Glossary of Signed and Gain Graphs and Allied Areas

by Thomas Zaslavsky

Department of Mathematical Sciences
Binghamton University
Binghamton, New York, U.S.A. 13902-6000

E-mail: `zaslav@math.binghamton.edu`

Version of 1998 July 21.

Contents

Key

[ ] : a (usually) rarely used term for which there is a preferred variant.

BASICS

partition (of a set)
Unordered class of pairwise disjoint, nonempty subsets whose union is the whole set. (The empty set has one partition, the empty one.)

partial partition (of a set)
Partition of any subset, including of the empty set.

bipartition (of a set)
Unordered pair of pairwise disjoint, possibly void subsets whose union is the whole set.

partial $\mathfrak{G}$-partition (of a set, where $\mathfrak{G}$ is a group)
An equivalence class of pairs $a = (\pi, \{a_B : B \to \mathfrak{G}\}_{B\in\pi})$, where $a \sim a'$ if there are constants γ_B, $B \in \pi$, so that $a'_B = \gamma_B a_B$ for each block $B \in \pi$.

GRAPHS

These definitions about graphs are intended not to be a glossary of graph theory but to clarify the special usages appropriate to signed, gain, and biased graphs.
I generally call a graph Γ.

Graph Elements

link
Edge with two distinct endpoints.

loop
Edge with two coincident endpoints.

ordinary edge
e:vw
A link or loop. To indicate that e is an ordinary edge with endpoints v, w one may write *e:vw*.

half edge
[spike] (Little??)
[lobe] (Aráoz et al.)
e:v
Edge with one endpoint (not labelled in a gain graph; considered to be like an unbalanced polygon). To indicate that e is a half edge with endpoint v one may write *e:v*.

loose edge
e:$\emptyset$
Edge with no endpoints (not labelled in a gain graph; always considered to be like a balanced polygon, although not technically a polygon). To indicate that e is a

loose edge one may write $e{:}\emptyset$.

edge end
: Each end of an edge is incident with exactly one vertex. A link or loop is considered to have two (distinguishable) ends (which in the case of a loop are incident with the same endpoint), a half edge one, and a loose edge none.

parallel edges
: Two or more edges with the same endpoints.

multiple edges (in a signed or gain graph)
: Two or more edges with the same endpoints and the same sign or gain. (Whether to count a negative loop and a half edge at the same vertex as multiple edges is not clear and may depend on the context. In matroid theory they should be considered multiple.)

multiple edges (in a biased graph)
: Two or more parallel links in which all digons are balanced, or two or more balanced loops at the same vertex, or two or more unbalanced edges (loops or half edges) at the same vertex.

directed edge
arc
: Edge to which a direction has been assigned.

bidirected edge
: Edge such that each end has been independently oriented. Thus a link or loop, with 2 ends, has 4 possible bidirections; a half edge has 2; a loose edge has 1 possible bidirection (since it has no ends to orient). If the two ends of a link or loop are directed coherently (that is, one end is directed into the edge and the other is directed out toward the endpoint), then the edge is considered to be an (ordinary) directed edge. Bidirection can be represented by signing the edge ends as follows: $+$ represents an end entering its vertex while $-$ represents an exiting end. (Some people follow the opposite convention.)

introverted edge
: Bidirected link or loop whose ends are both directed inward, away from the endpoints.

extroverted edge
: Bidirected link or loop whose ends are both directed outward, towards the endpoints.

Kinds of Graphs

ordinary graph
: Graph whose edges are links and loops only: no half or loose edges. Parallel edges are allowed.

simple graph
: Graph whose edges are links and having no parallel edges.

mixed graph

Graph in which edges may be directed (not bidirected!) or undirected. (These are naturally regarded as gain graphs with cyclic gain group of order 3 or more.)

bidirected graph (Edmonds)
[polarized graph] (Zítek and Zelinka)

Graph with bidirected edges (q.v.); equivalently,
graph with a signing of the edge ends (I like to use τ for such a signing); equivalently,
oriented signed graph (and in particular a digraph is an oriented all-positive graph).

empty graph

The graph with no vertices and no edges. (The empty graph is a graph. This is the most suitable definition for signed, gain, and biased graph theory. Beware competing definitions!)

Graph Structures

walk, trail, path, closed path

I follow Bondy and Murty, it Graph Theory with Applications, 1976. A walk goes from an initial to a final elements and allows arbitrary repetition. A trail is a walk that allows repeated vertices but not edges; a path is a trail that has no repeated vertex; a closed path is a nontrivial closed trail with no repeated vertex other than the endpoint. A loose edge cannot be part of a walk. I assume that a walk extends from a vertex to a vertex. (It may at times be desirable to broaden this definition to allow a half edge to be the initial or final element of a walk but this should be made explicit.)

trivial path, walk, trail

A path, walk, or trail of length 0.

path

A walk (q.v.) that is a path, or the graph of such a walk.

polygon
circle
graph circuit
[(simple) cycle]

Graph of a simple closed path (of length at least 1). This includes a loop, but not a loose edge. [I prefer to avoid the term "cycle" because it has so many other uses in graph theory—at least three at last count. See definition below.]
Sometimes, the edge set of such a graph.

linear subclass of polygons (in a graph)

A class of polygons such that no theta subgraph contains exactly two balanced polygons (and every loose edge constitutes a balanced polygon); the balanced polygons of a gain graph are such a class.

handcuff

A connected graph consisting of two vertex-disjoint polygons and a minimal (not minimum-length) connecting path (this is a loose handcuff), or of two polygons that meet at a single vertex (a tight handcuff or figure eight).

bicycle

A handcuff or theta graph.

cycle (in an undirected graph)

An element of the cycle space; equivalently,
an edge set with even degrees.

hole

A chordless polygon (usually, of length at least 4).

directed (of a walk in a digraph or mixed graph)

Every arc is traversed in its forward direction.

cycle (in a digraph)

The digraph of a directed walk around a polygon.

coherent (of a walk in a bidirected graph)

At each internal vertex of the walk, and also at the ends if it is a closed coherent walk, one edge enters and the other exits the vertex.

component (of a graph)
connected component

A maximal connected subgraph. A loose edge is a component, as is an isolated vertex.

vertex component (of a graph)
node component

A maximal connected subgraph that has a vertex. A loose edge is not a vertex component, but an isolated vertex is.

edge component (of a graph)

A maximal connected subgraph that contains an edge. A loose edge is an edge component but an isolated vertex is not.

edge cutpoint
edge cut-vertex

A vertex whose deletion (together with deletion of all incident edges) increases the number of connected components of the graph, or that is incident to more than one edge of which one (at least) is a loop or half edge.

vertex block

node block
block

A graph that is 2-connected.

A maximal subgraph (of a graph) that is 2-connected and has at least one vertex. (Thus a loose edge is not a vertex block and is not contained in any.)

edge block
block (when context shows that "edge block" is intended)

A connected graph that has no edge cutpoints. (It may consist of a loop or half edge and its supporting vertex or of a loose edge alone.)

A maximal edge-block subgraph (of a graph). (Thus an isolated vertex is not an edge block and is not contained in any.)

Graph Operations

Seidel switching (of a simple graph)
[graph switching] (Zaslavsky)
[switching] (Seidel)

Reversing the adjacencies between a vertex subset S and its complement; i.e., edges with one end in S and the other in S^c are deleted, while new edges are supplied joining each pair $x \in S$ and $y \in S^c$ that were nonadjacent in the original graph. (Since "switching" alone has a multitude of meanings, it is not recommended. The unambiguous term is the first one.) (Notation for the result of switching Γ by S is Γ^S, like conjugation in a group.)

vertex-switching (of a simple graph)

Seidel switching of a single vertex (Stanley).

Seidel switching (Ellingham; Krasnikov and Roditty).

i-switching (of a simple graph)

Seidel switching when i vertices are switched.

switching equivalence (of two simple graphs)

$\sim$

The relation between two simple graphs that one is obtained by Seidel-switching the other.

switching isomorphism (of two simple graphs)

$\simeq$

A combination of switching and isomorphism; equivalently,

a vertex bijection that is an isomorphism of one graph with a switching of the other.

switching class

$[\Gamma]$

An equivalence class of simple graphs under Seidel switching.

odd subdivision (of a graph Γ)
[even subdivision]

Unsigned graph underlying any all-negative subdivision of $-\Gamma$. That is, each edge is subdivided into an odd-length path. [The term "odd" arises because each edge of Γ is subdivided into an odd-length path. I recommend this term for compatibility with "odd Γ". "Even" arises from the fact that each edge is subdivided an even number of times.]

odd Γ (Gerards)

Unsigned graph underlying any antibalanced subdivision of $-\Gamma$. That is, each subdivided polygon has the same parity after subdivision as before. [Any odd subdivision of Γ is an odd Γ, but of course not conversely.]

Graph Invariants, etc.

number of (connected) components

$c(\ )$

The number of components of a graph or signed, gain, or biased graph, not counting loose edges.

odd girth

The length of a shortest odd polygon. Same as the negative girth of the all-negative signature.

oriented incidence matrix (of a graph Γ)
incidence matrix

A matrix whose rows are indexed by the vertices and whose columns are indexed by the edges. The entries in the column of edge e are 0 except at the endpoints of e. If e is a loop or loose edge, the entire column is 0. If $e{:}v$ is a half edge, the entry in row v is ± 1. If $e{:}vw$ is a link, the entries in rows v and w are ± 1 and have opposite signs. [From the signed-graphic standpoint this is an incidence matrix of the all-positive signature of Γ.]

unoriented incidence matrix (of a graph Γ)
[incidence matrix]

The matrix whose rows are indexed by the vertices and whose columns are indexed by the edges and whose (v, e) entry equals the number of ends of e that are incident with v. [From the standpoint of signed graph theory, this is an incidence matrix of the all-negative signing of Γ. Thus I prefer not to call it simply the "incidence matrix of Γ".]

demigenus (of a graph Γ) (Zaslavsky)
Euler genus (Archdeacon)

The smallest demigenus ($= 2 -$ Euler characteristic) of a compact surface in which Γ can be topologically embedded.

SIGNED, GAIN, AND BIASED GRAPHS

Notation

I generally call a graph Γ, a signed graph $\Sigma = (\Gamma, \sigma)$, a gain graph $\Phi = (\Gamma, \varphi)$, and a biased graph $\Omega = (\Gamma, \mathcal{B})$.

Basic Concepts of Signed Graphs and Their Orientations

signed graph
[sigraph]
[sigh! graph]

Σ, (Γ, σ)

Graph with edges labelled by signs (except that half and loose edges, if any, are unlabelled); equivalent to gain graph with 2-element gain group. Sometimes the "signs" are treated as numbers, $+1$ and -1, which are added; this is a weighted graph with unit weights and not (according to at least one person: me) a true signed graph. But that's okay.

signing (of a graph)
signature

A sign labelling of the edges (except half and loose edges).

underlying graph (of a signed, gain, or biased graph)

$|\Sigma|$ (or $||\Sigma||$), $||\Phi||$, $||\Omega||$

The graph alone, without signs, gains, or bias.

positive, negative subgraph (of a signed graph)

Σ_+, Σ_-

The unsigned spanning subgraph whose edge set is $E_+(\Sigma) = \sigma^{-1}(+)$, or $E_-(\Sigma) = \sigma^{-1}(-)$.

orientation of a signed graph

Direct a positive edge in the ordinary way; direct a negative edge at both ends so the ends both point inward or both outward (an 'introverted' or 'extroverted' edge); direct a half edge at its one end. (This gives a bidirected graph.)

polar graph (Zítek and Zelinka)

Graph where each vertex has two poles, each edge end being incident to one pole bidirected graph with the actual directions forgotten, remembering only the difference between the two directions at each vertex. This useful object is equivalent to a switching class of bidirected graphs.

sign, or gain, of a walk (in a signed, or gain, graph)

The product of the signs or gains of the edges in the walk, taken in the order and the direction that the walk traverses each edge. (It is undefined if the walk contains a half edge.)

sign, or gain, of a polygon (in a signed, or gain, graph)

The sign, or gain, of a walk that goes once around the polygon. It is determined only up to conjugation, but the most important property, whether the sign is positive, or the gain is the identity, is well determined.

positive, negative (of a walk or edge set)
[even, odd] (Gerards, Lovasz, Schrijver, et al.)

Having sign product $+$, or $-$. Edges in a walk are counted as many times as traversed. Analogous to even/odd length or even/odd cardinality in ordinary graphs.

balanced (of a subgraph or edge set) (Harary)
satisfied (Toulouse)
[bipartite] (Gerards, Lovasz, Schrijver, et al.)

A subgraph or edge set whose every polygon is positive (in a signed graph), has gain equal to the identity (in a gain graph), or belongs to the balanced-polygon class $\mathcal{B}(\Omega)$ (in a biased graph). The signed-graph analog of bipartiteness of ordinary graphs.

Harary bipartition (of a signed graph, necessarily balanced)

A bipartition of the vertex set so that an edge is negative iff its endpoints lie in different parts. (Reminder: A bipartition allows empty parts.)

frustrated (Toulouse)
unbalanced (Harary)

Not being balanced.

antibalanced (of a signed graph) (Harary)

The negative of a balanced signed graph; equivalently,
having each polygon sign equal to $+$ or $-$ depending on whether the length is even or odd, so that a polygon is balanced iff it has even length.

contrabalanced (of a signed, gain, or biased graph)

Having no balanced polygons (and no loose edges).

clusterable, k-clusterable (of a signed graph)
[k-balanced] (Doreian and Mrvar)

Vertex set partitionable into (k) parts such that every positive edge lies within a part and every negative edge goes between two parts.

clusterability
[generalized balance] (Doreian and Mrvar)

The property of being clusterable.

balancing vertex (of a signed, gain, or biased graph)
blocknode (Gerards)

Vertex of an unbalanced graph whose deletion leaves a balanced graph.

balancing edge (of a signed, gain, or biased graph)
Edge of an unbalanced graph whose deletion leaves a balanced graph.

balancing set (in a signed, gain, or biased graph)
balancing vertex set
balancing edge set
A set of vertices, edges, or edges and vertices (as appropriate to the context) whose deletion leaves a balanced graph.

balanced chord (of a balanced polygon in a signed, gain, or biased graph)
A chord whose union with the polygon is balanced.

pit (in a signed, gain, or biased graph)
A balanced polygon with no balanced chord. (Possibly, with a certain minimum length.)

Additional Basic Concepts of Gain and Biased Graphs

gain graph
[group-labelled graph]
[voltage graph]
Φ, (Γ, φ)
Graph whose edges (except half and loose edges) are labelled by elements of a group (the *gain group*), the gain of an edge, $\varphi(e)$, being inverted if the direction of traversal is reversed (thus, distinct from a weighted graph). (We call φ the *gain function*.) When necessary, one can indicate the direction of the gain by the notation $\varphi(e; v, w)$, where v and w are the endpoints of a link or loop e and the gain is read from v to w. ["Group-labelled" is ambiguous: it is also used for weights. "Voltage graph" should usually be reserved for surface embedding constructions where the voltage aspect is more significant. The differences among gains, voltages, and flows (q.v.) are in the questions asked. With a gain one looks at the product around a polygon and compares to the gain identity 1. With a voltage one is interested in the actual value of a polygon voltage product and such things as its order in the voltage group. With a flow one looks at the net inflows to vertices and compares to the group identity 0.]

biased graph
Ω, $(\Gamma, \mathcal{B})$
Graph together with a linear subclass $\mathcal{B} = \mathcal{B}(\Omega)$ of its polygons; these are called the "balanced" polygons. (That is, the number of balanced polygons in a theta subgraph is never exactly 2.) The "bias" is the unbalanced polygons, so more balanced is less biased.

balanced polygon (in a biased graph Ω)
A member of the distinguished linear subclass $\mathcal{B}(\Omega)$.

sign-biased graph
A biased graph whose bias arises from a signing of the underlying graph. Equivalent to being additively biased.

additively biased graph

A biased graph in which every theta subgraph contains an odd number of balanced polygons. Equivalent to being sign-biased.

compatibly biased signed graph

$(\Gamma, \sigma, \mathcal{B})$, $(\Sigma, \mathcal{B})$, (Ω, σ)

A graph with both signs and bias, in which every negative polygon is unbalanced in the bias. (Not the same as merely a graph that is both signed and biased, nor is it the same as a sign-biased graph, which is not a signed graph.)

weighted graph

Graph whose edges are labelled by elements of a group, often but not always additive; the weight of an edge is independent of the direction of traversal, in which respect this differs from a gain graph.

Examples (particular)

signed complete graph

Any signed K_n, $n \geq 1$. Thus it has no parallel edges, of whatever signs, and no loops.

complete signed graph

$\pm K_n^\circ$

Signed graph of order n consisting of all possible positive and negative links and negative loops, without multiple edges (that is, of the same sign) or positive loops.

complete signed link graph

$\pm K_n$

Same as the complete signed graph but without the loops.

complete $\mathfrak{G}$-gain graph

$\mathfrak{G}K_n^\bullet$

Gain graph of order n consisting of all possible links with all possible gains in $\mathfrak{G}$ and an unbalanced edge (a half edge or unbalanced loop) at each vertex. No multiple edges—that is, no parallel links having the same gain and no multiple unbalanced edges at the same vertex—and no balanced loops.

complete $\mathfrak{G}$-gain link graph

$\mathfrak{G}K_n$

Same as the complete $\mathfrak{G}$-gain graph but without the loops and/or half edges.

Examples (general)

biased graph of a graph

$\langle\Gamma\rangle$

The biased graph $(\Gamma, \mathcal{C}(\Gamma))$, where $\mathcal{C}(\Gamma)$ is the class of all polygons of Γ. It is balanced if and only if Γ has no half edges. Its bias and lift matroids both equal the graphic (polygon) matroid of Γ.

biased graph of a signed or gain graph

$\langle\Sigma\rangle$, $\langle\Phi\rangle$

The biased graph implied by a signed or gain graph, whose balanced polygons are the polygons with identity gain product. [I have in the past used bracket notation, $[\Sigma]$ and $[\Phi]$, but this was an error—not serious in the signed case because $\langle\Sigma\rangle$ determines $[\Sigma]$; but for gain groups larger than order 2, $\langle\Phi\rangle$ does not generally determine $[\Phi]$.]

full (of a signed, gain, or biased graph)

Having an unbalanced edge (a half edge or unbalanced loop) at every vertex.

filled graph (unsigned, signed, gain, biased)

$\Gamma^\bullet$, $\Phi^\bullet$, $\Omega^\bullet$

Γ, Φ, or Ω with an unbalanced edge adjoined to every vertex not already supporting one.

loop-filled graph

Γ°, Φ°, Ω°

Γ, Φ, or Ω with an unbalanced loop (negative, in a signed graph) adjoined to every vertex not already supporting one.

all-positive, or all-negative graph

$+\Gamma$, or $-\Gamma$

The signed graph obtained from an ordinary graph Γ by signing every edge $+$, or every edge $-$.

signed expansion (of an ordinary graph)

$\pm\Gamma$

The signed graph obtained from an ordinary graph Γ through replacing each edge by a positive and a negative copy of itself; equivalently,
the union of $+\Gamma$ and $-\Gamma$ (on the same vertex set—not a disjoint union). Same as $\{+,-\}\Gamma$.

full signed expansion (of a simple graph)

$\pm\Gamma^\bullet$

$\pm\Gamma$ with an unbalanced edge adjoined to every vertex.

looped signed expansion (of an ordinary graph)

$\pm\Gamma^\circ$

$\pm\Gamma$ with a negative loop adjoined to every vertex.

group expansion (of an ordinary graph)
$\mathfrak{G}$-expansion ($\mathfrak{G}$ denotes a group)

$\mathfrak{G}\Gamma$, or

$\mathfrak{G}\cdot\Gamma$

The gain graph obtained through replacing each edge of an ordinary graph Γ by one copy of itself for each element of a group $\mathfrak{G}$, having gain equal to the corresponding group element. That is, each edge $e{:}vw$ is replaced by edges (e,g): $vw, g\in\mathfrak{G}$, with gains $\varphi((e,g);v,w)=g$.

full group expansion (of a simple graph)
full $\mathfrak{G}$-expansion ($\mathfrak{G}$ denotes a group)

$\mathfrak{G}\Gamma^\bullet$

$\mathfrak{G}\Gamma$ with an unbalanced edge adjoined to every vertex.

biased expansion (of a simple graph)
(m-fold) biased expansion

$m \cdot \Gamma$

A combinatorial abstraction of the group expansion: a biased graph whose underlying graph is obtained through replacing each edge of Γ by m parallel edges and having $\mathcal{B}(m \cdot \Gamma)$ such that, for each polygon C of Γ, each edge $e \in C$, and each choice of one corresponding edge for every $f \in C \setminus e$, there is exactly one balanced polygon that consists of all the chosen corresponding edges and an edge corresponding to e.

antisymmetric signed digraph

Symmetric digraph signed so that every digon is negative.

periodic graph
dynamic graph (I think this is older usage)

Covering graph (q.v.) of a (finite) gain graph whose gains are in the additive group $\mathbb{Z}^d$ for some $d > 0$. The gain graph may also have costs, capacities, etc.; these are carried over to the covering graph. These graphs are studied, i.a., in computer science and percolation theory.

toroidal periodic graph

Same as a periodic graph except that the gains are taken modulo a d-dimensional integer vector α, i.e., the gains are in $\mathbb{Z}_\alpha := \mathbb{Z}_{\alpha_1} \times \cdots \times \mathbb{Z}_{\alpha_d}$.

static graph

The gain graph of a periodic graph.

poise gains (on a digraph or mixed graph)

Gains in $\mathbb{Z}$ whose value on a directed edge is 1 in the edge's direction (thus, -1 in the opposite direction) and on an undirected edge is 0.

modular poise gains (on a digraph or mixed graph)

Poise gains taken in $\mathbb{Z}_M$ where M is a positive integral modulus.

Structures

bias circuit (in a signed, gain, or biased graph)

A balanced polygon or contrabalanced handcuff or theta graph.

lift circuit (in a signed, gain, or biased graph)

A balanced polygon, or the union of two vertex-disjoint unbalanced polygons, or a contrabalanced tight handcuff or theta graph.

bias cycle (in a bidirected graph)

A circuit of the bias matroid, oriented (as a digraph, half edges included) so that it has no source or sink; equivalently, so it is the edge set of a coherent closed walk.

balanced partial partition (of V) (due to an edge set S in a signed, gain, or biased graph)

$\pi_{\mathrm{b}}(S)$

The set of vertex sets of balanced components of the spanning subgraph (V,S), that is, $\pi_{\mathrm{b}}(S) := \{V(Z) : Z \text{ is a balanced component of } (V,S)\}$.

Operations and Relations

switching (of a vertex set in a signed graph)

Reversing the signs of edges between a vertex subset S and its complement. Equivalently, switching by a selector which is the (signed) characteristic function of S, $\theta(v) := -$ if $v \in S$, $+$ if not. (Symbol for result of switching by S: Σ^S, like conjugation in a group.)

switching (of a vertex set S in a bidirected graph)
[reflection] (Ando, Fujishige, and Naitoh)

Reversing the signs of edge ends incident to vertices in S. Equivalently, switching by a selector which is the (signed) characteristic function of S, $\theta(v) := -$ if $v \in S$, $+$ if not. This has the effect of switching S in the associated signed graph. Note that in a bidirected graph, switching S and its complement are not equivalent.

switching (of a signed or gain graph by a selector θ)

Changing the gain function φ to φ^θ, where $\theta\colon V \to \mathfrak{G}$ (the gain group) and, for an edge $e{:}vw$, $\varphi^\theta(e;v,w) := \theta(v)^{-1}\varphi(e;v,w)\theta(w)$.

switching (of a bidirected graph by a selector θ)

Reversing the direction of each edge end at a vertex for which $\theta(v) = -$; equivalently,
replacing the signing of the edge ends, τ, by $\tau\theta(\text{end at } v) := \tau(\text{end})\theta(v)$.

selector (of a signed, gain, or bidirected graph)
switching function

A function $V \to \mathfrak{G}$ (the gain group; the sign group for a bidirected graph) used for switching.

switching equivalence (of signed, gain, or bidirected graphs)

$\sim$

The relation between two signed, gain, or bidirected graphs that one is obtained by switching the other.

switching class (of signed, gain, or bidirected graphs)

$[\Sigma]$, $[\Phi]$, $[B]$

An equivalence class of signed, gain, or bidirected graphs under switching.

isomorphism (of signed, gain, or bidirected graphs)

$\cong$

An isomorphism of underlying graphs that preserves the signs, gains, or bidirection.

switching isomorphism (of signed, gain, or bidirected graphs)

$\simeq$

Any combination of switching and isomorphism; thus,
an isomorphism of underlying graphs that preserves the signs, gains, or bidirection up to switching.

isomorphism (of switching classes of signed, gain, or bidirected graphs)

$\cong$

The residue on switching classes of switching isomorphism of signed, gain, or bidirected graphs.

subgraph (of a signed, gain, or biased graph)

A subgraph of the underlying graph: in a signed or gain graph each edge retains its sign or gain; in a biased graph, each polygon of the subgraph is balanced or not just as in the original graph.

deletion

The process or the result of deleting a (possibly void) set of vertices and/or edges from a graph. The result is a subgraph (q.v.).

restriction

$\Sigma|S$, $\Phi|S$, $\Omega|S$

The restriction to an edge set S is the spanning subgraph whose edge set is the specified set.

induced subgraph

$\Sigma{:}X$, $\Phi{:}X$, $\Omega{:}X$ (no space between symbols)

The subgraph induced by a vertex subset X. It has X as its vertex set and as its edges every non-loose edge whose endpoints are contained in X. The empty set X is permitted; it induces the empty graph (q.v.).

contraction (of a signed or gain graph)

Σ/S, Φ/S

A somewhat complex but fundamental operation. First switch so that every balanced component Z of S has identity gains. Then collapse the vertex set of each such component to a point; these points will be the vertices of the contraction. The edge set of the contraction will be S^c, the complement of S. The edge ends will be the ends of the retained edges whose incident vertex is in a balanced component of S; thus some links and half edges may become half or loose edges. (Formally, the vertex set of the contraction is the balanced partial partition (q.v.) of V due to S. Note that the contraction is defined only up to switching; that is, only contraction of a switching class is truly well defined.)

contraction (of a biased graph)

Ω/S

The underlying graph is similar to that above, but switching and gains are not involved. Instead, one defines the balanced polygons directly. A polygon of Ω/S is balanced if it has the form $C\backslash S$ where C is a balanced polygon of Ω. We write $\mathcal{B}/S$ for the class of balanced polygons of the contraction of $\Omega = (\Gamma, \mathcal{B})$.

minor
subcontraction

Any result of a sequence of deletions and contractions; equivalently, the result of a deletion followed by a contraction, or the reverse.

link contraction

Contraction of a balanced set. Equivalently (if the contracted set is finite), a series of contractions by links—that is, each contracted edge must be a link at the time of being contracted—combined with deletion of any or all of the balanced loops thereby formed.

link minor

The result of a sequence of deletions and link contractions; equivalently, the result of a deletion followed by a link contraction, or the reverse.

lift contraction

$\Sigma/_L S$, $\Phi/_L S$, $\Omega/_L S$

A link contraction, whose result is a signed, gain, or biased graph, or a contraction by an unbalanced edge set S, whose result is the plain graph obtained from contraction of the underlying graph (without signs, gains, or bias) by S. Note that it is necessary to distinguish between a (plain) graph, without signs, gains, or bias, and a signed, gain, or biased graph; even an all-positive signed graph, for instance, is different from a plain graph.

lift minor

The result of a sequence of deletions and lift contractions; equivalently, the result of a deletion followed by a lift contraction, or the reverse.

simple subdivision (of a signed or gain graph)

The process or result of replacing a link or loop, $e{:}vw$, by a path of length 2 whose sign or gain equals that of e. (I incline to doubt that it makes sense to subdivide a half edge, but there might be contexts in which it does.)

simple subdivision (of a biased graph)

The process or result of replacing a link or loop, $e{:}vw$, by a path of length 2 so that the balanced circles of the new biased graph are those of the old that do not contain e and those of the new that are obtained from old balanced circles through subdividing e. (I incline to doubt that it makes sense to subdivide a half edge.)

(multiple) subdivision

The process or result of any sequence (possibly null) of simple subdivisions.

negative-subdivision trick (of a signed graph)

Operation of replacing each positive edge by an all-negative path of length 2. The result is an all-negative signed graph in which positive/negative walks of the original signed graph become even/odd walks. Many (though not all) results about, e.g., odd polygons in ordinary graphs thereby generalize immediately to signed graphs. [This trick may be due to some combination of Gerards, Lovász, and Schrijver; at any rate I learned it from them.]

splitting a vertex (in a signed or gain graph)
vertex splitting

The operation or the result of replacing a vertex v by two vertices v' and v'' and a connecting edge e_v whose gain is the group identity ($+$, in a signed graph), and attaching each edge end incident to V to one of v' or v'', at will. (Note that the splitting, contracted by e_v, is the original signed or gain graph.)

splitting a vertex (in a biased graph Ω)
vertex splitting

The operation or the result of replacing a vertex v in Ω by two vertices v' and v'' and a connecting edge e_v, and attaching each edge end incident to V to one of v' or v'', at will, so that a circle C in the splitting is balanced if and only if it is a balanced circle in Ω or it contains e_v and C/e_v is balanced in Ω. (Note that the splitting, contracted by e_v, is Ω.)

proper vertex splitting

A vertex splitting in which both new vertices have degree at least 3.

split Σ, Φ, or Ω

A graph resulting from Σ, Φ, or Ω by any sequence (possibly null) of proper vertex splittings and subdivisions.

Covering or Derived Graphs

covering graph (of a gain graph with gain group $\mathfrak{G}$)
[derived graph]

$\tilde{\Phi}$

Graph with vertex set $\tilde{V} := V(\Phi) \times \mathfrak{G}$ and edge set $\tilde{E} := E \times \mathfrak{G}$; the endpoints of an edge $\tilde{e} = (e, g)$ are (v, g) and $(w, g\varphi(e; v, w)$ if e has endpoints v and w. We may think of the vertices of $\tilde{\Phi}$ as labelled by their group elements, but the edges are not intrinsically labelled (and indeed there is no labelling that is canonical under reversing edges). (The proper treatment of loose and half edges is unclear. I think that signed graphs have to be treated as special in this respect. For general gain graphs one should probably assume that there are only ordinary edges (links and loops). In a signed graph a half edge h at vertex v may become a single edge $\tilde{h}{:}\tilde{v}\tilde{v}^*$, where $\tilde{v}$ and $\tilde{v}^*$ are the vertices covering v; but the decision may depend on the requirements of the application.)

projection map (of a covering graph)

covering projection

The projection $p\colon \tilde\Phi \to \Phi$ of vertices and edges onto the first component.

signed covering graph (of a signed graph Σ)

$\tilde\Sigma$

The covering graph $\tilde\Sigma$, regarded as having the covering vertices distinguished as positive and negative. Thus Σ can be completely recovered from $\tilde\Sigma$.

double covering graph (of a signed graph Σ or an unsigned graph Γ)

$\tilde\Gamma$

A covering graph of Σ or of any signing of Γ, regarded as having the covering vertices not distinguished by sign. From $\tilde\Gamma$ the signature can be recovered up to switching.

Vertex Labels

vertex-signed graph
[marked graph] (Beineke and Harary)

Graph with signed vertices. [I prefer not to say "marked graph" because it is not self-explanatory and, indeed, is widely understood to be a Petri net, which is totally different.]

graph with vertex and edge signs
[net] (Cartwright and Harary)

Graph with vertex and edge signs. [I prefer not to use this term for the same general reason as I gave for "marked graph".]

consistent (of a vertex-signed graph)

Having the vertex-sign product around every polygon equal to $+$. [Usage is not entirely consistent. Some say "harmonious".]

harmonious (of a graph with vertex and edge signs)

Having the total sign product of vertices and edges around every polygon equal to $+$. [Usage is not entirely consistent. Some say "consistent".] (For a graph with vertex and edge signs, balance, consistency, and harmony are three different properties.)

Matrices

adjacency matrix (of a signed graph)

$A(\Sigma)$

The matrix $(a_{ij})_{n\times n}$, indexed by the vertex set, in which a_{ij} = the number of positive edges $-$ the number of negative edges between v_i and v_j. (A loop should possibly count twice but this depends on the application.)

incidence matrix (of a signed graph)

The incidence matrix of any orientation of the signed graph (see incidence matrix of a bidirected graph); equivalently,

A matrix whose rows are indexed by the vertices and whose columns are indexed by the edges. The entries in the column of edge e are 0 except at the endpoints of e. If e is a positive loop or loose edge, the entire column is 0. If $e{:}v$ is a half edge, the entry in row v is ± 1. If $e{:}vv$ is a negative loop, the entry in row v is ± 2. If $e{:}vw$ is a link, the entries in rows v and w are ± 1; they have opposite signs if e is positive, the same sign if it is negative.

incidence matrix (of a bidirected graph)
: The matrix whose rows are indexed by the vertices and whose columns are indexed by the edges, in which the (v, e) entry equals the number of incoming ends of e at v $-$ the number of outgoing ends.

Matroids

The points of the matroid are the edges (except for the extra point in the complete lift). A loose edge is a circuit and a half edge acts like an (unbalanced) loop (except in the polygon matroid).

polygon matroid (of a graph)
cycle matroid
graphic matroid
: $G(\Gamma)$
: Matroid whose circuits are the polygons of the graph.

bicircular matroid (of a graph)
: $B(\Gamma)$, $G(\Gamma, \emptyset)$
: Matroid whose circuits are the bicycles of the graph. A half edge acts like a loop.

even-cycle matroid (of a graph)
even-polygon matroid
[factor matroid] (Wagner)
: $G(-\Gamma)$
: Matroid whose circuits are the even polygons and the handcuffs with two odd polygons (and the loose edges). A half edge acts like a loop.

bias matroid (of a signed, gain, or biased graph)
: $G(\ )$
: Matroid whose circuits are the balanced polygons and contrabalanced bicycles. (This is not a purely matroidal construct; it depends on graph connectedness.)

lift matroid (of a signed, gain, or biased graph)
[even cycle matroid] (Gerards)
: $L(\ )$
: Matroid whose circuits are the balanced polygons and the contrabalanced theta subgraphs, tight handcuffs, and pairs of vertex-disjoint polygons. (This is a purely matroidal construct; it is a lift of the polygon matroid of the underlying graph.)

extended lift matroid (of a signed, gain, or biased graph)

[complete lift matroid] (Zaslavsky)
[extended even cycle matroid] (Gerards)

$L_0(\ )$

Matroid whose point set is $E_0 := E \cup \{e_0\}$, where e_0 is an extra point (not in the graph) which behaves like an unbalanced loop, and whose circuits are those of the lift matroid together with the sets $C \cup \{e_0\}$ where C is an unbalanced polygon. (A point set may be called "balanced in L_0" if it is a balanced edge set.)

flat (of a signed, gain, or biased graph)
closed set

An edge set that is closed in the bias matroid (or sometimes in the lift or extended lift matroid, depending on context; in the latter case it is any closed subset of the extended edge set E_0).

lattice of flats (of a matroid M)

$\operatorname{Lat} M$

The set of flats of M, ordered by inclusion.

semilattice of balanced flats (of a graph or a signed, gain, or biased graph)

$\operatorname{Lat}^{\mathrm{b}} \Gamma,\ \operatorname{Lat}^{\mathrm{b}} \Sigma,\ \operatorname{Lat}^{\mathrm{b}} \Phi,\ \operatorname{Lat}^{\mathrm{b}} \Omega$

The set of balanced flats in the matroid of a graph or a signed, gain, or biased graph, ordered by inclusion. (The matroid may be the bias, lift, or extended lift; the balanced flats are the same in all.)

Dowling matroid, geometry (of a group $\mathfrak{G}$)

$Q_n(\mathfrak{G})$ [Same as Dowling lattice; context tells the difference.]

The bias matroid (a.k.a. geometry) of the complete $\mathfrak{G}$-gain graph $\mathfrak{G}K_n^{\bullet}$.

Dowling lattice (of a group $\mathfrak{G}$)

$Q_n(\mathfrak{G})$

The lattice of flats of the Dowling matroid. When the group is trivial it is isomorphic to the partition lattice of $n+1$ objects and to the partial-partition lattice of n objects. In general it is the lattice of partial $\mathfrak{G}$-partitions of an n-element set.

Topology (of signed graphs)

orientation embedding (of a signed graph)

Embedding of $|\Sigma|$ into a surface so that every positive polygon has an orientable neighborhood but every negative polygon does not. (Undefined for signed graphs with loose or half edges; it is not yet clear to me how they should be treated.)

minimal surface

$S(\Sigma)$

The smallest compact surface (that is, the one with largest Euler characteristic) in which Σ has an orientation embedding.

demigenus (Zaslavsky)

Euler genus (Archdeacon)
: $d(\Sigma)$
: The smallest demigenus, i.e., 2 − Euler charateristic, of a compact surface in which Σ has an orientation embedding; equivalently,
the demigenus, i.e., 2 − Euler characteristic, of the minimal surface $S(\Sigma)$. (Latin plural: demigenera.)

maximum demigenus
: The largest demigenus of a compact surface in which Σ has a cellular orientation embedding.

demigenus range
: The set of all demigenera $d(S)$ of compact surfaces in which Σ has a cellular orientation embedding.

odd demigenus, even demigenus
: The smallest odd, or even, demigenus of a compact surface in which Σ has an orientation embedding.

projective planar
: Orientation embeddable in the projective plane.

projective planarity
: The property of being projective planar.

projectively outerplanar (POP)
: Orientation embeddable in the projective plane so that all vertices are on the boundary of one face.

projective outerplanarity (POP)
: The property of being projectively outerplanar.

cylindrical embedding (of a signed graph without loose or half edges)
: Embedding of $|\Sigma|$ in the cylinder (topologically equivalent to an annulus) so that every positive polygon is contractible and every negative polygon is noncontractible. Equivalently, a planar embedding of $|\Sigma|$ with two distinguished faces, signed so that a polygon is negative if and only if it separates the distinguished faces.

cylindrical
: Embeddable in the cylinder.

signed rotation system (of a graph)
: A rotation system and a signature for the graph. Equivalent to a rotation system for a signing of the graph but usually here the signature is a variable and the focus is on the underlying graph.

Coloring

color set
: C_κ
: The set $C^*_\kappa \cup \{0\}$. The gain group acts on it as follows: $0g = 0$ and $(\iota, h)g = (\iota, hg)$.

zero-free color set
: C^*_κ
: The set $\{1, 2, \dots, \kappa\} \times \mathfrak{G}$, where $\mathfrak{G}$ is a gain group (such as the sign group).

signed color
: An element of C_κ for a signed graph.

gain color
: An element of C_κ for a gain graph.

coloring/coloration (of a signed or gain graph) in κ colors
: A mapping $k : V \to C_\kappa$. Note that the number of colors is said to be κ, not the cardinality of the color set.

zero-free coloring/coloration (of a signed or gain graph) in κ colors
: A mapping $k : V \to C^*_\kappa$.

improper edge (in a coloration)
: An edge that is a loose edge, or a half edge whose vertex is colored 0, or an ordinary edge $e{:}vw$ whose endpoints are colored so that $k(w) = k(v)\varphi(e; v, w)$.

improper edge set
: $I(k)$
: The set of improper edges of a coloration.

proper coloring/coloration in κ colors
: Coloration with no improper edges.

Flows

flow (on a bidirected graph)
: A mapping $f : E \to \mathfrak{A}$, where $\mathfrak{A}$ is an abelian group, esp. the integers or the integers modulo M. [For the differences among gains, voltages, and flows see "gain graph".]

flow (on a signed graph)
: A flow on any arbitrary fixed orientation of Σ. The orientation is for notational convenience; by convention, if an edge is reoriented and the flow on that edge is negated, this is regarded as the same flow on Σ. (This is just as for ordinary graphs.)

net inflow to a vertex
: The sum of the flow values on the edges directed into the vertex and the negatives of the flow values on the edges directed away from the vertex. (This is for a

bidirected graph; on a signed graph one uses the convention of an arbitrary fixed orientation.)

circulation

A flow for which the net flow into every vertex is 0.

Invariants

negative girth (of a signed graph)

The length of a shortest negative polygon.

frustration index
[line index of (im)balance] (Harary)

$l(\ )$

Minimum number of edges that must be removed from a signed, gain, or biased graph in order to leave a balanced subgraph. (For a signed graph, negation may be used instead of deletion.)

(vertex) frustration number
(vertex) elimination number (Harary)

Minimum number of vertices that must be removed from a signed, gain, or biased graph in order to leave a balanced subgraph.

balanced induced subgraph number
[level of balance] (Sozański)

The largest order of a balanced induced subgraph; equivalently,
$|V|-$ the frustration number.

net sign (of a signed graph)

$\mathrm{ns}(\Sigma)$

$|E_+| - |E_-|$ (from Lee, Lucchese, and Chu (1987a)).

chromatic number (of a signed graph or a gain graph with finite gain group)

$\chi(\Sigma)$, $\chi(\Phi)$

The smallest number of colors with which a signed or gain graph can be properly colored.

zero-free chromatic number (of a signed graph or a gain graph with finite gain group)

$\chi^*(\Sigma)$, $\chi^*(\Phi)$

The smallest number of colors, excluding 0, with which a signed or gain graph can be properly colored.

chromatic polynomial (of a signed graph or a gain graph with finite gain group $\mathfrak{G}$)

$\chi_\Sigma(\lambda)$, $\chi_\Phi(\lambda)$

The polynomial whose value at $\lambda = \kappa|\mathfrak{G}| + 1$ is the number of proper κ-colorings of the signed or gain graph.

chromatic polynomial (of a biased graph)

$\chi_\Omega(\lambda)$

The polynomial that equals $\lambda^{b(\Omega)}$ times the characteristic polynomial of $\operatorname{Lat} G(\Omega)$. If $\Omega = \langle\Gamma\rangle$ or $\langle\Sigma\rangle$ or $\langle\Phi\rangle$, this definition agrees with that of the chromatic polynomial of Γ or Σ or Φ.

zero-free chromatic polynomial (of a gain graph with finite gain group $\mathfrak{G}$)
balanced chromatic polynomial

$\chi^*_\Sigma(\lambda),\ \chi^*_\Phi(\lambda)$

$[\chi^{\mathrm{b}}_\Sigma(\lambda),\ \chi^{\mathrm{b}}_\Phi(\lambda)]$

The polynomial whose value at $\lambda = \kappa|\mathfrak{G}|$ is the number of proper, zero-free κ-colorings of the signed or gain graph.

balanced chromatic polynomial (of a biased graph)
[zero-free chromatic polynomial]

$\chi^*_\Omega(\lambda)$

$[\chi^{\mathrm{b}}_\Omega(\lambda)]$

The polynomial that equals $\lambda^{c(\Omega)}$ times the characteristic polynomial of $\operatorname{Lat}^{\mathrm{b}}\Omega$. If $\Omega = \langle\Sigma\rangle$ or $\langle\Phi\rangle$, this definition agrees with that of the balanced chromatic polynomial of Σ or Φ.

CHEMISTRY

The molecules of interest seem to be hydrocarbons. The graphs are those of the carbon skeleta. Usage does not seem to be completely consistent.

Huckel
: Hückel (correctly).

Mobius
: Möbius (correctly).

Hückel graph
: Ordinary/all-positive/balanced; (sometimes) a balanced polygon.

Möbius graph
: Signed/not-all-positive/unbalanced; (sometimes) an unbalanced polygon.

annulene
cycle
: Polygon

Möbius system
anti-Hückel system
: molecule of an unbalanced cycle.

Hückel system
: molecule of balanced cycle.

PHYSICS

spin glass
: Signed graph (usually a lattice graph) with positive edge weights or bond energies (possibly all equal, in which case they can be ignored).

site
: Vertex.

bond
: Edge.

spin of a site
: Sign of a vertex: up or down, corresponding to $+$ or $-$.

state
: Assignment of spins to the sites, i.e., a vertex signing.

satisfied bond
: Edge whose endpoint sign product equals its edge sign.

frustrated bond
Unsatisfied bond.

frustration
That condition of a state of a spin glass in which the bonds are not all satisfied. In the constant-energy model, a state has minimum energy iff the number of frustrated bonds is minimal, i.e., equal to the frustration index of the signed graph.

SOCIAL SCIENCE

structural balance
Balance of a signed graph associated to a social structure like a small social group, a kinship or marriage system, etc.

Morishima matrix
Square matrix A for which there is a set S so that $a_{ij} \geq 0$ if $i, j \in S$ or $i, j \notin S$ and $a_{ij} \leq 0$ otherwise. Equivalently, of course, the signed digraph of A, regarded as an undirected signed graph, is balanced.