Packing Unit Squares in Squares: A Survey and New Results

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Abstract

Let $s(n)$ be the side of the smallest square into which we can pack n unit squares. We survey the best known packings for $n \leq 100$. We also improve the best known upper bounds for $s(n)$ when n = 26, 29, 37, 39, 50, 54, 69, 70, 85, 86, and 88, and we present relatively simple proofs for the values of $s(n)$ when n = 2, 3, 5, 8, 15, 24, and 35.

1 Introduction

The problem of packing equal circles in a square has been around for some 30 years and has seen much recent progress [1]. The problem of packing equal squares in a square is less well known. Results seem to be more difficult, as the computer-aided methods available for circles do not generalize for squares. We intend to give some packings which improve upon those in the literature, illustrate a technique for obtaining lower bounds, and exhibit the best known packings for less than one hundred squares.

Let $s(n)$ be the side of the smallest square into which we can pack n unit squares. It is clear that

$$\sqrt{n} \leq s(n) \leq \lceil \sqrt{n}\ \rceil,$$

the first inequality coming from area considerations, and the second coming from the facts that $s(n)$ is non-decreasing and $s(n^2) = n$. It is not hard to show that $s(2) = s(3) = 2$. It is a little harder to show that $s(5) = 2 + 1/\sqrt{2}$ [5]. Göbel says that Schrijver claims that Bajmóczy proved $s(7) = 3$ and therefore $s(8) = 3$ [5]. The proof has not been published. Walter Stromquist claims to have proved $s(6) = 3$ and $s(10) = 3 + 1/\sqrt{2}$. He also claims to know how to prove $s(14) = s(15) = 4$ and $s(24) = 5$. All his results are unverified. In Section 4 we prove the value of $s(n)$ for n =2, 3, 5, 8, 15, 24, and 35. These are all the known values of $s(n)$. There are many other good packings thought to be optimal, but as of yet no proofs.

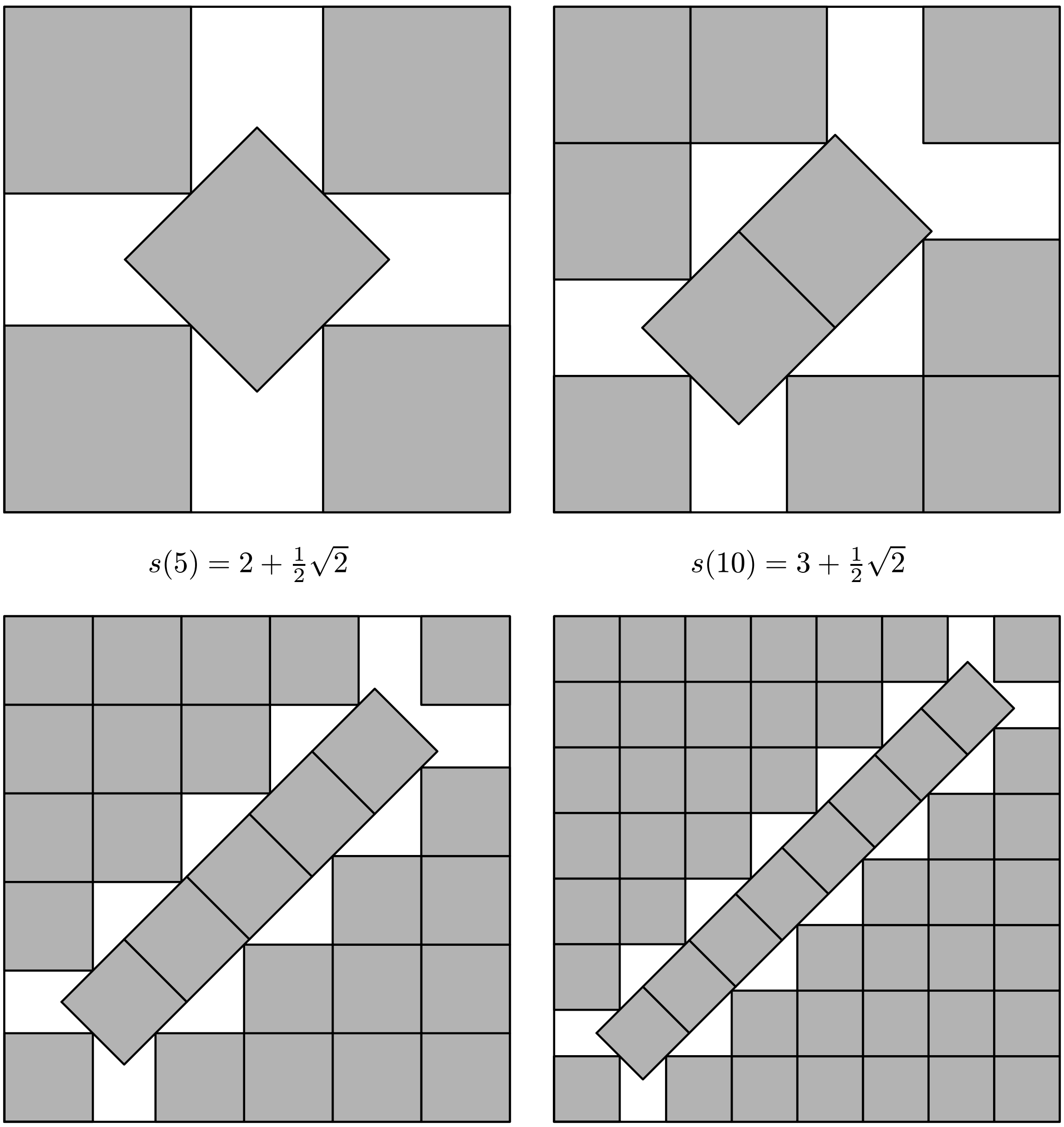
Previous results can be found in Section 2. Our improved packings appear in Section 3. Simple proofs of some values of $s(n)$ are in Section 4. A list of the best known upper bounds for $s(n)$ are given in Table 1 in the Appendix. Many of the results given are taken from unpublished letters.

2 Previous Results

Göbel was the first to publish on the subject. He found that

$$a^2 + a + 3 + \lfloor (a-1)\sqrt{2} \rfloor$$

squares can be packed in a square of side $a+1+\frac{1}{2}\sqrt{2}$ by placing a diagonal strip of squares at a 45° angle. This gives the best known packings for all values of a except for $a = 3$



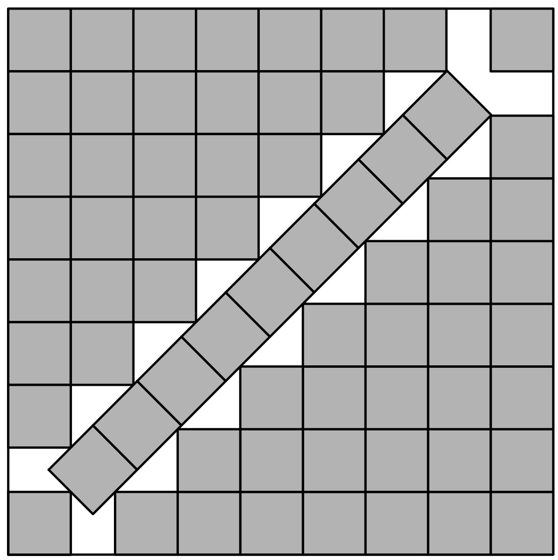
$s(5) = 2 + \frac{1}{2}\sqrt{2}$ $s(10) = 3 + \frac{1}{2}\sqrt{2}$

$s(27) \leq 5 + \frac{1}{2}\sqrt{2}$ $s(52) \leq 7 + \frac{1}{2}\sqrt{2}$

Figure 1.

(see Figures 1 and 2).

It is clear that $n + 2\lfloor s(n) \rfloor + 1$ squares can be packed in a square of side $s(n) + 1$ by packing n squares inside a square of side $s(n)$ and putting the other squares in an "L" around it. This modification of smaller packings gives optimal packings of 38 and 84 squares, and alternative optimal packings of 10 and 67 squares. Packings not containing an "L" of squares we will call *primitive* packings. We will only illustrate primitive packings.



$s(67) \leq 8 + \frac{1}{2}\sqrt{2}$

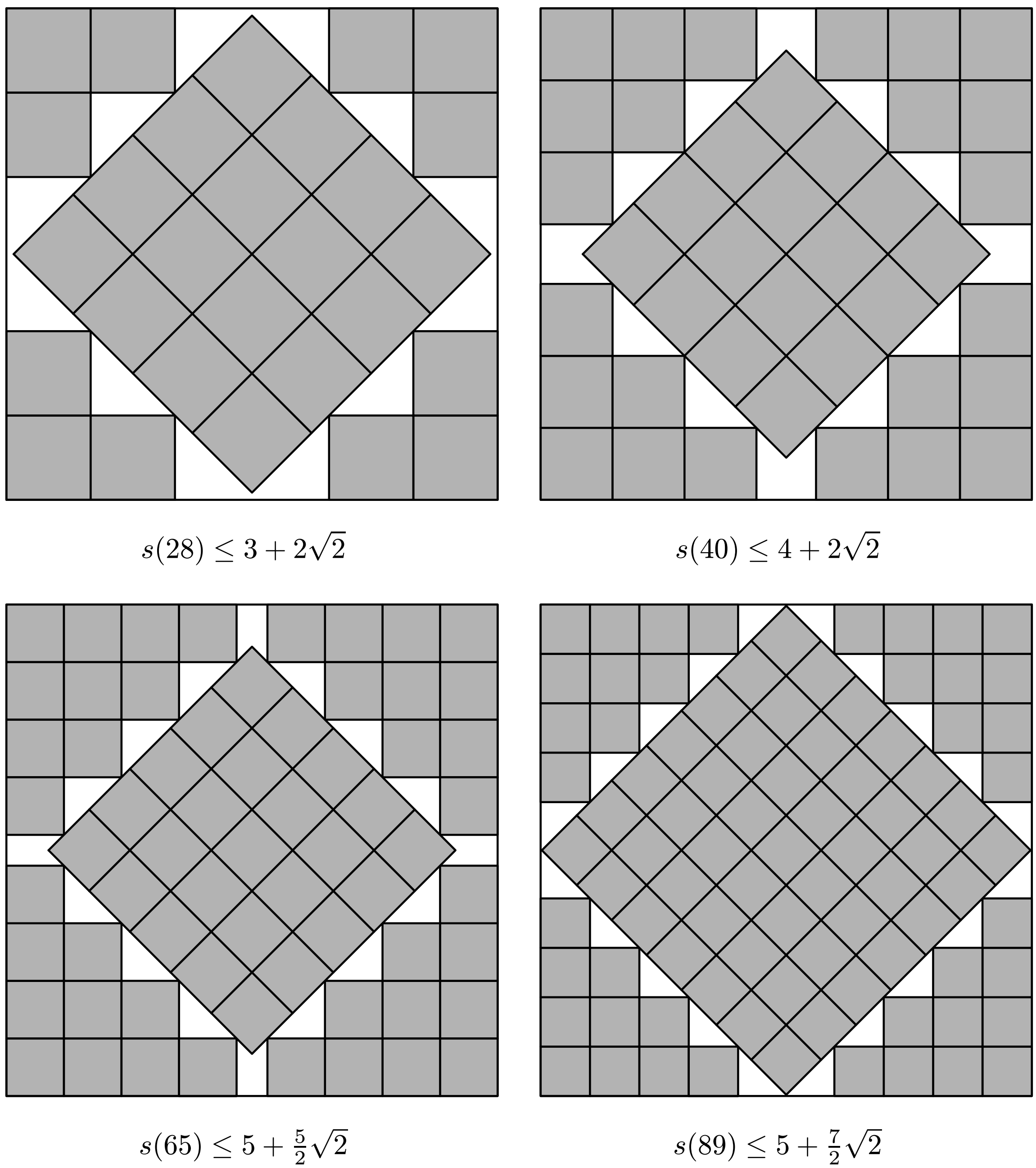
Figure 2.

Göbel also found that if integers a and b satisfied

$$a - 1 < \frac{b}{\sqrt{2}} < a + 1,$$

then $2a^2 + 2a + b^2$ squares can be packed inside a square of side $a + 1 + b/\sqrt{2}$. This is accomplished by placing a $b \times b$ square of squares at a 45° angle in the center. This gives the best known packings for 28, 40, 65, and 89 squares (see Figure 3).

By adding "L"s of squares around the packing of 28 squares, we get the best known packings of 37 and 50 squares. Adding "L"s around the packing of 40 squares gives the best known packings of 53 and 68 squares. Adding an "L" around the packing of 65 squares gives the best known packing of 82 squares.



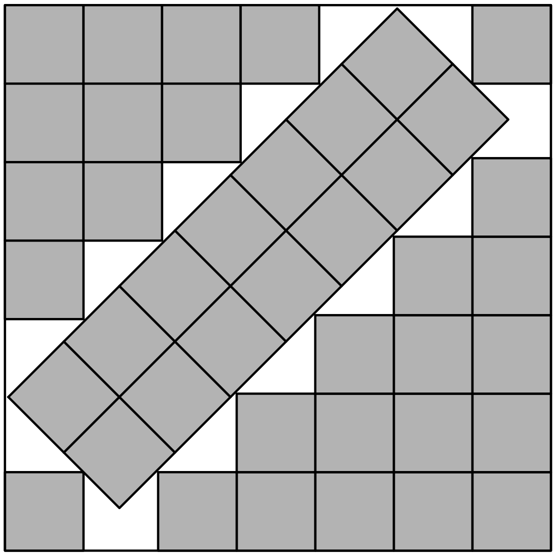
$s(28) \leq 3 + 2\sqrt{2}$ $s(40) \leq 4 + 2\sqrt{2}$

$s(65) \leq 5 + \frac{5}{2}\sqrt{2}$ $s(89) \leq 5 + \frac{7}{2}\sqrt{2}$

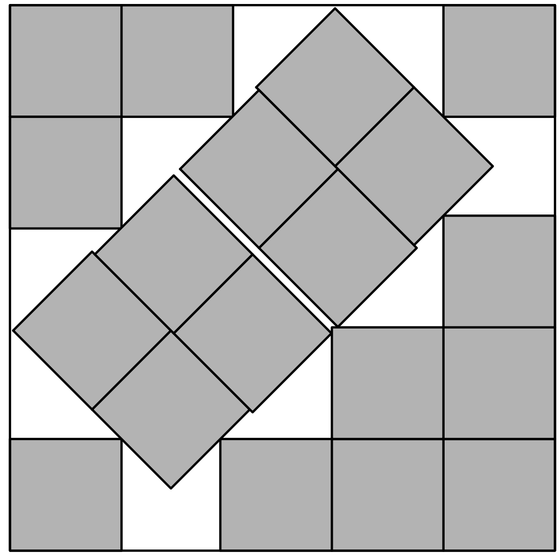
Figure 3.

Charles Cottingham, who improved some of Göbel's packings for $n \leq 49$, was the first to use diagonal strips of width 2. In 1979, he found the best known packing of 41 squares (see Figure 4). Although it is hard to see, the diagonal squares touch only the squares in the upper right and lower left corners.

Soon after Cottingham produced a packing of 19 squares with a diagonal strip of width 2, Robert Wainwright improved Cottingham's packing slightly (see Figure 4). This is still the best known packing of 19 squares [3].



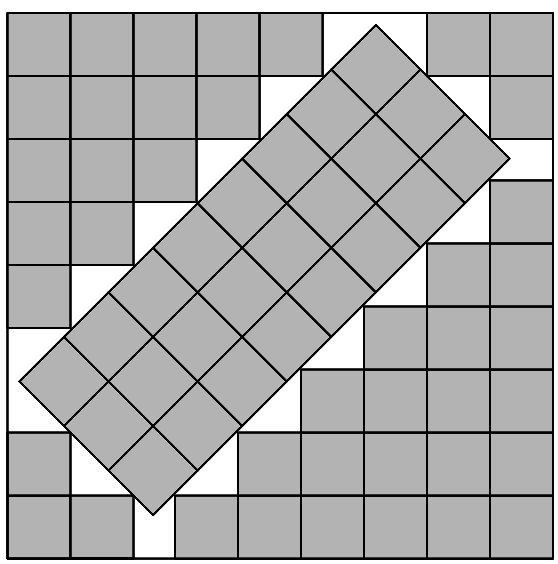
$s(41) \leq 2 + \frac{7}{2}\sqrt{2}$



$s(19) \leq 3 + \frac{4}{3}\sqrt{2}$

Figure 4.

In 1980, Evert Stenlund improved many of Cottingham's packings, and provided packings for $n \leq 100$. His packing of 66 squares uses a diagonal strip of width 3 (see Figure 5). In this packing, the diagonal squares touch only the squares in the upper right and lower left corners. Adding an "L" to this packing gives the best known packing of 83 squares.

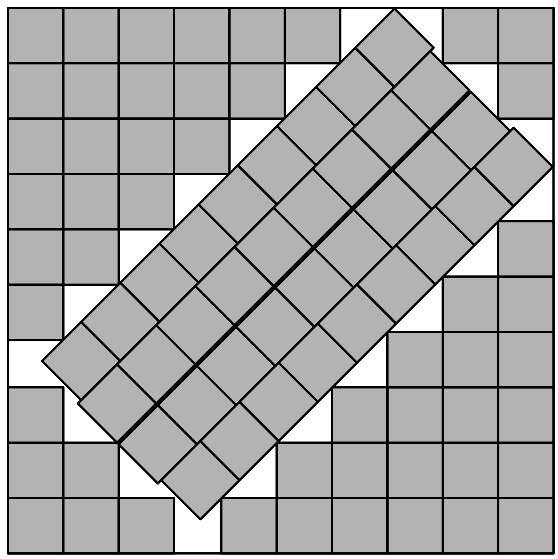


$s(66) \leq 3 + 4\sqrt{2}$

Figure 5.

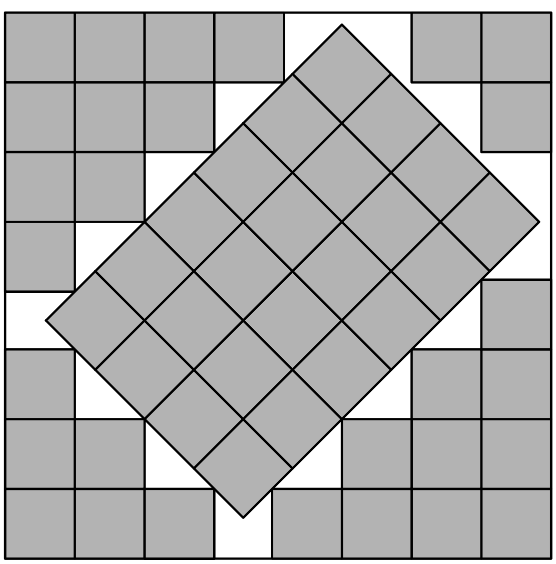
Note that a diagonal strip of width 2 or 3 must be off center in order to be optimal. Otherwise one could place at least as many squares by not rotating them.

Stenlund also modified a diagonal strip of width 4 to pack 87 squares (see Figure 6). There is a thin space between two of the diagonal strips. Compare this with the packing of 19 squares in Figure 4.

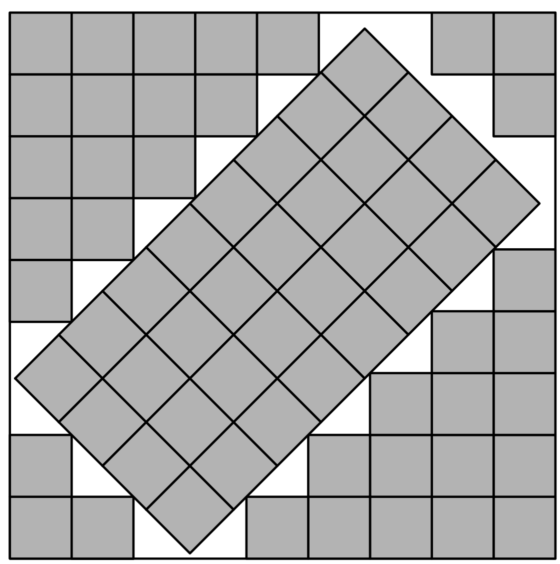


$s(87) \leq \frac{14}{3} + \frac{11}{3}\sqrt{2}$

Figure 6.



$s(53) \leq 5 + 2\sqrt{2}$



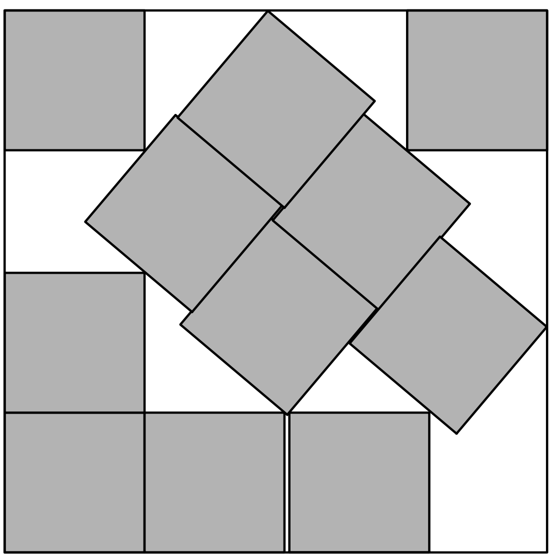
$s(68) \leq 6 + 2\sqrt{2}$

Figure 7.

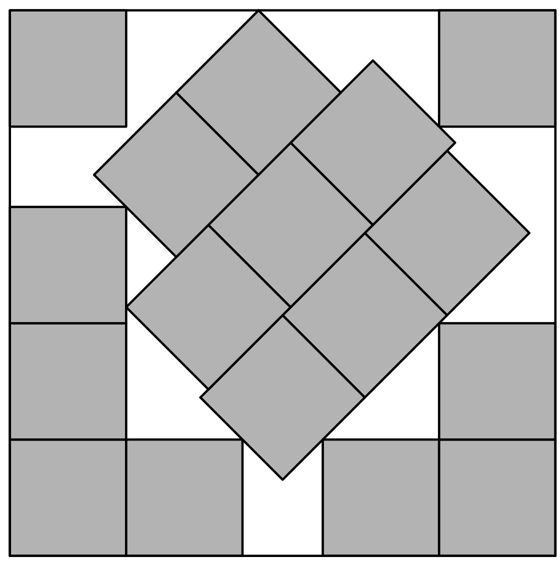
Diagonal strips of width 4 also give alternative optimal packings of 53 and 68 squares (see Figure 7).

The best known packings for many values of n are more complicated. Many seem to require packing with squares at angles other than 0° and 45°. In 1979, Walter Trump improved Göbel's packing of 11 squares (see Figure 8). Many people have independently discovered this packing. The original discovery has been incorrectly attributed to Gustafson and Thule [7]. The middle squares are tilted about 40.18194°, and there is a small gap between these squares.

In 1980, Pertti Hämäläinen improved Göbel's packing of 17 squares using a different arrangement of squares at a 45° angle (see Figure 8).



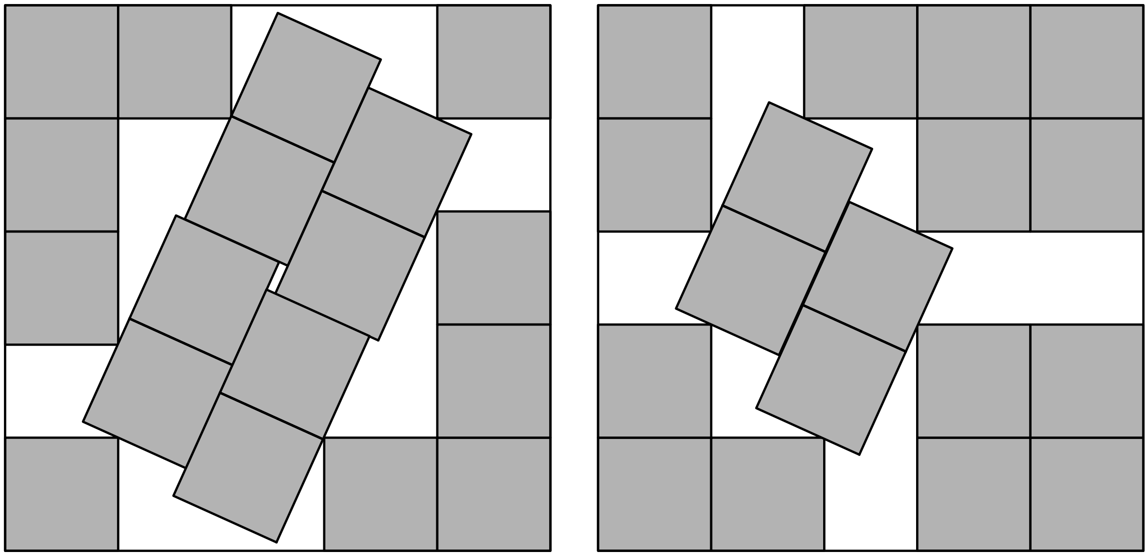
$s(11) \leq 3.8772$



$s(17) \leq \frac{7}{3} + \frac{5}{3}\sqrt{2}$

Figure 8.

By this time, Hämäläinen had already improved on Göbel's packing of 18 squares (see Figure 9). In 1981, Mats Gustafson found an alternative optimal packing of 18 squares (see Figure 9). The middle squares in these packings are tilted by an angle of $\sin^{-1}(\frac{\sqrt{7}-1}{4}) \approx 24.29518°$.



$s(18) \leq \frac{7}{2} + \frac{7}{2}\sqrt{2}$ $s(18) \leq \frac{7}{2} + \frac{7}{2}\sqrt{2}$

Figure 9.

In [2], Erdős and Graham define

$$W(s) = s^2 - \max\{n : s(n) \leq s\}.$$

Thus $W(s)$ is the wasted area in the optimal packing of unit squares into an $s \times s$ square. They show (by constructing explicit packings) that

$$W(s) = \mathcal{O}(s^{7/11}).$$

In [6], it is mentioned that Montgomery has improved this result to

$$W(s) = \mathcal{O}(s^{(3-\sqrt{3})/2+\epsilon})$$

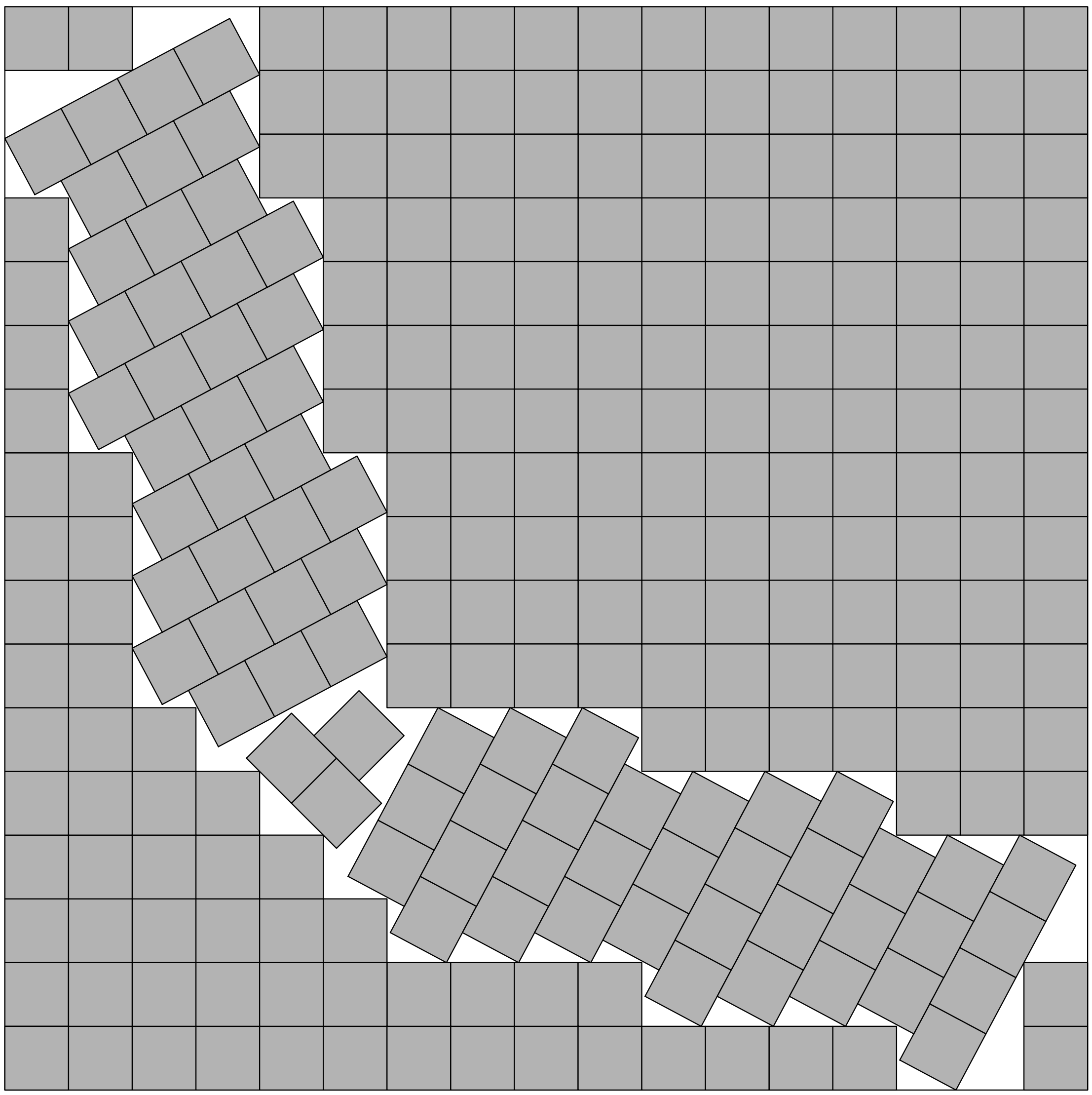
for every $\epsilon > 0$.

In [6], Roth and Vaughan establish a non-trivial lower bound for $W(s)$. They show that if $s\big(s - \lfloor s \rfloor\big) > \frac{1}{6}$, then

$$W(s) \geq 10^{-100}\sqrt{s \cdot |s - \lfloor s + .5 \rfloor|}.$$

This implies that $W(s) \neq \mathcal{O}(s^\alpha)$ when $\alpha < \frac{1}{2}$.

It is conjectured that $s(n^2 - n) = n$ whenever n is small. The smallest counterexample of this conjecture, due to Lars Cleemann, is $s(17^2 - 17) < 17$. 272 squares can be packed into a square of side 17 in such a way that the the square can be squeezed together slightly (see Figure 10). Three squares are tilted by an angle of 45°, and the other tilted squares are tilted by an angle of $\tan^{-1}(\frac{8}{15})$.



$s(272) < 17$

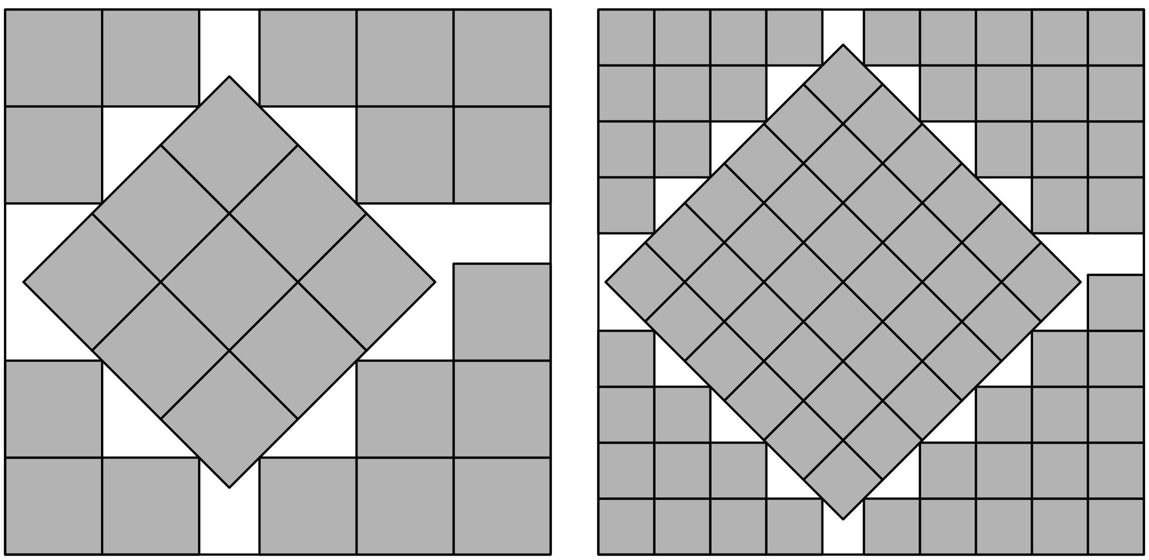
Figure 10.

3 New Packings

We can generalize the packings in Figure 3 by placing the central square a little off center. We can pack $2a^2+2a+b^2$ squares in a rectangle with sides

$$a+\frac{1}{2}+\frac{b}{\sqrt{2}} \quad \text{and} \quad a+\frac{3}{2}+\frac{b}{\sqrt{2}}.$$

Adding a column of squares to the side of this, we get a packing of $2a^2+4a+b^2+1$ squares in a square of side $a+\frac{3}{2}+b/\sqrt{2}$. This gives the best known packings for 26 and 85 squares (see Figure 11).



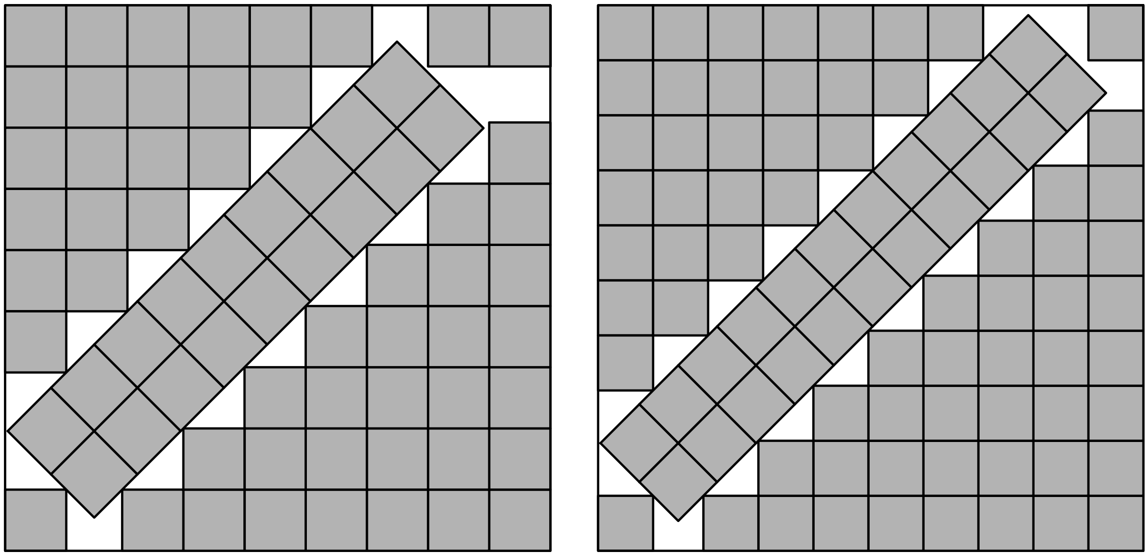
$s(26) \le \frac{7}{2}+\frac{3}{2}\sqrt{2}$ $\qquad$ $s(85) \le \frac{11}{2}+3\sqrt{2}$

Figure 11.

We can generalize Stenlund's packing of 41 squares in Figure 4 to packings of 70 and 88 squares (see Figure 12).

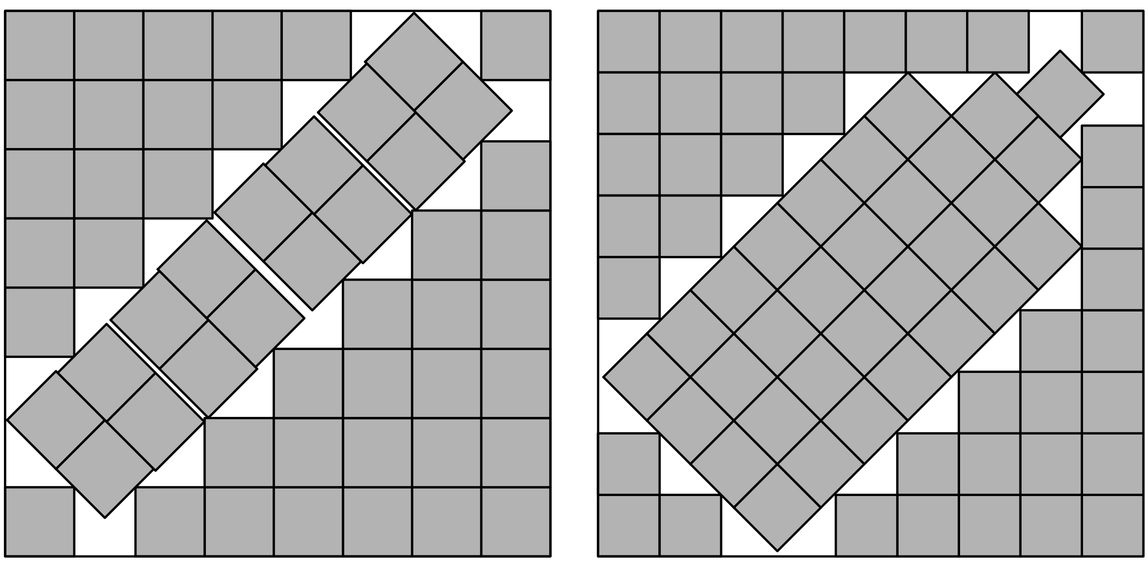
We can modify a diagonal strip of 2 squares to get an optimal packing of 54 squares (see Figure 13). Compare this with the packing of 19 squares in Figure 4. We can pack $9n^2+8n+2$ squares in a square of side $3n+\frac{4}{3}\sqrt{2}$ in this fashion.

We can also modify a strip of width 4 to get the best known packing of 69 squares by enlarging the bounding square and rearranging the upper right hand corner (see Figure 13). The diagonal squares touch only the squares in the upper right and lower left corners.



$s(70) \leq \frac{15}{2} + \sqrt{23}$ $\qquad$ $s(88) \leq \frac{17}{2} + \sqrt{2}$

Figure 12.

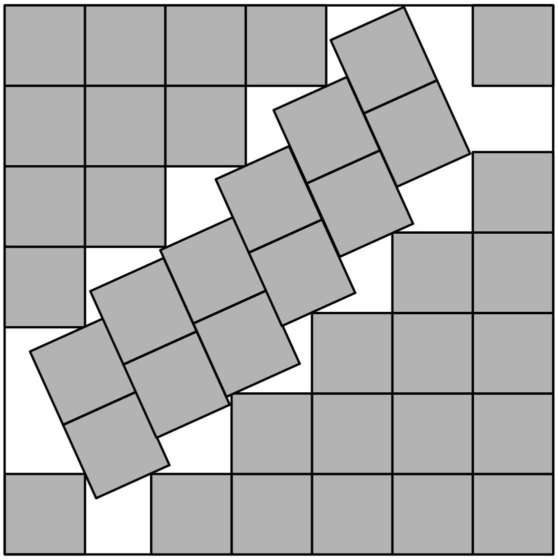


$s(54) \leq 6 + \frac{4}{3}\sqrt{2}$ $\qquad$ $s(69) \leq \frac{5}{2} + \frac{9}{2}\sqrt{2}$

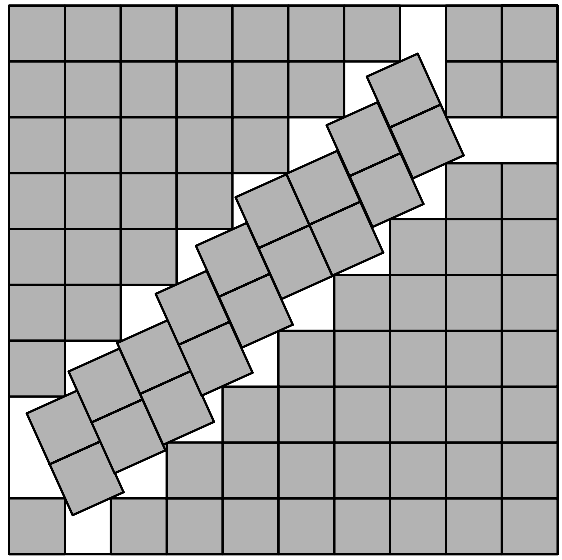
Figure 13.

We can generalize the packings in Figure 9 to provide the best known packings of 39 and 86 squares (see Figure 14). The angle of the tilted squares is the same as in Figure 9.

Our new packing of 29 squares uses a modified diagonal strip of width 2 (see Figure 15).

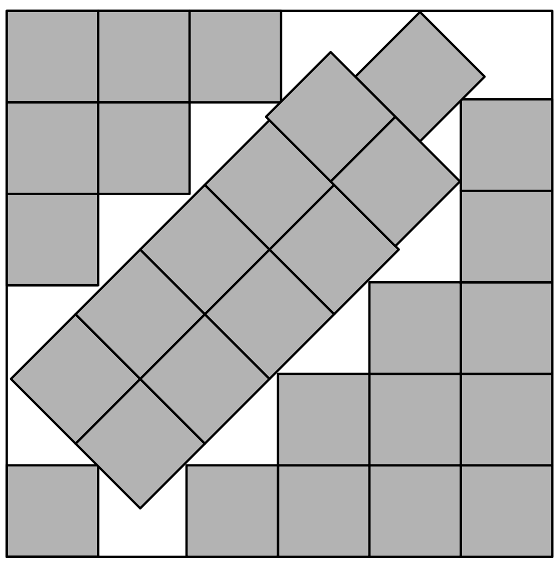


$s(39) \leq \frac{11}{2} + \frac{1}{2}\sqrt{7}$



$s(86) \leq \frac{17}{2} + \frac{1}{2}\sqrt{7}$

Figure 14.



$s(29) \leq 5.9665$

Figure 15.

The tilted squares contact the other squares in 3 places in the upper right and 1 place in the lower left. We label some important lengths in Figure 16. These lengths, the side of the square s, and the angle θ solve the following system of equations:

$$\begin{cases} 5\cos\theta + w\sin\theta = s - 2 \\ (5-x)\sin\theta + (2-w)\cos\theta = s - 2 \\ 2\sin\theta + x\cos\theta = s - 4 \\ 6\sin\theta + (1+y-w)\cos\theta = s - 1 \\ (1-z)\sin\theta + \cos\theta = s - 5 \\ (5+z)\cos\theta + (w-y)\sin\theta = s - 2 \end{cases}$$

The solution is $\theta \approx 44.994^\circ$ and $s \approx 5.9665$.

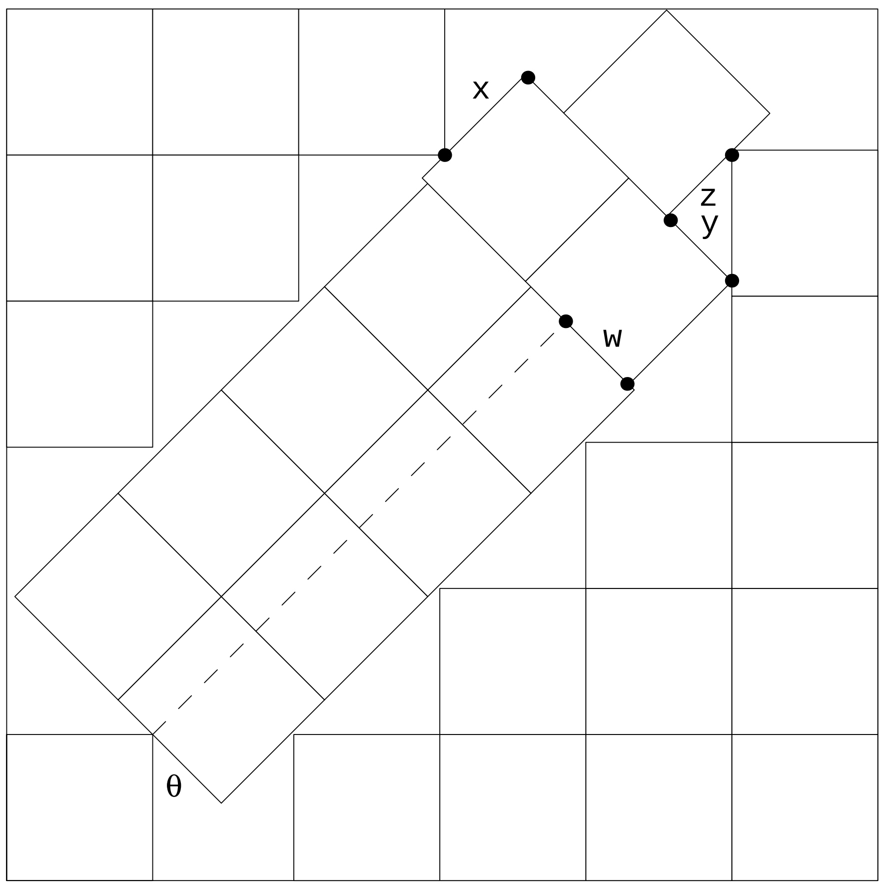
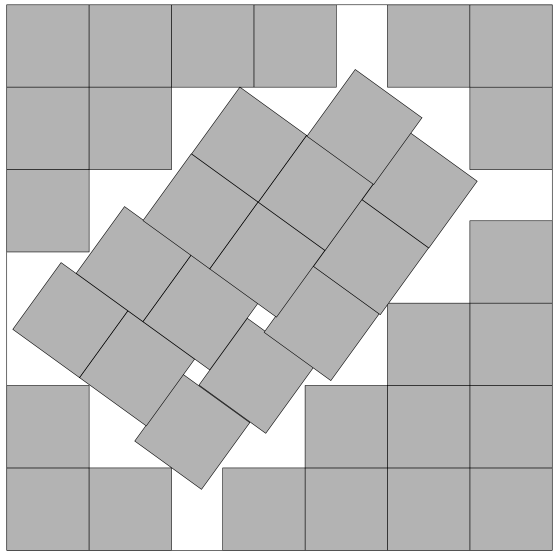


Figure 16.

Our new packing of 37 squares uses a modified diagonal strip of width 3 (see Figure 17).



$s(37) \leq 6.6213$

Figure 17.

The tilted squares contact the other squares in 3 places. Some important lengths are shown in Figure 18. These lengths, the side of the square s and the angle θ of the tilted squares satisfy the following system of equations:

$$\begin{cases} 4\sin\theta + x\cos\theta = s - 3 \\ y\sin\theta + 3\cos\theta = s - 4 \\ (3-x)\sin\theta + (4-y)\cos\theta = s - 3 \end{cases}$$

Solving for s gives

$$s = \frac{7 + 6\cos\theta + \cos 2\theta + 8\sin\theta + 3\sin 2\theta}{2 + \sin 2\theta},$$

which is minimized when $\theta \approx 51.1000°$ and $s \approx 6.6213$.

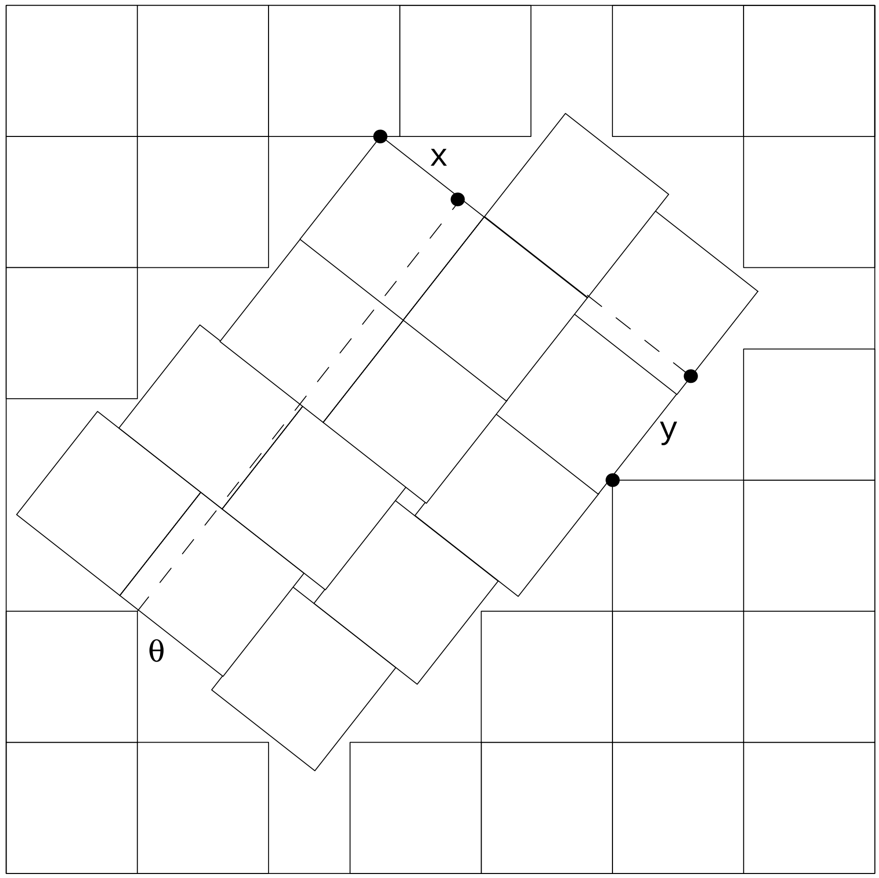


Figure 18.

Adding an "L" to this packing of 37 squares gives the best known packing of 50 squares.

Finally, we make the following conjectures:

Conjecture 1. If $s(n^2 - k) = n$, then $s\big((n+1)^2 - k\big) = n + 1$.

That is, if omitting k squares from an $n \times n$ square does not admit a smaller packing, then the same will be true for omitting k squares from any larger perfect square packing. This is true of all the best known packings.

Conjecture 2. $W(s) = \mathcal{O}(s^{1/2})$.

4 Lower Bounds

To show that $s(n) \geq k$, we will modify a method used by Walter Stromquist [8]. We will find a set P of $(n-1)$ points in a square S of side k so that any unit square in S contains an element of P (possibly on its boundary). Shrinking these by a factor of $(1-\epsilon/k)$ gives a set P' of $(n-1)$ points in a square S' of side $(k-\epsilon)$ so that any unit square in S' contains an element in P' in its interior. Therefore no more than $(n-1)$ non-overlapping squares can be packed into a square of side $(k-\epsilon)$, and $s(n) > k-\epsilon$. Since this is true for all $\epsilon > 0$, we must have $s(n) \geq k$.

We call P a set of *unavoidable* points in S. We now prove that certain sets of points are unavoidable.

Lemma 1. Any unit square inside the first quadrant whose center is in $[0,1]^2$ contains the point $(1,1)$.

Proof: It suffices to show that a unit square in the first quadrant that touches the x-axis and y-axis contains the point $(1,1)$. If the square is at an angle θ, it contains the points $(\sin\theta, 0)$ and $(0, \cos\theta)$ (see Figure 19). The two other corners of the square, $(\cos\theta, \cos\theta + \sin\theta)$ and $(\cos\theta + \sin\theta, \sin\theta)$, lie on the line $y - \sin\theta = -\cot\theta(x - \sin\theta - \cos\theta)$. In particular, when $x = 1$,

$$y = \frac{\sin^2\theta - \cos\theta(1-\sin\theta-\cos\theta)}{\sin\theta} = \frac{(1-\sin\theta)(1-\cos\theta)+\sin\theta}{\sin\theta} \geq 1. \blacksquare$$

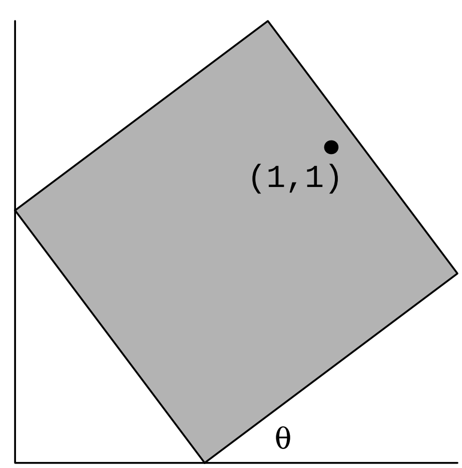


Figure 19.

This is enough to show

Theorem 1. $s(2) = s(3) = 2$.

Proof: Consider a unit square u in $[0,2]^2$. Since the center of u is either in $[0,1]^2$ or $[0,1] \times [1,2]$ or $[1,2] \times [0,1]$ or $[1,2]^2$, Lemma 1 shows that u contains the point $(1,1)$. That is, the set $P = \{(1,1)\}$ is unavoidable in $[0,2]^2$ (see Figure 20). ■

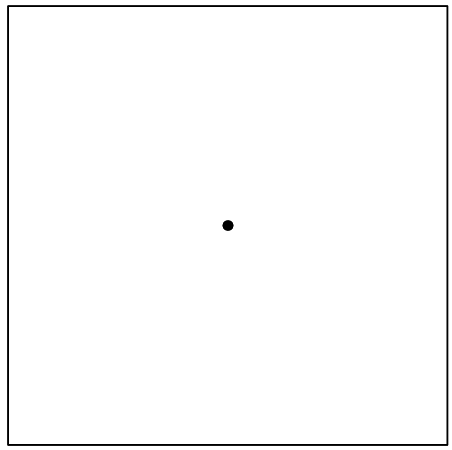


Figure 20. 1 unavoidable point in a square of side 2

Lemma 2. Let $0 < x \le 1$, $0 < y \le 1$, and $x + 2y < 2\sqrt{2}$. Then any unit square inside the first quadrant whose center is contained in $[1, 1+x] \times [0, y]$ contains either the point $(1, y)$ or the point $(1 + x, y)$.

Proof: It suffices to show that a unit square u whose center is contained in $[1, 1+x] \times [0, y]$ that contains the points $(1, y)$ and $(1+x, y)$ on its boundary contains a point on the x-axis. This is true if $(1, y)$ and $(1 + x, y)$ lie on the same side of u. If u is at an angle θ, then the lowest corner of the square is

$$(1 + x + (1 - x\sin\theta)\sin\theta - \cos\theta\,,\ y - (1 - x\sin\theta)\cos\theta - \sin\theta)$$

(see Figure 21). This point lies outside the first quadrant when $f(\theta) = \cos\theta + \sin\theta - x\sin\theta\cos\theta > y$. Since

$$f'(\theta) = (\cos\theta - \sin\theta)\big[1 - x(\cos\theta + \sin\theta)\big],$$

the critical points of $f(\theta)$ are

$$(\cos\theta, \sin\theta) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad \text{and} \quad (\cos\theta, \sin\theta) = \left(\frac{1 \pm \sqrt{2x^2 - 1}}{2x}, \frac{1 \mp \sqrt{2x^2 - 1}}{2x}\right).$$

Checking these 3 values and the endpoints, the global minimum of $f(\theta)$ occurs at $\theta = \frac{\pi}{4}$. Therefore, when $y < \sqrt{2} - \frac{x}{2}$, u contains some point of the x-axis. ■

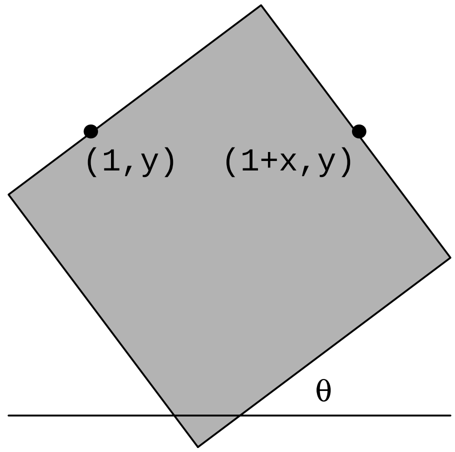


Figure 21.

Lemma 3. If the center of a unit square u is contained in $\triangle ABC$, and each side of the triangle has length no more than 1, then u contains A, B, or C.

Proof: The diagonals of u divide the plane into 4 regions, labeled clockwise as R_1, R_2, R_3, and R_4. These regions are closed, and intersect only on the diagonals. The points A, B, and C cannot all be on one side of either one of these diagonals, for then $\triangle ABC$ would not contain the center of u. Thus either both R_1 and R_3 contain vertices of the triangle, or both R_2 and R_4 do. In either case, two vertices of $\triangle ABC$ are closest to two opposite sides of u. Since the distance between these vertices is no more than 1, u must contain at least one of these points. ∎

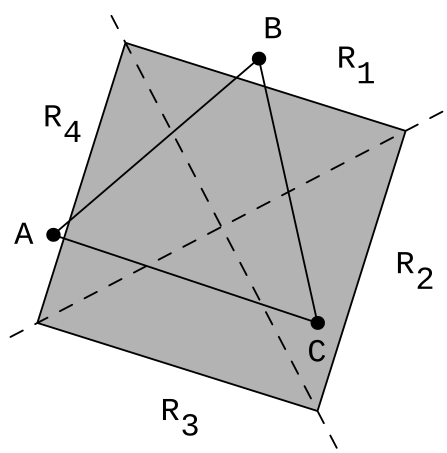


Figure 22.

Now we can show

Theorem 2. $s(5) = 2 + \frac{1}{2}\sqrt{2}$.

Proof: The set $P = \{(1,1), (1, 1+\frac{1}{2}\sqrt{2}), (1+\frac{1}{2}\sqrt{2}, 1), (1+\frac{1}{2}\sqrt{2}, 1+\frac{1}{2}\sqrt{2})\}$ is unavoidable in $\left[0, 2+\frac{1}{\sqrt{2}}\right]^2$. This follows from Lemma 1 if the center of the square is in a corner, from Lemma 2 if it is near a sides, and from Lemma 3 if it is in a triangle (see Figure 23). ∎

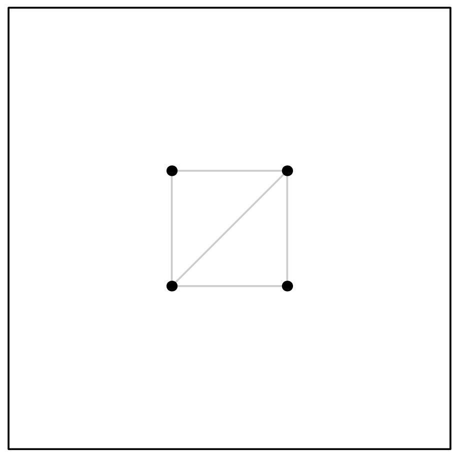


Figure 23. 4 unavoidable points in a square of side $2 + \frac{1}{2}\sqrt{2}$

Theorem 3. $s(8) = 3$.

Proof: The set $P = \{(.9, 1), (1.5, 1), (2.1, 1), (1.5, 1.5), (.9, 2), (1.5, 2), (2.1, 2)\}$ is unavoidable in $[0, 3]^2$ by Lemmas 1, 2, and 3 (see Figure 24). ∎

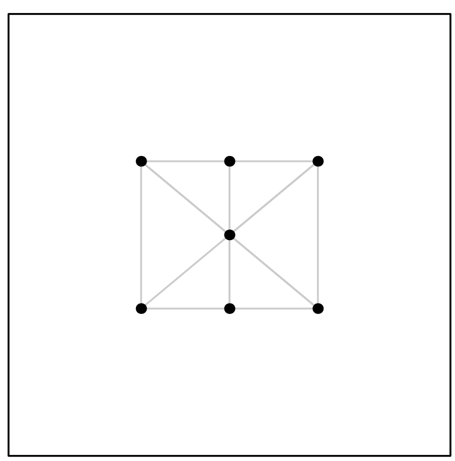


Figure 24. 7 unavoidable points in a square of side 3

Theorem 4. $s(24) = 5$.

Proof: The set

$$P = \{(1,1),(1.7,1),(2.5,1),(3.3,1),(4,1),(1,1.7),(2,1.7),(3,1.7),\\ (4,1.7),(1,2.5),(1.5,2.5),(2.5,2.5),(3.5,2.5),(4,2.5),(1,3.3),\\ (2,3.3),(3,3.3),(4,3.3),(1,4),(1.7,4),(2.5,4),(3.3,4),(4,4)\}$$

is unavoidable in $[0,5]^2$ by Lemmas 1, 2, and 3 (see Figure 25). ∎

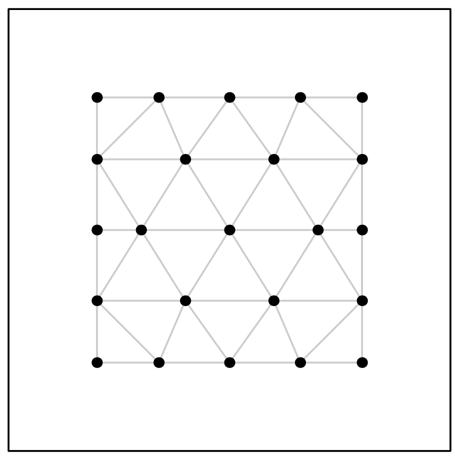


Figure 25. 23 unavoidable points in a square of side 5

Theorem 5. $s(35) = 6$.

Proof: The set

$$P = \{(1,.9),(2,.9),(3,.9),(4,.9),(5,.9),(1,1.725),(1.5,1.725),(2.5,1.725),(3.5,1.725),\\ (4.5,1.725),(5,1.725),(1,2.55),(2,2.55),(3,2.55),(4,2.55),(5,2.55),(1,3.375),\\ (1.5,3.375),(2.5,3.375),(3.5,3.375),(4.5,3.375),(5,3.375),(1,4.2),(2,4.2),\\ (3,4.2),(4,4.2),(5,4.2),(1,5),(1.6,5),(2.4,5),(3,5),(3.6,5),(4.4,5),(5,5)\}$$

is unavoidable in $[0,5]^2$ by Lemmas 1, 2, and 3 (see Figure 26). ∎

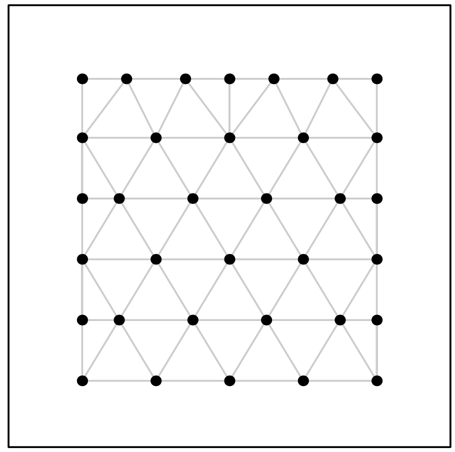


Figure 26. 34 unavoidable points in a square of side 6

Lemma 4. If the center of a unit square u is contained in the rectangle $R = [0,1] \times [0,.4]$, then u contains a vertex of R.

Proof: Let $A = (0,0)$, $B = (0,.4)$, $C = (1,0)$, and $D = (1,.4)$. It suffices to show that any u that contains A and B on its boundary and whose center is in R contains either C or D (see Figure 27). This is clearly the case if A and B lie on the same side of u. When $\theta = \frac{\pi}{4}$, u contains both C and D. It is easy to see that when $\theta < \frac{\pi}{4}$, u contains D, and when $\theta > \frac{\pi}{4}$, u contains C. ∎

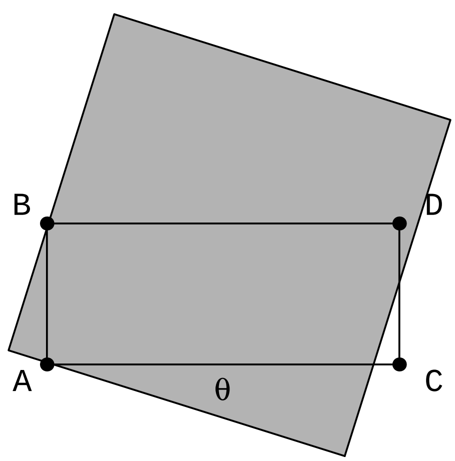


Figure 27.

Theorem 6. $s(15) = 4$.

Proof: The set

$$P = \{(1,1), (1.6,1), (2.4,1), (3,1), (1,1.8), (2,1.8), (3,1.8), (1,2.2), (2,2.2), (3,2.2), (1,3), (1.6,3), (2.4,3), (3,3)\}$$

is unavoidable in $[0,4]^2$ by Lemmas 1, 2, 3, and 4 (see Figure 28). ■

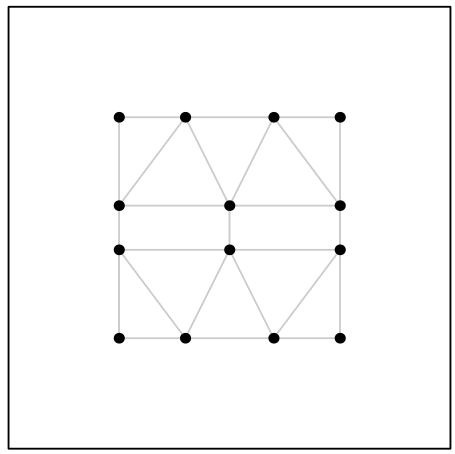


Figure 28. 14 unavoidable points in a square of side 4

Appendix

Table 1 contains a list of the best known upper bounds on $s(n)$ for $n \leq 100$. Only the values $n = 2, 3, 5, 6, 7, 8, 10, 15, 24, 35$ and n square have been proved. For each primitive packing, the Figure and the Author are given.

The type of each best known packing is given. If there is a rectangle at a 45° angle, the dimensions of the rectangle are given. An "M" means the rectangle was somehow modified. An "A" means the rectangle is at an angle other than 45°. An "O" means that one square from the tilted rectangle has been omitted. No type listed indicates that the best known packing is the trivial packing with all squares at 0° angles. More than one type indicates multiple packings are possible.

We conjecture that most of these packings are optimal. The packings most likely to be improved upon include $n = 50$ and $n = 51$. We also suspect that the trivial packing is not optimal for $n = 55$ and $n = 71$.

n	$s(n)$	Type	Figure	Author
1	1			
2–4	2			
5	$2+\frac{1}{2}\sqrt{2}\approx 2.7072$	1×1	Figure 1	Göbel
6–9	3			
10	$3+\frac{1}{2}\sqrt{2}\approx 3.7072$	$1\times 2, 1\times 1$	Figure 1	Göbel
11	≈ 3.8772	2×3AO	Figure 8	Trump
12–16	4			
17	$\frac{7}{3}+\frac{5}{3}\sqrt{2}\approx 4.6904$	3×3MO	Figure 8	Hämäläinen
18	$\frac{7}{2}+\frac{1}{2}\sqrt{7}\approx 4.8229$	2×4A, 4×4A	Figure 9	Hämäläinen
19	$3+\frac{4}{3}\sqrt{2}\approx 4.8857$	2×4M	Figure 4	Wainwright
20–25	5			
26	$\frac{7}{2}+\frac{3}{2}\sqrt{2}\approx 5.6214$	4×4	Figure 11	Friedman
27	$5+\frac{1}{2}\sqrt{2}\approx 5.7072$	1×5	Figure 1	Göbel
28	$3+2\sqrt{2}\approx 5.8285$	4×4	Figure 3	Göbel
29	≈ 5.9665	2×6AO	Figure 15	Friedman
30–36	6			
37	≈ 6.6213	3×5MO	Figure 17	Friedman
38	$6+\frac{1}{2}\sqrt{2}\approx 6.7072$	$1\times 6, 1\times 5$		
39	$\frac{11}{2}+\frac{1}{2}\sqrt{7}\approx 6.8229$	2×6A	Figure 14	Friedman
40	$4+2\sqrt{2}\approx 6.8285$	4×4	Figure 3	Göbel
41	$2+\frac{7}{2}\sqrt{2}\approx 6.9498$	2×7	Figure 4	Cottingham
42–49	7			
50	≈ 7.6213	3×5MO		
51–52	$7+\frac{1}{2}\sqrt{2}\approx 7.7072$	1×8	Figure 2	Göbel
53	$5+2\sqrt{2}\approx 7.8285$	$4\times 4, 4\times 6$	Figure 7	Stenlund
54	$6+\frac{4}{3}\sqrt{2}\approx 7.8857$	2×8M	Figure 13	Friedman
55–64	8			
65	$5+\frac{5}{2}\sqrt{2}\approx 8.5356$	5×5	Figure 3	Göbel
66	$3+4\sqrt{2}\approx 8.6569$	3×8	Figure 5	Stenlund
67	$8+\frac{1}{2}\sqrt{2}\approx 8.7072$	$1\times 9, 1\times 8$	Figure 2	Göbel
68	$6+2\sqrt{2}\approx 8.8285$	$4\times 4, 4\times 6, 4\times 8$	Figure 7	Stenlund
69	$\frac{5}{2}+\frac{9}{2}\sqrt{2}\approx 8.8640$	4×7M	Figure 13	Friedman
70	$\frac{15}{2}+\sqrt{2}\approx 8.9143$	2×9	Figure 12	Friedman
71–81	9			
82	$6+\frac{5}{2}\sqrt{2}\approx 9.5356$	5×5		
83	$4+4\sqrt{2}\approx 9.6569$	3×8		
84	$9+\frac{1}{2}\sqrt{2}\approx 9.7072$	$1\times 10, 1\times 9, 1\times 8$		
85	$\frac{11}{2}+3\sqrt{2}\approx 9.7427$	6×6	Figure 11	Friedman
86	$\frac{17}{2}+\frac{1}{2}\sqrt{7}\approx 9.8229$	2×9A	Figure 14	Friedman
87	$\frac{14}{3}+\frac{11}{3}\sqrt{2}\approx 9.8522$	4×9M	Figure 6	Stenlund
88	$\frac{17}{2}+\sqrt{2}\approx 9.9143$	2×11	Figure 12	Friedman
89	$5+\frac{7}{2}\sqrt{2}\approx 9.9498$	7×7	Figure 3	Stenlund
90–100	10			

Table 1. Best known upper bounds for $s(n)$

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