane proved that if a harmonious graph has an even number q of edges and the degree of every vertex is divisible by 2^k then q is divisible by 2^{k+1}. Thus, for example, a book with seven pages (i.e., the cartesian product of the complete bipartite graph $K_{1,7}$ and a path of length 1) is not harmonious. Liu and Zhang [388] have generalized this condition as follows: if a harmonious graph with q edges has degree sequence $d_1, d_2, \ldots, d_p$ then $\gcd(d_1, d_2, \ldots d_p, q)$ divides $q(q-1)/2$. They have also proved that every graph is a subgraph of a harmonious graph.

Over the past three decades in excess of 600 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments. In part because many of the papers have appeared in journals not widely available, frequently the same classes of graphs have been done by several authors. In this article, we survey what is known about numerous graph labeling methods. The author requests that he be sent preprints and reprints as well as corrections for inclusion in the updated versions of the survey.

Earlier surveys, restricted to one or two labeling methods, include [91], [106], [318], [229], and [231]. The extension of graceful labelings to directed graphs arose in the characterization of finite neofields by Hsu and Keedwell [288], [289]. The relationship between graceful digraphs and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings, near-complete mappings and neofields is discussed in [109], [110]. The connection between graceful labelings and perfect systems of difference sets is given in [94]. Labeled graphs serve as useful models for a broad range of applications such as: coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing and data base management–see [107], [108] and [549] for details. Terms and notation not defined below follow that used in [155] and [229].

2 Graceful and Harmonious Labelings

2.1 Trees

The Ringel-Kotzig conjecture that all trees are graceful has been the focus of many papers. Kotzig [291] has called the effort to prove it a "disease." Among the trees known to be graceful are: caterpillars [452] (a *caterpillar* is a tree with the property that the removal of its endpoints leaves a path); trees with at most 4 end-vertices [291], [616] and [298]; trees with diameter at most 5 [616] and [286]; trees with at most 27 vertices [21]; symmetrical trees (i.e., a rooted tree in which every level contains vertices of the same degree) [95], [439]; rooted trees where the roots have odd degree and the lengths of the paths from the root to the leaves differ by at most one and all the internal vertices have the same parity [145]; regular bamboo trees [462] (a rooted tree consisting of branches of equal length the endpoints of which are identified with end points of stars of equal size; and olive trees [433] and [2] (a rooted tree consisting of k branches, where the ith branch is a path of length i). Aldred, Širáň and Širáň [22] have proved that the number of graceful labelings of P_n grows at least as fast as $(5/3)^n$. They mention that this fact has an application to topological graph theory. Stanton and Zarnke [542] and Koh, Rogers and Tan [319] gave methods for combining graceful trees to yield larger graceful trees. Burzio and Ferrarese [133] have shown that the graph obtained from any graceful tree by subdividing every edge is also graceful. Morgan [415] has used Skolem sequences to construct classes of graceful trees. In 1979 Bermond [91] conjectured that lobsters are graceful (a *lobster* is a tree with the property that the removal of the endpoints leaves a caterpillar). Special cases of this conjecture have been done by Ng [426], by Wang, Jin, Lu and Zhang [579] and by Abhyanker [1]. Morgan [414] has shown that all lobsters with perfect matchings are graceful. Morgan and Rees [416] have used Skolem and Hooked-Skolem sequences to generate classes of graceful lobsters. Whether or not lobsters are harmonious seems to have attracted no attention thus far. Chen, Lü and Yeh [157] define a *firecracker* as a graph obtained from the concatenation of stars by linking one leaf from each. They also define a *banana tree* as a graph obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the stars). They proved that firecrackers are graceful and conjecture that banana trees are graceful. Various kinds of bananas trees have been shown to be graceful by Bhat-Nayak and Deshmukh [99], by Murugan and Arumugam [420], [422] and by Vilfred [570]. Despite the efforts of many, the graceful tree conjecture remains open even for trees with maximum degree 3. Aldred and McKay [21] used a computer to show that all trees with at most 26 vertices are harmonious. That caterpillars are harmonious has been shown by Graham and Sloane [249]. Cahit extended the notion of gracefulness to directed graphs in [146]. More specialized results about trees are contained in [91], [106], [318], [390], [140] and [297].

2.2 Cycle-Related Graphs

Cycle-related graphs have been the major focus of attention. Rosa [452] showed that the n-cycle C_n is graceful if and only if $n \equiv 0$ or 3 (mod 4) and Graham and Sloane [249] proved that C_n is harmonious if and only if $n \equiv 1$ or 3 (mod 4). Wheels $W_n = C_n + K_1$ are both graceful and harmonious – [218], [284] and [249]. Notice that a subgraph of a graceful (harmonious) graph need not be graceful (harmonious). The n-cone (also called the *n-point suspension* of C_m) $C_m + \overline{K_n}$ has been shown to be graceful when $m \equiv 0$ or 3 (mod 12) by Bhat-Nayak and Selvam [104]. When n is even and m is 2, 6 or 10 (mod 12) $C_m + \overline{K_n}$ violates the parity condition for a graceful graph. Bhat-Nayak and Selvam [104] also prove that the following cones are graceful: $C_4 + \overline{K_n}, C_5 + \overline{K_2}, C_7 + \overline{K_n}, C_9 + \overline{K_2}, C_{11} + \overline{K_n}$ and $C_{19} + \overline{K_n}$. The *helm* H_n is the graph obtained from a wheel by attaching a pendant edge at each vertex of the n-cycle. Helms have been shown to be graceful [34] and harmonious [240], [384], [385] (see also [388], [477], [380], [170] and [443]). Koh, et al. [320] define a *web* graph as one obtained by joining the pendant points of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. They asked whether such graphs are graceful. This was proved by Kang, Liang, Gao and Yang [303]. Yang has extended the notion of a web by iterating the process of adding pendant points and joining them to form a cycle and then adding pendant points to the new cycle. In his notation, $W(2, n)$ is the web graph whereas $W(t, n)$ is the generalized web with t n-cycles. Yang has shown that $W(3, n)$ and $W(4, n)$ are graceful (see [303]), Abhyanker and Bhat-Nayak [3] have done $W(5, n)$ and Abhyanker [1] has done $W(t, 5)$ for $5 \leq t \leq 13$. Gnanajothi [240] has shown that webs with odd cycles are harmonious. Seoud and Youssef [477] define a *closed helm* as the graph obtained from a helm by joining each pendant vertex to form a cycle and a *flower* as the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm. They prove that closed helms and flowers are harmonious when the cycles are odd. A *gear graph* is obtained from the wheel by adding a vertex between every pair of adjacent vertices of the cycle. Ma and Feng [393] have proved all gears are graceful. Liu [384] has shown that if two or more vertices are inserted between every pair of vertices of the n-cycle of the wheel W_n, the resulting graph is graceful. Liu [382] has also proved that the graph obtain from a gear graph by attaching one or more pendant points to each vertex between the cycle vertices is graceful.

Abhyanker [1] has investigated various unicyclic graphs. He proved that the unicyclic graphs obtained by identifying one vertex of C_4 with the root of the olive tree with $2n$ branches and identifying an adjacent vertex on C_4 with the end point of the path P_{2n-2} are graceful. He showed that if one attaches any number of pendent edges to these unicyclic graphs at the vertex of C_4 that is adjacent to the root of the olive tree but not adjacent to the end vertex of the attached path the resulting graphs are graceful. Likewise, he proved that the graph obtained by deleting the branch of length 1 from an olive tree with $2n$ branches and identifying the root of the edge deleted tree with a vertex of a cycle of the form C_{2n+3} is graceful. He also has a number of results similar to these.

Delorme, et al. [173] and Ma and Feng [392] showed that any cycle with a chord is graceful. This was first conjectured by Bodendiek, Schumacher and Wegner [118], who proved various special cases. Koh and Yap [321] generalized this by defining a *cycle with a P_k-chord* to be a cycle with the path P_k joining two nonconsecutive vertices of the cycle. They proved that these graphs are graceful when $k = 3$ and conjectured that all cycles with a P_k-chord are graceful. This was proved for $k \geq 4$ by Punnim and Pabhapote in 1987 [440]. Chen [162] obtained the same result except for three cases which were then handled by Gao [253]. Xu [595] proved that all cycles with a chord are harmonious except for C_6 in the case where the distance in C_6 between the endpoints of the chord is 2. The gracefulness of cycles with consecutive chords have also been investigated. For $3 \leq p \leq n - r$, let $C_n(p, r)$ denote the n-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the r chords $v_1v_p, v_1v_{p+1}, \ldots, v_1v_{p+r-1}$ have been added. Koh and others, [320] and [314], have handled the cases $r = 2, 3$ and $n - 3$ where n is the length of the cycle. Goh and Lim [242] then proved that all remaining cases are graceful. Moreover, Ma [391] has shown that $C_n(p, n-p)$ is graceful when $p \equiv 0, 3 \pmod 4$ and Ma, Liu and Liu [394] have proved other special cases of these graphs are graceful. Ma also proved that if one adds to the graph $C_n(3, n-3)$ any number k_i of paths of length 2 from the vertex v_1 to the vertex v_i for $i = 2, \ldots, n$, the resulting graph is graceful. Chen [162] has shown that apart from four exceptional cases, a graph consisting of three independent paths joining two vertices of a cycle is graceful. This generalizes the result that a cycle plus a chord is graceful. Liu [381] has shown that the n-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the chords v_1v_k and v_1v_{k+2} $(2 \leq k \leq n - 3)$ are adjoined is graceful.

In [171] Deb and Limaye use the notation $C(n, k)$ to denote the cycle C_n with k cords sharing a common endpoint. For certain choices of n and k there is a unique $C(n, k)$ graph and for other choices there is more than one graph possible. They call these *shell-type* graphs and they call the unique graph $C(n, n - 3)$ a *shell*. Notice that the shell $C(n, n-3)$ is the same as the fan F_{n-1}. Deb and Limaye define a *multiple shell* to be a collection of edge disjoint shells that have their apex in common. They show that a variety of multiple shells are harmonious and they conjecture that all multiple shells are harmonious.

Sethuraman and Dhavamani [480] use $H(n, t)$ to denote the graph obtained from the cycle C_n by adding t consecutive chords incident with a common vertex. If the common vertex is u and v is adjacent to u, then for $k \geq 1$, $n \geq 4$ and $1 \leq t \leq n-3$, Sethuraman and Dhavamani denote by $G(n, t, k)$ the graph obtained by taking the union of k copies of $H(n, k)$ with the edge uv identified. They conjecture that every graph $G(n, t, k)$ is graceful. They prove the conjecture for the case that $t = n - 3$.

For $i = 1, 2, \ldots, n$ let $v_{i,1}, v_{i,2}, \ldots, v_{i,2m}$ be the successive vertices of n copies of C_{2m}. Sekar [462] defines a chain of cycles $C_{2m,n}$ as the graph obtained by identifying $v_{i,m}$ and $v_{i+1,m}$ for $i = 1, 2, \ldots, n - 1$. He proves that $C_{6,2k}$ and $C_{8,n}$ are graceful for all k and all n.

Truszczyński [565] studied unicyclic graphs (i.e., graphs with a unique cycle) and proved several classes of such graphs are graceful. Among these are what he calls dragons.

A *dragon* is formed by joining the end point of a path to a cycle (Koh, et al. [320] call these *tadpoles*). This work led Truszczyński to conjecture that all unicyclic graphs except C_n, where $n \equiv 1$ or 2 (mod 4), are graceful. Guo [252] has shown that dragons are graceful when the length of the cycle is congruent to 1 or 2 (mod 4). In his Master's thesis, Doma [179] investigates the gracefulness of various unicyclic graphs where the cycle has up to 9 vertices. Because of the immense diversity of unicyclic graphs, a proof of Truszczyński's conjecture seems out of reach in the near future.

Cycles that share a common edge or a vertex have received some attention. Murugan and Arumugan [419] have shown that books with n pentagonal pages (i.e., nC_5 with an edge in common) are graceful when n is even and not graceful when n is odd. Let $C_n^{(t)}$ denote the one-point union of t cycles of length n. Bermond and others ([92] and [94]) proved that $C_3^{(t)}$ (that is, the friendship graph or Dutch t-windmill) is graceful if and only if $t \equiv 0$ or 1 (mod 4) while Graham and Sloane [249] proved $C_3^{(t)}$ is harmonious if and only if $t \not\equiv 2$ (mod 4). Koh et al. [315] conjecture that $C_n^{(t)}$ is graceful if and only if $nt \equiv 0$ or 3 (mod 4). Qian [442] verifies this conjecture for the case that $t = 2$ and n is even. Figueroa-Centeno, Ichishima and Muntaner-Batle [209] have shown that if $m \equiv 0$ (mod 4) then the one-point union of 2, 3 or 4 copies of C_m admits a special kind of graceful labeling called an α-valuation (see Section 3.1) and if $m \equiv 2$ (mod 4) then the one-point union of 2 or 4 copies of C_m admits an α-valuation. Bodendiek, Schumacher and Wegner [117] proved that the one-point union of any two cycles is graceful when the number of edges is congruent to 0 or 3 modulo 4. (The other cases violate the necessary parity condition.) Shee [498] has proved that $C_4^{(t)}$ is graceful for all t. Seoud and Youssef [475] have shown that the one-point union of a triangle and C_n is harmonious if and only if $n \equiv 1$ (mod 4) and that if the one-point union of two cycles is harmonious then the number of edges is divisible by 4. The question of whether this latter condition is sufficent is open. Figueroa-Centeno, Ichishima and Muntaner-Batle [209] have shown that if G is harmonious then the one-point union of an odd number of copies of G using the vertex labeled 0 as the shared point is harmonious. Sethuraman and Selvaraju [492] have shown that for a variety of choices of points the one point union of any number of non-isomorphic complete bipartite graphs is graceful. They raise the question of whether this is true for all choices of the common point.

Another class of cycle-related graphs is that of triangular cacti. A *triangular cactus* is a connected graph all of whose blocks are triangles. A *triangular snake* is a triangular cactus whose block-cutpoint-graph is a path (a triangular snake is obtained from a path $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \ldots, n-1$). Rosa [453] conjectured that all triangular cacti with $t \equiv 0$ or 1 (mod 4) blocks are graceful (the cases where $t \equiv 2$ or 3 (mod 4) fail to be graceful because of the parity condition.) Moulton [417] proved the conjecture for all triangular snakes. A proof of the general case (i.e., all triangular cacti) seems hopelessly difficult. Liu and Zhang [388] gave an incorrect proof that triangular snakes with an odd number of triangles are harmonious while triangular snakes with $n \equiv 2$ (mod 4) triangles are not harmonious. Xu [596] subsequently proved that triangular snakes are harmonious if and only if the number of triangles is not congruent to 2 (mod 4).

Defining an n-polygonal snake analogous to triangular snakes, Sekar [462] has shown that such graphs are graceful when $n \equiv 0 \pmod 4$, $(n \geq 8)$ and when $n \equiv 2 \pmod 4$ and the number of polygons is even. Gnanajothi [240, pp. 31–34], had earlier shown that quadrilateral snakes are graceful. Grace [248] has proved that K_4-snakes these are harmonious. Rosa [453] has also considered analogously defined quadrilateral and pentagonal cacti and examined small cases.

Several people have studied cycles with pendant edges attached. Frucht [218] proved that any cycle with a pendant edge attached at each vertex (i.e., a "crown") is graceful. Bu, Zhang and He [132] and Barrientos [76] have shown that any cycle with a fixed number of pendant edges adjoined to each vertex is graceful. Barrientos [76] proved that the graph obtained from a wheel by attaching one pendant edge to each vertex is graceful. Grace [247] showed that an odd cycle with one or more pendant edges at each vertex is harmonious and conjectured that an even cycle with one pendant edge attached at each vertex is harmonious. This conjecture has been proved by Liu and Zhang [387], Liu [384] and [385], Huang [290] and Bu [123]. Sekar [462] has shown that the graph obtained by attaching a path of fixed length to each vertex of a cycle is graceful. For any $n \geq 3$ and any t with $1 \leq t \leq n$, let C_n^{+t} denote the class of graphs formed by adding a single pendant edge to t vertices of a cycle of length n. Ropp [451] proved that for every n and t the class C_n^{+t} contains a graceful graph. Gallian and Ropp [229] conjectured that for all n and t, all members of C_n^{+t} are graceful. This was proved by Qian [442] and by Kang, Liang, Gao and Yang [303]. Of course, this is just a special case of the aforementioned conjecture of Truszczyński that all unicyclic graphs except C_n for $n \equiv 1$ or $2 \pmod 4$ are graceful. Sekar [462] proved that the graph obtained by identifying an endpoint of a star with a vertex of a cycle is graceful.

2.3 Product Related Graphs

Graphs that are cartesian products and related graphs have been the subject of many papers. That planar grids, $P_m \times P_n$, are graceful was proved by Acharya and Gill [15] in 1978 although the much simpler labeling scheme given by Maheo [397] in 1980 for $P_m \times P_2$ readily extends to all grids. In 1980 Graham and Sloane [249] proved ladders, $P_m \times P_2$, are harmonious when $m > 2$ and in 1992 Jungreis and Reid [301] showed that the grids $P_m \times P_n$ are harmonious when $(m, n) \neq (2, 2)$. A few people have looked at graphs obtained from planar grids in various ways. Kathiresan [305] has shown that graphs obtained from ladders by subdividing each step exactly once are graceful and that graphs obtained by appending an edge to each vertex of a ladder are graceful [307]. Acharya [13] has shown that certain subgraphs of grid graphs are graceful. Lee [338] defines a *Mongolian tent* as a graph obtained from $P_m \times P_n$, n odd, by adding one extra vertex above the grid and joining every other vertex of the top row of $P_m \times P_n$ to the new vertex. A *Mongolian village* is a graph formed by successively amalgamating copies of Mongolian tents with the same number of rows so that adjacent tents share a column. Lee proves that Mongolian tents and villages are graceful. A *Young tableau* is a subgraph of $P_m \times P_n$ obtained by retaining the first two rows of $P_m \times P_n$ and deleting vertices from the right hand end of other rows in such a way that the lengths of the successive rows form a nonincreasing sequence. Lee and K. C. Ng [346] have proved that all Young tableaus are graceful. Lee [338] has also defined a variation of Mongolian tents by adding an extra vertex above the top row of a Young tableau and joining every other vertex of that row to the extra vertex. He proves these graphs are graceful.

Prisms are graphs of the form $C_m \times P_n$. These can be viewed as grids on cylinders. In 1977 Bodendiek, Schumacher and Wegner [118] proved that $C_m \times P_2$ is graceful when $m \equiv 0 \pmod 4$. According to the survey by Bermond [91], T. Gangopadhyay and S. P. Rao Hebbare did the case that m is even about the same time. In a 1979 paper, Frucht [218] stated without proof that he had done all m. A complete proof of all cases and some related results were given by Frucht and Gallian [221] in 1988. In 1992 Jungreis and Reid [301] proved that all $C_m \times P_n$ are graceful when m and n are even or when $m \equiv 0 \pmod 4$. Yang and Wang have shown that the prisms $C_{4n+2} \times P_{4m+3}$ [602], $C_n \times P_2$ [600] and $C_6 \times P_m (m \geq 2)$ (see [602]) are graceful. Singh [515] proved that $C_3 \times P_n$ is graceful for all n. In their 1980 paper Graham and Sloane [249] proved that $C_m \times P_n$ is harmonious when n is odd and they used a computer to show $C_4 \times P_2$, the cube, is not harmonious. In 1992 Gallian, Prout and Winters [233] proved that $C_m \times P_2$ is harmonious when $m \neq 4$. In 1992, Jungreis and Reid [301] showed that $C_4 \times P_n$ is harmonious when $n \geq 3$. Huang and Skiena [292] have shown that $C_m \times P_n$ is graceful for all n when m is even and for all n with $3 \leq n \leq 12$ when m is odd. Abhyanker [1] proved that the graphs obtained from $C_{2m+1} \times P_5$ by adding a pendent edge to each vextex of the outercycle is graceful.

Torus grids are graphs of the form $C_m \times C_n$ $(m > 2, n > 2)$. Very little success has been achieved with these graphs. The graceful parity condition is violated for $C_m \times C_n$ when m and n are odd and the harmonious parity condition [249, Theorem 11] is violated for $C_m \times C_n$ when $m \equiv 1, 2, 3 \pmod 4$ and n is odd. In 1992 Jungreis and Reid [301] showed that $C_m \times C_n$ is graceful when $m \equiv 0 \pmod 4$ and n is even. A complete solution to both the graceful and harmonious torus grid problems will most likely involve a large number of cases.

There has been some work done on prism-related graphs. Gallian, Prout and Winters [233] proved that all prisms $C_m \times P_2$ with a single vertex deleted or single edge deleted are graceful and harmonious. The *Möbius ladder* M_n is the graph obtained from the ladder $P_n \times P_2$ by joining the opposite end points of the two copies of P_n. In 1989 Gallian [228] showed that all Möbius ladders are graceful and all but M_3 are harmonious. Ropp [451] has examined two classes of prisms with pendant edges attached. He proved that all $C_m \times P_2$ with a single pendant edge at each vertex are graceful and all $C_m \times P_2$ with a single pendant edge at each vertex of one of the cycles are graceful.

Another class of cartesian products that has been studied is that of books and "stacked" books. The *book* B_m is the graph $S_m \times P_2$ where S_m is the star with $m + 1$

vertices. In 1980 Maheo [397] proved that the books of the form B_{2m} are graceful and conjectured that the books B_{4m+1} were also graceful. (The books B_{4m+3} do not satisfy the graceful parity condition.) This conjecture was verified by Delorme [172] in 1980. Maheo [397] also proved that $L_n \times P_2$ and $B_{2m} \times P_2$ are graceful. Both Grace [246] and Reid (see [232]) have given harmonious labelings for B_{2m}. The books B_{4m+3} do not satisfy the harmonious parity condition [249, Theorem 11]. Gallian and Jungreis [232] conjectured that the books B_{4m+1} are harmonious. Gnanajothi [240] has verified this conjecture by showing B_{4m+1} has an even stronger form of labeling – see Section 4.1. Liang [372] also proved the conjecture. In their 1988 paper Gallian and Jungreis [232] defined a *stacked book* as a graph of the form $S_m \times P_n$. They proved that the stacked books of the form $S_{2m} \times P_n$ are graceful and posed the case $S_{2m+1} \times P_n$ as an open question. The *n-cube* $K_2 \times K_2 \times \cdots \times K_2$ (n copies) was shown to be graceful by Kotzig [325]—see also [397]. In 1986 Reid [446] found a harmonious labeling for $K_4 \times P_n$. Petrie and Smith [434] have investigated graceful labelings of graphs as an exercise in constraint satisfaction. They have shown that $K_m \times P_n$ is graceful for $(m, n) = (4, 2), (4, 3), (4, 4), (4, 5)$ and (5,2) but not is graceful for $(3, 3)$ and $(6, 2)$. The labeling for $K_5 \times P_2$ is the unique graceful labeling. They also considered the graph obtained by identifying the hubs of two copies of W_n. The resulting graph is not graceful when $n = 3$ but is graceful when n is 4 and 5.

The composition $G_1[G_2]$ is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(x_1, y_1), (x_2, y_2) | \ x_1x_2 \in E(G_1) \text{ or } x_1 = x_2 \text{ and } y_1y_2 \in E(G_2)\}$. The symmetric product $G_1 \oplus G_2$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and edge set $\{(x_1, y_1), (x_2, y_2) | \ x_1x_2 \in E(G_1) \text{ or } y_1y_2 \in E(G_2) \text{ but not both}\}$. Seoud and Youssef [476] have proved that $P_n \oplus \overline{K_2}$ is graceful when $n > 1$ and $P_n[P_2]$ is harmonious for all n. They also observe that the graphs $C_m \oplus C_n$ and $C_m[C_n]$ violates parity conditions for graceful and harmonious graphs when m and n are odd.

2.4 Complete Graphs

The questions of the gracefulness and harmoniousness of the complete graphs K_n have been answered. In each case the answer is positive if and only if $n \leq 4$ ([243], [514], [249], [96]). Both Rosa [452] and Golomb [243] proved that the complete bipartite graphs $K_{m,n}$ are graceful while Graham and Sloane [249] showed they are harmonious if and only if m or $n = 1$. Aravamudhan and Murugan [31] have shown that the complete tripartite graph $K_{1,m,n}$ is both graceful and harmonious while Gnanajothi [240, pp. 25–31] has shown that $K_{1,1,m,n}$ is both graceful and harmonious and $K_{2,m,n}$ is graceful. Some of the same results have been obtained by Seoud and Youssef [471] who also observed that when m, n and p are congruent to 2 (mod 4), $K_{m,n,p}$ violates the parity conditions for harmonious graphs.

Beutner and Harborth [96] show that $K_n - e$ (K_n with an edge deleted) is graceful only if $n \leq 5$, any $K_n - 2e$ (K_n with two edges deleted) is graceful only if $n \leq 6$ and any $K_n - 3e$ is graceful only if $n \leq 6$. They also determine all graceful graphs $K_n - G$

where G is $K_{1,a}$ with $a \leq n-2$ and where G is a matching M_a with $2a \leq n$. They give graceful labelings for $K_{1,m,n}, K_{2,m,n}, K_{1,1,m,n}$ and conjecture that these and $K_{m,n}$ are the only complete multipartite graphs that are graceful. They have verified this conjecture for graphs with up to 23 vertices via computer.

Define the *windmill* graphs $K_n^{(m)} (n > 3)$ to be the family of graphs consisting of m copies of K_n with a vertex in common. A necessary condition for $K_n^{(m)}$ to be graceful is that $n \leq 5$ – see [320]. Bermond [91] has conjectured that $K_4^{(m)}$ is graceful for all $m \geq 4$. This is known to be true for $m \leq 22$ [292]. Bermond, Kotzig and Turgeon [94] proved that $K_n^{(m)}$ is not graceful when $n = 4$ and $m = 2$ or 3 and when $m = 2$ and $n = 5$. In 1982 Hsu [287] proved that $K_4^{(m)}$ is harmonious for all m. Graham and Sloane [249] conjectured that $K_n^{(2)}$ is harmonious if and only if $n = 4$. They verified this conjecture for the cases that n is odd or $n = 6$. Liu [380] has shown that $K_n^{(2)}$ is not harmonious if $n = 2^a p_1^{a_1} \cdots p_s^{a_s}$ where $a, a_1, \ldots, a_s$ are positive integers and $p_1, \ldots, p_s$ are distinct odd primes and there is a j for which $p_j \equiv 3 \pmod 4$ and a_j is odd. He also shows that $K_n^{(3)}$ is not harmonious when $n \equiv 0 \pmod 4$ and $3n = 4^e(8k+7)$ or $n \equiv 5 \pmod 8$. Koh et al. [315] and Rajasingh and Pushpam [444] have shown that $K_{m,n}^{(t)}$, the one-point union of t copies of $K_{m,n}$, is graceful. Sethuraman and Selvaraju [488] have proved that the one-point union of graphs of the form K_{2,m_i} for $i = 1, 2, \ldots, n$ where the union is taken at a vertex from the partite set with 2 vertices is graceful if at most two of the m_i are equal. They conjecture that the restriction that at most two of the m_i are equal is not necessary. Koh et al. [320] introduced the notation $B(n, r, m)$ for the graph consisting of m copies of K_n with a K_r in common $(n \geq r)$. (We note that Guo [253] has used the notation $B(n, r, m)$ to denote three independent paths of lengths n, r and m joining two vertices.) Bermond [91] raised the question: "For which m, n and r is $B(n, r, m)$ graceful?" Of course, the case $r = 1$ is the same as $K_n^{(m)}$. For $r > 1, B(n, r, m)$ is graceful in the following cases: $n = 3,\ r = 2, m \geq 1$ [316]; $n = 4,\ r = 2, m \geq 1$ [172]; $n = 4,\ r = 3, m \geq 1$ (see [91]), [316]. Seoud and Youssef [471] have proved $B(3, 2, m)$ and $B(4, 3, m)$ are harmonious. Liu [379] has shown that if there is a prime p such that $p \equiv 3 \pmod 4$ and p divides both n and $n-2$ and the highest power of p that divides n and $n-2$ is odd, then $B(n, 2, 2)$ is not graceful. More generally, Bermond and Farhi [93] have considered the class of graphs consisting of m copies of K_n having exactly k copies of K_r in common. They proved such graphs are not graceful for n sufficiently large compared to r.

Sethuraman and Elumalai [482] have shown that $K_{1,m,n}$ with a pendent edge attached to each vertex is graceful and $K_{m,n}$ with a pendent edge attached at each vertex is graceful when m is even and $m \leq n \leq 2m+4$ and when m is odd and $m \leq n \leq 2m-1$. In [486] Sethuraman and Kishore determine the graceful graphs that are the union of n copies of K_4 with i edges deleted for $1 \leq i \leq 5$ with one edge in common. The only cases that are not graceful are those graphs where the members of the union are C_4 for $n \equiv 3 \bmod 4$ and where the members of the union are P_2. They conjecture that these two cases are the only instances of edge induced subgraphs of the union of n copies of

K_4 with one edge in common that are not graceful. Sethuraman and Selvaraju [494] have shown that union of any number of copies of K_4 with an edge deleted and one edge in common is harmonious.

2.5 Disconnected Graphs

There have been many papers dealing with graphs that are not connected. In 1975 Kotzig [324] considered graphs that are the disjoint union of r cycles of length s, denoted by rC_s. When $rs \equiv 1$ or 2 (mod 4), these graphs violate the parity condition and so are not graceful. Kotzig proved that when $r = 3$ and $s = 4k > 4$, then rC_s has a stronger form of graceful labeling called α-labeling (see §3.1) whereas when $r \geq 2$ and $s = 3$ or 5, rC_s is not graceful. In 1984 Kotzig [326] once again investigated the gracefulness of rC_s as well as graphs that are the disjoint union of odd cycles. For graphs of the latter kind he gives several necessary conditions. His paper concludes with an elaborate table that summarizes what was then known about the gracefulness of rC_s. He [264] has shown that graphs of the form $2C_{2m}$ and graphs obtained by connecting two copies of C_{2m} with an edge are graceful. Cahit [143] has shown that rC_s is harmonious when r and s are odd and Seoud, Abdel Maqsoud and Sheehan [463] noted that when r or s is even, rC_s is not harmonious. Seoud, Abdel Maqsoud and Sheehan [463] proved that $C_n \cup C_{n+1}$ is harmonious if and only if $n \geq 4$. They conjecture that $C_3 \cup C_{2n}$ is harmonious when $n \geq 3$. This conjecture was proved when Yang, Lu, and Zeng [599] showed that all graphs of the form $C_{2j+1} \cup C_{2n}$ are harmonious except for $(n, j) = (2, 1)$.

In 1978 Kotzig and Turgeon [329] proved that mK_n (i.e., the union of m disjoint copies of K_n) is graceful if and only if $m = 1$ and $n \leq 4$. Liu and Zhang [388] have shown that mK_n is not harmonious for n odd and $m \equiv 2$ (mod 4) and is harmonious for $n = 3$ and m odd. They conjecture that mK_3 is not harmonious when $m \equiv 0$ (mod 4). Bu and Cao [124] give some sufficient conditions for the gracefulness of graphs of the form $K_{m,n} \cup G$ and they prove that $K_{m,n} \cup P_t$ and the disjoint union of complete bipartite graphs are graceful under some conditions.

A *Skolem sequence* of order n is a sequence $s_1, s_2, \ldots, s_{2n}$ of $2n$ terms such that, for each $k \in \{1, 2, \ldots, n\}$, there exist exactly two subscripts $i(k)$ and $j(k)$ with $s_{i(k)} = s_{j(k)} = k$ and $|i(k) - j(k)| = k$. A Skolem sequence of order n exists if and only if $n \equiv 0$ or 1 (mod 4). Abrham [5] has proved that any graceful 2-regular graph of order $n \equiv 0$ (mod 4) in which all the component cycles are even or of order $n \equiv 3$ (mod 4), with exactly one component an odd cycle, can be used to construct a Skolem sequence of order $n + 1$. Also, he showed that certain special Skolem sequences of order n can be used to generate graceful labelings on certain 2-regular graphs.

In 1985 Frucht and Salinas [222] conjectured that $C_s \cup P_n$ is graceful if and only if $s + n \geq 7$ and they proved the conjecture for the case that $s = 4$. Frucht [220] did the case the $s = 3$ and the case that $s = 2n + 1$. Bhat-Nayak and Deshmukh [102] also did the case $s = 3$ and they have done the cases of the form $C_{2x+1} \cup P_{x-2\theta}$ where $1 \leq \theta \leq \lfloor (x - 2)/2 \rfloor$ [98]. Choudum and Kishore [164] have done the cases where

$s \geq 5$ and $n \geq (s+5)/2$ and Kishore [313] did the case $s = 5$. Gao and Liang [236] have done the following cases: $s > 4, n = 2$ (see also [235]); $s = 4k, n = k+2, n = k+3, n = 2k+2; s = 4k+1, n = 2k, n = 3k-1, n = 4k-1; s = 4k+2, n = 3k, n = 3k+1, n = 4k+1; s = 4k+3, n = 2k+1, n = 3k, n = 4k$. Seoud, Abdel Maqsoud and Sheehan [466] did the case that $s = 2k \ \ (k \geq 3)$ and $n \geq k+1$ as well as the cases where $s = 6, 8, 10, 12$ and $n \geq 2$. Shimazu [503] has handled the cases that $s \geq 5$ and $n = 2, s \geq 4$ and $n = 3$ and $s = 2n+2$ and $n \geq 2$. Liang [373] has done the following cases: $s = 4k, n = k+2, k+3, 2k+1, 2k+2, 2k+3, 2k+4, 2k+5; s = 4k-1, n = 2k, 3k-1, 4k-1; s = 4k+2, n = 3k, 3k+1, 4k+1; s = 4k+3, n = 2k+1, 3k, 4k$. Youssef [607] proved that $C_5 \cup S_n$ is graceful if and only if $n = 1$ or 2 and that $C_6 \cup S_n$ is graceful if and only if n is odd or $n = 2$ or 4.

Seoud and Youssef [478] have shown that $K_5 \cup K_{m,n}, K_{m,n} \cup K_{p,q} \ \ (m, n, p, q \geq 2), K_{m,n} \cup K_{p,q} \cup K_{r,s} \ \ (m, n, p, q, r, s \geq 2, \ \ (p,q) \neq (2,2))$, and $pK_{m,n} \ \ (m, n \geq 2, (m,n) \neq (2,2))$ are graceful. They also prove that $C_4 \cup K_{1,n} \ (n \neq 2)$ is not graceful whereas Choudum and Kishore [166], [313] have proved that $C_s \cup K_{1,n}$ is graceful for every $s \geq 7$ and $n \geq 1$. Lee, Quach and Wang [349] established the gracefulness of $P_s \cup K_{1,n}$. Seoud and Wilson [470] have shown that $C_3 \cup K_4, C_3 \cup C_3 \cup K_4$ and certain graphs of the form $C_3 \cup P_n$ and $C_3 \cup C_3 \cup P_n$ are not graceful. Abrham and Kotzig [10] proved that $C_p \cup C_q$ is graceful if and only if $p + q \equiv 0$ or 3 (mod 4). Zhou [618] proved that $K_m \cup K_n \ (n > 1, m > 1)$ is graceful if and only if $\{m, n\} = \{4, 2\}$ or $\{5, 2\}$. (C. Barrientos has called to my attention that $K_1 \cup K_n$ is graceful if and only if $n = 3$ or 4.) Shee [497] has shown that graphs of the form $P_2 \cup C_{2k+1} \ (k > 1), \ P_3 \cup C_{2k+1}, \ P_n \cup C_3$ and $S_n \cup C_{2k+1}$ all satisfy a condition that is a bit weaker than harmonious. Bhat-Nayak and Deshmukh [100] have shown that $C_{4t} \cup K_{1,4t-1}$ and $C_{4t+3} \cup K_{1,4t+2}$ are graceful.

In considering graceful labelings of the disjoint unions of two or three stars with e edges Yang and Wang [601] permitted the vertex labels to range from 0 to $e+1$ and 0 to $e+2$, respectively. With these definitions of graceful, they proved that $S_m \cup S_n$ is graceful if and only if m or n is even and that $S_m \cup S_n \cup S_k$ is graceful if and only if at least one of m, n or k is even $(m > 1, n > 1, k > 1)$.

Seoud and Youssef [474] investigated the gracefulness of specific families of the form $G \cup K_{m,n}$. They obtained the following results: $C_3 \cup K_{m,n}$ is graceful if and only if $m \geq 2$ and $n \geq 2$; $C_4 \cup K_{m,n}$ is graceful if and only if $m \geq 2$ and $n \geq 2$ or $\{m, n\} = \{1, 2\}$; $C_7 \cup K_{m,n}$ and $C_8 \cup K_{m,n}$ are graceful for all m and n; $mK_3 \cup nK_{1,r}$ is not graceful for all m, n and r; $K_i \cup K_{m,n}$ is graceful for $i \leq 4$ and $m \geq 2, n \geq 2$ except for $i = 2$ and $(m,n) = (2,2)$; $K_5 \cup K_{1,n}$ is graceful for all n; $K_6 \cup K_{1,n}$ is graceful if and only if n is different than 1 and 3.

Barrientos [78] has shown the following graphs are graceful: $C_6 \cup K_{1,2n+1}; C_m \cup K_{s,t}$ for $m \equiv 0$ or 3 (mod 4), $m \geq 11$ and $s \geq 1, t \geq 1; \bigcup_{i=1}^{t} K_{m_i,n_i}$ for $2 \leq m_i < n_i$; and $C_m \cup \bigcup_{i=1}^{t} K_{m_i,n_i}$ for $2 \leq m_i < n_i, m \equiv 0$ or 3 (mod 4), $m \geq 11$.

Youssef [608] has shown that if G is harmonious then mG and G^m are harmonious for all odd m. He asks the question of whether G is harmonious implies G^m is harmonious when $m \equiv 0 \pmod 4$.

2.6 Joins of Graphs

A few classes of graphs that are the join of graphs have been shown to be graceful and harmonious. Among these are fans $P_n + K_1$ [249] and double fans $P_n + \overline{K_2}$ [249]. More generally, Reid [446] proved that $P_n + \overline{K_t}$ is harmonious and Grace showed [247] that if T is any graceful tree, then $T + \overline{K_t}$ is also graceful. Fu and Wu [224] proved that if T is a graceful tree, then $T + S_k$ is graceful. Sethuraman and Selvaraju [493] have shown that $P_n + K_2$ is harmonious. They ask whether $S_n + P_n$ or $P_m + P_n$ is harmonious. Of course, wheels are of the form $C_n + K_1$ and are graceful and harmonious. Hebbare [267] showed that $S_m + K_1$ is graceful for all m. Shee [497] has proved $K_{m,n} + K_1$ is harmonious and observed that various cases of $K_{m,n} + K_t$ violate the harmonious parity condition in [249]. Liu and Zhang [388] have proved that $K_2 + K_2 + \cdots + K_2$ is harmonious. Yuan and Zhu [614] proved that $K_{m,n} + K_2$ is graceful and harmonious. Gnanajothi [240, pp. 80–127] obtained the following: $C_n + \overline{K_2}$ is harmonious when n is odd and not harmonious when $n \equiv 2, 4, 6 \pmod 8$; $S_n + \overline{K_t}$ is harmonious; and $P_n + \overline{K_t}$ is harmonious. Balakrishnan and Kumar [67] have proved that the join of $\overline{K}_n$ and two disjoint copies of K_2 is harmonious if and only if n is even. Bu [123] obtained partial results for the gracefulness of $K_n + \overline{K}_m$. Ramírez-Alfonsín [445] has proved that if G is graceful and $|V(G)| = |E(G)| = e$ and either 1 or e is not a vertex label then $G + \overline{K_t}$ is graceful for all t.

Seoud and Youssef [476] have proved: the join of any two stars is graceful and harmonious; the join of any path and any star is graceful; and $C_n + \overline{K_t}$ is harmonious for every t when n is odd. They also prove that if any edge is added to $K_{m,n}$ the resulting graph is harmonious if m or n is at least 2. Deng [174] has shown certain cases of $C_n + \overline{K_t}$ are harmonious. Seoud and Youssef [473] proved: the graph obtained by appending any number of edges from the two vertices of degree $n \geq 2$ in $K_{2,n}$ is not harmonious; dragons $D_{m,n}$ (i.e., P_m is appended to C_n) are not harmonious when $m+n$ is odd; and the disjoint union of any dragon and any number of cycles is not harmonious when the resulting graph has odd order. Youssef [607] has shown that if G is a graceful graph with p vertices and q edges with $p = q + 1$, then $G + S_n$ is graceful.

Sethuraman and Elumalai [484] have proved that for every graph G with p vertices and q edges the graph $G + K_1 + \overline{K_m}$ is graceful when $m \geq 2^p - p - 1 - q$. As a corollary they deduce that every graph is a vertex induced subgraph of a graceful graph. Balakrishnan and Sampathkumar [68] ask for which $m \geq 3$ is the graph $\overline{K_n} + mK_2$ graceful for all n. Bhat-Nayak and Gokhale [103] have proved that $\overline{K_n} + 2K_2$ is not graceful. Youssef [607] has shown that $\overline{K_n} + mK_2$ is graceful if $m \equiv 0$ or $1 \pmod 4$ and that $\overline{K_n} + mK_2$ is not graceful if n is odd and $m \equiv 2$ or $3 \pmod 4$.

Wu [589] calls a graceful graph with n edges ($n \geq 1$) and $n + 1$ vertices *vertex-saturated*. He proves that for every vertex-saturated graph G the join of G and $\overline{K_m}$ and the join of G and any star are graceful.

2.7 Miscellaneous Results

It is easy to see that P_n^2 is harmonious [247] while a proof that P_n^2 is graceful has been given by Kang, Liang, Gao and Yang [303]. (P_n^k, the kth power of P_n, is the graph obtained from P_n by adding edges that join all vertices u and v with $d(u,v) = k$.) This latter result proved a conjecture of Grace [247]. Seoud, Abdel Maqsoud and Sheeham [463] proved that P_n^3 is harmonious and conjecture that P_n^k is not harmonious when $k > 3$. However, Youssef [611] has observed that P_8^4 is harmonious. Gnanajothi [240, p. 50] has shown that the graph that consists of n copies of C_6 that have exactly P_4 in common is graceful if and only if n is even. For a fixed n, let v_{i1}, v_{i2}, v_{i3} and v_{i4} $(1 \leq i \leq n)$ be consecutive vertices of n 4-cycles. Gnanajothi [240, p. 35] also proves that the graph obtained by joining each v_{i1} to $v_{i+1,3}$ is graceful for all n and the generalized Petersen graph $P(n,k)$ is harmonious in all cases (see also [352]). ($P(n,k)$, where $n \geq 5$ and $1 \leq k \leq n$, has vertex set $\{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\}$ and edge set $\{a_ia_{i+1} \mid i = 0, 1, \ldots, n-1\} \cup \{a_ib_i \mid i = 0, 1, \ldots, n-1\} \cup \{b_ib_{i+k} \mid i = 0, 1, \ldots, n-1\}$ where all subscripts are taken modulo n [581]. The standard Petersen graph is $P(5,2)$.) The gracefulness of the generalized Petersen graphs appears to be an open problem.

Yuan and Zhu [614] proved that P_n^{2k} is harmonious when $1 \leq k \leq (n-1)/2$ and that P_n^{2k} has a stronger form of harmonious labeling (see Section 4.1) when $2k - 1 \leq n \leq 4k - 1$. Cahit [143] defines a *p-star* as the graph obtained by joining p disjoint paths of a fixed length k to single vertex. He proves all such graphs are harmonious when p is odd and when $k = 2$ and p is even.

Sethuraman and Selvaraju [487] define a graph H to be a *supersubdivision* of a graph G, if every edge uv of G is replaced by $K_{2,m}$ (m may vary for each edge) by identifying u and v with the two vertices in $K_{2,m}$ that form one of the two partite sets. Sethuraman and Selvaraju prove that every supersubdivision of a path is graceful and every cycle has some supersubdivision that is graceful. They conjecture that every supersubdivision of a star is graceful and that paths and stars are the only graphs for which every supersubdivison is graceful. In [491] Sethuraman and Selvaraju prove that every connected graph has some supersubdivision that is graceful. They pose the question as to whether this result is valid for disconnected graphs. They also ask if there is any graph other than $K_{2,m}$ that can be used to replace an edge of a connected graph to obtain a supersubdivision that is graceful. In [489] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions that have a strong form of graceful labeling called an α-labeling (see §3.1).

Kathiresan [306] uses the notation $P_{a,b}$ to denote the graph obtained by identifying the end points of b internally disjoint paths each of length a. He conjectures that $P_{a,b}$ is graceful except when a is odd and $b \equiv 2 \pmod 4$. He proves the conjecture for the case that a is even and b is odd. Sekar [462] has shown that $P_{a,b}$ is graceful when $a \neq 4r + 1, r > 1; b = 4m, m > r$. Kathiresan also shows that the graph obtained by identifying a vertex of K_n with any noncenter vertex of the star with $2^{n-1} - n(n-1)/2$ edges is graceful.

The graph T_n has $3n$ vertices and $6n - 3$ edges defined as follows. Start with a triangle T_1 with vertices $v_{1,1}, v_{1,2}$ and $v_{1,3}$. Then T_{i+1} consists of T_i together with three new vertices $v_{i+1,1}, v_{i+1,2}, v_{i+1,3}$ and edges $v_{i+1,1}v_{i,2}$, $v_{i+1,1}v_{i,3}$, $v_{i+1,2}v_{i,1}$, $v_{i+1,2}v_{i,3}$, $v_{i+1,3}v_{i,1}$, $v_{i+1,3}v_{i,2}$. Gnanajothi [240] proved that T_n is graceful if and only if n is odd. Sekar [462] proved T_n is graceful when n is odd and T_n with a pendant edge attached to the starting triangle is graceful when n is even.

For a graph G, the *splitting graph of* $G, S^1(G)$, is obtanied from G by adding for each vertex v of G a new vertex v^1 so that v^1 is adjacent to every vertex that is adjacent to v. Sekar [462] has shown that $S^1(P_n)$ is graceful for all n and $S^1(C_n)$ is graceful for $n \equiv 0, 1 \bmod 4$.

The total graph $T(P_n)$ has vertex set $V(P_n) \cup E(P_n)$ with two vertices adjacent whenever they are neighbors in P_n. Balakrishnan, Selvam and Yegnanarayanan [69] have proved that $T(P_n)$ is harmonious.

For any graph G with vertices $v_1, \ldots, v_n$ and a vector $\mathbf{m} = (m_1, \ldots, m_n)$ of positive integers the corresponding *replicated graph*, $R_{\mathbf{m}}(G)$, of G is defined as follows. For each v_i form a stable set S_i consisting of m_i new vertices $i = 1, 2, \ldots, n$ (recall a *stable* set S consists of a set of vertices such that there is not an edge v_iv_j for all pairs v_i, v_j in S); two stable sets $S_i, S_j, i \neq j$, form a complete bipartite graph if each v_iv_j is an edge in G and otherwise there are no edges between S_i and S_j. Ramírez-Alfonsín [445] has proved that $R_{\mathbf{m}}(P_n)$ is graceful for all $\mathbf{m}$ and all $n > 1$ (see §3.2 for a stronger result) and that $R_{(m,1,\ldots,1)}(C_{4n}), R_{(2,1,\ldots,1)}(C_n)$ $(n \geq 8)$ and, $R_{(2,2,1,\ldots,1)}(C_{4n})$ $(n \geq 12)$ are graceful.

For any permutation f on $1, \ldots, n$, the f*-permutation* graph on a graph $G, P(G, f)$, consists of two disjoint copies of G, G_1 and G_2, each of which has vertices labeled $v_1, v_2, \ldots, v_n$ with n edges obtained by joining each v_i in G_1 to $v_{f(i)}$ in G_2. In 1983 Lee (see [368]) conjectured that for all $n > 1$ and all permutations on $1, 2, \ldots, n$, the permutation graph $P(P_n, f)$ is graceful. Lee, Wang and Kiang [368] proved that $P(P_{2k}, f)$ is graceful when $f = (12)(34) \cdots (k, k+1) \cdots (2k-1, 2k)$. They conjectured that if G is a graceful nonbipartite graph with n vertices then for any permutation f on $1, 2, \ldots, n$, the permutation graph $P(G, f)$ is graceful. Some families of graceful permutation graphs are given in [341].

Gnanajothi [240, p. 51] calls a graph G *bigraceful* if both G and its line graph are graceful. She shows the following are bigraceful: P_m; $P_m \times P_n$; C_n if and only if $n \equiv 0, 3 \pmod 4$; S_n; K_n if and only if $n \leq 3$; and B_n if and only if $n \equiv 3 \pmod 4$. She also shows that $K_{m,n}$ is not bigraceful when $n \equiv 3 \pmod 4$. (Gangopadhyay and Hebbare [234] used the term "bigraceful" to mean a bipartite graceful graph.) Murugan and Arumugan [421] have shown that graphs obtained from C_4 by attaching two disjoint paths of equal length to two adjacent vertices are bigraceful.

Several well-known isolated graphs have been examined. Graceful labelings of the Petersen graph, the cube, the icosahedron and the dodecahedron can be found in [243] and [237]. On the other hand, Graham and Sloane [249] showed that all of these except the cube are harmonious. Winters [584] verified that the Grőtzsch graph (see [120, p. 118]), the Heawood graph (see [120, p. 236]) and the Herschel graph (see [120, p. 53]) are graceful. Graham and Sloane [249] determined all harmonious graphs with at most five vertices. Seoud and Youssef [475] did the same for graphs with six vertices.

2.8 Summary

The results and conjectures discussed above are summarized in the tables following. The letter G after a class of graphs indicates that the graphs in that class are known to be graceful; a question mark indicates that the gracefulness of the graphs in the class is an open problem; we put a "G" next to a question mark if the graphs have been conjectured to be graceful. The analogous notation with the letter H is used to indicate the status of the graphs with regard to being harmonious. The tables impart at a glimpse what has been done and what needs to be done to close out a particular class of graphs. Of course, there is an unlimited number of graphs one could consider. One wishes for some general results that would handle several broad classes at once but the experience of many people suggests that this is unlikely to occur soon. The Graceful Tree Conjecture alone has withstood the efforts of scores of people over the past three decades. Analogous sweeping conjectures are probably true but appear hopelessly difficult to prove.

Table 1: **Summary of Graceful Results**

Graph	*Graceful*
Trees	G if ≤ 27 vertices [21] G if symmetrical [95] G if at most 4 end-vertices [291] ?G Ringel-Kotzig
Cycles C_n	G iff $n \equiv 0, 3 \pmod 4$ [452]
Wheels W_n	G [218], [284]
Helms (see §2.2)	G [34]
Webs (see §2.2)	G [303]
Gears (see §2.2)	G [393]
Cycles with P_k-chord (see §2.2)	G [173], [392], [321], [440]
C_n with k consecutive chords (see §2.2)	G if $k = 2, 3, n-3$ [314], [320]
Unicyclic graphs	?G iff G $\neq C_n, n \equiv 1, 2 \pmod 4$ [565]
$C_n^{(t)}$ (see §2.2)	$n = 3$ G iff $t \equiv 0, 1 \pmod 4$ [92], [94] ?G if $nt \equiv 0, 3 \pmod 4$ [315] G if $n = 6, t$ even [315] G if $n = 4, t > 1$ [498] G if $t = 2$, $n \not\equiv 1 \pmod 4$ [442], [117]

Table 1: **continued**

Graph	*Graceful*
Triangular snakes (see §2.2)	G iff number of blocks $\equiv 0, 1 \pmod 4$ [417]
K_4-snakes (see §2.2)	?
Quadilateral snakes (see §2.2)	G [240], [442]
Crowns $C_n \odot K_1$	G [218]
Grids $P_m \times P_n$	G [15]
Prisms $C_m \times P_n$	G if $n = 2$ [221], [600] G if m even [292] G if m odd and $3 \leq n \leq 12$ [292] G if $m = 3$ [515] G if $m = 6$ see [602] G if $m \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 4$ [602]
Torus grids $C_m \times C_n$	G if $m \equiv 0 \pmod 4$, n even [301] not G if m, n odd (parity condition)
Vertex-deleted $C_m \times P_n$	G if $n = 2$ [233]
Edge-deleted $C_m \times P_n$	G if $n = 2$ [233]
Möbius ladders M_n (see §2.3)	G [228]
Stacked books $S_m \times P_n$ (see §2.3)	$n = 2$, G iff $m \not\equiv 3 \pmod 4$ [397], [172], [232] G if m even [232]
n-cube $K_2 \times K_2 \times \cdots \times K_2$	G [325]

Table 1: **continued**

Graph	*Graceful*
$K_4 \times P_n$	G if $n = 2, 3, 4, 5$ [434]
K_n	G iff $n \leq 4$ [243], [514]
$K_{m,n}$	G [452], [243]
$K_{1,m,n}$	G [31]
$K_{1,1,m,n}$	G [240]
Windmills $K_n^{(m)} (n > 3)$ (see §2.4)	G if $n = 4, m \leq 22$ [292] ?G if $n = 4, m \geq 4$ [91] G if $n = 4, 4 \leq m \leq 22$ [292] not G if $n = 4, m = 2, 3$ [91] not G if $(m, n) = (2, 5)$ [94] not G if $n > 5$ [320]
$B(n, r, m)$ $r > 1$ (see §2.4)	G if $(n, r) = (3, 2), (4, 3)$ [316], (4,2) [172]
mK_n (see §2.5)	G iff $m = 1, n \leq 4$ [329]
$C_s \cup P_n$	? G iff $s + n \geq 7$ [222] G if $s = 3$ [220], $s = 4$ [222], $s = 5$ [313] G if $s > 4, n = 2$ [236] G if $s = 2n + 1$ [220] G if $s = 2k, n \geq k + 1$ [466]
$C_p \cup C_q$	? G iff $p + q \equiv 0, 3 \pmod 4$ [222] G if $s = 2n + 1$ [220], $s \geq 5$ and $n \geq (s + 5)/2$ [164]

Table 1: **continued**

Graph	*Graceful*
Fans $F_n = P_n + K_1$	G [249]
Double fans $P_n + \overline{K_2}$	G [249]
t-point suspension $P_n + \overline{K_t}$ of P_n	G [247]
$S_m + K_1$	G [267]
t-point suspension of $C_n + \overline{K_t}$	G if $n \equiv 0$ or 3 (mod 12) [104] not G if t is even and $n \equiv 2, 6, 10$ (mod 12) G if $n = 4, 7, 11$ or 19 [104] G if $n = 5$ or 9 and $t = 2$ [104]
P_n^2 (see §2.7)	G [345]
Petersen $P(n, k)$ (see §2.7)	?
Caterpillars	G [452]
Lobsters	?G [91]

Table 2: **Summary of Harmonious Results**

Graph	*Harmonious*
Trees	H if ≤ 26 vertices [21] ?H [249]
Cycles C_n	H iff $n \equiv 1, 3 \pmod 4$ [249]
Wheels W_n	H [249]
Helms (see §2.2)	H [240], [385]
Webs (see §2.2)	H if cycle is odd
Gears (see §2.2)	?
Cycles with P_k-chord (see §2.2)	?
C_n with k consecutive chords (see §2.2)	?
Unicyclic graphs	?
$C_n^{(t)}$ (see §2.2)	$n = 3$ H iff $t \not\equiv 2 \pmod 4$ [249] H if $n = 4,\ t > 1$ [498]
Triangular snakes (see §2.2)	H if number of blocks is odd [596] not H if number of blocks $\equiv 2$ (mod 4) [596]
K_4-snakes (see §2.2)	H [248]
Quadrilateral snakes (see §2.2)	?
Crowns $C_n \odot K_1$	H [247], [387]
Grids $P_m \times P_n$	H iff $(m, n) \neq (2, 2)$ [301]

Table 2: **continued**

Graph	*Harmonious*
Prisms $C_m \times P_n$	H if $n = 2, m \neq 4$ [233] H if n odd [249] H if $m = 4$ and $n \geq 3$ [301]
Torus grids $C_m \times C_n$,	H if $m = 4$, $n > 1$ [301] not H if $m \not\equiv 0 \pmod 4$ and n odd [301]
Vertex-deleted $C_m \times P_n$	H if $n = 2$ [233]
Edge-deleted $C_m \times P_n$	H if $n = 2$ [233]
Möbius ladders M_n (see §2.3)	H iff $n \neq 3$ [228]
Stacked books $S_m \times P_n$ (see §2.3)	$n = 2$, H if m even [246], [446] not H $m \equiv 3 \pmod 4$, $n = 2$, (parity condition) H if $m \equiv 1 \pmod 4$, $n = 2$ [240]
n-cube $K_2 \times K_2 \times \cdots \times K_2$	not H if $n = 2, 3$ [249]
$K_4 \times P_n$	H [446]
K_n	H iff $n \leq 4$ [249]
$K_{m,n}$	H iff m or $n = 1$ [249]
$K_{1,m,n}$	H [31]
$K_{1,1,m,n}$	H [240]
Windmills $K_n^{(m)}$ $(n > 3)$ (see §2.4)	H if $n = 4$ [287] $m = 2$, ?H iff $n = 4$ [249] not H if $m = 2, n$ odd or 6 [249] not H for some cases $m = 3$ [380]

Table 2: **continued**

Graph	*Harmonious*
$B(n,r,m)$ $r>1$ (see §2.4)	$(n,r)=(3,2),(4,3)$ [471]
mK_n (see §2.5)	H $n=3$, m odd [388] not H for n odd, $m \equiv 2 \pmod 4$ [388]
$C_s \cup P_n$	?
Fans $F_n = P_n + K_1$ Double fans $P_n + \overline{K_2}$	H [249] H [249]
t-point suspension $P_n + \overline{K_t}$ of P_n	H [446]
$S_m + K_1$	H [240], [152]
t-point suspension $C_n + \overline{K_t}$ of C_n	H if n odd and $t=2$ [446], [240] not H if $n \equiv 2,4,6 \pmod 8$ and $t=2$ [240]
P_n^2 (see §2.7)	H [247], [387]
Petersen $P(n,k)$ (see §2.7)	H [240], [352]
Caterpillars	H [249]
Lobsters	?

3 Variations of Graceful Labelings

3.1 α-labelings

In [452] Rosa defined an *α-labeling* to be a graceful labeling with the additional property that there exists an integer k so that for each edge xy either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. (Other names for such labelings are *balanced* and *interlaced*.) It follows that such a k must be the smaller of the two vertex labels that yield the edge labeled 1. Also, a graph with an α-labeling is necessarily bipartite and therefore can not contain a cycle of odd length. Wu [590] has shown that a necessary condition for a bipartite graph with n edges and degree sequence $d_1, d_2, \ldots, d_p$ to have an α-labeling is that the $\gcd(d_1, d_2, \ldots, d_p, n)$ divides $n(n-1)/2$.

A common theme in graph labeling papers is to build up graphs that have desired labelings from pieces with particular properties. In these situations, starting with a graph that possesses an α-labeling is a typical approach. (See [152], [247], [157] and [301].) Moreover, Jungreis and Reid [301] showed how sequential labelings of graphs (see Section 4.1) can often be obtained by modifying α-labelings of the graphs.

Graphs with α-labelings have proved to be useful in the development of the theory of graph decompositions. Rosa [452], for instance, has shown that if G is a graph with q edges and has an α-labeling, then for every natural number p, the complete graph K_{2qp+1} can be decomposed into copies of G in such a way that the automorphism group of the decomposition itself contains the cyclic group of order p. In the same vein El-Zanati and Vanden Eynden [191] proved that if G has q edges and admits an α-labeling then $K_{qm,qn}$ can be partitioned into subgraphs isomorphic to G for all positive integers m and n. Although a proof of Ringel's conjecture that every tree has a graceful labeling has withstood many attempts, examples of trees that do not have α-labelings are easy to construct (see [452]).

As to which graphs have α-labelings, Rosa [452] observed that the n-cycle has an α-labeling if and only if $n \equiv 0 \pmod 4$ while P_n always has an α-labeling. Other familiar graphs that have α-labelings include caterpillars [452], the n-cube [323], B_{4n+1} (i.e., books with $4n+1$ pages) [232], $C_{2m} \cup C_{2m}$, and $C_{4m} \cup C_{4m} \cup C_{4m}$ for all $m > 1$ [324], $P_n \times Q_n$ [397], $K_{1,2k} \times Q_n$ [397], $C_{4m} \cup C_{4m} \cup C_{4m} \cup C_{4m}$ [335], $C_{4m} \cup C_{4n+2} \cup C_{4r+2}$, $C_{4m} \cup C_{4n} \cup C_{4r}$ when $m+n \leq r$ [10], $C_{4m} \cup C_{4n} \cup C_{4r} \cup C_{4s}$ when $m \geq n+r+s$ [6], $C_{4m} \cup C_{4n} \cup C_{4r+2} \cup C_{4s+2}$ when $m \geq n+r+s+1$ [6], $((m+1)^2+1)C_4$ for all m [617], k^2C_4 for all k [617], and $(k^2+k)C_4$ for all k [617]. Abrham and Kotzig [8] have shown that kC_4 has an α-labeling for $1 \leq k \leq 10$ and that if kC_4 has an α-labeling then so does $(4k+1)C_4, (5k+1)C_4$ and $(9k+1)C_4$. Eshghi [200] proved that $5C_{4k}$ has an α-labeling for all k.

Figueroa-Centeno, Ichishima and Muntaner-Batle [209] have shown that if $m \equiv 0 \pmod 4$ then the one-point union of 2, 3 or 4 copies of C_m admits an α-valuation and if $m \equiv 2 \pmod 4$ then the one-point union of 2 or 4 copies of C_m admits an α-valuation. They conjecture that the one-point union of n copies of C_m admits an α-valuation if and only if $mn \equiv 0 \pmod 4$.

Zhile [617] uses $C_m(n)$ to denote the connected graph all of whose blocks are C_m and whose block-cutpoint-graph is a path. He proves that for all positive integers m and n, $C_{4m}(n)$ has an α-labeling but $C_m(n)$ does not have an α-labeling when m is odd.

Abrham and Kotzig [10] have proved that $C_m \cup C_n$ has an α-labeling if and only if both m and n are even and $m + n \equiv 0 \pmod 4$. Kotzig [324] has also shown that $C_4 \cup C_4 \cup C_4$ does not have an α-labeling. He asked if $n = 3$ is the only integer such that the disjoint union of n copies of C_4 does not have an α-labeling. This was confirmed by Abrham and Kotzig in [9]. Eshghi [199] proved that every 2-regular bipartite graph with 3 components has an α-labeling if and only if the number of edges is a multiple of four except for $C_4 \cup C_4 \cup C_4$.

Jungreis and Reid [301] investigated the existence of α-labelings for graphs of the form $P_m \times P_n, C_m \times P_n$, and $C_m \times C_n$ (see also [231]). Of course, the cases involving C_m with m odd are not bipartite, so there is no α-labeling. The only unresolved cases among these three families are $C_{4m+2} \times P_{2n+1}$ and $C_{4m+2} \times C_{4n+2}$. All other cases result in α-labelings. Balakrishman [65] uses the notation $Q_n(G)$ to denote the graph $P_2 \times P_2 \times \cdots \times P_2 \times G$ where P_2 occurs $n-1$ times. Snevily [536] has shown that the graphs $Q_n(C_{4m})$ and the cycles C_{4m} with the path P_n adjoined at each vertex have α-labelings. He also has shown [537] that compositions of the form $G[\overline{K_n}]$ have an α-labeling whenever G does (see §2.3 for the definition of composition). Balakrishman and Kumar [66] have shown that all graphs of the form $Q_n(G)$ where G is $K_{3,3}, K_{4,4}$, or P_m have an α-labeling. Balakrishman [65] poses the following two problems. For which graphs G does $Q_n(G)$ have an α-labeling? For which graphs G does $Q_n(G)$ have a graceful labeling? Rosa [452] has shown that $K_{m,n}$ has an α-labeling (see also [75]). Barrientos [75] has shown that for n even the graph obtained from the wheel W_n by attaching a pendant edge at each vertex has an α-labeling. Qian [442] has proved that quadrilateral snakes have α-labelings. Fu and Wu [224] showed that if T is a tree that has an α-labeling with partite sets V_1 and V_2 then the graph obtained from T by joining new vertices $w_1, w_2, \ldots, w_k$ to every vertex of V_1 has an α-labeling. Similarly, they prove that the graph obtained from T by joining new vertices $w_1, w_2, \ldots, w_k$ to the vertices of V_1 and new vertices $u_1, u_2, \ldots, u_t$ to every vertex of V_2 has an α-labeling. They also prove that if one of the new vertices of either of these two graphs is replaced by a star and every vertex of the star is joined to the vertices of V_1 or the vertices of both V_1 and V_2, the resulting graphs have α-labelings. Fu and Wu [224] further show that if T is a tree with an α-labeling and the sizes of the two partite sets of T differ at by at most 1, then $T \times P_m$ has an α-labeling.

Barrientos [76] defines a *chain graph* as one with blocks $B_1, B_2, \ldots, B_m$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block-cutpoint graph is a path. He shows that if $B_1, B_2, \ldots, B_m$ are blocks that have α-labelings then there exists a chain graph G with blocks $B_1, B_2, \ldots, B_m$ that has an α-labeling. He also shows that if $B_1, B_2, \ldots, B_m$ are complete bipartite graphs, then any chain graph G obtained by concatenation of these blocks has an α-labeling.

Wu ([591] and [592]) has given a number of methods for constructing larger graceful graphs from graceful graphs. Let $G_1, G_2, \ldots, G_p$ be disjoint connected graphs. Let w_i be in G_i for $1 \leq i \leq p$. Let w be a new vertex not in any G_i. Form a new graph $\oplus_w(G_1, G_2, \ldots, G_p)$ by adjoining to the graph $G_1 \cup G_2 \cup \cdots \cup G_p$ the edges $ww_1, ww_2, \ldots, ww_p$. In the case where each of $G_1, G_2, \ldots, G_p$ is isomorphic to a graph G which has an α-labeling and each w_i is the isomorphic image of the same vertex in G_i, Wu shows that the resulting graph is graceful. If f is an α-labeling of a graph, the integer k with the property that for any edge uv either $f(u) \leq k < f(v)$ or $f(v) \leq k < f(u)$ is called the *boundary value* or *critical number* of f. Wu [591] has also shown that if $G_1, G_2, \ldots, G_p$ are graphs of the same order and have α-labelings where the labelings for each pair of graphs G_i and G_{p-i+1} have the same boundary value for $1 \leq i \leq n/2$, then $\oplus_w(G_1, G_2, \ldots, G_p)$ is graceful. In [589] Wu proves that if G has n edges and $n+1$ vertices and G has an α-labeling with boundary value λ, where $|n - 2\lambda - 1| \leq 1$, then $G \times P_m$ is graceful for all m.

Snevily [537] says that a graph G *eventually has an α-labeling* provided that there is a graph H, called a *host* of G, which has an α-labeling and that the edge set of H can be partitioned into subgraphs isomorphic to G. He defines the *α-labeling number of* G to be $G_\alpha = \min\{t : \text{there is a host } H \text{ of } G \text{ with } |E(H)| = t|G|\}$. Snevily proved that even cycles have α-labeling number at most 2 and he conjectured that every bipartite graph has an α-labeling number. This conjecture was proved by El-Zanati, Fu and Shiue [190]. There are no known examples of a graph G with $G_\alpha > 2$.

Given two bipartite graphs G_1 and G_2 with partite sets H_1 and L_1 and H_2 and L_2, respectively, Snevily [536] defines their *weak tensor product* $G_1 \overline{\bigotimes} G_2$ as the bipartite graph with vertex set $(H_1 \times H_2, L_1 \times L_2)$ and with edge $(h_1, h_2)(l_1, l_2)$ if $h_1 l_1 \in E(G_1)$ and $h_2 l_2 \in E(G_2)$. He proves that if G_1 and G_2 have α-labelings then so does $G_1 \overline{\bigotimes} G_2$. This result considerably enlarges the class of graphs known to have α-labelings.

The sequential join of graphs $G_1, G_2, \ldots, G_n$ is formed from $G_1 \cup G_2 \cup \cdots \cup G_n$ by adding edges joining each vertex of G_i with each vertex of G_{i+1} for $1 \leq i \leq n-1$. Lee and Wang [365] have shown that for all $n \geq 2$ and any positive integers $a_1, a_2, \ldots, a_n$ the sequential join of the graphs $\overline{K}_{a_1}, \overline{K}_{a_2}, \ldots, \overline{K}_{a_n}$ has an α-labeling.

In [229] Gallian and Ropp conjectured that every graph obtained by adding a single pendant edge to one or more vertices of a cycle is graceful. Qian [442] has proved this conjecture and in the case that the cycle is even he showns the graphs have an α-labeling. He further proves that for n even any graph obtained from an n-cycle by adding one or more pendant edges at some vertices has an α-labeling as long as at least one vertex has degree 3 and one vertex has degree 2.

For any tree $T(V, E)$ whose vertices are properly 2-colored Rosa and Širáň [454] define a *bipartite labeling* of T as a bijection $f : V \to \{0, 1, 2, \ldots, |E|\}$ for which there is a k such that whenever $f(u) \leq k \leq f(v)$, then u and v have different colors. They define the *α-size* of a tree T as the maximum number of distinct values of the induced edge labels $|f(u) - f(v)|, uv \in E$, taken over all bipartite labelings f of T. They prove that the α-size of any tree with n edges is at least $5(n+1)/7$ and that there exist trees whose α-size is at most $(5n+9)/6$. They conjectured that minimum of the α-sizes over all trees with n edges is asymptotically $5n/6$. This conjecture has been proved for trees of maximum degree 3 by Bonnington and Širáň [134]. Heinrich and Hell [268] defined the *gracesize* of a graph G with n vertices as the maximum, over all bijections $f\colon V(G) \to \{1, 2, \ldots, n\}$, of the number of distinct values $|f(u) - f(v)|$ over all edges uv of G. So, from Rosa and Širáň's result, the gracesize of any tree with n edges is at least $5(n + 1)/7$.

In [233] Gallian weakened the condition for an α-labeling somewhat by defining a *weakly α-labeling* as a graceful labeling for which there is an integer k so that for each edge xy either $f(x) \le k \le f(y)$ or $f(y) \le k \le f(x)$. Unlike α-labelings, this condition allows the graph to have an odd cycle, but still places a severe restriction on the structure of the graph; namely, that the vertex with the label k must be on every odd cycle. Gallian, Prout and Winters [233] showed that the prisms $C_n \times P_2$ with a vertex deleted have α-labelings. The same paper reveals that $C_n \times P_2$ with an edge deleted from a cycle has an α-labeling when n is even and a weakly α-labeling when $n > 3$.

A special case of α-labeling called strongly graceful was introduced by Maheo [397] in 1980. A graceful labeling f of a graph G is called *strongly graceful* if G is bipartite with two partite sets A and B of the same order s, the number of edges is $2t + s$, there is an integer k with $t - s \le k \le t + s - 1$ such that if $a \in A, f(a) \le k$, and if $b \in B, f(b) > k$, and there is an involution π which is an automorphism of G such that: π exchanges A and B and the s edges $a\pi(a)$ where $a \in A$ have as labels the integers between $t + 1$ and $t + s$. Maheo's main result is that if G is strongly graceful then so is $G \times Q_n$. In particular, she proved that $(P_n \times Q_n) \times K_2$, B_{2n} and $B_{2n} \times Q_n$ have strongly graceful labelings. El-Zanati and Vanden Eynden [192] call a strongly graceful labeling a *strong α-labeling*. They show that if G has a strong α-labeling, then $G \times P_n$ has an α-labeling. They show that $K_{m,2} \times K_2$ has a strong α-labeling and that $K_{m,2} \times P_n$ has an α-labeling. They also show that if G is a bipartite graph with one more vertex than the number of edges, and if G has an α-labeling such that the cardinalities of the sets of the corresponding bipartition of the vertices differ by at most 1, then $G \times K_2$ has a strong α-labeling and $G \times P_n$ has an α-labeling. El-Zanati and Vanden Eynden [192] also note that $K_{3,3} \times K_2$, $K_{3,4} \times K_2$, $K_{4,4} \times K_2$, and $C_{4k} \times K_2$ all have strong α-labelings. El-Zanati and Vanden Eynden proved that $K_{m,2} \times Q_n$ has a strong α-valuation and that $K_{m,2} \times P_n$ has an α-labeling for all n. They also prove that if G is a connected bipartite graph with partite sets of odd order such that in each partite set each vertex has the same degree, then $G \times K_2$ does not have a strong α-valuation. As a corollary they have that $K_{m,n} \times K_2$ does not have a strong α-valuation when m and n are odd.

An α-labeling f of a graph G is called *free* by El-Zanati and Vanden Eynden in [193] if the critical number k (in the definition of α-labeling) is greater than 2 and if neither 1 nor $k - 1$ is used in the labeling. Their main result is that the union of graphs with free α-labelings has an α-labeling. In particular, they show that $K_{m,n}$, $m > 1$, $n > 2$, has a free α-labeling. They also show that Q_n, $n \ge 3$, and $K_{m,2} \times Q_n$, $m > 1$, $n \ge 1$, have free α-labelings. El-Zanati [personal communication] has shown that the Heawood graph has a free α-labeling.

For connected bipartite graphs Grannell, Griggs and Holroyd [250] introduced a labeling that lies between α-labelings and graceful labelings. They call a vertex labeling f of a bipartite graph G with q edges and partitite sets D and U *gracious* if f is a bijection from the vertex set of G to $\{0, 1, \ldots, q\}$ such that the set of edge labels induced by $f(u) - f(v)$ for every edge uv with $u \in U$ and $v \in D$ is $\{1, 2, \ldots, q\}$. Thus a gracious labeling of G with partite sets D and U is a graceful labeling in which every vertex in D has a label lower than every adjacent vertex. They verified by computer that every tree of size up to 20 has a gracious labeling. This led them to conjecture that every tree has a gracious labeling. For any $k > 1$ and any tree T Grannell et al. say that T has a *gracious k-labeling* if the verticies of T can be partitioned into sets D and U in such a way that there is a function f from the verticies of G to the integers modulo k such that the edge labels induced by $f(u) - f(v)$ where $u \in U$ and $v \in D$ has the following properties: the number of edges labeled with 0 is one less than the number of verticies labeled with 0 and for each nonzero integer x the number of edges labeled with x is the same as the number of verticies labeled with x. They prove that every nontrivial tree has a k-gracious labeling for $k = 2, 3, 4, 5$ and that caterpillars are k-gracious for all $k \geq 2$.

The same labeling that is called gracious by Grannell, Griggs and Holroyd is called a *near α-labeling* by El-Zanati, Kenig and Vanden Eynden [194]. They prove that if G is a graph with n edges that has a near α-labeling then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x and a cyclic G-decomposition of $K_{n,n}$. They further prove that if G and H have near α-labelings, then so does their weak tensor product with respect to the corresponding vertex partitions. They conjecture that every tree has a near α-labeling.

Another kind of labelings for trees was introduced by Ringel, Llado and Serra [449] in an approach to proving their conjecture $K_{n,n}$ is edge-decomposable into n copies of any given tree with n edges. If T is a tree with n edges and partite sets A and B, they define a labeling f from the set of vertices to $\{1, 2, \ldots, n\}$ to be a *bigraceful* labeling of T if f restricted to A is injective, f restricted to B is injective, and the edge labels given by $f(y) - f(x)$ where yx is an edge with y in B and x in A is the set $\{0, 1, 2, \ldots, n-1\}$. (Notice that this terminology conflicts with that given in Section 2.7 In particular, the Ringel, Llado and Serra bigraceful does not imply the usual graceful.) Among the graphs that they show are bigraceful are: lobsters, trees of diameter at most 5, stars $S_{k,m}$ with k spokes of paths of length m, and complete d-ary trees for d odd. They also prove that if T is a tree then there is a vertex v and a nonnegative integer m such that the addition of m leaves to v results in a bigraceful tree. They conjecture that all trees are bigraceful.

3.2 k-graceful Labelings

A natural generalization of graceful graphs is the notion of k-graceful graphs introduced independently by Slater [529] in 1982 and by Maheo and Thuillier [398] in 1982. A graph G with q edges is *k-graceful* if there is labeling f from the vertices of G to $\{0, 1, 2, \ldots, q+$

$k-1\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is $\{k, k+1, \ldots, q+k-1\}$. Obviously, 1-graceful is graceful and it is readily shown that any graph that has an α-labeling is k-graceful for all k. Graphs that are k-graceful for all k are sometimes called *arbitrarily graceful.* Ng [427] has shown that there are graphs that are k-graceful for all k but do not have an α-labeling.

Results of Maheo and Thuillier [398] together with those of Slater [529] show that: C_n is k-graceful if and only if either $n \equiv 0$ or 1 (mod 4) with k even and $k \leq (n-1)/2$, or $n \equiv 3$ (mod 4) with k odd and $k \leq (n^2-1)/2$. Maheo and Thuillier [398] also proved that the wheel W_{2k+1} is k-graceful and conjectured that W_{2k} is k-graceful when $k \neq 3$ or $k \neq 4$. This conjecture was proved by Liang, Sun and Xu [374]. Kang [302] proved that $P_m \times C_{4n}$ is k-graceful for all k. Lee and Wang [364] showed that all pyramids, lotuses and diamonds are k-graceful and Liang and Liu [371] have shown that $K_{m,n}$ is k-graceful. Bu, Gao and Zhang [127] have proved that $P_n \times P_2$ and $(P_n \times P_2) \cup (P_n \times P_2)$ are k-graceful for all k. Acharya (see [13]) has shown that a k-graceful Eulerian graph with q edges must satisfy one of the following conditions: $q \equiv 0$ (mod 4), $q \equiv 1$ (mod 4) if k is even, or $q \equiv 3$ (mod 4) if k is odd. Bu, Zhang and He [132] have shown that an even cycle with a fixed number of pendant edges adjoined to each vertex is k-graceful.

Several authors have investigated the k-gracefulness of various classes of subgraphs of grid graphs. Acharya [11] proved that all 2-dimensional polyminoes that are convex and Eulerian are k-graceful for all k; Lee [338] showed that Mongolian tents and Mongolian villages are k-graceful for all k (see Section 2.3 for definitions); Lee and K. C. Ng [346] proved that all Young tableaus (see §2.3 for the defintions) are k-graceful for all k. (A special case of this is $P_n \times P_2$.) Lee and H. K. Ng [347] subsequently generalized these results on Young tableaus to a wider class of planar graphs.

Let $c, m, p_1, p_2, \ldots, p_m$ be positive integers. For $i = 1, 2, \ldots, m$, let S_i be a set of $p_i + 1$ integers and let D_i be the set of positive differences of the pairs of elements of S_i. If all these differences are distinct then the system $D_1, D_2, \ldots, D_m$ is called a *perfect system of difference sets starting at* c if the union of all the sets D_i is $c, c+1, \ldots, c-1+\sum_{i=1}^{m}\begin{pmatrix} p_i+1 \\ 2 \end{pmatrix}$. There is a relationship between k-graceful graphs and perfect systems of difference sets. A perfect system of difference sets starting with c describes a c-graceful labeling of a graph that is decomposable into complete subgraphs. A survey of perfect systems of difference sets is given in [4].

Acharya and Hegde [18] generalized k-graceful to (k, d)-*graceful labelings* by permitting the vertex labels to belong to $\{0, 1, 2, \ldots, k+(q-1)d\}$ and requiring the set of edge labels induced by the absolute value of the difference of labels of adjacent vertices to be $\{k, k+d, k+2d, \ldots, k+(q-1)d\}$. They also introduce an analog of α-labelings in the obvious way. Notice that a (1,1)-graceful labeling is a graceful labeling and a $(k, 1)$-graceful labeling is a k-graceful labeling. Bu and Zhang [131] have shown that $K_{m,n}$ is (k, d)-graceful for all k and d; for $n > 2$, K_n is (k, d)-graceful if and only if $k = d$ and $n \leq 4$; if $m_i, n_i \geq 2$ and $\max\{m_i, n_i\} \geq 3$, then $K_{m_1,n_1} \cup K_{m_2,n_2} \cup \cdots \cup K_{m_r,n_r}$ is

(k,d)-graceful for all k,d, and r; if G has an α-labeling, then G is (k,d)-graceful for all k and d; a k-graceful graph is a (kd,d)-graceful graph; a (kd,d)-graceful connected graph is k-graceful; and a (k,d)-graceful graph with q edges that is not bipartite must have $k \leq (q-2)d$.

Let T be a tree with adjacent vertices u_0 and v_0 and pendant vertices u and v such that the length of the path $u_0 - u$ is the same as the length of the path $v_0 - v$. Hegde and Shetty [278] call the graph obtained from T by deleting u_0v_0 and joining u and v is called an *elementary parallel transformation* of T. They say that a tree T is a T_p*-tree* if it can be transformed into a path by a sequence of elementary parallel transformations. They prove that every T_p-tree is (k,d)-graceful for all k and d and every graph obtained from a T_p-tree by subdividing each edge of the tree is (k,d)-graceful for all k and d.

Hegde [274] has proved the following: if a graph is (k,d)-graceful for odd k and even d, then the graph is bipartite; if a graph is (k,d)-graceful and contains C_{2j+1} as a subgraph, then $k \leq jd(q-j-1)$; K_n is (k,d)-graceful if and only if $n \leq 4$; C_{4t} is (k,d)-graceful for all k and d; C_{4t+1} is $(2t,1)$-graceful; C_{4t+2} is $(2t-1,2)$-graceful; and C_{4t+3} is $(2t+1,1)$-graceful.

Hegde [272] calls a (k,d)-graceful graph (k,d)-*balanced* if it has a (k,d)-graceful labeling f with the property that there is some integer m so that for every edge uv either $f(u) \leq m$ and $f(v) > m$ or $f(u) > m$ and $f(v) \leq m$. He proves that if a graph is $(1,1)$-balanced then it is (k,d)-graceful for all k and d and that every $(1,1)$-balanced graph is (k,k)-balanced for all k. He conjectures that all trees are (k,d)-balanced for some values of k and d.

Duan and Qi [186] use $G_t(m_1,n_1;m_2,n_2;\ldots;m_s,n_s)$ to denote the graph composed of the s complete bipartite graphs $K_{m_1,n_1}, K_{m_2,n_2}, \ldots, K_{m_s,n_s}$ that have only t $(1 \leq t \leq \min\{m_1,m_2,\ldots,m_s\})$ common vertices but no common edge and $G(m_1,n_1;m_2,n_2)$ to denote the graph composed of the complete bipartite graphs K_{m_1,n_1}, K_{m_2,n_2} with exactly one common edge. They prove that these graphs are k-graceful graphs for all k.

Slater [532] has extended the definition of k-graceful graphs to countable infinite graphs in a natural way. He proved that all countably infinite trees, the complete graph with countably many vertices and the countably infinite Dutch windmill is k-graceful for all k.

More specialized results on k-graceful labelings can be found in [338], [346], [347], [529], [126], [128], [127] and [156].

3.3 Skolem-Graceful Labelings

A number of authors have invented analogues of graceful graphs by modifying the permissible vertex labels. For instance, Lee (see [361]) calls a graph G with p vertices and q edges *Skolem-graceful* if there is an injection from the set of vertices of G to $\{1,2,\ldots,p\}$ such that the edge labels induced by $|f(x)-f(y)|$ for each edge xy are $1,2,\ldots,q$. A necessary condition for a graph to be Skolem-graceful is that $p \geq q+1$. Lee and Wui [369] have shown that a connected graph is Skolem-graceful if and only if it is a graceful tree.

Although the disjoint union of trees can not be graceful, they can be Skolem-graceful. Lee and Wui [369] prove that the disjoint union of 2 or 3 stars is Skolem-graceful if and only if at least one star has even size. In [165] Choudum and Kishore show that the disjoint union of k copies of the star $K_{1,2p}$ is Skolem graceful if $k \leq 4p+1$ and the disjoint union of any number of copies of $K_{1,2}$ is Skolem graceful. For $k \geq 2$, let $St(n_1, n_2, \ldots, n_k)$ denote the disjoint union of k stars with $n_1, n_2, \ldots, n_k$ edges. Lee, Wang and Wui [366] showed that the 4-star $St(n_1, n_2, n_3, n_4)$ is Skolem-graceful for some special cases and conjectured that all 4-stars are Skolem-graceful. Denham, Leu and Liu [175] proved this conjecture. Kishore [313] has shown that a necessary condition for $St(n_1, n_2, \ldots, n_k)$ to be Skolem graceful is that some n_i is even or $k \equiv 0$ or 1 (mod 4). He conjectures that each one of these conditions is sufficient. Choudum and Kishore [163] proved that all 5-stars are Skolem graceful.

Lee, Quach and Wang [349] showed that the disjoint union of the path P_n and the star of size m is Skolem-graceful if and only if $n = 2$ and m is even or $n \geq 3$ and $m \geq 1$. It follows from the work of Skolem [525] that nP_2, the disjoint union of n copies of P_2, is Skolem-graceful if and only if $n \equiv 0$ or 1 (mod 4). Harary and Hsu [259] studied Skolem-graceful graphs under the name *node-graceful*. Frucht [220] has shown that $P_m \cup P_n$ is Skolem-graceful when $m + n \geq 5$. Bhat-Nayak and Deshmukh [101] have shown that $P_{n_1} \cup P_{n_2} \cup P_{n_3}$ is Skolem-graceful when $n_1 < n_2 \leq n_3,\ n_2 = t(n_1 + 2) + 1$ and n_1 is even and when $n_1 < n_2 \leq n_3,\ n_2 = t(n_1 + 3) + 1$ and n_1 is odd. They also prove that the graphs of the form $P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_i}$ where $i \geq 4$ are Skolem-graceful under certain conditions. Youssef [607] proved that if G is Skolem-graceful, then $G + \overline{K_n}$ is graceful.

E. Mendelsohn [403] defined a *Skolem labeled* graph $G(V, E)$ as one for which there is a positive integer d and a function $L\colon V \to \{d, d+1, \ldots, d+m\}$, satisfying (a) there are exactly two vertices in V such that $L(v) = d + i$, $0 \leq i \leq m$; (b) the distance in G between any two vertices with the same label is the value of the label; and (c) if G' is a proper spanning subgraph of G, then L restricted to G' is not a Skolem labeled graph. Note that this definition is different from the Skolem-graceful labeling of Lee, Quach and Wang. He established the following: any tree can be embedded in a Skolem labeled tree with $O(v)$ vertices; any graph can be embedded as an induced subgraph in a Skolem labeled graph on $O(v^3)$ vertices; for $d = 1$, there is a Skolem or the minimum hooked Skolem (with as few unlabeled vertices as possible) labeling for paths and cycles; for $d = 1$, there is a minimum Skolem labeled graph containing a path or a cycle of length n as induced subgraph. In [402] Mendelsohn proves that the necessary conditions in [403] are sufficient for a Skolem or minimum hooked Skolem labeling of all trees consisting of edge-disjoint paths of the same length from some fixed vertex.

3.4 Odd Graceful Labelings

Gnanajothi [240, p. 182] defined a graph G with q edges to be *odd graceful* if there is an injection f from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q-1\}$. She proved that the class of odd graceful graphs lies between the class of graphs with α-labelings and the class of bipartite graphs by showing that every graph with an α-labeling has an odd graceful labeling and every graph with an odd cycle is not odd graceful. She also proved the following graphs are odd graceful: P_n; C_n if and only if n is even; $K_{m,n}$; combs $P_n \odot K_1$ (graphs obtained by joining a single pendant edge to each vertex of P_n); books; crowns $C_n \odot K_1$ (graphs obtained by joining a single pendant edge to each vertex of C_n) if and only if n is even; the disjoint union of copies of C_4; the one-point union of copies of C_4; $C_n \times K_2$ if and only if n is even; caterpillars; rooted trees of height 2; the graphs obtained from P_n ($n \geq 3$) by adding exactly two leaves at each vertex of degree 2 of P_n; the graphs consisting of vertices $a_0, a_1, \ldots a_n, b_0, b_1, \ldots b_n$ with edges a_ia_{i+1}, b_ib_{i+1} for $i = 0, \ldots, n-1$ and a_ib_i for $i = 1, \ldots, n-1$; the graphs obtained from a star by adjoining to each end vertex the path P_3 or by adjoining to each end vertex the path P_4. She conjectures that all trees are odd graceful and proves the conjecture for all trees with order up to 10. Barrientos [77] has extended this up to trees of order up to 12. Eldergill [188] generalized Gnanajothi's result on stars by showing that the graphs obtained by joining one end point from each of any odd number of paths of equal length is odd graceful. He also proved that the one-point union of any number of copies of C_6 is odd graceful. Kathiresan [308] has shown that ladders and graphs obtained from them by subdividing each step exactly once are odd graceful.

Sekar [462] has shown the following graphs are odd graceful: $C_m \odot P_n$ (the graph obtained by identifying an end point of P_n with every vertex of C_m) where $n \geq 3$ and m is even; $P_{a,b}$ when $a \geq 2$ and b is odd (see §2.7); $P_{2,b}$ and $b \geq 2$; $P_{4,b}$ and $b \geq 2$; $P_{a,b}$ when a and b are even and $a \geq 4$ and $b \geq 4$; $P_{4r+1,4r+2}$; $P_{4r-1,4r}$; all n-polygonal snakes with n even; $C_n^{(t)}$ (see §2.2); graphs obtained by beginning with C_6 and repeatedly forming the one-point union with additional copy of C_6 in succession; graphs obtained by beginning with C_8 and repeatedly forming the one-point union with additional copy of C_8 in succession; graphs obtained from even cycles by identifying a vertex of the cycle with the endpoint of a star; $C_{6,n}$ and $C_{8,n}$ (see §2.7); the splitting graph of P_n (see §2.7) the splitting graph of C_n, n even; lobsters, banana trees and regular bamboo trees (see §2.1).

Barrientos [77] has shown that all disjoint unions of caterpillars are odd graceful and all trees of diameter 5 are odd graceful. He conjectures that every bipartite graph is odd graceful.

Seoud, Diab and Elsakhawi [467] have shown that a connected r-partite graph is odd graceful if and only if $r = 2$ and that the join of any two connected graphs is not odd graceful.

3.5 Graceful-like Labelings

As a means of attacking graph decomposition problems, Rosa [452] invented another analogue of graceful labelings by permitting the vertices of a graph with q edges to assume labels from the set $\{0, 1, \ldots, q+1\}$, while the edge labels induced by the absolute value of the difference of the vertex labels are $\{1, 2, \ldots, q-1, q\}$ or $\{1, 2, \ldots, q-1, q+1\}$. He calls these *$\hat{\rho}$-labelings*. Frucht [220] used the term *nearly graceful labeling* instead of *$\hat{\rho}$-labelings*. Frucht [220] has shown that the following graphs have nearly graceful labelings with edge labels from $\{1, 2, \ldots, q-1, q+1\}$: $P_m \cup P_n$; $S_m \cup S_n$; $S_m \cup P_n$; $G \cup K_2$ where G is graceful; and $C_3 \cup K_2 \cup S_m$ where m is even or $m \equiv 3 \pmod{14}$. Seoud and Elsakhawi [468] have shown that all cycles are nearly graceful. Barrientos [74] proved that C_n is nearly graceful with edge labels $1, 2, \ldots, n-1, n+1$ if and only if $n \equiv 1$ or $2 \pmod 4$. Rosa [453] conjectured that triangular snakes with $t \equiv 0$ or $1 \pmod 4$ blocks are graceful and those with $t \equiv 2$ or $3 \pmod 4$ blocks are nearly graceful (a parity condition ensures that the graphs in the latter case cannot be graceful). Moulton [417] proved Rosa's conjecture while introducing the slightly stronger concept of *almost graceful* by permitting the vertex labels to come from $\{0, 1, 2, \ldots, q-1, q+1\}$ while the edge labels are $\{1, 2, \ldots, q-1, q\}$, or $\{1, 2, \ldots, q-1, q+1\}$. Seoud and Elsakhawi [468] have shown that the following graphs are almost graceful: $C_n; P_n + \overline{K_m}; P_n + K_{1,m}; K_{m,n}; K_{1,m,n}; K_{2,2,m}; K_{1,1,m,n}$; ladders; and $P_n \times P_3$ $(n \geq 3)$.

Barrientos [74] calls a graph a *kC_n-snake* if it is a connected graph with k blocks whose block-cutpoint graph is path and each of the k blocks is isomorphic to C_n. (When $n > 3$ and $k > 3$ there is more than one kC_n-snake.) If a kC_n-snake where the path of minimum length that contains all the cut-vertices of the graph has the property that the distance between any two consecutive cut-vertices is $\lfloor n/2 \rfloor$ it is called *linear*. Barrientos proves that kC_4-snakes are graceful and that the linear kC_6-snakes are graceful when k is even. When k is odd he proves that the linear kC_6-snake is nearly graceful. Barrientos further proves that kC_8-snakes and kC_{12}-snakes are graceful in the cases where the distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph are all even and that certain cases of kC_{4n}-snakes and kC_{5n}-snakes are graceful (depending on the distances between consecutive vertices of the path of minimum length that contains all the cut-vertices of the graph). Barrientos [78] also has shown that $C_m \cup K_{1,n}$ is nearly graceful when $m = 3, 4, 5, 6$.

Yet another kind of labeling introduced by Rosa in his 1967 paper [452] is a ρ-valuation. A *ρ-valuation* of a graph is an injection from the vertices of the graph with q edges to the set $\{0, 1, \ldots, 2q\}$, where if the edge labels induced by the absolute value of the difference of the vertex labels are $a_1, a_2, \ldots, a_q$, then $a_i = i$ or $a_i = 2q + 1 - i$. Rosa [452] proved that a cyclic decomposition of the edge set of the complete graph K_{2q+1} into subgraphs isomorphic to a given graph G with q edges exists if and only if G has a ρ-valuation. (A decomposition of K_n into copies of G is called *cyclic* if the automorphism group of the decomposition itself contains the cyclic group of order n.) It is known that every graph with at most 11 edges has a ρ-labeling and that all lobsters have a ρ-labeling (see [150]). Donovan, El-Zanati, Vanden Eyden and Sutinuntopas [180] prove that rC_m has a ρ-labeling (or a more restrictive labeling) when $r \leq 4$. They conjecture that every 2-regular graph has a ρ-labeling. Caro, Roditty and Schőnheim [150] provide a construction for the adjacency matrix for every graph that has a ρ-labeling. They ask the following question: If H is a connected graph having a ρ-labeling and q edges and

G is a new graph with q edges constructed by breaking H up into disconnected parts does G also have a ρ-labeling?

In their investigation of cyclic decompositions of complete graphs El-Zanati, Vanden Eynden and Punnim [195] introduced two kinds of labelings. They say a bipartite graph G with n edges and partite sets A and B has a *θ-labeling* h if h is a one-to-one function from $V(G)$ to $\{0, 1, \ldots, 2n\}$ such that $\{h(b) - h(a)| \ ab \in E(G), a \in A, b \in B\} = \{1, 2, \ldots, n\}$. They call h a *ρ^+-labeling* of G if h is a one-to-one function from $V(G)$ to $\{0, 1, \ldots, 2n\}$ and the integers $h(x) - h(y)$ are distinct modulo $2n + 1$ over all ordered pairs (x, y) where xy is an edge in G, and $h(b) > h(a)$ whenever $a \in A, b \in B$ and ab is an edge in G. Note that θ-labelings are ρ^+-labelings and ρ^+-labelings are ρ-labelings. They prove that if G is a bipartite graph with n edges and a ρ^+-labeling, then for every positive integer x there is a cyclic G-decomposition of K_{2nx+1}. They prove the following graphs have ρ^+-labelings: trees of diameter at most 5, C_{2n}, lobsters, and comets (that is, graphs obtained from stars by replacing each edge by a path of some fixed length). They also prove that the disjoint union of graphs with α-labelings have a θ-labeling and conjecture that all forests have ρ-labelings.

Blinco, El-Zanati and Vanden Eynden [105] call a non-bipartite graph *almost-bipartite* if the removal of some edge results in a bipartite graph. For these kinds of graphs G they call a labeling h a *γ-labeling* of G if the following conditions are met: h is a ρ-labeling; G is tripartite with vertex tripartition A, B, C with $C = \{c\}$ and $\overline{b} \in B$ such that $\{\overline{b}, c\}$ is the unique edge joining an element of B to c; if $\{a, v\}$ is an edge of G with $a \in A$, then $h(a) < h(v)$; and $h(c) - h(\overline{b}) = n$. They prove that if an almost-bipartite graph G with n edges has a γ-labeling then there is a cyclic G-decomposition of K_{2nx+1} for all x. They prove that all odd cycles with more than 3 vertices have a γ-labeling and that $C_3 \cup C_{4m}$ has a γ-labeling if and only if $m > 1$.

In [105] Blinco, El-Zanati and Vanden Eynden consider a slightly restricted ρ^+-labeling for a bipartite graph with partite sets A and B by requiring that there exists a number λ with the property that $\rho^+(a) \leq \lambda$ for all $a \in A$ and $\rho^+(b) > \lambda$ for all $b \in B$. They denote such a labeling by ρ^{++}. They use this knid of labeling to show that if G is a 2-regular graph of order n in which each component has even order then there is a cyclic G-decomposition of K_{2nx+1} for all x. They also conjecture that every bipartite graph has a ρ-labeling and every 2-regular graph has a ρ-labeling.

Dufour [187] and Eldergill [188] have some results on the decomposition of the complete graph using labeling methods. Balakrishnan and Sampathkumar [68] showed that for each positive integer n the graph $\overline{K_n} + 2K_2$ admits a ρ-valuation. Balakrishnan [65] asks if it is true that $\overline{K_n} + mK_2$ admits a ρ-valuation for all n and m. Frondek [214], [215] and Fronček and Kubesa [217] have introduced several kinds of labelings for the purpose of proving the existence of special kinds of decompositions of complete graphs into spanning trees.

For graphs with the property $p = q+1$, Frucht [220] has introduced a stronger version of almost graceful graphs by permitting as vertex labels $\{0, 1, \ldots, q - 1, q + 1\}$ and as edge labels $\{1, 2, \ldots, q\}$. He calls such a labeling *pseudograceful.* Frucht proved that

P_n $(n \geq 3)$, combs, sparklers (i.e., graphs obtained by joining an end vertex of a path to the center of a star), $C_3 \cup P_n$ $(n \neq 3)$, and $C_4 \cup P_n$ $(n \neq 1)$ are pseudograceful while $K_{1,n}$ $(n \geq 3)$ is not. Kishore [313] proved that $C_s \cup P_n$ is pseudograceful when $s \geq 5$ and $n \geq (s+7)/2$ and that $C_s \cup S_n$ is pseudograceful when $s = 3, s = 4$, and $s \geq 7$. Seoud and Youssef [478] and [474] extended the definition of pseudograceful to all graphs with $p \leq q+1$. They proved that K_m is pseudograceful if and only if $m = 1, 3$ or 4 [474]; $K_{m,n}$ is pseudograceful when $n \geq 2$ and $P_m + \overline{K_n}$ $(m \geq 2)$ [478] is pseudograceful. They also proved that if G is pseudograceful, then $G \cup K_{m,n}$ is graceful for $m \geq 2$ and $n \geq 2$ and $G \cup K_{m,n}$ is pseudograceful for $m \geq 2, n \geq 2$ and $(m, n) \neq (2, 2)$ [474]. They ask if $G \cup K_{2,2}$ is pseudograceful whenever G is. Youssef [607] has shown that if H is pseudograceful and G has an α-labeling with k being the smaller vertex label of the edge labeled with 1 and if either $k+2$ or $k-1$ is not a vertex label of G, then $G \cup H$ is graceful.

McTavish [400] has investigated labelings where the vertex and edge labels are from $\{0, \ldots, q, q+1\}$. She calls these *$\tilde{\rho}$-labelings*. Graphs that have $\tilde{\rho}$-labelings include cycles and the disjoint union of P_n or S_n with any graceful graph.

Frucht [220] has made an observation about graceful labelings that yields nearly graceful analogs of α-labelings and weakly α-labelings in a natural way. Suppose $G(V, E)$ is a graceful graph with the vertex labeling f. For each edge xy in E, let $[f(x), f(y)]$ (where $f(x) \leq f(y)$) denote the interval of real numbers r with $f(x) \leq r \leq f(y)$. Then the intersection $\cap[f(x), f(y)]$ over all edges $xy \in E$ is a unit interval, a single point, or empty. Indeed, if f is an α-labeling of G then the intersection is a unit interval; if f is a weakly α-labeling, but not an α-labeling, then the intersection is a point; and, if f is graceful but not a weakly α-labeling, then the intersection is empty. For nearly graceful labelings, the intersection also gives three distinct classes.

Singh and Devaraj [521] call a graph G with p vertices and q edges *triangular graceful* if there is an injection f from $V(G)$ to $\{0, 1, 2, \ldots, T_q\}$ where T_q is the qth triangular number and the labels induced on each edge uv by $|f(u) - f(v)|$ are the first q triangular numbers. They prove the following graphs are trianglar graceful: paths, level 2 rooted trees, olive trees (see Section 2.1 for the definition), complete n-ary trees, double stars, caterpillars, C_{4n}, C_{4n} with pendent edges, the one-point union of C_3 and P_n, and unicyclic graphs that have C_3 as the unique cycle. They prove that wheels, helms, flowers (see §2.2 for the definition) and K_n with $n \geq 3$ are not triangular graceful. They conjecture that all trees are triangular graceful.

Van Bussel [567] considered two kinds of relaxations of graceful labelings as applied to trees. He called a labeling *range-relaxed graceful* it is meets the same conditions as a graceful labeling except the range of possible vertex labels and edge labels are not restricted to the number of edges of the graph (the edges are distinctly labeled but not necessarily labeled 1 to the number of edges). Similarly, he calls a labeling *vertex-relaxed graceful* if it satisfies the conditions of a graceful labeling while permitting repeated vertex labels. He proves that every tree T with q edges has a range-relaxed graceful labeling with the vertex labels in the range $0, 1, \ldots, 2q - diameter(T)$ and that every tree on n vertices has a vertex-relaxed graceful labeling such that the number of distinct vertex labels is strictly greater than $n/2$.

Sekar [462] calls an injective function ϕ from the vertices of a graph with q edges to $\{0, 1, 3, 4, \ldots, 3q-2\}$ *one modulo three graceful* if the edge labels induced by labeling each edge uv with $|\phi(u) - \phi(v)|$ is $\{1, 4, 7, \ldots, 3q-2\}$. He proves that the following graphs are one modulo three graceful: P_m; C_n if and only if $n \equiv 0 \bmod 4$; $K_{m,n}$; $C_{2n}^{(2)}$ (the one-point union of two copies of C_{2n}); $C_n^{(t)}$ for $n = 4$ or 8 and $t > 2$; $C_6^{(t)}$ and $t \geq 4$; caterpillars, stars, lobsters; banana trees, rooted trees of height 2; ladders; the graphs obtained by identifying the endpoints of any number of copies of P_n; the graph obtained by attaching pendent edges to each endpoint of two identical stars and then identifying one endpoint from each of these graphs; the graph obtained by identifying a vertex of C_{4k+2} with an endpoint of a star; n-polygonal snakes (see §2.2) for $n \equiv 0 \pmod 4$; n-polygonal snakes for $n \equiv 0 \pmod 4$; n-polygonal snakes for $n \equiv 2 \pmod 4$ where the number of polygons is even; crowns $C_n \otimes K_1$ for n even; $C_{2n} \otimes P_m (C_{2n}$ with P_m attached at each vertex of the cycle) for $m \geq 3$; chains of cycles (see §2.2) of the form $C_{4,m}, C_{6,2m}$ and $C_{8,m}$. He conjectueres that every one modulo three graceful graph is graceful.

3.6 Cordial Labelings

Cahit [138] has introduced a variation of both graceful and harmonious labelings. Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. Call f a *cordial labeling* of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Cahit [139] proved the following: every tree is cordial; K_n is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all m and n; the friendship graph $C_3^{(t)}$ (i.e., the one-point union of t 3-cycles) is cordial if and only if $t \not\equiv 2 \pmod 4$; all fans are cordial; the wheel W_n is cordial if and only if $n \not\equiv 3 \pmod 4$ (see also [185]); maximal outerplanar graphs are cordial; and an Eulerian graph is not cordial if its size is congruent to 2 (mod 4). Kuo, Chang and Kwong [334] determine all m and n for which mK_n is cordial.

A *k-angular cactus* is a connected graph all of whose blocks are cycles with k vertices. In [139] Cahit proved that a k-angular cactus with t cycles is cordial if and only if $kt \not\equiv 2 \pmod 4$. This was improved by Kirchherr [311] who showed any cactus whose blocks are cycles is cordial if and only if the size of the graph is not congruent to 2 (mod 4). Kirchherr [312] also gave a characterization of cordial graphs in terms of their adjacency matrices. Ho, Lee and Shee [283] proved: $P_n \times C_{4m}$ is cordial for all m and all odd n; the composition G [267] and H is cordial if G is cordial and H is cordial and has odd order and even size (see §2.3 for definition of composition); for $n \geq 4$ the composition $C_n[K_2]$ is cordial if and only if $n \not\equiv 2 \pmod 4$; the Cartesian product of two cordial graphs of even size is cordial. The same authors [282] showed that a unicyclic graph is cordial unless it is C_{4k+2} and that the generalized Petersen graph (see §2.7 for definition) $P(n, k)$ is cordial if and only if $n \not\equiv 2 \pmod 4$. Du [185] determines the maximal number of edges in a cordial graph of order n and gives a necessary condition for a k-regular graph to be cordial.

Seoud and Abdel Maqusoud [464] proved that if G is a graph with n vertices and m edges and every vertex has odd degree then G is not cordial when $m+n \equiv 2 \pmod 4$. They also prove the following: for $m \geq 2$, $C_n \times P_m$ is cordial except for the case $C_{4k+2} \times P_2$; P_n^2 is cordial for all n; P_n^3 is cordial if and only if $n \neq 4$; and P_n^4 is cordial if and only if $n \neq 4, 5$ or 6. Seoud, Diab and Elsakhawi [467] have proved the following graphs are cordial: $P_n + P_m$ for all m and n except $(m,n) = (2,2)$; $C_m + C_n$ if $m \not\equiv 0 \pmod 4$ and $n \neq 2 \pmod 4$; $P_n + K_{1,m}$ for all n and odd m; $C_n + K_{1,m}$ for $n \not\equiv 3 \pmod 4$ and odd m except $(n,m) = (3,1)$; $C_n + \overline{K_m}$ when n is odd and when n is even and m is odd; $K_{1,m,n}$; $K_{2,2,m}$; the n-cube; books B_n if and only if $n \not\equiv 3 \pmod 4$; $B(3,2,m)$ for all m; $B(4,3,m)$ if and only if m is even; and $B(5,3,m)$ if and only if $m \not\equiv 1 \pmod 4$ (see §2.4 for the notation $B(n,r,m)$).

Youssef [610] has proved the following: If G and H are cordial and one has even size, then $G \cup H$ is cordial; if G and H are cordial and both have even size, then $G + H$ is cordial; if G and H are cordial and one has even size and one of either has even order, then $G + H$ is cordial; $C_m \cup C_n$ is cordial if and only if $m + n \not\equiv 2 \pmod 4$; mC_n is cordial if and only if $mn \not\equiv 2 (\mathrm{mod}\ 4)$; $C_m + C_n$ is cordial if and only if $(m,n) \neq (3,3)$ and $\{m \pmod 4, n \pmod 4\} \not\equiv \{0,2\}$; if P_n^k is cordial, then $n \geq k + 1 + \sqrt{k-2}$. He conjectures that this latter condition is also sufficient. He confirms the conjecture for $k = 5, 6, 7, 8$ and 9.

Lee and Liu [344] have shown that the complete n-partite graph if and only if at most three of its partite sets have odd cardinality (see also [185]). Lee, Lee and Chang [336] prove the following graphs are cordial: the Cartesian product of an arbitrary number of paths; the Cartesian product of two cycles if and only if at least one of them is even; and the Cartesian product of an arbitrary number of cycles if at least one of them has length a multiple of 4 or at least two of them are even.

Shee and Ho [499] have investigated the cordiality of the one-point union of n copies of various graphs. For $C_m^{(n)}$, the one-point union of n copies of C_m, they proved:

(i) If $m \equiv 0 \pmod 4$, then $C_m^{(n)}$ is cordial for all n;

(ii) If $m \equiv 1$ or $3 \pmod 4$, then $C_m^{(n)}$ is cordial if and only if $n \not\equiv 2 \pmod 4$;

(iii) If $m \equiv 2 \pmod 4$, then $C_m^{(n)}$ is cordial if and only if n is even.

For $K_m^{(n)}$, the one-point union of n copies of K_m, Shee and Ho [499] prove:

(i) If $m \equiv 0 \pmod 8$, then $K_m^{(n)}$ is not cordial for $n \equiv 3 \pmod 4$;

(ii) If $m \equiv 4 \pmod 8$, then $K_m^{(n)}$ is not cordial for $n \equiv 1 \pmod 4$;

(iii) If $m \equiv 5 \pmod 8$, then $K_m^{(n)}$ is not cordial for all odd n;

(iv) $K_4^{(n)}$ is cordial if and only if $n \not\equiv 1 \pmod 4$;

(v) $K_5^{(n)}$ is cordial if and only if n is even;

(vi) $K_6^{(n)}$ is cordial if and only if $n > 2$;

(vii) $K_7^{(n)}$ is cordial if and only if $n \not\equiv 2 \pmod 4$;

(viii) $K_n^{(2)}$ is cordial if and only if n has the form p^2 or $p^2 + 1$.

Benson and Lee [86] have investigated the regular windmill graphs $K_m^{(n)}$ and determined precisely which ones are cordial for $m < 14$.

For $W_m^{(n)}$, the one-point union of n copies of the wheel W_m with the common vertex being the center, Shee and Ho [499] show:

(i) If $m \equiv 0$ or 2 (mod 4), then $W_m^{(n)}$ is cordial for all n;

(ii) If $m \equiv 3$ (mod 4), then $W_m^{(n)}$ is cordial if $n \not\equiv 1$ (mod 4);

(iii) If $m \equiv 1$ (mod 4), then $W_m^{(n)}$ is cordial if $n \not\equiv 3$ (mod 4).

For all n and all $m > 1$ Shee and Ho [499] prove $F_m^{(n)}$, the one-point union of n copies of the fan $F_m = P_m + K_1$ with the common point of the fans being the center, is cordial. The *flag* Fl_m is obtained by joining one vertex of C_m to an extra vertex called the *root*. Shee and Ho [499] show all $Fl_m^{(n)}$, the one-point union of n copies of Fl_m with the common point being the root, are cordial.

Andar, Boxwala and Limaye [25] and [28] have proved the following graphs are cordial: helms; closed helms; generalized helms obtained by taking a web and attaching pendent vertices to all the vertices of the outermost cycle in the case that the number cycles is even; flowers (see §2.2), which are obtained by joining the vertices of degree one of a helm to the central vertex; sunflower graphs, which are obtained by taking a wheel with the central vertex v_0 and the n-cycle $v_1, v_2, \ldots, v_n$ and additional vertices $w_1, w_2, \ldots, w_n$ where w_i is joined by edges to v_i, v_{i+1}, where $i+1$ is taken modulo n, and multiple shells (see §2.2).

For a graph G and a positive integer t, Andar, Boxwala and Limaye [26] define the t-uniform homeomorph $P_t(G)$ of G as the graph obtained from G by replacing every edge of G by vertex disjoint paths of length t. They prove that if G is cordial and t is odd, then $P_t(G)$ is cordial; if $t \equiv 2$ (mod 4) a cordial labeling of G can be extended to a cordial labeling of $P_t(G)$ if and only if the number of edges labeled 0 in G is even; and when $t \equiv 0$ (mod 4) a cordial labeling of G can be extended to a cordial labeling of $P_t(G)$ if and only if the number of edges labeled 1 in G is even. In [27] Ander et al. prove that $P_t(K_{2n})$ is cordial for all $t \geq 2$ and that $P_t(K_{2n+1})$ is cordial if and only if $t \equiv 0$ (mod 4) or t is odd and $n \not\equiv 2$ (mod 4) or $t \equiv 2$ (mod 4) and n is even. In [29] Andar et al. define a t-ply graph $P_t(u, v)$ as a graph consisting of t internally disjoint paths joining vertices u and v. They prove that $P_t(u, v)$ is cordial except when it Eulerian and the number of edges is congruent to 2 (mod 4).

For a binary labeling g of a graph G let $v_g(j)$ denote the number of vertices labeled with j and $e_g(j)$ denote the number edges labeled with j. Then $i(G) = \min\{|e_g(0) - e_g(1)|\}$ taken over all binary labelings g of G with $|v_g(0) - v_g(1)| \leq 1$. In [30] Andar et al. show that a cordial labeling of G can be extended to a cordial labeling of the graph obtained from G by attaching $2m$ pendant edges at each vertex of G. They also prove that a cordial labeling g of a graph G with p vertices can be extended to a cordial labeling of the graph obtained from G by attaching $2m+1$ pendant edges at each vertex of G if and only if G does not satisfy either of the conditions: (1) G has an even number of edges and $p \equiv 2$ (mod 4); (2) G has an odd number of edges and either $p \equiv 1$ (mod 4) with $e_g(1) = e_g(0) + i(G)$ or $n \equiv 3$ (mod 4) and $e_g(0) = e_g(1) + i(G)$.

For graphs $G_1, G_2, \ldots, G_n$ $(n \geq 2)$ that are all copies of a fixed graph G, Shee and Ho [500] call a graph obtained by adding an edge from G_i to G_{i+1} for $i = 1, \ldots, n-1$ a *path-union* of G (the resulting graph may depend on how the edges are chosen). Among their results they show the following graphs are cordial: path-unions of cycles; path-unions of n copies of K_m when $m = 4, 6$ or 7; path-unions of three or more copies of K_5; and path-unions of two copies of K_m if and only if $m-2, m$ or $m+2$ is a perfect square. They also show that there exist cordial path-unions of wheels, fans, unicyclic graphs, Petersen graphs, trees and various compositions.

Lee and Liu [344] give the following general construction for the forming of cordial graphs from smaller cordial graphs. Let H be a graph with an even number of edges and a cordial labeling such that the vertices of H can be divided into t parts $H_1, H_2, \ldots, H_t$ each consisting of an equal number of vertices labeled 0 and vertices labeled 1. Let G be any graph and $G_1, G_2, \ldots, G_t$ be any t subsets of the vertices of G. Let (G, H) be the graph that is the disjoint union of G and H augmented by edges joining every vertex in G_i to every vertex in H_i for all i. Then G is cordial if and only if (G, H) is. From this it follows that: all generalized fans $F_{m,n} = \overline{K_m} + P_n$ are cordial; the generalized bundle $B_{m,n}$ is cordial if and only if m is even or $n \not\equiv 2 \pmod 4$ ($B_{m,n}$ consists of $2n$ vertices $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ with an edge from v_i to u_i and $2m$ vertices $x_1, x_2, \ldots x_m, y_1, y_2, \ldots, y_m$ with x_i joined to v_i and y_i joined to u_i); if m is odd a generalized wheel $W_{m,n} = \overline{K_m} + C_n$ is cordial if and only if $n \not\equiv 3 \pmod 4$. If m is even, $W_{m,n}$ is cordial if and only if $n \not\equiv 2 \pmod 4$; a complete k-partite graph is cordial if and only if the number of parts with an odd number of vertices is at most 3.

Sethuraman and Selvaraju [494] have shown that certain cases of the union of any number of copies of K_4 with one or more edges deleted and one edge in common are cordial. Youssef [611] has shown that the kth power of C_n is cordial for all n when $k \equiv 2 \pmod 4$ and for all even n when $k \equiv 0 \pmod 4$.

Cahit [144] calls a graph *H-cordial* if it is possible to label the edges with the numbers from the set $\{1, -1\}$ in such a way that, for some k, at each vertex v the algebraic sum of the labels on the edges incident with v is either k or $-k$ and the inequalities $|v(k) - v(-k)| \leq 1$ and $|e(1) - e(-1)| \leq 1$ are also satisfied, where $v(i)$ and $e(j)$ are, respectively, the number of vertices labeled with i and the number of edges labeled with j. He calls a graph *H_n-cordial* if it is possible to label the edges with the numbers from the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ in such a way that, at each vertex v the algebraic sum of the labels on the edges incident with v is in the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ and the inequalities $|v(i) - v(-i)| \leq 1$ and $|e(i) - e(-i)| \leq 1$ are also satistied for each i with $1 \leq i \leq n$. Among Cahit's results are: $K_{n,n}$ is H-cordial if and only if $n > 2$ and n is even and $K_{m,n}, m \neq n$, is H-cordial if and only if $n \equiv 0 \pmod 4$, m is even and $m > 2, n > 2$. Unfortunately, Ghebleh and Khoeilar [238] have shown that other statements in Cahit's paper are incorrect. In particular, Cahit states that K_n is H-cordial if and only if $n \equiv 0 \pmod 4$; W_n is H-cordial if and only if $n \equiv 1 \pmod 4$; and K_n is H_2-cordial if and only if $n \equiv 0 \pmod 4$ while Ghebleh and Khoeilar instead prove that K_n is H-cordial if and only if $n \equiv 0$ or $3 \pmod 4$ and $n \neq 3; W_n$ is H-cordial if and only if n is odd; and K_n is

H_2-cordial if $n \equiv 0$ or 3 (mod 4) and K_n is not H_2-cordial if $n \equiv 1$ (mod 4). Ghebleh and Khoeilar also prove every wheel has an H_2-cordial labeling.

By allowing 0 as the possible induced vertex label of an H-cordial labeling Cahit [144] studies semi-H-cordiality of trees. He also generalizes H-cordial labelings.

Cahit and Yilmaz [147] call a graph E_k-*cordial* if it is possible to label the edges with the numbers from the set $\{0, 1, 2, \ldots, k-1\}$ in such a way that, at each vertex v, the sum modulo k of the labels on the edges incident with v satisfies the inequalities $|v(i) - v(j)| \leq 1$ and $|e(i) - e(j)| \leq 1$, where $v(s)$ and $e(t)$ are, respectively, the number of vertices labeled with s and the number of edges labeled with t. Obviously, E_2-cordial is the same as cordial. Cahit and Yilmaz prove the following graphs are E_3-cordial: P_n $(n \geq 3)$; stars S_n if and only if $n \not\equiv 1$ (mod 3); K_n $(n \geq 3)$; C_n $(n \geq 3)$; friendship graphs; and fans F_n $(n \geq 3)$. They also prove that S_n $(n \geq 2)$ is E_k-cordial if and only if $n \not\equiv 1$ (mod k) when k is odd or $n \not\equiv 1$ mod $2k$ when k is even and $k \neq 2$.

Hovey [285] has introduced a simultaneous generalization of harmonious and cordial labelings. For any Abelian group A (under addition) and graph $G(V, E)$ he defines G to be A-*cordial* if there is a labeling of V with elements of A so that for all a and b in A when the edge ab is labeled with $f(a) + f(b)$, the number of vertices labeled with a and the number of vertices labeled b differ by at most one and the number of edges labeled with a and the number labeled with b differ by at most one. In the case where A is the cyclic group of order k, the labeling is called k-*cordial*. With this definition we have: $G(V, E)$ is harmonious if and only if G is $|E|$-cordial; G is cordial if and only if G is 2-cordial.

Hovey has obtained the following: caterpillars are k-cordial for all k; all trees are k-cordial for $k = 3, 4$ and 5; odd cycles with pendant edges attached are k-cordial for all k; cycles are k-cordial for all odd k; for k even, C_{2mk+j} is k-cordial when $0 \leq j \leq \frac{k}{2} + 2$ and when $k < j < 2k$; $C_{(2m+1)k}$ is not k-cordial; K_m is 3-cordial; and, for k even, K_{mk} is k-cordial if and only if $m = 1$.

Hovey advances the following conjectures: all trees are k-cordial for all k; all connected graphs are 3-cordial; and C_{2mk+j} is k-cordial if and only if $j \neq k$, where k and j are even and $0 \leq j < 2k$. The last conjecture was verified by Tao [555]. This result combined with those of Hovey show that for all positive integers k the n-cycle is k-cordial with the exception that k is even and $n = 2mk + k$. Tao also proved that the crown with $2mk + j$ vertices is k-cordial unless $j = k$ is even and for $4 \leq n \leq k$ and the wheel W_n is k-cordial unless $k \equiv 5$ (mod 8) and $n = (k + 1)/2$.

In [489] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph G and successively form supersubdivisions (see §2.7 for the definition) that are cordial in that case that every edge in G is replaced by $K_{2,m}$ where m is even. Sethuraman and Selvaraju [490] also prove that the one edge union of k copies of shell graphs $C(n, n-3)$ (see §2.2) is cordial for all $n \geq 4$ and all k and that the one vertex union of any number of copies of $K_{m,n}$ is cordial.

Cairnie and Edwards [148] have determined the computational complexity of cordial and k-cordial labelings. They prove a conjecture of Kirchherr [312] that deciding whether a graph admits a cordial labeling is NP-complete. As a corollary, this result implies that the same problem for k-cordial labelings is NP-complete. They remark that even the restricted problem of deciding whether connected graphs of diameter 2 have a cordial labeling is also NP-complete.

In [154] Chartrand, Lee and Zhang introduced the notion of ramdomly cordial as follows. Let f be a labeling from $V(G)$ to $\{0,1\}$ and for each edge xy define $f^*(xy) = |f(x) - f(y)|$. For $i = 0$ and 1 let $n_i(f)$ denote the number of vertices v with $f(v) = i$ and $m_i(f)$ denote the number of edges e with $f^*(e) = i$. They call a such a labeling f *friendly* if $|n_0(f) - n_1(f)| \leq 1$. A graph G for which every friendly labeling is cordial is called *randomly cordial.* They prove that a connected graph of order $n \geq 2$ is randomly cordial if and only if $n = 3$ and $G = K_3$, or n is even and $G = K_{1,n-1}$.

3.7 k-equitable Labelings

In 1990 Cahit [140] proposed the idea of distributing the vertex and edge labels among $\{0, 1, \ldots, k-1\}$ as evenly as possible to obtain a generalization of graceful labelings as follows. For any graph $G(V, E)$ and any positive integer k, assign vertex labels from $\{0, 1, \ldots, k-1\}$ so that when the edge labels induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the number of edges labeled with i and the number of edges labeled with j differ by at most one. Cahit has called a graph with such an assignment of labels *k-equitable*. Note that $G(V, E)$ is graceful if and only if it is $|E| + 1$-equitable and $G(V, E)$ is cordial if and only if it is 2-equitable. Cahit [139] has shown the following: C_n is 3-equitable if and only if $n \not\equiv 3 \pmod 6$; a triangular snake with n blocks is 3-equitable if and only if n is even; the friendship graph $C_3^{(n)}$ is 3-equitable if and only if n is even; an Eulerian graph with $q \equiv 3 \pmod 6$ edges is not 3-equitable; and all caterpillars are 3-equitable [139]. Cahit [139] further gives a proof that W_n is 3-equitable if and only if $n \not\equiv 3 \pmod 6$ but Youssef [609] proved that W_n is 3-equitable for all $n \geq 4$. Youssef [607] also proved that if G is a k-equitable Eulerian graph with q edges and $k \equiv 2$ or $3 \pmod 4$ then $q \not\equiv k \pmod{2k}$. Cahit conjectures [139] that a triangular cactus with n blocks is 3-equitable if and only if n is even. In [140] Cahit proves that every tree with fewer than five end vertices has a 3-equitable labeling. He conjectures that all trees are k-equitable [141]. In 1999 Speyer and Szaniszló [541] proved Cahit's conjecture for $k = 3$.

In [465] Seoud and Abdel Maqsoud prove: a graph with n vertices and q edges in which every vertex has odd degree is not 3-equitable if $n \equiv 0 \pmod 3$ and $q \equiv 3 \pmod 6$; all fans except $P_2 + \overline{K_1}$ are 3-equitable; all double fans except $P_4 + \overline{K_2}$ are 3-equitable; P_n^2 is 3-equitable for all n except 3; $K_{1,1,n}$ is 3-equitable if and only if $n \equiv 0$ or $2 \pmod 3$; $K_{1,2,n}, n \geq 2$, is 3-equitable if and only if $n \equiv 2 \pmod 3$; $K_{m,n},\ 3 \leq m \leq n$, is 3-equitable if and only if $(m, n) = (4, 4)$; $K_{1,m,n},\ 3 \leq m \leq n$, is 3-equitable if and only if $(m, n) = (3, 4)$.

Szaniszló [554] has proved the following: P_n is k-equitable for all k; K_n is 2-equitable if and only if $n = 1, 2$ or 3; K_n is not k-equitable for $3 \leq k < n$; S_n is k-equitable for all k; $K_{2,n}$ is k-equitable if and only if $n \equiv k - 1 \pmod{k}$, or $n \equiv 0, 1, 2, \ldots, \lfloor k/2 \rfloor - 1 \pmod{k}$, or $n = \lfloor k/2 \rfloor$ and k is odd. She also proves that C_n is k-equitable if and only if k meets all of the following conditions: $n \neq k$; if $k \equiv 2, 3 \pmod{4}$, then $n \neq k - 1$; if $k \equiv 2, 3 \pmod{4}$ then $n \not\equiv k \pmod{2k}$.

Vickrey [569] has determined the k-equitablity of complete multipartite graphs. He shows that for $m \geq 3$ and $k \geq 3$, $K_{m,n}$ is k-equitable if and only if $K_{m,n}$ is one of the following graphs: $K_{4,4}$ for $k = 3$; $K_{3,k-1}$ for all k; or $K_{m,n}$ for $k > mn$. He also shows that when k is less than or equal to the number of edges in the graph and at least 3, the only complete multipartite graphs that are k-equitable are $K_{kn+k-1,2,1}$ and $K_{kn+k-1,1,1}$. Partial results on the k-equitability of $K_{m,n}$ were obtained by Krussel [332].

As a corollary of the result of Cairnie and Edwards [148] on the computational complexity of cordially labeling graphs, it follows that the problem of finding k-equitable labelings of graphs is NP-complete as well.

Seoud and Abdel Maqsoud [464] call a graph *k-balanced* if the vertex labels can be selected from $\{0, 1, \ldots, k-1\}$ so that the number of edges labeled i and the number of edges labeled j induced by the absolute value of the differences of the vertex labels differ by at most 1. They prove that P_n^2 is 3-balanced if and only if $n = 2, 3, 4$ or 6; for $k \geq 4$, P_n^2 is not k-balanced if $k \leq n - 2$ or $n + 1 \leq k \leq 2n - 3$; for $k \geq 4$, P_n^2 is k-balanced if $k \geq 2n - 2$; for $k, m, n \geq 3$, $K_{m,n}$ is k-balanced if and only if $k \geq mn$; for $m \leq n$, $K_{1,m,n}$ is k-balanced if and only if (i) $m = 1, n = 1$ or 2, and $k = 3$; (ii) $m = 1$ and $k = n + 1$ or $n + 2$; or (iii) $k \geq (m + 1)(n + 1)$.

Bloom has used the term k-equitable to describe another kind of labeling (see [585] and [586]). He calls a graph *k-equitable* if the edge labels induced by the absolute value of the difference of the vertex labels have the property that every edge label induced occurs exactly k times. A graph of order n is called *minimally k-equitable* if the vertex labels are 1, 2,..., n and it is k-equitable. Both Bloom and Wojciechowski [585], [586] proved that C_n is minimally k-equitable if and only if k is a proper divisor of n. Barrientos and Hevia [80] proved that if G is k-equitable of size $q = kw$ (in the sense of Bloom) then $\delta(G) \leq w$ and $\Delta(G) \leq 2w$. Barrientos, Dejter and Hevia [79] have shown that forests of even size are 2-equitable. They also prove that for $k = 3$ or $k = 4$ a forest of size kw is k-equitable if and only if its maximum degree is at most $2w$ and that if 3 divides $mn+1$, then the double star $S_{m,n}$ is 3-equitable if and only if $q/3 \leq m \leq \lfloor (q-1)/2 \rfloor$. ($S_{m,n}$ is K_2 with m pendant edges attached at one end and n pendant edges attached at the other end.) They discuss the k-equitability of forests for $k \geq 5$ and characterize all caterpillars of diameter 2 that are k-equitable for all possible values of k. Acharya and Bhat-Nayak [14] have shown that coronas of the form $C_{2n} \odot K_1$ are minimally 4-equitable. In [73] Barrientos proves that the one-point union of a cycle and a path (dragons) and the disjoint union of a cycle and a path are k-equitable for all k that divide the size of the graph. Barrientos and Havia [80] have shown the following: $C_n \times K_2$ is 2-equitable when n is even; books B_n $(n \geq 3)$ are 2-equitable when n is odd; the vertex union of k-equitable graphs is k-equitable; wheels W_n are 2-equitable when $n \not\equiv 3 \pmod{4}$. They conjecture that W_n is 2-equitable when $n \equiv 3 \pmod 4$ except when $n = 3$. Their 2-equitable labelings of $C_n \times K_2$ and the n-cube utilized graceful labelings of those graphs.

Bhat-Nayak and M. Acharya [97] have proved the following: the crowns $C_{2n} \odot K_1$ are minimally 2-equitable, minimally $2n$-equitable, minimally 4-equitable, and minimally n-equitable; the crowns $C_{3n} \odot K_1$ are minimally 3-equitable, minimally $3n$-equitable, minimally n-equitable, and minimally 6-equitable; the crowns $C_{5n} \odot K_1$ are minimally 5-equitable, minimally $5n$-equitable, minimally n-equitable, and minimally 10-equitable; the crowns $C_{2n+1} \odot K_1$ are minimally $(2n+1)$-equitable; and that the graphs P_{kn+1} are k-equitable.

In [75] Barrientos calls a k-equitable labeling *optimal* if the vertex labels are consecutive integers and *complete* if the induced edge labels are $1, 2, \ldots, w$ where w is the number of distinct edge labels. Note that a graceful labeling is a complete 1-equitable labeling. Barrientos proves that $C_m \odot nK_1$ (that is, an m-cycle with n pendant edges attached at each vertex) is optimal 2-equitable when m is even, $C_3 \odot nK_1$ is complete 2-equitable when n is odd and that $C_3 \odot nK_1$ is complete 3-equitable for all n. He also shows that $C_n \odot K_1$ is k-equitable for every proper divisor k of the size $2n$. Barrientos and Havia [80] have shown that the n-cube ($n \geq 2$) has a complete 2-equitable labeling and that $K_{m,n}$ has a complete 2-equitable labeling when m or n is even. They conjecture that every tree of even size has an optimal 2-equitable labeling.

3.8 Hamming-graceful Labelings

Mollard, Payan and Shixin [413] introduced a generalization of graceful graphs called Hamming-graceful. A graph $G = (V, E)$ is called *Hamming-graceful* if there exists an injective labeling g from V to the set of binary $|E|$-tuples such that $\{d(g(v), g(u)) | uv \in E\} = \{1, 2, \ldots, |E|\}$ where d is the Hamming distance. Shixin and Yu [511] have shown that all graceful graphs are Hamming-graceful; all trees are Hamming-graceful; C_n is Hamming-graceful if and only if $n \equiv 0$ or $3 \pmod 4$; if K_n is Hamming-graceful, then n has the form k^2 or $k^2 + 2$; K_n is Hamming-graceful for $n = 2, 3, 4, 6, 9, 11, 16$, and 18. They conjecture that K_n is Hamming-graceful for n of the form k^2 and $k^2 + 2$ for $k \geq 5$.

4 Variations of Harmonious Labelings

4.1 Sequential and Strongly c-harmonious Labelings

Chang, Hsu and Rogers [152] and Grace [246], [247] have investigated subclasses of harmonious graphs. Chang et al. define an injective labeling f of a graph G with q vertices to be *strongly c-harmonious* if the vertex labels are from $\{0, 1, \ldots, q-1\}$ and the edge labels induced by $f(x) + f(y)$ for each edge xy are $c, \ldots, c+q-1$. Grace called such a labeling *sequential*. In the case of a tree, Chang et al. modify the definition to permit exactly one vertex label to be assigned to two vertices while Grace allows the vertex labels to range from 0 to q with no vertex label used twice. By taking the edge labels of a sequentially labeled graph with q edges modulo q, we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. Grace [247] proved that caterpillars, caterpillars with a pendant edge, odd cycles with zero or more pendant edges, trees with α-labelings, wheels W_{2n+1}, and P_n^2 are sequential. Liu and Zhang [387] finished off the crowns $C_{2n} \odot K_1$. (The case $C_{2n+1} \odot K_1$ was a special case of Grace's results. Liu [384] proved crowns are harmonious.) Bu [123] also proved that crowns are sequential as are all even cycles with m pendant edges attached at each vertex. Figueroa-Centeno, Ichishima and Muntaner-Batle [208] proved that all cycles with m pendant edges attached at each vertex are sequential.

Singh has proved the following: $C_n \odot K_2$ is sequential for all odd $n > 1$ [517]; $C_n \odot P_3$ is sequential for all odd n [518]; $K_2 \odot C_n$ (each vertex of the cycle is joined by edges to the end points of a copy of K_2) is sequential for all odd n [518]; helms H_n are sequential when n is even [518]; and $K_{1,n} + K_2, K_{1,n} + \overline{K}_2$, and ladders are sequential [519]. Both Grace [246] and Reid (see [232]) have found sequential labelings for the books B_{2n}. Jungreis and Reid [301] have shown the following graphs are sequential: $P_m \times P_n \;\; (m, n) \neq (2, 2)$; $C_{4m} \times P_n \;\; (m, n) \neq (1, 2)$; $C_{4m+2} \times P_{2n}$; $C_{2m+1} \times P_n$; and, $C_4 \times C_{2n} \;\; (n > 1)$. The graphs $C_{4m+2} \times C_{2n+1}$ and $C_{2m+1} \times C_{2n+1}$ fail to satisfy a necessary parity condition given by Graham and Sloane [249]. The remaining cases of $C_m \times P_n$ and $C_m \times C_n$ are open. Gallian, Prout and Winters [233] proved that all graphs $C_n \times P_2$ with a vertex or edge deleted are sequential.

Gnanajothi [240, pp. 68–78] has shown the following graphs are sequential: $K_{1,m,n}$; mC_n, the disjoint union of m copies of C_n, if and only if m and n are odd; books with triangular pages or pentagonal pages; and books of the form B_{4n+1}, thereby answering a question and proving a conjecture of Gallian and Jungreis [232]. Sun [545] has also proved that B_n is sequential if and only if $n \not\equiv 3 \pmod 4$.

Yuan and Zhu [614] have shown that mC_n is sequential when m and n are odd. Although Graham and Sloane [249] proved that the Möbius ladder M_3 is not harmonious, Gallian [228] established that all other Möbius ladders are sequential (see §2.3 for the definition). Chung, Hsu and Rogers [152] have shown that $K_{m,n} + K_1$, which includes $S_m + K_1$, is sequential. Seoud and Youssef [473] proved that if G is sequential and has the same number of edges as vertices, then $G + \overline{K_n}$ is sequential for all n.

Singh and Varkey [523] call a graph with q edges *odd sequential* if the vertices can be labeled with distinct integers from the set $\{0, 1, 2, \ldots, q\}$ or, in the case of a tree from the set $\{0, 1, 2, \ldots, 2q - 1\}$, so that the edge labels induced by addition of the labels of the endpoints take on the values $\{1, 3, 5, \ldots, 2q - 1\}$. They prove that combs, grids, stars and rooted trees of level 2 are odd sequential while odd cycles are not. Singh and Varkey call a graph G *bisequential* if both G and its line graph have a sequential labeling. They prove paths and cycles are bisequential.

Among the strongly 1-harmonious (also called *strongly harmonious*) are: fans F_n with $n \geq 2$ [152]; wheels W_n with $n \not\equiv 2 \pmod 3$ [152]; $K_{m,n} + K_1$ [152]; French windmills $K_4^{(t)}$ [287], [304]; the friendship graphs $C_3^{(n)}$ if and only if $n \equiv 0$ or $1 \pmod 4$ [287], [304];

$C_{4k}^{(t)}$ [546]; and helms [443].

Seoud, Diab and Elsakhawi [467] have shown that the following graphs are strongly harmonious: $K_{m,n}$ with an edge joining two vertices in the same partite set; $K_{1,m,n}$; the composition $P_n[P_2]$ (see §2.3 for definition); $B(3,2,m)$ and $B(4,3,m)$ for all m (see §2.4 for notation); P_n^2 $(n \geq 3)$; and P_n^3 $(n \geq 3)$. Seoud et al. [467] have also proved: B_{2n} is strongly $2n$-harmonious; P_n is strongly $\lfloor n/2 \rfloor$-harmonious; ladders L_{2k+1} are strongly $(k+1)$-harmonious; and that if G is strongly c-harmonious and has an equal number of vertices and edges, then $G + \overline{K_n}$ is also strongly c-harmonious.

Sethuraman and Selvaraju [493] have proved that the graph obtained by adjoining two complete bipartite graphs at one edge is graceful and strongly harmonious. They ask whether these results extend to any number of complete bipartite graphs.

Acharya and Hegde [18] have generalized sequential labelings as follows. Let G be a graph with q edges and let k and d be positive integers. A labeling f of G is said to be (k,d)-*arithmetic* if the vertex labels are distinct nonnegative integers and the edge labels induced by $f(x)+f(y)$ for each edge xy are $k, k+d, k+2d, \ldots, k+(q-1)d$. They obtained a number of necessary conditions for various kinds of graphs to have a (k,d)-arithmetic labeling. The case where $k = 1$ and $d = 1$ was called *additively graceful* by Hegde [269]. Hegde [269] showed: K_n is additively graceful if and only if $n = 2, 3$ or 4; every additively graceful graph except K_2 or $K_{1,2}$ contains a triangle; and a unicyclic graph is additively graceful if and only if it is a 3-cycle or a 3-cycle with a single pendant edge attached. Jinnah and Singh [299] noted that P_n^2 is additively graceful. Hegde [270] proved that if G is strongly k-indexable, then G and $G + \overline{K_n}$ are (kd,d)-arithmetic; Bu and Shi [129] proved the conjecture of Acharya and Hegde [18] that K_n is not (k,d)-arithmetic for $n \geq 5$. They also proved that $K_{m,n}$ is (k,d)-arithmetic when k is not of the form id for $1 \leq i \leq n-1$. Acharya and Hegde [18] showed that for all $d \geq 1$ and all $r \geq 0$ the following: $K_{m,n,1}$ is $(d+2r,d)$-arithmetic; C_{4t+1} is $(2dt+2r,d)$-arithmetic; C_{4t+2} is not (k,d)-arithmetic for any values of k and d; C_{4t+3} is $((2t+1)d+2r,d)$-arithmetic; W_{4t+2} is $(2dt+2r,d)$-arithmetic; and W_{4t} is $((2t+1)d+2r,d)$-arithmetic. They conjecture that C_{4t+1} is $(2dt+2r,d)$-arithmetic for some r and that C_{4t+3} is $((2dt+d+2r,d)$-arithmetic for some r. Hegde and Shetty [277] proved the following: the generalized web $W(t,n)$ (see §2.2) is $((n-1)d/2,d)$-arithmetic and $(3n-1)d/2$-arithmetic for odd n; the join of the generalized web $W(t,n)$ with the center removed and $\overline{K_p}$ where n is odd is $((n-1)d/2,d)$-arithmetic.

Yu [612] proved that a necessary condition for C_{4t+1} to be (k,d)-arithmetic is that $k = 2dt + r$ for some $r \geq 0$ and a necessary condition for C_{4t+3} to be (k,d)-arithmetic is that $k = (2t+1)d + 2r$ for some $r \geq 0$. These conditions were conjectured by Acharya and Hegde [18]. Singh proved that the graph obtained by subdividing every edge of the ladder L_n is $(5,2)$-arithmetic [516] and that the ladder L_n is $(n,1)$-arithmetic [520]. He also proves that $P_m \times C_n$ is $((n-1)/2,1)$-arithmetic when n is odd [520].

A graph is called *arithmetic* if it is (k,d)-arithmetic for some k and d. Singh and Vilfred [524] showed that various classes of trees are arithmetic. Singh [520] has proved that the union of an arithmetic graph and an arithmetic bipartite graph is arithmetic.

He conjectures that the union of arithmetic graphs is arithmetic. He provides an example to show that the converse is not true.

Acharya and Hegde [18] introduced a stronger form of sequential labeling by calling a graph with p vertices and q edges *strongly k-indexable* if there is an injective function from V to $\{0, 1, 2, \ldots, p-1\}$ such that the set of edge labels induced by adding the vertex labels is $\{k, k+1, k+2, \ldots, k+q-1\}$. Strongly 1-indexable graphs are simply called *strongly indexable*. Notice that for trees and unicyclic graphs the notions of sequential labelings and strongly k-indexable labelings coincide. Acharaya and Hegde prove that the only nontrivial regular graphs that are strongly indexable are K_2, K_3 and $K_2 \times K_3$ and that every strongly indexable graph has exactly one nontrivial component that is either a star or has a triangle. Acharya and Hegde [18] call a graph with p vertices *indexable* if there is an injective labeling of the vertices with labels from $\{0, 1, 2, \ldots, p-1\}$ such that the edge labels induced by addition of the vertex labels are distinct. They conjecture that all unicyclic graphs are indexable. This conjecture was proved by Arumugam and Germina [32] who also proved that all trees are indexable. Bu and Shi [130] also proved that all trees are indexable and that all uncyclic graphs with the cycle C_3 are indexable. Hegde [270] has shown the following: every graph can be embedded as an induced subgraph of an indexable graph; if a connected graph with p vertices and q edges ($q \geq 2$) is (k, d)-indexable then $d \leq 2$; $P_m \times P_n$ is indexable for all m and n; if G is a connected $(1, 2)$-indexable graph, then G is a tree; the minimum degree of any $(k, 1)$-indexable graph with at least two vertices is at most 3; a caterpillar with partite sets of orders a and b is strongly $(1, 2)$-indexable if and only if $|a-b| \leq 1$; in a connected strongly k-indexable graph with p vertices and q edges, $k \leq p-1$, and if a graph with p vertices and q edges is (k, d)-indexable, then $q \leq (2p-3-k+d)/d$. As a corollary of the latter, it follows that K_n ($n \geq 4$) and wheels are not (k, d)-indexable. Hegde and Shetty [277] proved that for n odd the generalized web graph $W(t, n)$ with the center removed is strongly $(n-1)/2$-indexable.

Let T be a tree with adjacent edges u_0 and v_0 and suppose that there are two pendant edges u and v of T so that the lengths of the paths $u_0 - u$ and $v_0 - v$ are eqaul. The tree obtained from T by deleting the edge u_0v_0 and joining u and v is called an *elementary parallel transformation* of T. Any tree that can be reduced to a path by a sequence of elementary parallel transformations is called a T_p-tree. Hedge and Shetty [277] have shown that every T_p-tree with q edges and every tree obtained by subdividing every edge of a T_p-tree exactly once is $(k+(q-1)d, d)$-arithmetic for all k and d. Hegde and Shetty [279] define a *level joined planar grid* as follows. Let u be a vertex of $P_m \times P_n$ of degree 2. For every every pair of distinct vertices v and w which do not have degree 4, introduce an edge between v and w provided that the distance from u to v equals the distance from u to w. They prove that every level joined planar grid is strongly indexable.

Section 5.2 of this survey includes a discussion of a labeling method called super edge-magic. In 2002 Hegde and Shetty [279] showed that a graph has a strongly k-indexable labeling if and only if it has a super edge-magic labeling.

4.2 Elegant Labelings

An *elegant labeling* f of a graph G with q edges is an injective function from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that when each edge xy is assigned the label $f(x)+f(y)$ (mod $q+1$) the resulting edge labels are distinct and nonzero. This notion was introduced by Chang, Hsu and Rogers in 1981 [152]. Note that in contrast to the definition of a harmonious labeling, it is not necessary to make an exception for trees. While the cycle C_n is harmonious if and only if n is odd, Chang et al. [152] proved that C_n is elegant when $n \equiv 0$ or 3 (mod 4) and not elegant when $n \equiv 1$ (mod 4). Chang et al. further showed that all fans are elegant and the paths P_n are elegant for $n \not\equiv 0$ (mod 4). Cahit [137] then showed that P_4 is the only path that is not elegant. Balakrishnan, Selvam and Yegnanarayanan [70] have proved numerous graphs are elegant. Among them are $K_{m,n}$ and the mth-subdivision graph of $K_{1,2n}$. They prove that the bistar $B_{n,n}$ (K_2 with n pendant edges at each endpoint) is elegant if and only if n is even. They also prove that every simple graph is a subgraph of an elegant graph and that several families of graphs are not elegant. Deb and Limaye [169] have shown that triangular snakes are elegant if and only if the number of triangles is not equal to 3 (mod 4). In the case where the number of triangles is 3 (mod 4) they show the triangular snakes satisfy a weaker condition they call *semi-elegant* whereby the edge label 0 is permitted. In [170] Deb and Limaye define a graph G with q edges to be *near-elegant* if there is an injective function f from the vertices of G to the set $\{0, 1, \ldots, q\}$ such that when each edge xy is assigned the label $f(x) + f(y)$ (mod $(q + 1)$) the resulting edge labels are distinct and not equal to q. Thus, in a near-elegent labeling, instead of 0 being the missing value in the edge labels, q is the missing value. Deb and Limaye show that triangular snakes where the number of triangles is 3 (mod 4) are near-elegant. For any positive integers $\alpha \leq \beta \leq \gamma$ where β is at least 2 the *theta graph* $\theta_{\alpha,\beta,\gamma}$ consists of three edge disjoint paths of lengths α, β and γ having the same end points. Deb and Limaye [170] provide elegant and near-elegnat labelings for some theta graphs where $\alpha = 1, 2$ or 3. Seoud and Elsakhawi [468] have proved that the following graphs are elegant: $K_{1,m,n}; K_{1,1,m,n}; K_2 + \overline{K_m}; K_3 + \overline{K_m}$; and $K_{m,n}$ with an edge joining two vertices of the same partite set.

Sethuraman and Elumalai [484] have proved that for every graph G with p vertices and q edges the graph $G + K_1 + \overline{K_m}$ is graceful when $m \geq 2^p - p - q$. As a corollary they deduce that every graph is a vertex induced subgraph of a elegant graph. In [489] Sethuraman and Selvaraju present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions that have a strong form of elegant labeling.

Sethuraman and Elumalai [483] define a graph H to be a $K_{1,m}$-star extension of a graph G with p vertices and q edges at a vertex v of G where $m > p-1-deg(v)$ if H is obtained from G by merging the center of the star $K_{1,m}$ with v and merging $p-1-deg(v)$ pendent vertices of $K_{1,m}$ with the $p - 1 - deg(v)$ nonadjacent vertices of v in G. They prove that for every graph G with p vertices and q edges and every vertex v of G and every $m \geq 2^{p-1} - 1 - q$ there is a $K_{1,m}$-star extension of G that is both graceful and harmonious. In the case where $m \geq 2^{p-1} - q$ they show that G has a $K_{1,m}$-star extension that is elegant.

Sethuraman and Selvaraju [494] have shown that certain cases of the union of any number of copies of K_4 with one or more edges deleted and one edge in common are elegant.

Gallian extended the notion of harmoniousness to arbitrary finite Abelian groups as follows. Let G be a graph with q edges and H a finite Abelian group (under addition) of order q. Define G to be *H-harmonious* if there is an injection f from the vertices of G to H such that when each edge xy is assigned the label $f(x) + f(y)$ the resulting edge labels are distinct. When G is a tree, one label may be used on exactly two vertices. Beals, Gallian, Headley and Jungreis [82] have shown that if H is a finite Abelian group of order $n > 1$ then C_n is H-harmonious if and only if H has a non-cyclic or trivial Sylow 2-subgroup and H is not of the form $Z_2 \times Z_2 \times \cdots \times Z_2$. Thus, for example, C_{12} is not Z_{12}-harmonious but is $(Z_2 \times Z_2 \times Z_3)$-harmonious. Analogously, the notion of an elegant graph can be extended to arbitrary finite Abelian groups. Let G be a graph with q edges and H a finite Abelian group (under addition) with $q+1$ elements. We say G is *H-elegant* if there is an injection f from the vertices of G to H such that when each edge xy is assigned the label $f(x) + f(y)$ the resulting set of edge labels is the non-identity elements of H. Beals et al. [82] proved that if H is a finite Abelian group of order n with $n \neq 1$ and $n \neq 3$, then C_{n-1} is H-elegant using only the non-identity elements of H as vertex labels if and only if H has either a non-cyclic or trivial Sylow 2-subgroup. This result completed a partial characterization of elegant cycles given by Chang, Hsu and Rogers [152] by showing that C_n is elegant when $n \equiv 2 \pmod 4$. Mollard and Payan [412] also proved that C_n is elegant when $n \equiv 2 \pmod 4$ and gave another proof that P_n is elegant when $n \neq 4$.

For a graph $G(V, E)$ and an Abelian group H Valentin [566] defines a *polychrome* labeling of G by H to be a bijection f from V to H such that the edge labels induced by $f(uv) = f(v) + f(u)$ are distinct. Valentin investigates the existence of polychrome labelings for paths and cycles for various Abelian groups.

4.3 Felicitous Labelings

Another generalization of harmonious labelings are felicitous labelings. An injective function f from the vertices of a graph G with q edges to the set $\{0, 1, \ldots, q\}$ is called *felicitous* if the edge labels induced by $f(x) + f(y) \pmod q$ for each edge xy are distinct. This definition first appeared in a paper by Lee, Schmeichel and Shee in [352] and is attributed to E. Choo. Balakrishnan and Kumar [67] proved the conjecture of Lee, Schmeichel and Shee [352] that every graph is a subgraph of a felicitous graph by showing the stronger result that every graph is a subgraph of a sequential graph. Among the graphs known to be felicitous are: C_n except when $n \equiv 2 \pmod 4$ [352]; $K_{m,n}$ when $m, n > 1$ [352]; $P_2 \cup C_{2n+1}$ [352]; $P_2 \cup C_{2n}$ [559]; $P_3 \cup C_{2n+1}$ [352]; $S_m \cup C_{2n+1}$ [352]; K_n if and only if $n \leq 4$ [485]; $P_n + \overline{K_m}$ [485]; the friendship graph $C_3^{(n)}$ for n odd [352]; $P_n \cup C_3$ [501]; $P_n \cup C_{n+3}$ [559]; and the one-point union of an odd cycle and a caterpillar

[501]. Shee [497] conjectured that $P_m \cup C_n$ is felicitous when $n > 2$ and $m > 3$. Lee, Schmeichel and Shee [352] ask for which m and n is the one-point union of n copies of C_m felicitous. They showed that the case where mn is twice an odd integer is not felicitous. In contrast to the situation for felicitous labelings, we remark that C_{4k} and $K_{m,n}$ where $m, n > 1$ are not harmonious and the one-point union of an odd cycle and a caterpillar is not always harmonious. Lee, Schmeichel and Shee [352] conjecture that the n-cube is felicitous. This conjecture was proved by Figueroa-Centeno and Ichishima in 2001 [204].

Balakrishnan, Selvam and Yegnanarayanan [69] obtained numerous results on felicitous labelings. The *wreath product*, $G * H$, of graphs G and H has vertex set $V(G) \times V(H)$ and (g_1, h_1) is adjacent to (g_2, h_2) whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. They define $H_{n,n}$ as the graph with vertex set $\{u_1, \ldots, u_n; v_1, \ldots, v_n\}$ and edge set $\{u_iv_j | \ 1 \leq i \leq j \leq n\}$. They let $\langle K_{1,n} : m\rangle$ denote the graph obtained by taking m disjoint copies of $K_{1,n}$, and joining a new vertex to the centers of the m copies of $K_{1,n}$. They prove the following are felicitous: $H_{n,n}; P_n * \overline{K_2}; \langle K_{1,m} : m\rangle; \langle K_{1,2} : m\rangle$ when $m \not\equiv 0 \pmod 3$ or $m \equiv 3 \pmod 6$ or $m \equiv 6 \pmod{12}$; $\langle K_{1,2n} : m\rangle$ for all m and $n \geq 2$; $\langle K_{1,2t+1} : 2n+1\rangle$ when $n \geq t$; P_n^k when $k = n-1$ and $n \not\equiv 2 \pmod 4$ or $k = 2t$ and $n \geq 3$ and $k < n-1$; the join of a star and $\overline{K_n}$; and graphs obtained by joining two end vertices or two central vertices of stars with an edge. Yegnanarayanan [603] conjectures that the graphs obtained from an even cycle by attaching n new vertices to each vertex of the cycle is felicitous. This conjecture was verified by Figueroa-Centeno, Ichishima and Muntaner-Batle in [208]. In [489] Sethuraman and Selvaraju [494] have shown that certain cases of the union of any number of copies of K_4 with 3 edges deleted and one edge in common are felicitous. Sethuraman and Selvaraju [489] present an algorithm that permits one to start with any non-trivial connected graph and successively form supersubdivisions (see §2.7) that have a felcitious labeling.

Figueroa-Centeno, Ichishima and Muntaner-Batle [209] define a felicitous graph to be *strongly felicitous* if there exists an integer k so that for every edge uv $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$. For a graph with p vertices and q edges with $q \geq p-1$ they show that G is strongly felicitous if and only if G has an α-valuation (see §3.1). They also show that for graphs G_1 and G_2 with strongly felicitous labelings f_1 and f_2 the graph obtained from G_1 and G_2 by identifying the vertices u and v such that $f_1(u) = 0 = f_2(v)$ is strongly felicitous and that the one-point union of two copies of C_m where $m \geq 4$ and m is even is strongly felicitous. As a corollary they have that the one-point union n copies of C_m where m is even and at least 4 and $n \equiv 2 \pmod 4$ is felicitous. They conjecture that the one-point union of n copies of C_m is felicitous if and only if $mn \equiv 0, 1$ or $3 \pmod 4$. In [212] Figueroa-Centeno, Ichishima and Muntaner-Batle prove that $2C_n$ is strongly felicitous if and only if n is even and at least 4. They conjecture [212] that mC_n is felicitous if and only if $mn \not\equiv 2 \pmod 4$ and that $C_m \cup C_n$ is felicitous if and only if $m + n \not\equiv 2 \pmod 4$.

Chang, Hsu and Rogers [152] have given a sequential counterpart to felecitous labelings. They call a graph *strongly c-elegant* if the vertex labels are from $\{0, 1, \ldots, q\}$

and the edge labels induced by addition are $\{c, c+1, \ldots, c+q-1\}$. (A strongly 1-elegant labeling has also been called a *consecutive* labeling.) Notice that every strongly c-elegant graph is felicitous and that strongly c-elegant is the same as $(c,1)$-arithmetic in the case where the vertex labels are from $\{0, 1, \ldots, q\}$. Results on strongly c-elegant graphs are meager. Chang et al. [152] have shown: K_n is strongly 1-elegant if and only if $n = 2, 3, 4$; C_n is strongly 1-elegant if and only if $n = 3$; and a bipartite graph is strongly 1-elegant if and only if it is a star. Shee [498] has proved that $K_{m,n}$ is strongly c-elegant for a particular value of c and obtained several more specialized results pertaining to graphs formed from complete bipartite graphs.

Seoud and Elsakhawi [469] have shown: $K_{m,n}$ $(m \leq n)$ with an edge joining two vertices of the same partite set is strongly c-elegant for $c = 1, 3, 5, \ldots, 2n+2$; $K_{1,m,n}$ is strongly c-elegant for $c = 1, 3, 5, \ldots, 2m$ when $m = n$, and for $c = 1, 3, 5, \ldots, m+n+1$ when $m \neq n$; $K_{1,1,m,m}$ is strongly c-elegant for $c = 1, 3, 5, \ldots, 2m+1$; $P_n + \overline{K_m}$ is strongly $\lfloor n/2 \rfloor$-elegant; $C_m + \overline{K_n}$ is strongly c-elegant for odd m and all n for $c = (m-1)/2, (m-1)/2+2, \ldots, 2m$ when $(m-1)/2$ is even and for $c = (m-1)/2, (m-1)/2+2, \ldots, 2m-(m-1)/2$ when $(m-1)/2$ is odd; ladders L_{2k+1} $(k > 1)$ are strongly $(k+1)$-elegant; and $B(3,2,m)$ and $B(4,3,m)$ (see §2.4 for notation) are strongly 1-elegant and strongly 3-elegant for all m; the composition $P_n[P_2]$ (see §2.3) is strongly c-elegant for $1, 3, 5, \ldots, 5n-6$ when n is odd and for $c = 1, 3, 5, \ldots, 5n-5$ when n is even; P_n is strongly $\lfloor n/2 \rfloor$-elegant; P_n^2 is strongly c-elegant for $c = 1, 3, 5, \ldots, q$ where q is the number of edges of P_n^2; and P_n^3 $(n > 3)$ is strongly c-elegant for $c = 1, 3, 5, \ldots, 6k-1$ when $n = 4k$, $c = 1, 3, 5, \ldots, 6k+1$ when $n = 4k+1$, $c = 1, 3, 5, \ldots, 6k+3$ when $n = 4k+2$, $c = 1, 3, 5, \ldots, 6k+5$ when $n = 4k+3$.

5 Magic-type Labelings

5.1 Magic Labelings

Motivated by the notion of magic squares in number theory, magic labelings were introduced by Sedláček [460] in 1963. Responding to a problem raised by Sedláček, Stewart ([543] and [544]) studied various ways to label the edges of a graph in the mid 60s. Stewart calls a connected graph *semi-magic* if there is a labeling of the edges with integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v. (Berge [87] used the term "regularisable" for this notion.) A semi-magic labeling where the edges are labeled with distinct positive integers is called a *magic* labeling. Stewart calls a magic labeling *supermagic* if the set of edge labels consists of consecutive positive integers. The classic concept of an $n \times n$ magic square in number theory corresponds to a supermagic labeling of $K_{n,n}$. Stewart [543] proved the following: K_n is magic for $n = 2$ and all $n \geq 5$; $K_{n,n}$ is magic for all $n \geq 3$; fans F_n are magic if and only if n is odd and $n \geq 3$; wheels W_n are magic for $n \geq 4$; W_n with one spoke deleted is magic for $n = 4$ and for $n \geq 6$. Stewart [543] also proved that $K_{m,n}$ is semi-magic if and only if $m = n$. In [544] Stewart proved that K_n is supermagic for $n \geq 5$ if and only if $n > 5$ and $n \not\equiv 0 \pmod 4$. Sedláček [461] showed that Möbius ladders M_n (see §2.3 for the definition) are supermagic when $n \geq 3$ and n is odd and that $C_n \times P_2$ is magic, but not supermagic, when $n \geq 4$ and n is even. Shiu, Lam and Lee [507] have proved: the composition of C_m and $\overline{K}_n$ is supermagic when $m \geq 3$ and $n \geq 2$; the complete m-partite graph $K_{n,n,\ldots,n}$ is supermagic when $n \geq 3,\ m > 5$ and $m \not\equiv 0 \pmod 4$; and if G is an r-regular supermagic graph, then so is the composition of G and $\overline{K}_n$ for $n \geq 3$. Ho and Lee [280] showed that the composition of K_k and the null graph with n vertices is supermagic for $k = 3$ or 5 and $n = 2$ or n odd. Bača, Holländer and Lih [56] have found two families of 4-regular supermagic graphs. Shiu, Lam and Cheng [504] prove that for $n \geq 2, mK_{n,n}$ is supermagic if and only if n is even or both m and n are odd. Ivančo [293] gives a characterization of all supermagic regular complete multipartite graphs. He proves that Q_n is supermagic if and only if $n = 1$ or n is even and greater than 2 and that $C_n \times C_n$ and $C_{2m} \times C_{2n}$ are supermagic. He conjectures that $C_m \times C_n$ is supermagic for all m and n. Trenklér [561] has proved that a connected magic graph with p vertices and q edges other than P_2 exits if and only if $5p/4 < q \leq p(p-1)/2$. In [547] Sun, Guan and Lee give an efficient algorithm for finding a magic labeling of a graph.

Trenklér [562] extended the definition of supermagic graphs to include hypergraphs and proved that the complete k-uniform n-partite hypergraph is supermagic if $n \neq 2$ or 6 and $k \geq 2$ (see also [563]).

Sedláček [461] also proves that graphs obtained from an odd cycle $u_1, u_2, \ldots, u_m, u_{m+1}, v_m, \ldots, v_1$ ($m \geq 2$) by joining each u_i to v_i and v_{i+1} and u_1 to v_{m+1}, u_m to v_1 and v_1 to v_{m+1} are magic. Trenklér and Vetchý [564] have shown that if G has order at least 5 then G^i is magic for all $i \geq 3$ and G^2 is magic if and only if G is not P_5 and G does not have a 1-factor whose every edge is incident with an end-vertex of G. Seoud and Abdel Maqsoud [465] proved that $K_{1,m,n}$ is magic for all m and n and that P_n^2 is magic for all n. Characterizations of regular magic graphs were given by Dood [183] and necessary and sufficient conditions for a graph to be magic were given in [295], [296] and [177]. Some sufficient conditions for a graph to be magic are given in [181], [560] and [418]. The notion of magic graphs was generalized in [182] and [456].

In 1976 Sedláček [461] defined a connected graph with at least two edges to be *pseudo-magic* if there exists a real-valued function on the edges with the property that distinct edges have distinct values and the sum of the values assigned to all the edges incident to any vertex is the same for all vertices. Sedláček proved that when $n \geq 4$ and n is even, the Möbius ladder M_n is not pseudo-magic and when $m \geq 3$ and m is odd, $C_m \times P_2$ is not pseudo-magic.

Kong, Lee and Sun [322] used the "magic labeling" for a labeling of the edges with nonnegative integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v. In particular, the edge labels need not be distinct. They let $M(G)$ denote the set of all such labelings of G. For any L in $M(G)$, they let $s(L) = \max\{L(e): e \text{ in } E\}$ and define the *magic strength* of G as $m(G) = \min\{s(L): L \text{ in } M(G)\}$. To distinguish these notions from ones with the same names and notation introduced in the next section for labelings from the set of vertices and edges we call the Kong, Lee and Sun version the *edge magic strength* and use $em(G)$ for $\min\{s(L)\colon L$ in $M(G)\}$ instead of $m(G)$. Kong, Lee and Sun [322] use $DS(k)$ to denote the graph obtained by taking two copies of $K_{1,k}$ and connecting the k pairs of corresponding leafs. They show: for $k > 1$, $em(DS(k)) = k-1$; $em(P_k+K_1)$ is 1 for $k = 1$ or 2, k if k is even and greater than 2 and 0 if k is odd and greater than 1; for $k \geq 3$, $em(W(k)) = k/2$ if k is even and $em(W(k)) = (k-1)/2$ if k is odd; $em(P_2 \times P_2) = 1, em(P_2 \times P_n) = 2$ if $n > 3$, $em(P_m \times P_n) = 3$ if m or n is even and greater than 2; $em(C_3^{(n)}) = 1$ if $n = 1$ (Dutch windmill – see §2.4) and $em(C_3^{(n)}) = 2n-1$ if $n > 1$. They also prove that if G and H are magic graphs then $G \times H$ is magic and $em(G \times H) = \max\{em(G), em(H)\}$ and that every connected graph is an induced subgraph of a magic graph (see also [196] and [206]). They conjecture that almost all connected graphs are not magic. In [350] Lee, Saba and Sun show that the edge magic strength of P_n^k is 0 when k and n are both odd. Sun and Lee [548] show that the Cartesian, conjunctive, normal, lexicographic, and disjunctive products of two magic graphs are magic and the sum of two magic graphs is magic. They also determine the magic strengths of the products and sum in terms of the magic strengths of the components graphs.

S. M. Lee and colleagues [367] and [342] call a graph G *k-magic* if there is a labeling from the edges of G to the set $\{1, 2, \ldots, k-1\}$ such that for each vertex v of G the sum of all edges incident with v is a constant independent of v. The set of all k for which G is k-magic is denoted by IM(G) and called the *integer-magic spectrum* of G. In [367] Lee and Wong investigate the integer-magic spectrum of powers of paths. They prove: IM(P_4^2) is $\{4, 6, 8, 10, \ldots\}$; for $n > 5$, IM(P_n^2) is the set of all positive integers except 2; for all odd $d > 1$, IM(P_{2d}^d) is the set of all positive integers except 1; IM(P_4^3) is the set of all positive integers; for all odd $n \geq 5$, IM(P_n^3) is the set of all positive integers except 1 and 2; and for all even $n \geq 6$, IM(P_n^3) is the set of all positive integers except 2. They conjecture that for $k > 3$, IM(P_n^k) is the set of all positive integers when $n = k+1$; the set of all positive integers except 1 and 2 when n and k are odd and $n \geq k$; the set of all positive integers except 1 and 2 when n and k are even and $k \geq n/2$; the set of all positive integers except 2 when n is even and k is odd and $n \geq k$; and the set of all positive integers except 2 when n and k are even and $k \leq n/2$.

In [342] Lee et al. investigated the integer-magic spectrum of trees obtained by joining the centers of two disjoint stars $K_{1,m}$ and $K_{1,n}$ with an edge. They denote these graphs by ST(m, n). Among their results are: IM(ST(m, n)) is the empty set when $|m-n| = 1$; IM(ST$(2m, 2m)$) is the set of all positive integers; IM(ST$(2m+1, 2m+1)$) $(m \geq 1)$ is the set of all positive integers except 2; IM(W_{2n+1}) is the set of all positive integers; IM(W_{2n}) $(n > 1)$ is the set of all positive integers except 2; IM$(C_{2n} \cdot K_1)$ is the set of all positive integers except 2; IM$(C_{2n+1} \cdot K_1)$ is the set of all even positive integers; IM$(P_m \times P_n)$ $(m, n) \neq (2, 2)$ is the set of all positive integers except 2; IM$(P_2 \times P_2)$ is the set of all positive integers and IM$(P_n + K_1)$ $(n > 2)$ is the set of all positive integers except 2; and IM$(K_{1,k+1})$ $(k > 2)$ is the set of all multiples of k.

Lee et al. use the notation $C_m@C_n$ for the graph obtained by starting with C_m and attaching paths P_n to C_m by identifying the endpoints of the paths with each successive pairs of verticies of C_m. They prove that $\mathrm{IM}(C_m @ C_n)$ is the set of all positive integers if m or n is even and $\mathrm{IM}(C_m @ C_n)$ is the set of all even positive integers if m and n are odd.

Specialized results about the integer-magic spectra of amalgamations of stars and cycles are given by Lee and Salehi in [351].

The table following summarizes the state of knowledge about magic-type labelings. In the table **SM** means semi-magic; **M** means magic; **SPM** means supermagic. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovar and Tereza Kovarova.

Table 3: **Summary of Magic Labelings**

Graph	*Types*	*Notes*
K_n	M SPM	if $n = 2$, $n \geq 5$ [543] for $n \geq 5$ iff $n > 5$, $n \not\equiv 0 \pmod 4$ [544]
$K_{m,n}$	SM	if $n \geq 3$ [543]
$K_{n,n}$	M	if $n \geq 3$ [543]
Fans F_n	M	iff n is odd, $n \geq 3$ [543]
Wheels W_n	M	if $n \geq 4$ [543]
Wheels with one spoke deleted	M	if $n = 4$, $n \geq 6$ [543]
Möbius ladders M_n	SPM	if $n \geq 3$, n is odd [461]
$C_n \times P_2$	M not SPM	for $n \geq 4$, n even [461]
Composition of C_m and $\overline{K}_n$	SPM	if $m \geq 3$, $n \geq 2$ [507]
$K\underbrace{n, n, \ldots, n}_{p}$	SPM	$n \geq 3$, $p > 5$ and $p \not\equiv 0 \pmod 4$ [507]
Composition of r-regular SPM graph and $\overline{K}_n$	SPM	if $n \geq 3$ [507]
Composition of K_k and $\overline{K}_n$	SPM	if $k = 3$ or 5, $n = 2$ or n odd [280]
$mK_{n,n}$	SPM	for $n \geq 2$ iff n is even or both n and m are odd [504]

Table 3: **continued**

Graph	*Types*	*Notes*
Q_n	SPM	iff $n = 1$ or $n > 2$ even [293]
$C_m \times C_n$	SPM	$m = n$ or m, n even [293]
$C_m \times C_n$	SPM?	for all m and n [293]
connected (p, q)-graph other than P_2	M	iff $5p/4 < q \leq p(p-1)/2$ [561]
G^i	M	$\|G\| \geq 5$, $i \geq 3$ [564]
G^2	M	$G \neq P_5$ and G does not have a 1-factor whose every edge is incident with an end-vertex of G
$K_{1,m,n}$	M	for all m, n [465]
P_n^2	M	for all n [465]
$G \times H$	M	iff G and H are magic [322]

5.2 Edge-magic Total and Super Edge-magic Labelings

In 1970 Kotzig and Rosa [327] defined a magic labeling of a graph $G(V, E)$ as a bijection f from $V \cup E$ to $\{1, 2, \dots, |V \cup E|\}$ such that for all edges xy, $f(x) + f(y) + f(xy)$ is constant. To distinguish between this usage from that of Stewart we will call this labeling an *edge-magic total* labeling. Kotzig and Rosa proved: $K_{m,n}$ has an edge-magic total labeling for all m and n; C_n has an edge-magic total labeling for all $n \geq 3$ (see also [241], [450], [90] and [196]); and the disjoint union of n copies of P_2 has an edge-magic total labeling if and only if n is odd. They further state that K_n has an edge-magic total labeling if and only if $n = 1, 2, 3, 5$ or 6 (see [328], [168] and [196]) and ask whether all trees have edge-magic total labelings. Wallis et al. [578] enumerate every edge-magic total labeling of complete graphs. They also prove that the following graphs are edge-magic total: paths, crowns, complete bipartite graphs, and cycles with a single edge attached to one vertex. Enomoto, Llado, Nakamigana and Ringel [196] prove that all complete bipartite graphs are edge-magic total. They also show that wheels W_n are not edge-magic total when $n \equiv 3 \pmod 4$ and conjectured that all other wheels are edge-magic total. This conjecture was proved when $n \equiv 0, 1 \pmod 4$ by Phillips, Rees and Wallis [435] and when $n \equiv 6 \pmod 8$ by Slamin, Bača, Lin, Miller and Simanjuntak [526]. Fukuchi [227] verified all cases of the conjecture independently of the work of others. Slamin et al. further show that all fans are edge-magic total. Ringel and Llado [448] prove that a graph with p vertices and q edges is not edge-magic total if q is even and $p+q \equiv 2 \pmod 4$ and each vertex has odd degree. Ringel and Llado conjecture that trees are edge-magic total. In [35] Babujee, Baskar Rao present algorithms for producing edge-magic total labelings of trees with a minimum number of pendent vertices and trees with a maximum number of pendent vertices.

Beardon [84] extends the notion of edge-magic total to countable infinite graphs $G(V, E)$ (that is, $V \cup E$ is countable). His main result is that a countably infinite tree that processes an infinite simple path has a bijective edge-magic total labeling using the integers as labels. He asks whether all countably infinite trees have an edge-magic total labeling with the integers as labels and whether the graph with the integers as vertices and an edge joining every two distinct vertices has a bijective edge-magic total labeling using the integers. Balakrishnan and Kumar [67] proved that the join of $\overline{K_n}$ and two disjoint copies of K_2 is edge-magic total if and only if $n = 3$. Yegnanarayanan [604] has proved the following graphs have edge-magic total labelings: nP_3 where n is odd; $P_n + K_1$; $P_n \times C_3$ $(n \geq 2)$; the crown $C_n \odot K_1$; and $P_m \times C_3$ with n pendant vertices attached to each vertex of the outermost C_3. He conjectures that for all n, $C_n \odot \overline{K_n}$, the n-cycle with n pendant vertices attached at each vertex of the cycle, and nP_3 have edge-magic total labelings. In fact, Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle [212] have proved the stronger statement that for all $n \geq 3$, the corona of $C_n \odot \overline{K_m}$ admits an edge-magic labeling where the set of vertex labels is $\{1, 2, \dots, |V|\}$ Yegnanarayanan [604] also introduces several variations of edge-magic labelings and provides some results about them. Kotzig [577] provides some necessary conditions for graphs with an even number of edges in which every vertex has odd degree to have an edge-magic total labeling. Craft and Tesar [168] proved that an r-regular graph with r odd and $p \equiv 4 \pmod 8$ vertices can not be edge-magic total. Wallis [575] proved that if G is an edge-magic total r-regular graph with p vertices and q edges with $r = 2^t s + 1$ $(t > 0)$ and q even, then 2^{t+2} divides p. Figueroa-Centeno, Ichishima and Muntaner-Batle [207] and Ichishima [207] have proved the following graphs are edge-magic total: $P_4 \cup nK_2$ for n odd; $P_3 \cup nK_2$; $P_5 \cup nK_2$; nP_i for n odd and $i = 3, 4, 5$; $2P_n$; $P_1 \cup P_2 \cup \cdots \cup P_n$; $mK_{1,n}$; $K_{1,n} \cup K_{1,n+1}$; $C_m \odot nK_1$; $K_1 \odot nK_2$ for n even; W_{2n}; $K_2 \times \overline{K}_n$, nK_3 for n odd; binary trees, generalized Petersen graphs (see also [428]), ladders (see also [582]), books, fans, and odd cycles with pendant edges attached to one vertex. Enomoto et al. [196] conjecture that if G is a graph of order $n + m$ that contains K_n, then G is not edge-magic total for $n \gg m$. Wijaya and Baskoro [582] proved that $P_m \times C_n$ is edge-magic total for odd n at least 3. Ngurah and Baskoro [428] state that $P_2 \times C_n$ is not edge-magic total. Hegde and Shetty [275] have shown that every T_p-tree (see §4.2 for the definition) is edge-magic total. Wallis [575] proves that a cycle with one pendent edge is edge-magic total. In [575] Wallis poses a large number of research problems about edge-magic total graphs. Avadayappan, Jeyanthi and Vasuki [33] define the *magic strength* of a graph G as the minimum of all constants over all edge-magic total labelings of G. We denote this by $emt(G)$. They use the notation $< K_{1,n} : 2 >$ for the tree obtained from the bistar $B_{n,n}$ by subdividing the edge joining the two stars. They prove: $emt(P_{2n}) = 5n + 1$; $emt(P_{2n+1}) = 5n + 3$; $emt(< K_{1,n} : 2 >) = 4n+9$; $emt(B_{n,n}) = 5n+6$; $emt((2n+1)P_2) = 9n+6$; $emt(C_{2n+1}) = 5n+4$; $emt(C_{2n}) = 5n + 2$; $emt(K_{1,n}) = 2n + 4$; $emt(P^2) = 3n$; and $emt(K_{n,m}) \leq (m + 2)(n + 1)$ where $n \leq m$.

Hegde and Shetty [278] define the *maximum magic strength* of a graph G as the maximum constant over all edge-magic total labelings of G. We use $eMt(G)$ to denote the maximum magic strength of G. Hegde and Shetty call a graph G with p vertices *strong magic* if $emt(G) = eMt(G)$; *ideal magic* if $1 \leq eMt(G) - emt(G) \leq p$; and *weak magic* if $eMt(G) - emt(G) > p$. They prove that for an edge-magic total graph G with p vertices and q edges, $eMt(G) = 3(p+q+1) - emt(G)$. Using this result they obtain: P_n is ideal magic for $n > 2$; $K_{1,1}$ is strong magic, $K_{1,2}$ and $K_{1,3}$ are ideal magic, and $K_{1,n}$ is weak magic for $n > 3$; $B_{n,n}$ is ideal magic; $(2n+1)P_2$ is strong magic; cycles are ideal magic; and the generalized web $W(t, 3)$ (see §2.2) with the central vertex deleted is weak magic.

Enomoto et al. [196] call an edge-magic total labeling *super edge-magic* if the set of vertex labels is $\{1, 2, \ldots, |V|\}$ (Wallis [575] calls these labelings *strongly edge-magic*). They prove the following: C_n is super edge-magic if and only if n is odd, caterpillars are super edge-magic; $K_{m,n}$ is super edge-magic if and only if $m = 1$ or $n = 1$; and K_n is super edge-magic if and only if $n = 1, 2$ or 3. They also prove that if a graph with p vertices and q edges is super edge-magic then, $q \leq 2p - 3$. Enomoto et al. [196] conjecture that every tree is super edge-magic. Lee and Shan [360] have verified this conjectures for trees with up to 17 vertices with a computer. Kotzig and Rosa's ([327] and [328]) proof that nK_2 is edge-magic total when n is odd actually shows that it is super edge-magic. Kotzig and Rosa also prove that every caterpillar is super-edge magic. Figueroa-Centeno, Ichishima and Muntaner-Batle prove the following: if G is a bipartite or tripartite (super) edge-magic graph, then nG is (super) edge-magic when n is odd [211]; if m is a multiple of $n+1$, then $K_{1,m} \cup K_{1,n}$ is super edge-magic [211]; $K_{1,2} \cup K_{1,n}$ is super edge-magic if and only if n is a multiple of 3; $K_{1,m} \cup K_{1,n}$ is edge-magic if and only if mn is even [211]; $K_{1,3} \cup K_{1,n}$ is super edge-magic if and only if n is a multiple of 4 [211]; $P_m \cup K_{1,n}$ is super edge-magic when $m \geq 4$ [211]; $2P_n$ is super edge-magic if and only if n is not 2 or 3; $2P_{4n}$ is super edge-magic for all n [211]; $K_{1,m} \cup 2nK_2$ is super edge-magic for all m and n [211]; $C_3 \cup C_n$ is super edge-magic if and only if $n \geq 6$ and n is even [212]; $C_4 \cup C_n$ is super edge-magic if and only if $n \geq 5$ and n is odd [212]; $C_5 \cup C_n$ is super edge-magic if and only if $n \geq 5$ and n is even [212]; if m is even and at least 6 and n is odd and satisfies $n \geq m/2+2$, then $C_m \cup C_n$ is super edge-magic [212]; $C_4 \cup P_n$ is super edge-magic if and only if $n \neq 3$ [212]; $C_5 \cup P_n$ is super edge-magic if $n \geq 4$ [212]; if m is even and at least 6 and $n \geq m/2+2$, then $C_m \cup P_n$ is super edge-magic [212]; and $P_m \cup P_n$ is super edge-magic if and only if $(m, n) \neq (2, 2)$ or (3,3) [212]. They conjecture [211] that $K_{1,m} \cup K_{1,n}$ is super egde-magic only when m is a multiple of $n+1$ and they prove that if G is a super edge-magic graph with p vertices and q edges with $p \geq 4$ and $q \geq 2p-4$, then G contains triangles. In [212] Figueroa-Centeno et al. conjecture that $C_m \cup C_n$ is super edge-magic if and only if $m+n \geq 9$ and $m+n$ is odd.

Lee and Kong [345] use $\mathrm{St}(a_1, a_2, \ldots, a_n)$ to denote the disjoint union of the n stars $\mathrm{St}(a_1)$, $\mathrm{St}(a_2)$, ... , $\mathrm{St}(a_n)$. They prove the following graphs are super edge-magic: $\mathrm{St}(m, n)$ where $n \equiv 0 \bmod (m+1)$; $\mathrm{St}(1, 1, n)$; $\mathrm{St}(1, 2, n)$; $\mathrm{St}(1, n, n)$; $\mathrm{St}(2, 2, n)$;$\mathrm{St}(2, 3, n)$; $\mathrm{St}(1, 1, 2, n)$ $(n \geq 2)$; $\mathrm{St}(1, 1, 3, n)$; $\mathrm{St}(1, 2, 2, n)$; and $\mathrm{St}(2, 2, 2, n)$. They conjecture that $\mathrm{St}(a_1, a_2, \ldots, a_n)$ is super edge-magic when $n > 1$ is odd.

Enomoto, Masuda and Nakamigawa [197] have proved that every graph can be embedded in a connected super edge-magic graph as an induced subgraph. Slamin et al. [526] proved that the friendship graph consisting of n triangles is super edge-magic if and only if n is 3, 4, 5 or 7. Fukuchi proved [226] the generalizied Petersen graph $P(n, 2)$ (see §2.7 for the definition) is super edge-magic if n is odd and at least 3. Baskoro and Ngurah [81] showed that nP_3 is super edge-magic for $n \geq 4$ and n even.

Hegde and Shetty [279] showed that a graph is super edge-magic if and only if it is strongly k-indexable (see §4.1). Figueroa-Centeno, Ichishima and Muntaner [206] proved that a graph is super edge-magic if and only if it is strongly 1-harmonious and that a super edge-magic graph is cordial. They also proved that P_n^2 and $K_2 \times C_{2n+1}$ are super edge-magic. In [207] Figueroa-Centeno et al. show that the following graphs are super edge-magic: $P_3 \cup kP_2$ for all k; kP_n when k is odd; and $k(P_2 \cup P_n)$ when k is odd and $n = 3$ or $n = 4$; fans F_n if and only if $n \leq 6$. They conjecture that kP_2 is not super edge-magic when k is even. This conjecture has been proved by Z. Chen [160] who showed that kP_2 is super edge-magic if and only if k is odd. Figueroa-Centeno et al. provide a strong necessary condition for a book to have a super edge-magic labeling and conjecture that for $n \geq 5$ the book B_n is super edge-magic if and only if n is even or $n \equiv 5 \pmod 8$. They prove that every tree with an α-labeling is super edge-magic.

Yokomura (see [196]) has shown that $P_{2m+1} \times P_2$ and $C_{2m+1} \times P_m$ are super edge-magic (see also [206]). In [208], Figueroa-Centeno et al. proved that if G is a (super) edge-magic 2-regular graph, then $G \odot \overline{K}_n$ is (super) edge-magic and that $C_m \odot \overline{K}_n$ is super edge-magic. Fukuchi [225] shows how to recursively create super edge-magic trees from certains kinds of existing super edge-magically labelled trees.

Lee and Lee [343] investigate the existence of total edge-magic lableings and super edge-magic labelings of unicylic graphs. They obtain a variety of positive and negative results and conjecture that all unicyclic are total edge-magic.

Shiu and Lee [509] investigated edge labelings of multigraphs. Given a multigraph G with p vertices and q edges they call a bijection from the set of edges of G to $\{1, 2, \ldots, q\}$ with the property that for each vertex v the sum of all edge labels incident to v is a constant independent of v a *supermagic* labeling of G. They use $K_2[n]$ to denote the multigraph consisting of n edges joining 2 vertices and $mK_2[n]$ to denote the disjoint union of m copies of $K_2[n]$. They prove that for m and n at least 2, $mK_2[n]$ is supermagic if and only if n is even or if both m and n are odd.

In 1970 Kotzig and Rosa [327] defined the *edge-magic deficiency*, $\mu(G)$, of a graph G as the minimum n such that $G \cup nK_1$ is edge-magic total. If no such n exists they define $\mu(G) = \infty$. In 1999 Figueroa-Centeno, Ichishima and Muntaner-Batle [210] extended this notion to *super edge-magic deficiency*, $\mu_s(G)$, is the analogous way. They prove the following: $\mu_s(nK_2) = \mu(nK_2) = n - 1 \pmod 2$; $\mu_s(C_n) = 0$ if n is odd; $\mu_s(C_n) = 1$ if $n \equiv 0 \pmod 4$; $\mu_s(C_n) = \infty$ if $n \equiv 2 \pmod 4$; $\mu_s(K_n) = \infty$ if and only if $n \geq 5$; $\mu_s(K_{m,n}) \leq (m-1)(n-1)$; $\mu_s(K_{2,n}) = n - 1$; and $\mu_s(F)$ is finite for all forests F. They also prove that if a graph G has q edges with $q/2$ odd and every vertex is even, then $\mu_s(G) = \infty$.

In [213] Figueroa-Centeno et al. proved that $\mu_s(P_m \cup K_{1,n}) = 1$ if $m = 2$ and n is odd or $m = 3$ and is not congruent to 0 mod 3 while in all other cases $\mu_s(P_m \cup K_{1,n}) = 0$. They also proved that $\mu_s(2K_{1,n} = 1$ when n is odd and $\mu_s(2K_{1,n} \leq 1$ when n is even. They conjecture that $\mu_s(2K_{1,n} = 1$ in all cases. Other results in [213] are: $\mu_s(P_m \cup P_n) = 1$ when $(m, n) = (2, 2)$ or $(3, 3)$ and $\mu_s(P_m \cup P_n) = 0$ when $(m, n) \neq (2, 2)$ or $(3, 3)$; $\mu_s(K_{1,m} \cup K_{1,n}) = 0$ when mn is even and $\mu_s(K_{1,m} \cup K_{1,n}) = 1$ when mn is odd; $\mu(P_m \cup K_{1,n}) = 1$ when $m = 2$ and n is odd and $\mu(P_m \cup K_{1,n}) = 0$ in all other cases; $\mu(P_m \cup P_n) = 1$ when $(m, n) = (2, 2)$ and $\mu(P_m \cup P_,n) = 0$ in all other cases; $\mu_s(2C_n) = 1$ when n is even and ∞ when n is odd; $\mu_s(3C_n) = 0$ when n is odd; $\mu_s(3C_n) = 1$ when $n \equiv 0 \pmod 4$; $\mu_s(3C_n) = \infty$ when $n \equiv 2 \pmod 4$; and $\mu_s(4C_n) = 1$ when $n \equiv 0$ (mod 4). They conjecture the following: $\mu_s(mC_n) = 0$ when mn is odd; $\mu_s(mC_n) = 1$ when $mn \equiv 0 \pmod 4$; $\mu_s(mC_n) = \infty$ when $mn \equiv 2 \pmod 4$; $\mu_s(2K_{1,n}) = 1$; if F is a forest with two components, then $\mu(F) \leq 1$ and $\mu_s(F) \leq 1$.

Z. Chen [160] has proven: the join of K_1 with any subgraph of a star is super edge-magic; the join of two nontrivial graphs is super edge-magic if and only if at least one of them has exactly two vertices and their union has exactly one edge; and if a k-regular graph is super edge-magic, then $k \leq 3$. Chen also obtained the following conditions: there is a connected super edge-magic graph with p vertices and q edges if and only if

$p-1 \leq q \leq 2p-3$; there is a connected 3-regular super edge-magic graph with p vertices if and only if $p \equiv 2 \pmod 4$; if G is a k-regular edge-magic total graph with p vertices and q edges then $(p+q)(1+p+q) \equiv 0 \pmod{2d}$ where $d = \gcd(k-1, q)$. As a corollary of the last result, Chen observes that $nK_2 + nK_2$ is not edge-magic total.

Another labeling that has been called "edge-magic" was introduced by Lee, Seah and Tan in 1992 [358]. They defined a graph $G = (V, E)$ to be *edge-magic* if there exists a bijection $f\colon E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $f^+\colon V \to N$ defined by $f^+(u) = \sum_{(u,v)\in E} f(u, v) \pmod{|V|}$ is a constant map. Lee conjectured that a cubic graph with p vertices is edge-magic if and only if $p \equiv 2 \pmod 4$. Lee, Pigg and Cox [348] verified this conjecture for several classes of cubic graphs. Shiu and Lee [509] showed that the conjecture is not true for multigraphs and disconnected graphs. Lee, Seah and Tan [358] establish that a necessary condition for a multigraph with p vertices and q edges to be edge-magic is that p divides $q(q+1)$ and exhibit several new classes of cubic edge-magic graphs. They also proved: $K_{n,n}$ $(n \geq 3)$ is edge-magic, K_n is edge-magic for $n \equiv 1, 2 \pmod 4$ and for $n \equiv 3 \pmod 4$ $(n \geq 7)$. Lee, Seah and Tan further proved that following graphs are not edge-magic: all trees except P_2, all unicyclic graphs, and K_n where $n \equiv 0 \pmod 4$. Schaffer and Lee [459] have proved that $C_m \times C_n$ is always edge-magic. Lee, Tong and Seah [363] have conjectured that the total graph of a (p, p)-graph is edge-magic if and only if p is odd. They prove this conjecture for cycles.

For any graph G and any positive integer k the graph $G[k]$, called the *k-fold* G, is the hypergraph obtained from G by replacing each edge of G with k parallel edges. Lee, Seah and Tan [358] proved that for any graph G with p vertices and q edges, $G[2p]$ is edge-magic and, if p is odd, $G[p]$ is edge-magic. Shiu, Lam and Lee [508] show that if G is an $(n+1, n)$-multigraph, then G is edge-magic if and only if n is odd and G is isomorphic to the disjoint union of K_2 and $(n-1)/2$ copies of $K_2[2]$. They also prove that if G is a $(2m+1, 2m)$-multigraph and k is at least 2, then $G[k]$ is edge-magic if and only if $2m+1$ divides $k(k-1)$. For a $(2m, 2m-1)$-multigraph G and k at least 2, they show that $G[k]$ is edge-magic if $4m$ divides $(2m-1)k((2m-1)k+1)$ or if $4m$ divides $(2m+k-1)k$. In [506] Shiu, Lam and Lee characterize the (p, p)-multigraphs that are edge-magic as $mK_2[2]$ or the disjoint union of $mK_2[2]$ and two particular multigraphs or the disjoint union of $K_2, mK_2[2]$ and 4 particular multigraphs. They also show for every $(2m+1, 2m+1)$-multigraph G, $G[k]$ is edge-magic for all k at least 2. Lee, Seah and Tan [358] prove that the multigraph $C_n[k]$ is edge-magic for $k \geq 2$.

The table following summarizes what is known about edge-magic total labelings. We use **EMT** for edge-magic total and **SEM** for super edge-magic labelings. A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovar and Tereza Kovarova.

Table 4: **Summary of Edge-magic total Labelings**

Graph	*Types*	*Notes*
$K_{m,n}$	EMT	for all m and n [327]
C_n	EMT	for $n \geq 3$ [327] [241] [450] [90]
$\bigcup_n P_2$	EMT	iff n odd [327]
K_n	EMT	iff $n = 1, 2, 3, 4, 5$ or 6 [328] [168] [196] enumerate all EMT of K_n [578]
Trees	EMT?	[328] [448]
P_n	EMT	[578]
Crowns $C_n \odot K_1$	EMT	[578]
$K_{m,n}$	EMT	[578]
C_n with a single edge attached to one vertex	EMT	[578]
Wheels W_n	not EMT EMT EMT EMT	if $n \equiv 3 \pmod 4$ [196] if $n \equiv 0, 1 \pmod 4$ [435] if $n \equiv 6 \pmod 8$ [526] if $n \equiv 0, 1, 2 \pmod 4$ [226]
Fans	EMT	[526]
(p, q)-graph	not EMT	if q even and $p + q \equiv 2 \pmod 4$ [448]
nP_3	EMT	if b is odd [604]
$P_n + K_1$	EMT	[604]

Table 4: **continued**

Graph	*Types*	*Notes*
$P_n + K_1$	EMT	[604]
$P_n \times C_3$	EMT	$n \geq 2$ [604]
Crown $C_n \odot K_1$	EMT	[604]
nP_3	EMT?	[604]
r-regular graph	not EMT	r odd and $p \equiv 4 \pmod 8$ [168]
G r-regular (p,q)-graph		if $r = 2^t s + 1 (t > 0)$ and q even then $2^t + 2$ divides p [575]
$P_4 \cup nK_2$	EMT	n odd [206] [207]
$P_3 \cup nK_2$ and $P_5 \cup nK_2$	EMT	[206] [207]
nP_i	EMT	n odd, $i = 3, 4, 5$ [206][207]
$2P_n$	EMT	[206] [207]
$P_1 \cup P_2 \cup \cdots \cup P_n$	EMT	[206] [207]
$mK_{1,n}$	EMT	[206] [207]
$K_{1,n} \cup K_{1,n+1}$	EMT	[206] [207]
$C_m \odot \overline{K}_n$	EMT	[206] [207]

Table 4: **continued**

Graph	*Types*	*Notes*
$K_1 \odot nK_2$	EMT	n even [206] [207]
W_{2n}	EMT	[206] [207]
$K_2 \times \overline{K}_n$	EMT	[206] [207]
nK_3	EMT	n odd [206] [207]
binary trees	EMT	[206] [207]
generalized Petersen graph $P(m, n)$	EMT	[206] [207] [428]
ladders	EMT	[206] [207]
books	EMT	[206] [207]
fans	EMT	[206] [207]
odd cycle with pendant edges attached to one vertex	EMT	[206] [207]
$P_m \times C_n$	EMT	n odd $n \geq 3$ [582]
$P_m \times P_2$	EMT	m odd $m \geq 3$ [582]
$P_2 \times C_n$	not EMT	[428]
$K_{1,m} \cup K_{1,n}$	EMT	iff mn is even [211]

Table 5: **Summary of Super Edge-magic Labelings**

Graph	*Types*	*Notes*
C_n	SEM	iff n is odd [196]
Caterpillars	SEM	[196] [327] [328]
$K_{m,n}$	SEM	iff $m = 1$ or $n = 1$ [196]
K_n	SEM	iff $n = 1, 2$ or 3 [196]
Trees	SEM?	[196]
nK_2	SEM	if n odd [327] [328]
nG	SEM	if G is a bipartite or tripartite SEM graph and n odd [211]
$K_{1,m} \cup K_{1,n}$	SEM	if m is a multiple of $n + 1$ [211]
$K_{1,m} \cup K_{1,n}$	SEM?	iff m is a multiple of $n + 1$ [211]
$K_{1,2} \cup K_{1,n}$	SEM	iff n is a multiple of 3 [211]
$K_{1,3} \cup K_{1,n}$	SEM	iff n is a multiple of 4 [211]
$P_m \cup K_{1,n}$	SEM	if $m \geq 4$ is even [211]
$2P_n$	SEM	iff n is not 2 or 3 [211]
$2P_{4n}$	SEM	for all n [211]
$K_{1,m} \cup 2nK_{1,2}$	SEM	for all m and n [211]

Table 5: **continued**

Graph	*Types*	*Notes*
$C_3 \cup C_n$	SEM	iff $n \geq 6$ even [212]
$C_4 \cup C_n$	SEM	iff $n \geq 5$ odd [212]
$C_5 \cup C_n$	SEM	iff $n \geq 5$ even [212]
$C_m \cup C_n$	SEM	if $m \geq 6$ even and n odd $n \geq m/2+2$ [212]
$C_m \cup C_n$ conjecture	SEM?	iff $m+n \geq 9$ and $m+n$ odd [212]
$C_4 \cup P_n$	SEM	iff $n \neq 3$ [212]
$C_5 \cup P_n$	SEM	if $n \neq 4$ [212]
$C_m \cup P_n$	SEM	if $m \geq 6$ even and $n \geq m/2+2$ [212]
$P_m \cup P_n$	SEM	iff $(m,n) \neq (2,2)$ or $(3,3)$ [212]
Corona $C_n \odot \overline{K}_m$	SEM	$n \geq 3$ [212]
$St(a_1,\ldots,a_n)$		is a disjoint union of stars (see §5.2)
$St(m,n)$	SEM	$n \equiv 0 \pmod{m+1}$ [345]
$St(1,k,n)$	SEM	$k=1,2$ or n [345]
$St(2,k,n)$	SEM	$k=2,3$ [345]
$St(1,1,k,n)$	SEM	$k=2,3$ [345]
$St(k,2,2,n)$	SEM	$k=1,2$ [345]
$St(a_1,\ldots,a_n)$	SEM?	for $n>1$ odd [345]
Friendship graph of n triangles	SEM	iff $n=3,4,5$ or 7 [526]

Table 5: **continued**

Graph	*Types*	*Notes*
Generalized Petersen graph $P(n, 2)$ (see §2.7)	SEM	if $n \geq 3$ odd [225]
nP_3	SEM	if $n \geq 4$ even [81]
P_n^2	SEM	[206]
$K_2 \times C_{2n+1}$	SEM	[206]
$P_3 \cup kP_2$	SEM	for all k [207]
kP_n	SEM	if k is odd [207]
$k(P_2 \cup P_n)$	SEM	if k is odd and $n = 3, 4$ [207]
Fans F_n	SEM	iff $n \leq 6$ [207]
kP_2	SEM	iff k is odd [160]
Book B_n	SEM?	iff n even or $n \equiv 5 \pmod 8$[207]
Tree with α labeling	SEM	[207]
$P_{2m+1} \times P_2$ $P_{2m+1} \times P_m$	SEM SEM	[196] [206] [196] [206]
$G \odot \overline{K}_n$	EMT/SEM	if G is EMT/SEM 2-regular graph [208]
$C_m \odot \overline{K}_n$	SEM	[208]

Table 5: **continued**

Graph	*Types*	*Notes*
join of K_1 with any subgraph of a star	SEM	[160]
join of two nontrivial graphs one has two vertices and their union has exactly one edge	SEM	[160]
if G is k-regular SEM graph		then $k \leq 3$ [160]
G is connected (p, q)-graph	SEM	G exists iff $p - 1 \leq q \leq 2p - 3$ [160]
G is connected 3-regular graph on p vertices	SEM	iff $p \equiv 2 \pmod 4$ [160]
$nK_2 + nK_2$	not SEM	[160]

5.3 Vertex-magic Total Labelings and Totally Magic Labelings

MacDougall, Miller, Slamin and Wallis [395] introduced the notion of a vertex-magic total labeling in 1999. For a graph $G(V, E)$ an injective mapping f from $V \cup E$ to the set $\{1, 2, \ldots, |V| + |E|\}$ is a *vertex-magic total labeling* if there is a constant k so that for every vertex v, $f(v) + \sum f(vu) = k$ where the sum is over all vertices u adjacent to v. They prove that the following graphs have vertex-magic total labelings: C_n; P_n for $n > 2$; $K_{m,m}$ for $m > 1$; $K_{m,m} - e$ for $m > 2$; and K_n for n odd. They also prove that when $n > m+1$, $K_{m,n}$ does not have a vertex-magic total labeling. They conjecture that $K_{m,m+1}$ has a vertex-magic total labeling for all m and that K_n has vertex-magic total labeling for all $n \geq 3$. Lin and Miller [377] have shown that $K_{m,m}$ is vertex-magic total for all $m > 1$ and that K_n is vertex-magic total for all $n \equiv 0 \pmod 4$. Phillips, Rees and Wallis [436] generalized the Lin and Miller result by proving that $K_{m,n}$ is vertex-magic total if and only if m and n differ by at most 1. Miller, Bača, and MacDougall [407] have proved that the generalized Petersen graphs $P(n, k)$ (see Section 2.7 for the definition) are vertex-magic total when n is even and $k \leq n/2 - 1$. They conjecture that all $P(n, k)$ are vertex-magic total when $k \leq (n-1)/2$ and all prisms $C_n \times P_2$ are vertex-magic total. Bača, Miller and Slamin [64] proved the first of these conjectures (see also [527] for partial results) while Slamin and Miller prove the second. MacDougall et al. ([395] and [396]) have shown: W_n has a vertex-magic total labeling if and only if $n \leq 11$; fans F_n have a vertex-magic total labelings if and only if $n \leq 10$; freindship graphs have vertex-magic total labelings if and only if the number of triangles is at most 3; $K_{m,n}$ $(m > 1)$ has a vertex-magic total labeling if and only if m and n differ by at most 1. Wallis [575] proved: if G and H have the same order and $G \cup H$ is vertex-magic total then so is $G + H$; if the disjoint union of stars is vertex-magic total then the average size of the stars is less than 3; if a tree has n internal vertices and more than $2n$ leaves then it does not have a vertex-magic total labeling. Wallis [576] has shown that if G is a regular graph of even degree that has a vertex-magic total labeling then the graph consisting of an odd number of copies of G is vertex-magic total. He also proved that if G is a regular graph of odd degree (not K_1) that has a vertex-magic total labeling then the graph consisting of any number of copies of G is vertex-magic total.

Fronček, Kovář and Kovářová [216] proved that $C_n \times C_{2m+1}$ and $K_5 \times C_{2n+1}$ are vertex-magic total. Kovář in [330] furhermore proved some general results about products of certain regular vertex-magic total graphs. In particular if G is a $2r + 1$ vertex-magic total graph which can be factored into an $(r + 1)$-regular graph and an r-regular graph then $G \times K_5$ and $G \times C_n$ for n even are also vertex-magic total. He also proved that taking G an r-regular vertex-magic total graph and H a $2s$-regular supermagic graph which can be factored into two s-regular factors then their Cartesian product $G \times H$ is vertex-magic total if either r is odd or r is even and $|H|$ is odd.

Beardon [83] has shown that a necessary condition for a graph with p vertices and q edges and a vertex of degree d to be vertex-magic total is $(d+2)^2 \leq (14q^2 + 16q + 4)/p$. When the graph is connected we have the stronger condition $(d+2)^2 \leq (7q^2 + 11q + 4)/p$. As a corollary it follows that the following are not vertex-magic total: wheels W_n when

$n \geq 12$, fans F_n when $n \geq 11$, and friendship graphs $C_3^{(n)}$ when $n \geq 4$.

Wood [587] generalizes vertex-magic and edge-magic labelings by requiring only that the labels be positive integers rather than consecutive positive integers. He gives upper bounds for the minimum values of the magic constant and the largest label for complete graphs, forests, and arbitrary graphs.

Exoo et al. [203] call a function λ a *totally magic labeling* of a graph G if λ is both an edge-magic and a vertex-magic labeling of G. A graph with such a labeling is called *totally magic*. Among their results are: P_3 is the only connected totally magic graph that has a vertex of degree 1; the only totally magic graphs with a component K_1 are K_1 and $K_1 \cup P_3$; the only totally magic complete graphs are K_1 and K_3; the only totally magic complete bipartite graph is $K_{1,2}$; nK_3 is totally magic if and only if n is odd; $P_3 \cup nK_3$ is totally magic if and only if n is even.

J. McSorley and W. Wallis [399] examine the possible totally magic labelings of a union of an odd number of triangles and determine the spectrum of possible values for the sum of the label on a vertex and the labels on its incident edges and the sum of an edge label and the labels of the endpoints of the edge for all known totally magic graphs.

In the table we use following abbreviations

VMT vertex-magic total labeling

TM totally magic labeling

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovar and Tereza Kovarova.

Table 6: **Summary of vertex-magic total labelings**

Graph	*Labeling*	*Notes*
C_n	VMT	[395]
P_n	VMT	$n > 2$ [395]
$K_{m,m}$	VMT	$m > 1$ [395][377]
$K_{m,m} - e$	VMT	$m > 2$ [395]
$K_{m,n}$	VMT	iff $\|m - n\| \leq 1$ [436][395][396]
K_n	VMT	for n odd [395] for $n \equiv 2 \pmod 4$, $n > 2$ [377]
Petersen $P(n, k)$	VMT	[64]
prisms $C_n \times P_2$	VMT	[527]
W_n	VMT	iff $n \leq 11$ [395][396]
F_n	VMT	iff $n \leq 10$ [395][396]
friendship graphs (see §5.3)	VMT	iff # of triangles ≤ 3 [395][396]
$G + H$	VMT	$\|V(G)\| = \|V(H)\|$ and $G \cup H$ is VMT [575]
unions of stars	VMT	[575]
Tree with n internal vertices and more than $2n$ leaves	not VMT	[575]
nG	VMT	n odd, G regular of even degree, VMT [576]
nG	VMT	G is regular of odd degree, VMT, but not K_1 [576]

Table 6: **continued**

Graph	*Labeling*	*Notes*
$C_n \times C_{2m+1}$	VMT	[216]
$K_5 \times C_{2n+1}$	VMT	[216]
$G \times C_{2n}$	VMT	G $2r+1$-regular VMT (see §5.3) [330]
$G \times K_5$	VMT	G $2r+1$-regular VMT (see §5.3) [330]
$G \times H$	VMT	G r-regular VMT, r odd or r even and $\|H\|$ odd, H $2s$-regular supermagic [330]
P_3	TM	the only connected TM graph with vertex of degree 1 [203]
K_n	TM	iff $n = 1, 3$ [203]
$K_{m,n}$	TM	iff $K_{m,n} = K_{1,2}$ [203]
nK_3	TM	iff n is odd [203]
$P_3 \cup nK_3$	TM	iff n is even [203]

5.4 1-vertex magic vertex labeling

In 2001, Simanjuntak, Rodgers and Miller [513] defined a *1-vertex magic vertex labeling* of $G(V, E)$ as a bijection from V to $\{1, 2, \ldots, |V|\}$ with the property that there is a constant k such that at any vertex v the sum $\sum f(u)$ taken over all neighbors of v is k. Among their results are: $H \times \overline{K}_{2k}$ has a 1-vertex-magic vertex labeling for any regular graph H; the symmetric complete multipartite graph with p parts, each of which contains n vertices, has a 1-vertex-magic vertex labeling if and only if whenever n is odd, p is also odd, and if $n = 1$, then $p = 1$; P_n has a 1-vertex-magic vertex labeling if and only if $n = 1$ or 3; C_n has a 1-vertex-magic vertex labeling if and only if $n = 4$; K_n has a 1-vertex-magic vertex labeling if and only if $n = 1$; W_n has a 1-vertex-magic vertex labeling if and only if $n = 4$; a tree has a 1-vertex-magic vertex labeling if and only if it is P_1 or P_3; and r-regular graphs with r odd do not have a 1-vertex-magic vertex labeling.

In the table we use the abbreviation 1VM for 1-vertex magic vertex labeling. The table was prepared by Petr Kovar and Tereza Kovarova.

Table 7: **Summary of 1-vertex magic vertex labelings**

Graph	*Labeling*	*Notes*
$H \times \overline{K}_{2k}$	1VM	H is regular [513]
symmetric $K_{\underbrace{n, n, \ldots, n}_{p}}$	1VM	iff whenever n is odd also p is odd , and for $n = 1$ also $p = 1$ [513]
P_n	1VM	iff $n = 1$ or $n = 3$ [513]
C_n	1VM	iff $n = 4$ [513]
K_n	1VM	iff $n = 1$ [513]
W_n	1VM	iff $n = 4$ [513]
tree T	1VM	iff $T = P_1$ or P_3 [513]
r – regular graph	not 1VM	r is odd [513]

5.5 Magic Labelings of Type (a, b, c)

A magic-type method for labeling the vertices, edges and faces of a planar graph was introduced by Lih [376] in 1983. Lih defines a *magic labeling of type* (1,1,0) of a planar graph $G(V, E)$ as an injective function from $\{1, 2, \ldots, |V| + |E|\}$ to $V \cup E$ with the property that for each interior face the sum of the labels of the vertices and the edges surrounding that face is some fixed value. Similarly, Lih defines a *magic labeling of type* $(1, 1, 1)$ of a planar graph $G(V, E)$ with face set F as an injective function from $\{1, 2, \ldots, |V| + |E| + |F|\}$ to $V \cup E \cup F$ with the property that for each interior face the sum of the labels of the face and the vertices and the edges surrounding that face is some fixed value. Lih calls a labeling involving the faces of a plane graph *consecutive* if for every integer s the weights of all s-sided faces constitute a set of consecutive integers. Lih gave consecutive magic labelings of type $(1, 1, 0)$ for wheels, friendship graphs, prisms and some members of the Platonic family. In [40] Bača shows that the cylinders $C_n \times P_m$ have magic labelings of type $(1, 1, 0)$ when $m \geq 2, n \geq 3, n \neq 4$. Bača has given magic labelings of type $(1, 1, 1)$ for fans [36], ladders [36], planar bipyramids (that is, 2-point suspensions of paths) [36], grids [43], hexagonal lattices [42], Möbius ladders [38] and $P_n \times P_3$ [39]. Bača [37], [46], [44], [39], [45] and Bača and Holländer [54] give magic labelings of type $(1, 1, 1)$ and type $(1, 1, 0)$ for certain classes of convex polytopes. Bača [41] also provides consecutive and magic labelings of type $(0, 1, 1)$ (that is, an injective function from $\{1, 2, \ldots, |E| + |F|\}$ to $E \cup F$ with the property that for each interior face the sum of the labels of the face and the edges surrounding that face is some fixed value) and a consecutive labeling of type $(1, 1, 1)$ for a kind of planar graph with hexagonal faces.

A *magic labeling of type* (1,0,0) of a planar graph G with vertex set V is an injective function from $\{1, 2, \ldots, |V|\}$ to V with the property that for each interior face the sum of the labels of the vertices surrounding that face is some fixed value. Kathiresan, Muthuvel and Nagasubbu [309] define a *lotus inside a circle* as the graph obtained from the cycle with consecutive vertices $a_1, a_2, \ldots, a_n$ and the star with central vertex b_0 and end vertices $b_1, b_2, \ldots, b_n$ by joining each b_i to a_i and a_{i+1} $(a_{n+1} = a_1)$. They prove that these graphs $(n \geq 5)$ and subdivisions of ladders have consecutive labelings of type $(1, 0, 0)$. Devaraj [178] proves that graphs obtained by subdividing each edge of a ladder exactly the same number of times has a magic labeling of type $(1, 0, 0)$.

In the table we use following abbreviations

M(x,x,x) magic labeling of type (x, x, x)

CM(x,x,x) consecutive magic labeling of type (x, x, x)

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovar and Tereza Kovarova.

Table 8: **Summary of magic labelings of type** (a, b, c)

Graph	*Labeling*	*Notes*
W_n	CM(1,1,0)	[376]
friendship graphs	CM(1,1,0)	[376]
prisms	CM(1,1,0)	[376]
cylinders $C_n \times P_m$	M(1,1,0)	$m \geq 2$, $n \geq 3$, $n \neq 4$ [40]
fans F_n	M(1,1,1)	[36]
ladders	M(1,1,1)	[36]
planar bipyramids (see §5.3)	M(1,1,1)	[36]
grids	M(1,1,1)	[43]
hexagonal lattices	M(1,1,1)	[42]
Möbius ladders	M(1,1,1)	[38]
$P_n \times P_3$	M(1,1,1)	[39]
certain classes of convex polytopes	M(1,1,1) M(1,1,0)	[37], [46], [44], [39], [45], [54]
kind of planar graph with hexagonal faces	M(0,1,1) CM(0,1,1) CM(1,1,1)	[41]
lotus inside a circle (see §5.3)	CM(1,0,0)	$n \geq 5$ [309]
subdivisions of ladders	M(1,0,0) CM(1,0,0)	[178] [309]

5.6 Antimagic Labelings

Bača, et al. [51] introduced the notion of a (a, d)-vertex-antimagic total labeling in 2000. For a graph $G(V, E)$, an injective mapping f from $V \cup E$ to the set $\{1, 2, \ldots, |V|+|E|\}$ is a *(a, d)-vertex-antimagic total labeling* if the set $\{f(v)+\sum f(vu)\}$ where the sum is over all vertices u adjacent to v for all v in G is $\{a, a+d, a+2d, \ldots, a+(|V|-1)d\}$. Among their results are: every super-magic graph has an $(a, 1)$-vertex-antimagic total labeling; every (a, d)-antimagic graph $G(V, E)$ is $(a+|E|+1, d+1)$-vertex-antimagic total; and, for $d > 1$, every (a, d)-antimagic graph $G(V, E)$ is $(a+|V|+|E|, d-1)$-vertex-antimagic total. They also show that paths and cycles have (a, d)-vertex-antimagic total labelings for a wide variety of a and d. In [52] Bača et al. use their results in [51] to obtain numerous (a, d)-vertex-antimagic total labelings for prisms, antiprisms and generalized Petersen graphs. Lin, Miller, Simanjuntak and Slamin [378] have shown that for $n > 20$, W_n has no (a, d)-vertex-antimagic total labeling.

Simanjuntak, Bertault and Miller [512] define an *(a,d)-edge-antimagic vertex labeling* for a graph $G(V, E)$ as an injective mapping f from V onto the set $\{1, 2, \ldots, |V|\}$ such that the set $\{f(u)+f(v)|uv \in E\}$ is $\{a, a+d, a+2d, \ldots, a+(|E|-1)d\}$. (The equivalent notion of *(a,d)-indexable* labeling was defined by Hegde in 1989 in his Ph. D. thesis–see [270].) Similarly, Simanjuntak et al. define an *(a,d)-edge-antimagic total labeling* for a graph $G(V, E)$ as an injective mapping f from $V \cup E$ onto the set $\{1, 2, \ldots, |V| + |E|\}$ such that the set $\{f(v) + \sum f(vu)|uv \in E\}$ where v ranges over all of V is $\{a, a + d, a+2d, \ldots, a+(|V|-1)d\}$. Among their results are: C_{2n} has no (a, d)-edge-antimagic vertex labeling; C_{2n+1} has a $(n + 2, 1)$-edge-antimagic vertex labeling and a $(n + 3, 1)$-edge-antimagic vertex labeling; P_{2n} has a $(n + 2, 1)$-edge-antimagic vertex labeling; P_n has a $(3, 2)$-edge-antimagic vertex labeling; C_n has a $(2n + 2, 1)$- and a $(3n + 2, 1)$-edge-antimagic total labeling; C_{2n} has a $(4n + 2, 2)$- and a $(4n + 3, 2)$-edge-antimagic total labeling; C_{2n+1} has a $(3n + 4, 3)$- and a $(3n + 5, 3)$-edge-antimagic total labeling; P_{2n+1} has a $(3n+4, 2)$-, a $(3n+4, 3)$-, a $(2n+4, 4)$-, a $(5n+4, 2)$-, a $(3n+5, 2)$-, and a $(2n+6, 4)$-edge-antimagic total labeling; P_{2n} has a $(6n, 1)$- and a $(6n + 2, 2)$-edge-antimagic total labeling; and several parity conditions for (a, d)-edge-antimagic total labelings. They conjecture: paths have no (a, d)-edge-antimagic vertex labelings with $d > 2$; C_{2n} has a $(2n + 3, 4)$- or a $(2n + 4, 4)$-edge-antimagic total labeling; C_{2n+1} has a $(n + 4, 5)$- or a $(n + 5, 5)$-edge-antimagic total labeling; and cycles have no (a, d)-antimagic total labelings with $d > 5$.

Bača, Lin, Miller and Simanjuntak [58] prove that a graph with v vertices and e edges that has an (a, d)-edge-antimagic vertex labeling must satisfy $d(e - 1) \leq 2v - 1 - a \leq 2v - 4$. As a consequence, they obtain: for every path there is no (a, d)-edge-antimagic vertex labeling with $d > 2$; for every cycle there is no (a, d)-edge-antimagic vertex labeling with $d > 1$; for K_n $(n > 1)$ there is no (a, d)-edge-antimagic vertex labeling (the cases for $n = 2$ and $n = 3$ are handled individually); $K_{n,n}$ $(n > 3)$ has no (a, d)-edge-antimagic vertex labeling; for every wheel there is no (a, d)-edge-antimagic vertex labeling; for every generalized Petersen graph there is no (a, d)-edge-antimagic vertex labeling with $d > 1$.

Bača et al. [58] show that a graph with an (a, d)-edge-antimagic total labeling must satisfy $d(e-1) \leq 2v-1-a \leq 2v-4$. As a consequence, they obtain: for every path there is no (a, d)-edge-antimagic total labeling with $d > 2$; for every cycle there is no (a, d)-edge-antimagic total labeling with $d > 1$; for every complete graph K_n there is no (a, d)-edge-antimagic total labeling with $n > 3$; $K_{n,n}$ has no (a, d)-edge-antimagic total labeling with $n > 3$; for every wheel there is no (a, d)-edge-antimagic total labeling; for every generalized Petersen graph there is no (a, d)-edge-antimagic total labeling with $d > 1$.

An $a, d)$-edge-magic total labeling of $G(V, E)$ is called a *super (a,d)-edge-magic* if the vertex labels are $\{1, 2, \ldots, |V(G)|\}$ and the edge labels are $\{|V(G)|+1, |V(G)|+2, \ldots, |V(G)|+|E(G)|\}$. Ngurah and Baskoro [428] have shown that for n odd and at least 3 the generalized Petersen graphs $P(n, 1)$ and $P(n, 2)$ have $((5n+5)/2, 2)$-edge-antimagic total labelings and $P(n, m), n \geq 3, 1 \leq m < n/2$ has a super $(4n+2, 1)$-edge-antimagic total labeling.

Hartsfield and Ringel [262] introduced antimagic graphs in 1990. A graph with q edges is called *antimagic* if its edges can be labeled with $1, 2, \ldots, q$ so that the sums of the labels of the edges incident to each vertex are distinct. Among the antimagic graphs are [262]: P_n $(n \geq 3)$, cycles, wheels, and K_n $(n \geq 3)$. Hartsfield and Ringel conjecture that every tree except P_2 is antimagic and, moreover, every connected graph except P_2 is antimagic. Alon, Kaplan, Lev, Roditty and Yuster [23] use probabilistic methods and analytic number theory to show that this conjecture is true for all graphs with n vertices and minmum degree $\Omega(\log n)$. They also prove that if G is a graph with $n \geq 4$ vertices and $\Delta(G) \geq n-2$, then G is antimagic and all complete partite graphs except K_2 are antimagic.

The concept of an (a, d)-antimagic labelings was introduced by Bodendiek and Wagner [112] in 1993. A connected graph $G = (V, E)$ is said to be (a, d)*-antimagic* if there exist positive integers a, d and a bijection $f\colon E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $g_f\colon V \to N$, defined by $g_f(v) = \sum\{f(u, v)\colon (u, v) \in E(G)\}$, is injective and $g_f(V) = \{a, a+d, \ldots, a+(|V|-1)d\}$. (In [378] these are called (a, d)*-vertex-antimagic edge labelings*). They prove ([114] and [115]) the Herschel graph is not (a, d)-antimagic and obtain both positive and negative results about (a, d)-antimagic labelings for various cases of graphs called *parachutes* $P_{g,b}$. ($P_{g,b}$ is the graph obtained from the wheel W_{g+p} by deleting p consecutive spokes.) In [53] Bača and Holländer prove that necessary conditions for $C_n \times P_2$ to be (a, d)-antimagic are $d = 1, a = (7n+4)/2$ or $d = 3, a = (3n+6)/2$ when n is even and $d = 2, a = (5n+5)/2$ or $d = 4, a = (n+7)/2$ when n is odd. Bodendiek and Walther [113] conjectured that $C_n \times P_2$ $(n \geq 3)$ is $((7n+4)/2, 1)$-antimagic when n is even and is $((5n+5)/2, 2)$-antimagic when n is odd. These conjectures were verified by Bača and Holländer [53] who further proved that $C_n \times P_2$ $(n \geq 3)$ is $((3n+6)/2, 3)$-antimagic when n is even. Bača and Holländer [53] conjecture that $C_n \times P_2$ is $((n+7)/2, 4)$-antimagic when n is odd and at least 7. Bodendiek and Walther [113] also conjectured that $C_n \times P_2$ $(n \geq 7)$ is $((n+7)/2, 4)$-antimagic. Bača and Holländer [55] prove that the generalized Petersen graph $P(n, 2)$ is $((3n+6)/2, 3)$-antimagic for

$n \equiv 0 \pmod 4$, $n \geq 8$ (see §2.7 for the definition). Bodendiek and Walther [116] proved that the following graphs are not (a,d)-antimagic: even cycles; paths of even order; stars; $C_3^{(k)}$; $C_4^{(k)}$; trees of odd order at least 5 that have a vertex that is adjacent to three or more end vertices; n-ary trees with at least two layers when $d = 1$; $K_{3,3}$; the Petersen graph; and K_4. They also prove: P_{2k+1} is $(k,1)$-antimagic; C_{2k+1} is $(k+2,1)$-antimagic; if a tree of odd order $2k+1$ $(k > 1)$ is (a,d)-antimagic, then $d = 1$ and $a = k$; if K_{4k} $(k \geq 2)$ is (a,d)-antimagic, then d is odd and $d \leq 2k(4k-3)+1$; if K_{4k+2} is (a,d)-antimagic, then d is even and $d \leq (2k+1)(4k-1)+1$; and if K_{2k+1} $(k \geq 2)$ is (a,d)-antimagic, then $d \leq (2k+1)(k-1)$. Lin, Miller, Simanjuntak and Slamin [378] show that no wheel W_n $(n > 3)$ has an (a,d)-antimagic labeling.

Yegnanarayanan [604] introduces several variations of antimagic labelings and provides some results about them.

The *antiprism* on $2n$ vertices has vertex set $\{x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}\}$ and edge set $\{x_{j,i}, x_{j,i+1}\} \cup \{x_{1,i}, x_{2,i}\} \cup \{x_{1,i}, x_{2,i-1}\}$ (subscripts are taken modulo n). For $n \geq 3$ and $n \not\equiv 2 \pmod 4$ Bača [48] gives $(6n+3,2)$-antimagic labelings and $(4n+4,4)$-antimagic labelings for the antiprism on $2n$ vertices. He conjectures that for $n \equiv 2 \pmod 4$, $n \geq 6$, the antiprism on $2n$ vertices has a $(6n+3,2)$-antimagic labeling and a $(4n+4,4)$-antimagic labeling.

Nicholas, Somasundaram and Vilfred [430] prove the following: If $K_{m,n}$ where $m \leq n$ is (a,d)-antimagic then d divides $((m-n)(2a+d(m+n-1)))/4 + dmn/2$; if $m+n$ is prime, then $K_{m,n}$ where $n > m > 1$ is not (a,d)-antimagic; if $K_{n,n+2}$ is (a,d)-antimagic, then d is even and $n+1 \leq d < (n+1)^2/2$; if $K_{n,n+2}$ is (a,d)-antimagic and n is odd, then a is even and d divides a; if $K_{n,n+2}$ is (a,d)-antimagic and n is even, then d divides $2a$; if $K_{n,n}$ is (a,d)-antimagic, then n and d are even and $0 < d < n^2/2$; if G has order n and is unicylic and (a,d)-antimagic, then $(a,d) = (2,2)$ when n is even and $(a,d) = (2,2)$ or $(a,d) = ((n+3)/2, 1)$; a cycle with m pendant edges attached at each vertex is (a,d)-antimagic if and only if $m = 1$; the graph obtained by joining an endpoint of P_m with one vertex of the cycle C_n is $(2,2)$-antimagic if $m = n$ or $m = n-1$; if $m+n$ is even the graph obtained by joining an endpoint of P_m with one vertex of the cycle C_n is (a,d)-antimagic if and only if $m = n$ or $m = n-1$. They conjecture that for n odd and at least 3, $K_{n,n+2}$ is $((n+1)(n^2-1)/2, n+1)$-antimagic and they have obtained several results about (a,d)-antimagic labelings of caterpillars.

Bača [47] defines a connected plane graph G with edge set E and face set F to be *(a,d)-face antimagic* if there exist positive integers a and d and a bijection $g\colon E \to \{1, 2, \ldots, |E|\}$ such that the induced mapping $\psi_g\colon F \to \{a, a+d, \ldots, a+(|F(G)|-1)d\}$ is also a bijection where for a face f, $\psi_g(f)$ is the sum of all $g(e)$ for all edges e surrounding f. Bača [47] and Bača and Miller [61] describe (a,d)-face antimagic labelings for a certain classes of convex polytopes. In [60] Bača and Miller define the class Q_n^m of convex polytopes with vertex set $\{y_{j,i} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, m+1\}$ and edge set $\{y_{j,i}y_{j,i+1} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, m+1\} \cup \{y_{j,i}y_{j+1,i} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\} \cup \{y_{j,i+1}y_{j+1,i} : 1+1, 2, \ldots, n; j = 1, 2, \ldots, m, j \text{ odd}\} \cup \{y_{j,i}y_{j+1,i+1} : i = 1, 2, \ldots, n; j = 1, 2, \ldots, m, j \text{ even}\}$ where $y_{j,n+1} = y_{j,1}$. They prove that for m odd,

$m \geq 3, n \geq 3, Q_n^m$ is $(7n(m+1)/2+2,1)$-face antimagic and when m and n are even, $m \geq 4, n \geq 4, Q_n^m$ is $(7n(m+1)/2+2,1)$-face antimagic. They conjecture that when n is odd, $n \geq 3$, and m is even, then Q_n^m is $((5n(m+1)+5)/2,2)-$face antimagic and $((n(m+1)+7)/2,4)$-face antimagic. They further conjecture that when n is even, $n > 4, m > 1$ or n is odd, $n > 3$ and m is odd, $m > 1$ then Q_n^m is $(3n(m+1)/2+3,3)$-face antimagic. In [49] Bača proves that for n even and at least 4 the prism $C_n \times P_2$ is $(6n+3,2)$-face antimagic and $(4n+4,4)$-face antimagic. He also conjectures that $C_n \times P_2$ is $(2n+5,6)$-face antimagic. In [57] Bača, Lin and Miller investigate (a,d)-face antimagic labelings of the convex polytopes $P_{m+1} \times C_n$. They show that if these graphs are (a,d)-face antimagic then either $d=2$ and $a=3n(m+1)+3$ or $d=4$ and $a=2n(m+1)+4$, or $d=6$ and $a=n(m+1)+5$. They also prove that if n is even, $n \geq 4$ and $m \equiv 1 \pmod 4$, $m \geq 3$, then $P_{m+1} \times C_n$ has a $(3n(m+1)+3,2)$-face antimagic labeling and if n is even, $n \geq 4$ and m is odd, $m \geq 3$, or if $n \equiv 2 \pmod 4$, $n \geq 6$ and m is even, $m \geq 4$, then $P_{m+1} \times C_n$ has a $(3n(m+1)+3,2)$-face antimagic labeling and a $(2n(m+1)+4,4)$-face antimagic labeling. They conjecture that $P_{m+1} \times C_n$ has $(3n(m+1)+3,2)$ and $(2n(m+1)+4,4)$-face antimagic labelings when $m \equiv 0 \pmod 4$, $n \geq 4$ and for m even and $m \geq 4$ and that $P_{m+1} \times C_n$ has a $(n(m+1)+5,6)$-face antimagic labeling when n is even and at least 4.

For a plane graph G, Bača and Miller [62] call a bijection h from $V(G) \cup E(G) \cup F(G)$ to $\{1,2,\ldots,|V(G)|+|E(G)| \cup |F(G)|\}$ a *d-antimagic labeling of type* $(1,1,1)$ if for every number s the set of s-sided face weights is $W_s = \{a_s, a_s+d, a_s+2d, \ldots, a_s+(f_s-1)d\}$ for some integers a_s and d, where f_s is the number of s-sided faces (W_s varies with s). They show that the prisms $C_n \times P_2$ $(n \geq 3)$ have a 1-antimagic labeling of type $(1,1,1)$ and that for $n \equiv 3 \pmod 4$ $C_n \times P_2$ have a d-antimagic labeling of type $(1,1,1)$ for $d = 2,3,4$ and 6. They conjecture for all $n \geq 3$, $C_n \times P_2$ has a d-antimagic labeling of type $(1,1,1)$ for $d = 2,3,4,5$ and 6. This conjecture has been proved for the case $d = 3$ and $n \neq 4$ by Bača, Miller and Ryan [63] (the case $d = 3$ and $n = 4$ is open). They also prove that for $n \geq 4$ the antiprism on $2n$ vertices has a d-antimagic labeling of type $(1,1,1)$ for $d = 1,2$ and 4. They conjecture the result holds for $d = 3,5$ and 6 as well.

Bača, Baskoro and Miller and [50] have proved that hexagonal planar honeycomb graphs with an even number of columns have a 2-antimagic and 4-antimagic labelings of type $(1,1,1)$. They conjecture that these honeycombs also have d-antimagic labelings of type $(1,1,1)$ for $d = 1,3$ and 5. They pose the odd number of columns case for $1 \leq d \leq 5$ as an open problem.

Sonntag [538] has extended the notion of antimagic labelings to hypergraphs. He shows that certain classes of cacti, cycle and wheel hypergraphs have antimagic labelings. In [59] Bača et al. survey results on antimagic, edge-magic total and vertex-magic total labelings.

Figueroa-Centeno, Ichishima and Muntaner-Batle [205] have introduced multiplicative analogs of magic and antimagic labelings. They define a graph G of size q to be *product magic* if there is a labeling f from $E(G)$ onto $\{1,2,\ldots,q\}$ such that, at each vertex v, the product of the labels on the edges incident with v is the same. They call a graph G of size q *product antimagic* if there is a labeling f from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that the products of the labels on the edges incident at each vertex v are distinct. They prove that a graph of size q is product magic if and only if $q \leq 1$ (that is, if and only if it is $K_2, \overline{K_n}$ or $K_2 \cup \overline{K_n}$); P_n $(n \geq 4)$ is product antimagic; every 2-regular graph is product antimagic; and, if G is product antimagic, then so are $G + K_1$ and $G \odot \overline{K}_n$. They conjecture that a connected graph of size q is product antimagic if and only if $q \geq 3$. They also define a graph G with p vertrices and q edges to be *product edge-magic* if there is a labeling f from $V(G) \cup E(G)$ onto $\{1, 2, \ldots, p+q\}$ such that $f(u) \cdot f(v) \cdot f(uv)$ is a constant for all edges uv and *product edge-antimagic* if there is a labeling f from $V(G) \cup E(G)$ onto $\{1, 2, \ldots, p+q\}$ such that for all edges uv the products $f(u) \cdot f(v) \cdot f(uv)$ are distinct. They prove $K_2 \cup \overline{K}_n$ is product edge-magic, a graph of size q without isolated vertices is product edge-magic if and only if $q \leq 1$ and that every graph other than K_2 and $K_2 \cup \overline{K}_n$ is product edge-antimagic.

In the table following we use these abbreviations

A antimagic labeling

(a, d)**-VAT** (a, d)-vertex-antimagic total labeling

(a, d)**-EAV** (a, d)-edge-antimagic vertex labeling

(a, d)**-EAT** (a, d)-edge-antimagic total labeling

(a, d)**-VAE** (a, d)-antimagic labeling

(a, d)**-FA** (a, d)-face antimagic labeling

d**-AT** d-antimagic labeling of type $(1, 1, 1)$

A question mark following an abbreviation indicates that the graph is conjectured to have the corresponding property. The table was prepared by Petr Kovar and Tereza Kovarova.

Table 9: **Summary of antimagic labelings**

Graph	*Labeling*	*Notes*
P_n	(a, d)-VAT	wide variety of a and d [51]
C_n	(a, d)-VAT	wide variety of a and d [51]
generalized Petersen graph $P(n, k)$	(a, d)-VAT	[52]
prisms $C_n \times P_2$	(a, d)-VAT	[52]
antiprisms	(a, d)-VAT	[52]
W_n	not (a, d)-VAT	for $n > 20$ [378]
P_n	$(3, 2)$-EAV not (a, d)-EAV	[512] with $d > 2$ [512]
P_{2n}	$(n + 2, 1)$-EAV	[512]
C_n	not (a, d)-EAV	with $d > 1$ [58]
C_{2n}	not (a, d)-EAV	[512]
C_{2n+1}	$(n + 2, 1)$-EAV $(n + 3, 1)$-EAV	[512] [512]
K_n	not (a, d)-EAV	for $n > 1$ [58]
$K_{n,n}$	not (a, d)-EAV	for $n > 3$ [58]
W_n	not (a, d)-EAV	[58]
generalized Petersen graph $P(n, k)$	not (a, d)-EAV	with $d > 1$ [58]
P_n	not (a, d)-EAT	with $d > 2$ [58]

Table 9: **continued**

Graph	*Labeling*	*Notes*
P_{2n}	$(6n, 1)$-EAT	[512]
	$(6n + 2, 2)$-EAT	[512]
P_{2n+1}	$(3n + 4, 2)$-EAT	[512]
	$(3n + 4, 3)$-EAT	[512]
	$(2n + 4, 4)$-EAT	[512]
	$(5n + 4, 2)$-EAT	[512]
	$(3n + 5, 2)$-EAT	[512]
	$(2n + 6, 4)$-EAT	[512]
C_n	$(2n + 2, 1)$-EAT	[512]
	$(3n + 2, 1)$-EAT	[512]
	not (a, d)-EAT	with $d > 5$ [58]
C_{2n}	$(4n + 2, 2)$-EAT	[512]
	$(4n + 3, 2)$-EAT	[512]
	$(2n + 3, 4)$-EAT?	[512]
	$(2n + 4, 4)$-EAT?	[512]
C_{2n+1}	$(3n + 4, 3)$-EAT	[512]
	$(3n + 5, 3)$-EAT	[512]
	$(n + 4, 5)$-EAT?	[512]
	$(n + 5, 5)$-EAT?	[512]
K_n	not (a, d)-EAT	with $d > 5$ [58]
$K_{n,n}$	not (a, d)-EAT	with $d > 5$ [58]
W_n	not (a, d)-EAT	with $d > 4$ [58]
generalized Petersen	not (a, d)-EAT	with $d > 4$ [58]
graph $P(n, k)$	$((5n + 5)/2, 2)$-EAT	for n odd, $n \geq 3$ and $k = 1, 2$[428]
	super $(4n + 2, 1)$-EAT	for $n \geq 3$, and $1 \leq k \leq n/2$ [428]
Trees	$(a, 1)$-EAT?	[58]

Table 9: **continued**

Graph	*Labeling*	*Notes*
P_n	A	for $n \geq 3$ [262]
C_n	A	[262]
W_n	A	[262]
K_n	A	for $n \geq 3$ [262]
every connected graph except P_2	A	[262]
Hershel graph	not (a,d)-VAE	[112] [114]
parachutes $P_{g,b}$ (see §5.6)	(a,d)-VAE	for certain classes [112] [114]
prisms $C_n \times P_2$	$((7n+4)/2,1)$-VAE $((5n+5)/2,2)$-VAE $((3n+6)/2,3)$-VAE $((n+7)/2,4)$-VAE?	$n \geq 3$, n even [113] [53] $n \geq 3$, n odd [113] [53] $n \geq 3$, n even [53] $n \geq 7$, [114] [53]
generalized Petersen graph $P(n,2)$	$((3n+6)/2,3)$-VAE	$n \geq 8$, $n \equiv 0 (\bmod 4)$ [55]
C_n	not (a,d)-VAE	n even [116]
C_{2n+1}	not $(n+2,1)$-VAE	n even [116]
P_{2n}	not (a,d)-VAE	[116]
P_{2n+1}	$(n,1)$-VAE	[116]
stars	not (a,d)-VAE	[116]
$C_3^{(k)}, C_4^{(k)}$	not (a,d)-VAE	[116]
$K_{n,n+2}$	$\left(\frac{(n+1)(n^2-1)}{2}, n+1\right)$-VAE	$n \geq 3$, n odd [116]

Table 9: **continued**

Graph	*Labeling*	*Notes*
$K_{3,3}$	not (a,d)-VAE	[116]
K_4	not (a,d)-VAE	[116]
Petersen graph	not (a,d)-VAE	[116]
W_n	not (a,d)-VAE	$n > 3$ [378]
antiprism on $2n$ vertices (see §5.6)	$(6n+3,2)$-VAE $(4n+4,4)$-VAE $(2n+5,6)$-VAE? $(6n+3,2)$-VAE? $(4n+4,4)$-VAE?	$n \geq 3$, $n \not\equiv 2(\bmod 4)$ [48] $n \geq 3$, $n \not\equiv 2(\bmod 4)$ [48] $n \geq 4$ [48] $n \geq 6$, $n \not\equiv 2(\bmod 4)$ [48] $n \geq 6$, $n \not\equiv 2(\bmod 4)$ [48]
Q_n^m (see §5.6)	$(7n(m+1)/2+2,1)$-FA $(7n(m+1)/2+2,1)$-FA $((5n(m+1)+5)/2,2)$-FA? $((n(m+1)+7)/2,4)$-FA? $(3n(m+1)/2+3,3)$-FA? $(3n(m+1)/2+3,3)$-FA?	$m \geq 3$, $n \geq 3$, m odd [60] $m \geq 4$, $n \geq 4$, m,n even [60] $m \geq 2$, $n \geq 3$, m even, n odd [60] $m \geq 2$, $n \geq 3$, m even, n odd [60] $m > 1$, $n > 4$, n even [60] $m > 1$, $n > 3$, m odd, n odd [60]
$C_n \times P_2$	$(6n+3,2)$-FA $(4n+4,4)$-FA $(2n+5,6)$-FA?	$n \geq 4$, n even [49] $n \geq 4$, n even [49] [49]
$P_{m+1} \times C_n$	$(3n(m+1)+3,2)$-FA $(3n(m+1)+3,2)$-FA and $(2n(m+1)+4,4)$-FA $(3n(m+1)+3,2)$-FA? $(2n(m+1)+4,4)$-FA? $(n(m+1)+5,6)$-FA?	$n \geq 4$, n even and [57] $m \geq 3$, $m \equiv 1(\bmod 4)$, $n \geq 4$, n even and [57] $m \geq 3$, m odd, [57] or $n \geq 6$, $n \equiv 2(\bmod 4)$ and $m \geq 4$, m even $m \geq 4$, $n \geq 4$, $m \equiv 0(\bmod 4)$ [57] $m \geq 4$, $n \geq 4$, $m \equiv 0(\bmod 4)$ [57] $n \geq 4$, n even [57]

Table 9: **continued**

Graph	*Labeling*	*Notes*
$P_n \times P_2$	d-AT	with $d = 1$ [62]
	d-AT	with $d = 2, 3, 4$ and 6 [62] for $n \equiv 3 \pmod 4$
	d-AT?	with $d = 2, 3, 4, 5, 6$ for $n \geq 3$ [62]
	d-AT	with $d = 3$ for $n \geq 4$ [63]
antiprism on $2n$ vertices	d-AT	with $d = 1, 2$ and 4 for $n \geq 4$ [63]
	d-AT?	with $d = 3, 5$ and 6 for $n \geq 4$ [63]
honeycomb graphs with even number of columns	d-AT	with $d = 3, 4$ [50]
	d-AT?	with $d = 1, 3$ and 5 [50]

6 Miscellaneous Labelings

6.1 Sum Graphs

In 1990, Harary [256] introduced the notion of a sum graph. A graph $G(V, E)$ is called a *sum graph* if there is an bijective labeling f from V to a set of positive integers S such that $xy \in E$ if and only if $f(x) + f(y) \in S$. Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain isolated vertices. In 1991 Harary, Hentzel and Jacobs [258] defined a *real sum graph* in an analogous way by allowing S to be any finite set of positive real numbers. However, they proved that every real sum graph is a sum graph. Bergstrand et al. [89] defined a *product graph* analogous to a sum graph except that 1 is not permitted to belong to S. They proved that every product graph is a sum graph and vice versa.

For a connected graph G, let $\sigma(G)$, the *sum number of* G, denote the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph (some authors use $s(G)$ for the sum number of G). A labeling that makes G together with $\sigma(G)$ isolated points a sum graph is called an *optimal sum graph* labeling. Ellingham [189] proved the conjecture of Harary [256] that $\sigma(T) = 1$ for every tree $T \neq K_1$. Smyth [534] proved that there is no graph G with e edges and $\sigma(G) = 1$ when $n^2/4 < e \leq n(n-1)/2$. Smyth [535] conjectures that the disjoint union of graphs with sum number 1 has sum number 1. More generally, Kratochvil, Miller and Nguyen [331] conjecture that $\sigma(G \cup H) \leq \sigma(G) + \sigma(H) - 1$. Hao [255] has shown that if $d_1 \leq d_2 \leq \cdots \leq d_n$ is the degree sequence of a graph G then $\sigma(G) > \max(d_i - i)$ where the maximum is taken over all i. Bergstand et al. [88] proved that $\sigma(K_n) = 2n - 3$.

Hartsfield and Smyth [263] claimed to have proved that $\sigma(K_{m,n}) = \lceil 3m+n-3 \rceil/2$ when $n \geq m$ but Yan and Liu [597] found counterexamples to this assertion when $m \neq n$. Pyatkin [441], Liaw, Kuo and Chang [375], Wang and Liu [580] and He et al. [266] have shown that for $2 \leq m \leq n$, $\sigma(K_{m,n}) = \lceil \frac{n}{p} + \frac{(p+1)(m-1)}{2} \rceil$ where $p = \lceil \sqrt{\frac{2n}{m-1} + \frac{1}{4}} - \frac{1}{2} \rceil$ is the unique integer such that $\frac{(p-1)p(m-1)}{2} < n \leq \frac{(p+1)p(m-1)}{2}$.

Miller et al. [409] proved that $\sigma(W_n) = \frac{n}{2}+2$ for n even and $\sigma(W_n) = n$ for $n \geq 5$ and n odd (see also [552]). Miller, Ryan and Smyth [410] prove that the complete n-partite graph on n sets of 2 nonadjacent vertices has sum number $4n-5$ and obtain upper and lower bounds on the complete n-partite graph on n sets of m nonadjacent vertices. Gould and Rödl [245] investigated bounds on the number of isolated points in a sum graph. A group of six undergraduate students [244] proved that $\sigma(K_n - \text{edge}) \leq 2n-4$. The same group of six students also investigated the difference between the largest and smallest labels in a sum graph, which they called the *spum*. They proved spum of K_n is $4n-6$ and the spum of C_n is at most $4n-10$. Kratochvil, Miller and Nguyen [331] have proved that every sum graph on n vertices has a sum labeling such that every label is at most 4^n.

At a conference in 2000 Miller [404] posed the following two problems. Given any graph G, does there exist an optimal sum graph labeling that uses the label 1? Find a class of graphs G that have sum number of the order $|V(G)|^s$ for $s > 1$. (Such graphs were shown to exist for $s = 2$ by Gould and Rödl in [245]).

Chang [151] generalized the notion of sum graph by permitting $x = y$ in the definition of sum graph. He calls graphs that have this kind of labeling *strong sum graphs* and uses $i^*(G)$ to denote the minimum positive integer m such that $G \cup mK_1$ is a strong sum graph. Chang proves that $i^*(K_n) = \sigma(K_n)$ for $n = 2, 3$ and 4 and $i^*(K_n) > \sigma(K_n)$ for $n \geq 5$. He further shows that for $n \geq 5$, $3n^{\log_2 3} > i^*(K_n) \geq 12\lfloor n/5 \rfloor - 3$.

In 1994 Harary [257] generalized sum graphs by permitting S to be any set of integers. He calls these graphs *integral sum graphs*. Unlike sum graphs, integral sum graphs need not have isolated vertices. Sharary [496] has shown that C_n and W_n are integral sum graphs for all $n \neq 4$. Chen [159] proved that trees obtained from a star by extending each edge to a path and trees all of whose vertices of degree not 2 are at least distance 4 apart are integral sum graphs. He conjectures that all trees are integral sum graphs. In [159] and [161] Chen gives methods for constructing new connected integral sum graphs from given integral sum graphs by identifying verticies. Chen [161] has shown that every graph is an induced subgraph of a connected integral sum graph. Chen [161] calls a vertex of a graph *saturated* if it is adjacent to every other vertex of the graph. He proves that every integral sum graph except K_3 has at most two saturated vertices and gives the exact structure of all integral sum graphs that have exactly two saturated vertices. Chen [161] also proves that a connected integral sum graph with $p > 1$ vertices and q edges and no saturated vertices satisfies $q \leq p(3p-2)/8 - 2$. Wu, Mao and Le [588] proved that mP_n are integral sum graphs. They also show that the conjecture of Harary [257] that the sum number of C_n equals the integral sum number of C_n if and only if $n \neq 3$ or 5 is false and that for $n \neq 4$ or 6 the integral sum number of C_n is at most 1.

B. Xu [594] has shown that the following are integral sum graphs: the union of any three stars; $T \cup K_{1,n}$ for all trees T; mK_3 for all m; and the union of any number of integral sum trees; Xu also proved that if $2G$ and $3G$ are integral sum graphs, then so is mG for all $m > 1$. Xu poses the question as to whether all disconnected forests are integral sum graphs. Nicholas and Somasundaram [429] prove that all banana trees (see Section 2.1) and the union of any number of stars are integral sum graphs.

Liaw, Kuo and Chang [375] proved that all caterpillars are intergral sum graphs (see also [588] and [594] for some special cases of caterpillers). This shows that the assertion by Harary in [257] that $K(1,3)$ and $S(2,2)$ are not integral sum graphs is incorrect. They also prove that all cycles except C_4 are integral sum graphs and they conjecture that every tree is an integral sum graph. Singh and Santhosh show that the crowns $C_n \odot K_1$ are integral sum graphs for $n \geq 4$ [522] and that the subdivsion graphs of $C_n \odot K_1$ are integral sum graphs for $n \geq 3$ [457].

Melnikov and Pyatkin [401] have shown that every 2-regular graph except C_4 is an integral sum graph and that for every positive integer r there exists an r-regular integral sum graph. They also show that the cube is not an integral sum graph. For any integral sum graph G, Melnikov and Pyatkin define the *integral radius of* G as the smallest natural number $r(G)$ that has all its vertex labels in the interval $[-r(G), r(G)]$. For the family of all integral sum graphs of order n they use $r(n)$ to denote maximum integral radius among all members of the family. Two questions they raise are: Is there a constant C such that $r(n) \leq Cn$; for $n > 2$, is $r(n)$ equal to the $(n-2)$th prime.

The concepts of sum number and integral sum number have been extended to hypergraphs. Sonntag and Teichert [539] prove that every hypertree (i.e., every connected, non-trivial, cycle-free hypergraph) has sum number 1 provided that a certain cardinality condition for the number of edges is fulfilled. In [540] the same authors prove that for $d \geq 3$ every d-uniform hypertree is an integral sum graph and that for $n \geq d+2$ the sum number of the complete d-uniform hypergraph on n vertices is $d(n-d)+1$. They also prove that the integral sum number for the complete d-uniform hypergraph on n vertices is 0 when $d = n$ or $n-1$ and is between $(d-1)(n-d-1)$ and $d(n-d)+1$ for $d \leq n-2$. They conjecture that for $d \leq n-2$ the sum number and the integral sum number of the complete d-uniform hypergraph are equal.

Teichert [557] proves that hypercycles have sum number 1 when each edge has cardinality at least 3 and that hyperwheels have sum number 1 under certain restrictions for the edge cardinalities. Teichert [556] determined an upper bound for the sum number of the d-partite complete hypergraph $K^d_{n_1,\ldots,n_d}$. In [558] Teichert defines the *strong hypercycle* $\mathcal{C}^d_n$ to be the d-uniform hypergraph with the same vertices as C_n where any d consecutive vertices of C_n form an edge of $\mathcal{C}^d_n$. He proves that for $n \geq 2d+1 \geq 5$, $\sigma(\mathcal{C}^d_n) = d$ and for $d \geq 2$, $\sigma(\mathcal{C}^d_{d+1}) = d$. He also shows that $\sigma(\mathcal{C}^3_5) = 3; \sigma(\mathcal{C}^3_6) = 2$ and he conjectures that $\sigma(\mathcal{C}^d_n) < d$ for $d \geq 4$ and $d+2 \leq n \leq 2d$.

The *integral sum number*, $\zeta(G)$, of G, is the minimum number of isolated vertices that must be added to G so that the resulting graph is an integral sum graph. Thus, by definition, G is a integral sum graph if and only if $\zeta(G) = 0$. Harary [257] conjectured that for $n \geq 4$ the integral sum number $\zeta(K_n) = 2n-3$. This conjecture was verified by Chen [158], by Sharary [496] and by B. Xu [594]. Yan and Liu proved: $\zeta(K_n - E(K_r)) = n-1$ when $n \geq 6, n \equiv 0 \pmod 3$ and $r = 2n/3 - 1$ [598]; $\zeta(K_{m.m}) = 2m-1$ for $m \geq 2$ [598]; $\zeta(K_n - \text{edge}) = 2n-4$ for $n \geq 4$ [598], [594]; if $n \geq 5$ and $n-3 \geq r$, then $\zeta(K_n - E(K_r)) \geq n-1$ [598]; if $\lceil 2n/3 \rceil - 1 > r \geq 2$, then $\zeta(K_n - E(K_r)) \geq 2n - r - 2$ [598]; and if $2 \leq m < n$, and $n = (i+1)(im - i + 2)/2$, then $\sigma(K_{m,n}) = \zeta(K_{m,n}) = (m-1)(i+1)+1$ while if $(i+1)(im-i+2)/2 < n < (i+2)[(i+1)m - i + 1]/2$, then $\sigma(K_{m,n}) = \zeta(K_{m,n}) = \lceil ((m-1)(i+1)(i+2) + 2n)/(2i+2) \rceil$ [598].

Nagamochi, Miller and Slamin [423] have determined upper and lower bounds on the sum number a graph. For most graphs $G(V,E)$ they show that $\sigma(G) = \Omega(|E|)$. He et al. [265] investigated $\zeta(K_n - E(K_r))$ where $n \geq 5$ and $r \geq 2$. They proved that $\zeta(K_n - E(K_r)) = 0$ when $r = n$ or $n-1$; $\zeta(K_n - E(K_r)) = n-2$ when $r = n-2$; $\zeta(K_n - E(K_r)) = n-1$ when $n-3 \geq r \geq \lceil 2n/3 \rceil - 1$; $\zeta(K_n - E(K_r)) = 3n - 2r - 4$ when $\lceil 2n/3 \rceil - 1 > r \geq n/2$; $\zeta(K_n - E(K_r)) = 2n-4$ when $\lceil 2n/3 \rceil - 1 \geq n/2 > r \geq 2$. Moreover, they prove that if $n \geq 5, r \geq 2$ and $r \neq n-1$, then $\sigma(K_n - E(K_r)) = \zeta(K_n - E(K_r))$.

In [431] Nicholas and Vilfred define the *edge reduced sum number* of a graph as the minimum number of edges whose removal from the graph results in a sum graph. They show that for K_n, $n \geq 3$, this number is $(n(n-1)/2 + \lfloor n/2 \rfloor)/2$. They ask for a characterization of graphs for which the edge reduced sum number is the same as its sum number. They conjecture that an integral sum graph of order p and size q exists if and only if $q \leq 3(p^2-1)/8 - \lfloor (p-1)/4 \rfloor$ when p is odd and $q \leq 3(3p-2)/8$ when p is even. They also define the *edge reduced integral sum number* in an analogous way and conjecture that for K_n this number is $(n-1)(n-3)/8 + \lfloor (n-1)/4 \rfloor$ when n is odd and $n(n-2)/8$ when n is even.

Alon and Scheinermann [24] generalized sum graphs by replacing the condition $f(x) + f(y) \in S$ with $g(f(x), f(y)) \in S$ where g is an arbitrary symmetric polynomial. They called a graph with this property a *g-graph* and proved that for a given symmetric polynomial g not all graphs are g-graphs. On the other hand, for every symmetric polynomial g and every graph G there is some vertex labeling so that G together with at most $|E(G)|$ isolated vertices is a g-graph.

Boland, Laskar, Turner, and Domke [119] investigated a modular version of sum graphs. They call a graph $G(V,E)$ a *mod sum graph* (MSG) if there exists a positive integer n and an injective labeling from V to $\{1, 2, \ldots, n-1\}$ such that $xy \in E$ if and only if $f(x) + f(y) \pmod n = f(z)$ for some vertex z. Obviously, all sum graphs are mod sum graphs. However, not all mod sum graphs are sum graphs. Boland et al. [119] have shown the following graphs are MSG: all trees on 3 or more vertices; all cycles on 4 or more vertices; and all $K_{2,n}$. They further proved that K_p $(p \geq 2)$ is not MSG (see also [239]) and that W_4 is MSG. They conjecture that W_p is MSG for $p \geq 4$. This conjecture was refuted by Sutton, Miller, Ryan and Slamin [553] who proved that for

$n \neq 4$, W_n is not MSG (the case where n is prime had been proved in 1994 by Ghoshal et al. [239]). In the same paper Sutton et al. also showed that for $n \geq 3$, $K_{n,n}$ is not MSG. Ghoshal, Laskar, Pillone and Fricke [239] proved that every connected graph is an induced subgraph of a connected MSG graph and any graph with n vertices and at least two vertices of degree $n-1$ is not MSG. Sutton et al. define the *mod sum number*, $\rho(G)$, of a connected graph G to be the least integer r such that $G + \overline{K_r}$ is MSG. Sutton and Miller [551] define the cocktail party graph $H_{m,n}$, $m, n \geq 2$, as the graph with a vertex set $V = \{v_1, v_2, v_3, \ldots, v_{mn}\}$ partitioned into n independent sets $V = \{I_1, I_2, \ldots, I_n\}$ each of size m such that $v_i v_j \in E$ for all $i, j \in \{1, \ldots, mn\}$ where $i \in I_p$, $j \in I_q$, $p \neq q$. The graphs $H_{m,n}$ can be used to model relational database management systems (see [549]). Sutton and Miller prove that $H_{m,n}$ is not MSG for $n > m \geq 3$ and $\rho(K_n) = n$ for $n \geq 4$. In [550] Sutton, Draganova and Miller prove that for n odd and $n \geq 5$, $\rho(W_n) = n$ and when n is even, $\rho(W_n) = 2$. Draganova [184] has shown that for $n \geq 5$ and n odd, $\rho(F_n) = n$. She poses as an open problem the determination of the mod sum number of the t-point suspension of C_n. Wallace [574] has proved that $K_{m,n}$ is MSG when n is even and $n \geq 2m$ or when n is odd and $n \geq 3m-3$ and that $\rho(K_{m,n}) = m$ when $3 \leq m \leq n < 2m$. He also proves that the complete m-partite $K_{n_1,n_2,\ldots,n_m}$ is not MSG when there exist n_i and n_j such that $n_i < n_j < 2n_i$. He poses the following conjectures: $\rho(K_{m,n}) = n$ when $3m-3 > n \geq m \geq 3$; if $K_{n_1,n_2,\ldots,n_m}$ where $n_1 > n_2 > \cdots > n_m$, is not MSG then $(m-1)n_m \leq \rho(K_{n_1,n_2,\ldots,n_m}) \leq (m-1)n_1$; if G has n vertices then $\rho(G) \leq n$; determining the mod sum number of a graph is NP-complete (Sutton has observed that Wallace probably meant to say 'NP-hard'); Miller [404] has asked if it is possible for the mod sum number of a graph G be of the order $|V(G)|^2$.

Grimaldi [251] has investigated labeling the vertices of a graph $G(V, E)$ with n vertices with distinct elements of the ring Z_n so that $xy \in E$ whenever $(x+y)^{-1}$ exists in Z_n

In his 2001 Ph.D. thesis Sutton [549] introduced two methods of graph labelings with applications to storage and manipulation of relational database links specifically in mind. He calls a graph $G = (V_p \cup V_i, E)$ a *sum* graph* of $G_p = (V_p, E_p)$ if there is an injective labeling λ of the vertices of G with non-negative integers with the property that $uv \in E_p$ if and only if $\lambda(u)+\lambda(v) = \lambda(z)$ for some vertex $z \in G$. The *sum* number*, $\sigma^*(G_p)$, is the minimum cardinality of a set of new vertices V_i (members of V_i are called *incidentals*) such that there exists a sum* graph of G_p on the set of vertices $V_p \cup V_i$. A *mod sum* graph* of G_p is defined in the identical fashion except the sum $\lambda(u) + \lambda(v)$ is taken modulo n where the vertex labels of G are restricted to $\{0, 1, 2, \ldots, n-1\}$. The *mod sum* number*, $\rho^*(G_p)$, of a graph G_p is defined in the analogous way. Sum* graphs are a generalization of sum graphs and mod sum* graphs are a generalization of mod sum graphs. Sutton shows that every graph is an induced subgraph of a connected sum* graph.

The following table summarizing what is known about sum graphs, mod sum graphs, sum* graphs and mod sum* graphs is reproduced from Sutton's Ph. D. thesis [549]. The results on sum* and mod sum* graphs are found in [549]. Sutton [549] poses the following conjectures: $\rho(H_{m,n}) \leq mn$ for $m, n \geq 2$, $\sigma^*(G_p) \leq |V_p|$, $\rho^*(G_p) \leq |V_p|$.

Table 10: **Summary of sum graph Labelings**

Graph	$\sigma(G)$	$\rho(G)$	$\sigma^*(G)$	$\rho^*(G)$
$K_2 = S_1$	1	1	0	0
Stars, S_n, $n \geq 2$	1	0	0	0
Trees, T_n, $n \geq 3$ when $T_n \neq S_n$	1	0	1	0
Cycle, C_3	1	0	1	0
Cycles, C_4	3	0	2	0
Cycles, C_n, $n \geq 4$	2	0	2	0
Wheels, W_4	4	0	2	0
Wheels, W_n, $n \geq 5$, n odd	n	n	2	0
Wheels, W_n, $n \geq 6$, n even	$\frac{n}{2}+2$	2	2	0
Fans, F_n, $n \geq 5$, n odd	?	n	1	0
Complete graphs, K_n, $n \geq 3$	$2n-3$	n	$n-2$	0
Cocktail party graphs, $H_{2,n}$	$4n-5$	0	?	0
Complete symmetric bipartite graphs, $K_{n,n}$	$\lceil \frac{4n-3}{2} \rceil$	?	?	?
Complete bipartite graphs, $K_{m,n}$ $2nm \geq n \geq 3$	?	n	?	?
Complete bipartite graphs, $K_{m,n}$ $m \geq 3n-3$, $n \geq 3$, m odd	?	0	?	0
Complete bipartite graphs, $K_{m,n}$ $m \geq 2n$, $n \geq 3$, m even	?	0	?	0

6.2 Prime and Vertex Prime Labelings

The notion of a prime labeling originated with Entringer and was introduced in a paper by Tout, Dabboucy and Howalla (see [370]). A graph with vertex set V is said to have a *prime labeling* if its vertices are labeled with distinct integers $1, 2, \ldots, |V|$ such that for each edge xy the labels assigned to x and y are relatively prime. Around 1980, Entringer conjectured that all trees have a prime labeling. So far, there has been little progress towards proving this conjecture. Among the classes of trees known to have prime labelings are: paths, stars, caterpillars, complete binary trees, spiders (i.e., trees with a one vertex of degree at least 3 and with all other vertices with degree at most 2) and all trees of order up 50 (see [437], [438] and [223]).

Other graphs with prime labelings include all cycles and the disjoint union of C_{2k} and C_n [176]. The complete graph K_n does not have a prime labeling for $n \geq 4$ and W_n is prime if and only if n is even (see [370]).

Seoud, Diab and Elsakhawi [467] have shown the following graphs are prime: fans; helms; flowers (see §2.2); stars; $K_{2,n}$; and $K_{3,n}$ unless $n = 3$ or 7. They also shown that $P_n + \overline{K_m}$ $(m \geq 3)$ is not prime.

For m and n at least 3, Seoud and Youssef [472] define $S_n^{(m)}$, the (m, n)*-gon star*, as the graph obtained from the cycle C_n by joining the two end vertices of the path P_{m-2} to every pair of consecutive vertices of the cycle such that each of the end vertices of the path is connected to exactly one vertex of the cycle. Seoud and Youssef [472] have proved the following graphs have prime labelings: books, $S_n^{(m)}, C_n \odot P_m, P_n + \overline{K}_2$ if and only if $n = 2$ or n is odd, and $C_n \odot K_1$ with a complete binary tree of order $2^k - 1$ $(k \geq 2)$ attached at each pendant vertex. They also prove that every spanning subgraph of a prime graph is prime and every graph is a subgraph of a prime graph. They conjecture that all unicycle graphs have prime labelings. Seoud and Youssef [472] prove the following graphs are not prime: $C_m + C_n; C_n^2$ for $n \geq 4$; P_n^2 for $n = 6$ and for $n \geq 8$; and Möbius ladders M_n for n even. They also give an exact formula for the maximum number of edges in a prime graph of order n and an upper bound for the chromatic number of a prime graph.

Youssef [606] has shown that helms, the union of stars $S_m \cup S_n$, and the union of cycles and stars $C_m \cup S_n$ are prime. He has also proved: $K_m \cup P_n$ is prime if and only if m is at most 3 or if $m = 4$ and n is odd; $K_n \odot K_1$ is prime if and only if $n \leq 7$; $K_m \cup S_n$ is prime if and only if the number of primes less than or equal to $m + n + 1$ is at least m; and that the complement of every prime graph with odd order at least 21 and every even order graph of order at least 16 is not prime.

Salmasian [455] has shown that every tree with n vertices $(n \geq 50)$ can be labeled with n integers between 1 and $4n$ so that every two adjacent vertices have relatively prime labels. Pikhurko [438] has improved this by showing that for any $c > 0$ there is an N such that any tree of order $n > N$ can be labeled with n integers between 1 and $(1 + c)n$ so that adjacent labels are relatively prime.

Varkey and Singh (see [568]) have shown the following graphs have prime labelings:

ladders, crowns, cycles with a chord, books, one point unions of C_n, cycles with a chord, $L_n + K_1$. Varkey [568] has shown that graph obtained by connecting two points with internally disjoint paths of equal length are prime. Varkey defines a *twig* as a graph obtained from a path by attaching exactly two pendent edges to each internal vertex of the path. He proves that twigs obtained from a path of odd length (at least 3) and lotus inside a circle (see Section 5.1) graphs are prime.

Given a collection of graphs $G_1, \ldots, G_n$ and some fixed vertex v_i from each G_i, Lee, Wui and Yeh [370] define $Amal\{(G_i, v_i)\}$, the almagamation of $\{(G_i, v_i)|\ i = 1, \ldots, n\}$, as the graph obtained by taking the union of the G_i and identifying $v_1, v_2, \ldots, v_n$. Lee et al. [370] have shown $Amal\{(G_i, v_i)\}$ has a prime labeling when G_i are paths and when G_i are cycles. They also showed that the almagamation of any number of copies of W_n, n odd, with a common vertex is not prime. They conjecture that for any tree T and v from T, the almagamation of two or more copies of T with v in common is prime. They further conjecture that the almagamation of two or more copies of W_n that share a common point is prime when n is even ($n \neq 4$). Vilfred, Somasundaram and Nicholas [573] have proved this conjecture for the case that $n \equiv 2 \pmod 4$ where the central vertices are identified.

Vilfred, Somasundaram and Nicholas [573] have also proved the following: helms are prime; the grid $P_m \times P_n$ is prime when $m \leq 3$ and n is a prime greater than m; the ladder $P_2 \times P_n$ is prime in the cases that $2n+1, n+1$ or $n+2$ is prime; the double cone $C_n + \overline{K_2}$ is prime only for $n = 3$; the double fan $P_2 \times \overline{K_2}(n \neq 2)$ is prime if and only if n is odd; every cycle with a P_k-chord is prime. They conjecture that the grid $P_m \times P_n$ is prime when n is prime and $n > m$.

For any finite collection $\{G_i, u_i v_i\}$ of graphs G_i, each with a fixed edge $u_i v_i$, Carlson [149] defines the edge amalgamation $Edgeamal\{(G_i, u_i v_i)\}$ as the graph obtained by taking the union of all the G_i and identifying their fixed edges. The case where all the graphs are cycles she calls *generalized books*. She proves that all generalized books are prime graphs. Moreover, she shows that graphs obtained by taking the union of cycles and identifying in each cycle the path P_n are also prime. Carlson also proves that C_m-snakes are prime.

A dual of prime labelings has been introduced by Deretsky, Lee and Mitchem [176]. They say a graph with edge set E has a *vertex prime labeling* if its edges can be labeled with distinct integers $1, \ldots, |E|$ such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is 1. Deretsky, Lee and Mitchem show the following graphs have vertex prime labelings: forests; all connected graphs; $C_{2k} \cup C_n$; $C_{2m} \cup C_{2n} \cup C_{2k+1}$; $C_{2m} \cup C_{2n} \cup C_{2t} \cup C_k$; and $5C_{2m}$. They further prove that a graph with exactly two components, one of which is not an odd cycle, has a vertex prime labeling and a 2-regular graph with at least two odd cycles does not have a vertex prime labeling. They conjecture that a 2-regular graph has a vertex prime labeling if and only if it does not have two odd cycles. Let $G = \bigcup_{i=1}^{t} C_{2n_i}$ and $N = \sum_{i=1}^{t} n_i$. In [121] Borosh, Hensley and Hobbs proved that there is a positive constant n_0 such that the conjecture of Deretsky et al. is true for the cases that (i) G is the disjoint union of at most seven cycles, or (ii) G is a union of cycles all of the same even length $2n$ if $n \leq 150\,000$ or if $n \geq n_0$, or (iii) $n_i \geq (\log N)^{4 \log\log\log n}$ for all $i = 1, \ldots, t$, or (iv) each C_{2n_i} is repeated at most n_i times. They end their paper with a discussion of graphs whose components are all even cycles, and of graphs with some components that are not cycles and some components that are odd cycles.

6.3 Edge-graceful Labelings

In 1985, Lo [389] introduced the notion of edge-graceful graphs. A graph $G(V, E)$ is said to be *edge-graceful* if there exists a bijection f from E to $\{1, 2, \ldots, |E|\}$ so that the induced mapping f^+ from V to $\{0, 1, \ldots, |V| - 1\}$ given by $f^+(x) = \sum\{f(xy)|xy \in E\}$ (mod $|V|$) is a bijection. Note that an edge-graceful graph is anti-magic (see §5.6). A necessary condition for a graph with p vertices and q edges to be edge-graceful is that $q(q+1) \equiv p(p+1)/2$ (mod p). Lee [340] notes that this necessary condition extends to any multigraph with p vertices and q edges. Lee, Lee and Murthy [337] proved that K_n is edge-graceful if and only if $n \not\equiv 2$ (mod 4). (An edge-graceful labeling for K_n for $n \not\equiv 2$ (mod 4) in [389] was incorrect.) Lee [340] notes that a multigraph with $p \equiv 2$ (mod 4) vertices is not edge-graceful and conjectures that this condition is sufficient for the edge-gracefulness of connected graphs. Lo proved that all odd cycles are edge-graceful and Wilson and Riskin [583] proved the Cartesian product of any number of odd cycles is edge-graceful. Lee [339] has conjectured that all trees of odd order are edge-graceful. Small [533] has proved that spiders (see §5.2 for the definition) of odd degree with the property that the distance from the vertex of degree greater than 2 to each end vertex is the same are edge-graceful. Keene and Simoson [310] proved that all spiders of odd order with exactly three end vertices are edge-graceful. Cabaniss, Low and Mitchem [135] have shown that regular spiders of odd order are edge-graceful. Lee and Seah [354] have shown that $K_{n,n,\ldots,n}$ is edge-graceful if and only if n is odd and the number of partite sets is either odd or a multiple of 4. Lee and Seah [353] have also proved that C_n^k (the kth power of C_n) is edge-graceful for $k < \lfloor n/2 \rfloor$ if and only if n is odd and C_n^k is edge-graceful for $k \geq \lfloor n/2 \rfloor$ if and only if n is a multiple of 4 or n is odd (see also [135]). Lee, Seah and Wang [359] gave a complete characterization of edge-graceful P_n^k graphs. Shiu, Lam and Cheng [505] proved that the composition of the path P_3 and any null graph of odd order is edge-graceful. Shiu, Lee and Schaffer [510] investigated the edge-gracefulness of multigraphs derived from paths, combs and spiders obtained by replacing each edge by k parallel edges. Lee and Seah [355] have also investigated edge-gracefulness of various multigraphs.

Lee and Seah (see [340]) define a *sunflower* graph $SF(n)$ as the graph obtained by starting with an n-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ and creating new vertices $w_1, w_2, \ldots, w_n$ with w_i connected to v_i and v_{i+1} (v_{n+1} is v_1). In [356] they prove that $SF(n)$ is edge-graceful if and only if n is even. In the same paper they prove that C_3 is the only triangular snake that is edge-graceful. Lee and Seah [353] prove that for $k \leq n/2$, C_n^k is edge-graceful if and only if n is odd and, for $k \geq n/2$, C_n^k is edge-graceful if and only if $n \not\equiv 2 \pmod 4$. Lee, Seah and Lo (see [340]) have proved that for n odd, $C_{2n} \cup C_{2n+1}, C_n \cup C_{2n+2}$ and $C_n \cup C_{4n}$ are edge-graceful. They also show that for odd k and odd n, kC_n is edge-graceful. Lee and Seah (see [340]) prove that the generalized Petersen graph $P(n,k)$ (see Section 2.7) is edge-graceful if and only if n is even and $k < n/2$. In particular, $P(n,1) = C_n \times P_2$ is edge-graceful if and only if n is even. Lee and Schaffer [459] proved that $C_m \times C_n$ $(m > 2, n > 2)$ is edge-graceful if and only if m and n are odd. They also showed that if G and H are edge-graceful regular graphs of odd order then $G \times H$ is edge-graceful and that if G and H are edge-graceful graphs where G is c-regular of odd order m and H is d-regular of odd order n, then $G \times H$ is edge-magic if $\gcd(c, mn) = \gcd(d, m) = 1$. They further show that if H has odd order, is $2d$-regular and edge-graceful with $\gcd(d, m) = 1$, then $C_{2m} \times H$ is edge-magic and if G is odd-regular, edge-graceful of even order m which is not divisible by 3, and G can be partitioned into 1-factors, then $G \times C_m$ is edge-graceful.

In 1987 Lee (see [357]) conjectured that $C_{2m} \cup C_{2n+1}$ is edge-graceful for all m and n except for $C_4 \cup C_3$. Lee, Seah and Lo [357] have proved this for the case that $m = n$ and m is odd. They also prove: the disjoint union of an odd number copies of C_m is edge-graceful when m is odd; $C_n \cup C_{2n+2}$ is edge-graceful; and $C_n \cup C_{4n}$ is edge-graceful for n odd.

Kendrick and Lee (see [340]) proved that there are only finitely many n for which $K_{m,n}$ is edge-graceful and they completely solve the problem for $m = 2$ and $m = 3$. Ho, Lee and Seah [281] use $S(n; a_1, a_2, \dots, a_k)$ where n is odd and $1 \le a_1 \le a_2 \le \cdots \le a_k < n/2$ to denote the (n, nk)-multigraph with vertices $v_0, v_1, \dots, v_{n-1}$ and edge set $\{v_i v_j |\, i \ne j, i - j \equiv a_t \pmod n \text{ for } t = 1, 2, \dots, k\}$. They prove that all such multigraphs are edge-graceful. Lee and Pritikin (see [340]) prove that the Möbius ladders of order $4n$ are edge-graceful. Lee, Tong and Seah [363] have conjectured that the total graph of a (p,p)-graph is edge-graceful if and only if p is even. They have proved this conjecture for cycles.

Kuang, Lee, Mitchem and Wang [333] have conjectured that unicyclic graphs of odd order are edge-graceful. They have verified this conjecture in the following cases: graphs obtained by identifying the end point of a path P_m with a vertex of C_n when $m + n$ is even; crowns with one pendant edge deleted; graphs obtained from crowns by identifying an endpoint of P_m, m odd, with a vertex of degree 1; amalgamations of a cycle and a star obtained by identifying the center of the star with a cycle vertex where the resulting graph has odd order; graphs obtained from C_n by joining a pendant edge to $n-1$ of the cycle vertices and two pendant edges to the remaining cycle vertex.

In 1997 Yilmaz and Cahit [605] introduced a weaker version of edge-graceful called E-cordial. Let G be a graph with vertex set V and edge set E and let f a function from E to $\{0,1\}$. Define f on V by $f(v) = \sum\{f(uv) | uv \in E\} \pmod 2$. The function f is called an *E-cordial* labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph that admits an E-cordial labeling is called *E-cordial*. Yilmaz and Cahit prove the following graphs are E-cordial: trees with n vertices if and only if $n \not\equiv 2 \pmod 4$; K_n if and only if $n \not\equiv 2 \pmod 4$; $K_{m,n}$ if and only if $m+n \not\equiv 2 \pmod 4$; C_n if and only if $n \not\equiv 2 \pmod 4$; regular graphs of degree 1 on $2n$ vertices if and only if n is even; friendship graphs $C_3^{(n)}$ for all n (see §2.2); fans F_n if and only if $n \not\equiv 1 \pmod 4$; and wheels W_n if and only if $n \not\equiv 1 \pmod 4$. They observe that graphs with $n \equiv 2 \pmod 4$ vertices can not be E-cordial. They generalize E-cordial labelings to E_k-cordial $(k > 1)$ labelings by replacing $\{0,1\}$ by $\{0,1,2,\ldots,k-1\}$. Of course, E_2-cordial is the same as E-cordial.

6.4 Line-graceful Labelings

Gnanajothi [240] has defined a concept similar to edge-graceful. She calls a graph with n vertices *line-graceful* if it is possible to label its edges with $0,1,2,\ldots,n$ so that when each vertex is assigned the sum modulo n of all the edge labels incident with that vertex the resulting vertex labels are $0,1,\ldots,n-1$. A necessary condition for the line-gracefulness of a graph is that its order is not congruent to $2 \pmod 4$. Among line-graceful graphs are (see [240, pp. 132–181]) P_n if and only if $n \not\equiv 2 \pmod 4$; C_n if and only if $n \not\equiv 2 \pmod 4$; $K_{1,n}$ if and only if $n \not\equiv 1 \pmod 4$; $P_n \odot K_1$ (combs) if and only if n is even; $(P_n \odot K_1) \odot K_1$ if and only if $n \not\equiv 2 \pmod 4$; (in general, if G has order n, $G \odot H$ is the graph obtained by taking one copy of G and n copies of H and joining the ith vertex of G with an edge to every vertex in the ith copy of H); mC_n when mn is odd; $C_n \odot K_1$ (crowns) if and only if n is even; mC_4 for all m; complete n-ary trees when n is even; $K_{1,n} \cup K_{1,n}$ if and only if n is odd; odd cycles with a chord; even cycles with a tail; even cycles with a tail of length 1 and a chord; graphs consisting of two triangles having a common vertex and tails of equal length attached to a vertex other than the common one; the complete n-ary tree when n is even; trees for which exactly one vertex has even degree. She conjectures that all trees with $p \not\equiv 2 \pmod 4$ vertices are line-graceful and proved this conjecture for $p \le 9$.

Gnanajothi [240] has investigated the line-gracefulness of several graphs obtained from stars. In particular, the graph obtained from $K_{1,4}$ by subdividing one spoke to form a path of even order (counting the center of the star) is line-graceful; the graph obtained from a star by inserting one vertex in a single spoke is line-graceful if and only if the star has $p \not\equiv 2 \pmod 4$ vertices; the graph obtained from $K_{1,n}$ by replacing each spoke with a path of length m (counting the center vertex) is line-graceful in the following cases: $n = 2$; $n = 3$ and $m \not\equiv 3 \pmod 4$; m is even and $mn + 1 \equiv 0 \pmod 4$.

Gnanajothi studied graphs obtained by joining disjoint graphs G and H with an edge. She proved such graphs are line-graceful in the following circumstances: $G = H$; $G = P_n, H = P_m$ and $m+n \not\equiv 0 \pmod 4$; and $G = P_n \odot K_1$, $H = P_m \odot K_1$ and $m+n \not\equiv 0 \pmod 4$.

6.5 Radio Labelings

In 2001 Chartrand, Erwin, Zhang and Harary [153] were motivated by regulations for channel assignments of FM radio stations to introduce radio labelings of graphs. A *radio labeling* of a connected graph G is an injection c from the vertices of G to the natural numbers such that

$$d(u,v) + |c(u) - c(v)| \geq 1 + diam(G)$$

for every two distinct vertices u and v of G. The *radio number* of $c, rn(c)$, is the maximum number assigned to any vertex of G. The *radio number* of $G, rn(G)$, is the minimum value of $rn(c)$ taken over all radio labelings c of G. Among the results of Chartrand et al. are: for $k \geq 2, rn(C_{2k+1} \leq k^2 + 1$; for $k \geq 3, rn(C_{2k} \leq k^2 - k + 2$; for $k \geq 6, rn(C_k) \geq 3\lceil n/2 - 1\rceil; rn(C_3) = 3; rn(C_4) = 5; rn(C_6) = 8; rn(C_7) = 10; rn(C_8) = 14; rn(K_{n_1,n_2,\ldots,n_k}) = n_1 + n_2 + \cdots + n_k + k - 1$. They also prove that if G is a connected graph of order n and diameter 2, then $n \leq rn(G) \leq 2n - 2$ and that for every pair of integers k and n with $n \leq k \leq 2n - 2$, there exists a connected graph of order n and diameter 2 with $rn(G) = k$. They further provide a characterization of connected graphs of order n and diameter 2 with prescribed radio number.

Zhang [615] proved the following: for $k \geq 2, rn(C_{4k+1} \geq 2k^2 + 2k + 1$; for k even and $k \geq 2, rn(C_{4k+2}) = 2k^2 + 5k + 2$; for k odd and $k \geq 2,\ 2k^2 + 5k + 2 \leq rn(C_{4k+2}) \leq 2k^2 + 5k + 3; rn(C_{4k+3} = 2k^2 + 2k + 3; rn(C_{4k+4} \geq 2k^2 + 6k + 4$; and for $k \geq 3,\ rn(C_{2k+1}) \leq k^2 - \lfloor d/2 \rfloor + 2$.

6.6 Representations of Graphs modulo n

In 1989 Erdős and Evans [198] defined a *representation* modulo n of a graph G with vertices $\{v_1, v_2, \ldots, v_r\}$ as a set $\{a_1, \ldots, a_r\}$ of distinct, nonnegative integers each less than n satisfying $\gcd(a_i - a_j, n) = 1$ if and only if v_i is adjacent to v_j. They proved that every finite graph can be represented modulo some positive integer. The *representation number*, Rep(G), is smallest such integer. Obviously the representation number of a graph is prime if and only if a graph is complete. Evans, Fricke, Maneri, McKee and Perkel [201] have shown that a graph is representable modulo a product of a pair of distinct primes if and only if the graph does not contain an induced subgraph isomorphic to $K_2 \cup 2K_1, K_3 \cup K_1$, or the complement of a chordless cycle of length at least five. Nešetřil and Pultr [425] showed that every graph can be represented modulo a product of some set of distinct primes. Evans et al. [201] proved that if G is representable modulo n and p is a prime divisor of n, then $p \geq \chi(G)$. Evans, Isaak and Narayan [202] determined representation numbers for specific families as follows (here we use q_i to denote the ith prime and for any prime p_i we use $p_{i+1}, p_{i+2}, \ldots, p_{i+k}$ to denote the next k primes larger than p_i): $\mathrm{Rep}(P_n) = 2 \cdot 3 \cdot \cdots \cdot q_{\lceil \log_2(n-1) \rceil}$; $\mathrm{Rep}(C_4) = 4$ and for $n \geq 3$, $\mathrm{Rep}(C_{2n}) = 2 \cdot 3 \cdot \cdots \cdot q_{\lceil \log_2(n-1) \rceil + 1}$; $\mathrm{Rep}(C_5) = 3 \cdot 5 \cdot 7 = 105$ and for $n \geq 4$ and not a power of 2, $\mathrm{Rep}(C_{2n+1}) = 3 \cdot 5 \cdot \cdots \cdot q_{\lceil \log_2 n \rceil + 1}$; if $m \geq n \geq 3$, then $\mathrm{Rep}(K_m - P_n) = p_i p_{i+1}$ where p_i is the smallest prime greater than or equal to $m - n = \lceil n/2 \rceil$; if $m \geq n \geq 4$, and p_i is the smallest prime greater than or equal to $m-n = \lceil n/2 \rceil$ then $\mathrm{Rep}(K_m - C_n) = q_i q_{i+1}$ if n is even and $\mathrm{Rep}(K_m - C_n) = q_i q_{i+1} q_{i+2}$ if n is odd; if $n \le m-1$, then $\mathrm{Rep}(K_m - K_{1,n}) = p_s p_{s+1} \cdots p_{s+n-1}$ where p_s is the smallest prime greater than or equal to $m-1$; $\mathrm{Rep}(K_m)$ is the smallest prime greater than or equal to m; $\mathrm{Rep}(nK_2) = 2 \cdot 3 \cdot \cdots q_{\lceil \log_2 n \rceil + 1}$; if $n, m \ge 2$, then $\mathrm{Rep}(nK_m) = p_i p_{i+1} \cdots p_{i+m-1}$, where p_i is the smallest prime satisfying $p_i \ge m$, if and only if there exists a set of $n-1$ mutually orthogonal Latin squares of order m; $\mathrm{Rep}(mK_1) = 2m$; if $t \le (m-1)!$, then $\mathrm{Rep}(K_m + tK_1) = p_s p_{s+1} \cdots p_{s+m-1}$ where p_s is the smallest prime greater than or equal to m.

In [424] Narayan asked for the values of $\mathrm{Rep}(C_{2^k+1})$ when $k \ge 3$ and $\mathrm{Rep}(G)$ when G is a complete multipartite graph or a disjoint union of complete graphs. He also asked about the behavior of the representation number for ramdom graphs.

6.7 k-sequential Labelings

In 1981 Bange, Barkauskas and Slater [71] defined a *k-sequential* labeling f of a graph $G(V, E)$ as one for which f is a bijection from $V \cup E$ to $\{k, k+1, \ldots, |V \cup E| + k - 1\}$ such that for each edge xy in E, $f(xy) = |f(x) - f(y)|$. This generalized the notion of *simply sequential* where $k = 1$ introduced by Slater. Bange, Barkauskas and Slater showed that cycles are 1-sequential and if G is 1-sequential then $G + K_1$ is graceful. Hegde [271] proved that every graph can be embedded as an induced subgraph of a 1-sequential graph. Hegde and Shetty [275] have shown that every T_p-tree (see §4.2 for the definition) is 1-sequential. In [528], Slater proved: K_n is 1-sequential if and only if $n \le 3$; for $n \ge 2$, K_n is not k-sequential for any $k \ge 2$; and $K_{1,n}$ is k-sequential if and only if k divides n. Acharya and Hegde [16] proved: If G is k-sequential then k is at most the independence number of G; P_{2n} is n-sequential for all n and P_{2n+1} is both n-sequential and $(n+1)$-sequential for all n; $K_{m,n}$ is k-sequential for $k = 1, m$ and n; $K_{m,n,1}$ is 1-sequential; and the join of any caterpillar and $\overline{K_t}$ is 1-sequential. Acharya [11] showed that if $G(E, V)$ is an odd graph with $|E| + |V| \equiv 1$ or $2 \pmod 4$ when k is odd or $|E| + |V| \equiv 2$ or $3 \pmod 4$ when k is even, then G is not k-sequential. Acharya also observed that as a consequence of results of Bermond, Kotzig and Turgeon [94] we have: mK_4 is not k-sequential for any k when m is odd and mK_2 is not k-sequential for any odd k when $m \equiv 2$ or $3 \pmod 4$ or for any even k when $m \equiv 1$ or $2 \pmod 4$. He further noted that $K_{m,n}$ is not k-sequential when k is even and m and n are odd, while $K_{m,k}$ is k-sequential for all k. Acharya [11] points out that the following result of Slater's [529] for $k = 1$ linking k-graceful graphs and k-sequential graphs holds in general: A graph is k-sequential if and only if $G + v$ has a k-graceful labeling f with $f(v) = 0$. Slater [528] also proved that a k-sequential graph with p vertices and $q > 0$ edges must satisfy $k \le p - 1$. Hegde [271] proved that every graph can be embedded as an induced subgraph of a simply sequential graph. In [11] Acharya conjectured that if G is a connected k-sequential graph of order p with $k > \lfloor p/2 \rfloor$, then $k = p - 1$ and $G = K_{1,p-1}$ and that, except for $K_{1,p-1}$, every tree in which all vertices are odd is k-sequential for all odd positive integers $k \le p/2$. Hegde [271] gave counterexamples for both of these conjectures.

6.8 Binary Labelings

In 1996 Caccetta and Jia [136] introduced binary labelings of graphs. Let $G = (V, E)$ be a graph. A mapping $f : E \mapsto \{0,1\}^m$ is called an M-*coding* if the induced mapping $g : V \mapsto \{0,1\}^m$, defined as $g(v) = \sum_{u\in V,\, uv\in E} f(uv)$ is injective, where the summation is modulo 2. An M-coding is called *positive* if the zero vector is not assigned to an edge and a vertex of G. Cacetta and Jia show that the minimal m for a positive M-coding equals $k+1$ if $|V| \in \{2^k, 2^k - 2, 2^k - 3\}$ and k otherwise, where $k = \lceil \log_2 |V| \rceil$.

6.9 Average Labelings

In 1997 Harminc [260] introduced a new kind of labeling in an effort to characterize forests and graphs without edges. Let $G = (V, E)$ be a graph. A mapping $f : V \mapsto N$ is called *average labeling* if for any $(v, u), (u, w) \in E$ one has $f(u) = (f(v) + f(w))/2$. A labeling is called *nontrivial* if any connected component of G (excluding isolated vertices) has at least two differently labeled vertices. Harminc provides three results towards the characterization of hereditary graphs properties in terms of average labelings. In particular, all maximal connected subgraphs of G are exactly paths (i.e., G is a linear forest) if and only if there exists a nontrivial average labeling of G. He also characterizes forests and graphs without edges by introducing a bit more complicated average-type labelings. In 2001 Harminc and Soták [261] gave a characterization of all non-complete connected graphs that have a non-trivial average labeling.

6.10 Sequentially Additive Graphs

Bange, Barkauskas and Slater [72] defined a *k-sequentially additive labeling* f of a graph $G(V, E)$ to be a bijection from $V \cup E$ to $\{k, \ldots, k + |V \cup E| - 1\}$ such that for each edge $xy, f(xy) = f(x) + f(y)$. They proved: K_n is 1-sequentially additive if and only if $n \leq 3$; C_{3n+1} is not k-sequentially additive for $k \equiv 0$ or 2 (mod 3); C_{3n+2} is not k-sequentially additive for $k \equiv 1$ or 2 (mod 3); C_n is 1-sequentially additive if and only if $n \equiv 0$ or 1 (mod 3); and P_n is 1-sequentially additive. They conjecture that all trees are 1-sequentially additive. Hegde [272] proved that $K_{1,n}$ is k-sequentially additive if and only if k divides n.

Acharya and Hegde [18] have generalized k-sequentially additive labelings by allowing the image of the bijection to be $\{k, k+d, \ldots, (k+|V\cup E|-1)d\}$. They call such a labeling *additively (k, d)-sequential.*

6.11 Divisor Graphs

G. Santhosh and G. Singh [458] call a graph $G(V, E)$ a *divisor graph* if V is a set of integers and $uv \in E$ if and only if u divides v or vice versa. They prove the following are divisor graphs: trees, mK_n, induced subgraphs of divisor graphs, $H_{m,n}$ (see Section 5.6), the one-point union of complete graphs of different orders, complete bipartite graphs, W_n for n even and $n > 2$, and $P_n + \overline{K}_t$. They also prove that C_n ($n \geq 4$) is a divisor graph if and only if n is even and if G is a divisor graph then for all n so is $G + K_n$.

6.12 Strongly Multiplicative Graphs

Beineke and Hegde [85] call a graph with p vertices *strongly multiplicative* if the vertices of G can be labeled with distinct integers $1, 2, \ldots, p$ so that the labels induced on the edges by the product of the end vertices are distinct. They prove the following graphs are strongly multiplicative: trees; cycles; wheels; K_n if and only if $n \leq 5$; $K_{r,r}$ if and only if $r \leq 4$; and $P_m \times P_n$. They then consider the maximum number of edges a strongly multiplicative graph on n vertices can have. Denoting this number by $\lambda(n)$, they show that $\lambda(4r) \leq 6r^2$, $\lambda(4r+1) \leq 6r^2+4r$, $\lambda(4r+2) \leq 6r^2+6r+1$, and $\lambda(4r+3) \leq 6r^2+10r+3$. It remains an open problem to find a nontrivial lower bound for $\lambda(n)$.

Seoud and Zid [479] prove the following graphs are multiplicative: Wheels; rK_n for all r and n at most 5; rK_n for $r \geq 2$ and $n = 6$ or 7; rK_n for $r \geq 3$ and $n = 8$ or 9; $K_{4,r}$ for all r; the corona of P_n and K_m for all n and $2 \leq m \leq 8$.

6.13 Strongly $\star$-graphs

A variation of strong multiplicity of graphs is a stronly $\star$-graph. A graph of order n is said to be a *strongly $\star$-graph* if its vertices can be assigned the values $1, 2, \ldots, n$ in such a way that, when an edge whose vertices are labeled i and j is labeled with the value $i + j + ij$, all edges have different labels. C. Adiga and D. Somashekara [20] have shown that all trees, cycles, and grids are strongly $\star$-graphs. They further consider the problem of determining the maximum number of edges in any strongly $\star$-graph of given order and relate it to the corresponding problem for strongly multiplicative graphs.

6.14 Sigma Labelings

Vilfred and Jinnah [572] call a labeling f from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$ a *sigma labeling* if for every vertex u the sum of all $f(v)$ such that v is adjacent to u is a constant independent of u. This notion was first introduced by Vilfred in his Ph. D. thesis in 1994. In [572] Vilfred and Jinnah give a number of necessary conditions for a graph to have a sigma labeling. One of them is that if u and v are vertices of a graph with a sigma labeling then the order of the symmetric difference of $N(u)$ and $N(v)$ is not 1 or 2. This condition rules out a large class of graphs as having sigma labelings. Vilfred and Jinnah raise a number of open questions: Does there exist connected graphs that have sigma labelings other than complete multipartite graphs (in [571] it is shown that $K_{2,2,\ldots,2}$ has a sigma labeling); Which complete multipartite graphs have sigma labelings; Is it true that $P_m \times C_n$ ($m > 1$) does not have a sigma labeling; Is every graph an induced subgraph of a graph with a sigma labeling (thye show that every graph is a subgraph of a graph with a sigma labeling)?

6.15 Set Graceful and Set Sequential Graphs

The notions of set graceful and set sequential graphs were introduced in by Acharaya in 1983. A graph is called *set graceful* if there is an assignment of nonempty subsets of a finite set to the vertices and edges of the graph so that the value given to each edge is the symmetric difference of the sets assigned to the endpoints of the edge, the assignment of sets to the vertices is injective and the assigment to the edges is bijective. A graph is called *set sequential* if there is an assignment of nonempty subsets of a finite set to the vertices and edges of the graph so that the value given to each edge is the symmetric difference of the sets assigned to the endpoints of the edge and the the assignment of sets to the vertices and the edges is bijective. The following has been shown: no cycle is set sequential [17]; a necessary condition for K_n to be set sequential is the n has the form $(\sqrt{2^{m+3}+7}-1)/2$ for some m [17]; a necessary condition for $K_{a,b,c}$ to be set sequential is that a, b and c cannot have the same parity; $K_{2,b,c}$ is not set sequential when b and c are odd [274]; P_n $(n > 3)$ is not set graceful [274]; no theta graph is set graceful [274]; the complete nontrivial n-ary tree is set sequential if and only if $n + 1$ is a power of 2 and the number of levels is 1 [274]; a tree is set sequential graceful if and only if it is set graceful [274]; every graph can be embedded as an induced subgraph of a connected set sequential graph [274]; every graph can be embedded as an induced subgraph of a connected set graceful graph [274].

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