Moore graphs and beyond: A survey of the degree/diameter problem

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Abstract

The degree/diameter problem is to determine the largest graphs or digraphs of given maximum degree and given diameter.

General upper bounds – called Moore bounds – for the order of such graphs and digraphs are attainable only for certain special graphs and digraphs. Finding better (tighter) upper bounds for the maximum possible number of vertices, given the other two parameters, and thus attacking the degree/diameter problem 'from above', remains a largely unexplored area. Constructions producing large graphs and digraphs of given degree and diameter represent a way of attacking the degree/diameter problem 'from below'.

This survey aims to give an overview of the current state-of-the-art of the degree/diameter problem. We focus mainly on the above two streams of research.

However, we could not resist mentioning also results on various related problems. These include considering Moore-like bounds for special types of graphs and digraphs, such as vertex-transitive, Cayley, planar, bipartite, and many others, on the one hand, and related properties such as connectivity, regularity, and surface embeddability, on the other hand.

Contents

1 Preface to Second Edition

The first edition of this dynamic survey of the degree/diameter problem was published in 2005. During the intervening 6 or 7 years there has been a great deal of activity in this research area so that now it is high time to produce a second edition of this survey.

As we shall see, there is a wealth of new theoretical results. These appear in the appropriate places within the main text. In some cases, the new results have justified the creation of new sections or subsections, as is for example the case of mixed Moore graphs, or approximations of Moore graphs and digraphs.

Pleasingly, we find that there is now a more widespread knowledge and appreciation of the degree/diameter and related problems, together with their associated Moore bounds. This trend is not confined only to graph theorists. Indeed, since 2005, we have found many new references in computer science, in particular, parallel and distributed processing [112], [113], [130], [235], optical networks [88], interconnection networks [345], [346], [315], [354], [193], [214], [219], P2P networks [194], [195], [234], [197], [198], [281], [295], etc.

We only list some of the references as an illustration of the growing practical importance of the degree/diameter and related problems; however, we do not intend to include the details in this survey since we aim to concentrate on the theoretical aspects.

In the second edition we have kept the basic structure of the survey the same as in the first edition, except that we added several new topics and we separated some subtopics as they have become more popular or important during the last 7 years. However, the overall aims of the survey remain the same: to gather together material relevant to the degree/diameter problem and to popularise it amongst researchers in graph theory.

An interesting new development occurred in 2009 when Loz, Pérez-Rosés and Pineda-Villavicencio created CombinatoricsWiki

`http://combinatoricswiki.org`

a project aiming to present the latest concepts, results, conjectures and references in various topics of Combinatorics. At present CombinatoricsWiki contains material about the following topics: enumeration of Latin squares and rectangles, the cage problem or degree/girth problem, the degree/diameter problem, the maximum degree-and-diameter-bounded subgraph problem, minor-closed classes of matroids, Ramsey theory, extremal C_t-free graphs.

Concerning the degree/diameter problem, CombinatoricsWiki contains the latest information on the best current upper and lower bounds as follows. (a) Undirected case: general graphs, Cayley graphs, bipartite graphs, vertex-transitive graphs, arc-transitive graphs, planar graphs, and toroidal graphs. (b) Directed case: general digraphs, and vertex-symmetric digraphs.

CombinatoricsWiki relies on the whole research community to keep its content accurate and up to date. Any researcher wanting to update a section can do so once he/she is registered. For a researcher to register he/she needs to contact one of the moderators of the relevant section. We believe the CombinatoricsWiki project provides an excellent service and deserves to be supported by the research community in the degree/diameter and other research areas in graph theory and combinatorics.

We would like to also mention the existence of a workshop series IWONT (International Workshop On Network Topologies) which is largely devoted to the degree/diameter problem and related research areas. The establishment of the workshop coincided with the publication of the first edition of this survey in 2005.

Finally, we would like to thank all those who have sent us their new papers on this topic for this second edition of the survey and we would appreciate receiving more papers as they emerge before the next edition.

2 Introduction

The topology of a network (such as a telecommunications, multiprocessor, or local area network, to name just a few) is usually modelled by a graph in which vertices represent 'nodes' (stations or processors) while undirected or directed edges stand for 'links' or other types of connections.

In the design of such networks, there are a number of features that must be taken into account. The most common ones, however, seem to be limitations on the vertex degrees and on the diameter. The network interpretation of these two parameters is obvious: The degree of a vertex is the number of the connections attached to a node, while the diameter indicates the largest number of links that must be traversed in order to transmit a message between any two nodes.

What is then the largest number of nodes in a network with a limited degree and diameter? If links are modelled by undirected edges, this leads to the

- *Degree/Diameter Problem*: Given natural numbers Δ and D, find the largest possible number of vertices $n_{\Delta,D}$ in a graph of maximum degree Δ and diameter $\leqslant D$.

The statement of the directed version of the problem differs only in that 'degree' is replaced by 'out-degree'. We recall that the out-degree of a vertex in a digraph is the number of directed edges leaving the vertex. We thus arrive at the

- *(Directed) Degree/Diameter Problem*: Given natural numbers d and k, find the largest possible number of vertices $n_{d,k}$ in a digraph of maximum out-degree d and diameter $\leqslant k$.

Research activities related to the degree/diameter problem fall into two main streams. On the one hand, there are proofs of non-existence of graphs or digraphs of order close to the general upper bounds, known as the Moore bounds. On the other hand, there is a great deal of activity in the constructions of large graphs or digraphs, furnishing better lower bounds on $n_{\Delta,D}$ (resp., $n_{d,k}$).

Since the treatments of the undirected and directed cases have been quite different, we divide further exposition into two parts. Part 1 deals with the undirected case and Part 2 with the directed one.

We first discuss the existence of Moore graphs (Section 3.1) and Moore digraphs (Section 4.1). These are graphs and digraphs which attain the so called Moore bound, giving the theoretical maximum for the order of a graph (resp., digraph) of a given diameter and maximum degree (resp., out-degree).

Then we present known results on the existence of graphs (Section 3.2) and digraphs (Section 4.2) whose order is 'close' to the Moore bound, whenever the Moore bound cannot be attained.

The question of regularity of graphs close to the Moore bound is much more interesting for directed graphs than for undirected ones, and so we include a section (4.3) on this topic only in Part 2.

The next two sections (3.3 and 4.4) are then devoted to the constructions of large graphs and digraphs. In Sections 3.4 and 4.5 we introduce and discuss several restricted versions of the degree/diameter problem for graphs and digraphs.

Various related topics are listed in Sections 3.5 and 4.6. In Section 4.7 we deal with the topic of mixed graphs, also known as partially directed graphs. Finally, in the Conclusion, we present a short list of some of the interesting open problems in the area.

3 Part 1: Undirected graphs

3.1 Moore graphs

There is a straightforward upper bound on the largest possible order (i.e., the number of vertices) $n_{\Delta,D}$ of a graph G of maximum degree Δ and diameter D. Trivially, if $\Delta = 1$ then $D = 1$ and $n_{1,1} = 2$; in what follows we therefore assume that $\Delta \geqslant 2$.

Let v be a vertex of the graph G and let n_i, for $0 \leqslant i \leqslant D$, be the number of vertices at distance i from v. Since a vertex at distance $i \geqslant 1$ from v can be adjacent to at most $\Delta - 1$ vertices at distance $i + 1$ from v, we have $n_{i+1} \leqslant (\Delta - 1)n_i$, for all i such that $1 \leqslant i \leqslant D-1$. With the help of $n_1 \leqslant \Delta$, it follows that $n_i \leqslant \Delta(\Delta-1)^{i-1}$, for $1 \leqslant i \leqslant D$. Therefore,

$$\begin{aligned} n_{\Delta,D} = \sum_{i=0}^{D} n_i &\leqslant 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{D-1} \\ &= 1 + \Delta(1 + (\Delta-1) + \cdots + (\Delta-1)^{D-1}) \\ &= \begin{cases} 1 + \Delta\,\frac{(\Delta-1)^D - 1}{\Delta - 2} & \text{if } \Delta > 2 \\ 2D + 1 & \text{if } \Delta = 2 \end{cases} \end{aligned} \tag{1}$$

The right-hand side of (1) is called the *Moore bound* and is denoted by $M_{\Delta,D}$. The bound was named after Edward Forrest Moore who first proposed the problem, as mentioned in [205]. A graph whose order is equal to the Moore bound $M_{\Delta,D}$ is called a *Moore graph*; such a graph is necessarily regular of degree Δ.

The study of Moore graphs was initiated by Hoffman and Singleton. Their pioneering paper [205] was devoted to Moore graphs of diameter 2 and 3. In the case of diameter $D = 2$, they proved that Moore graphs exist for $\Delta = 2, 3, 7$ and possibly 57 but for no other degrees, and that for the first three values of Δ the graphs are unique. For $D = 3$ they showed that the unique Moore graph is the heptagon (for $\Delta = 2$). The proofs exploit eigenvalues and eigenvectors of the adjacency matrix (and its principal submatrices) of graphs.

Because of the importance of the result we include an outline of the proof of rarity of Moore graphs in the case of diameter two. Let A be the adjacency matrix of a graph of degree $\Delta \geqslant 2$, diameter $D = 2$ and order

$$n = M_{\Delta,2} = 1 + \Delta + \Delta(\Delta - 1) = \Delta^2 + 1.$$

In such a graph, any two distinct vertices are connected by a *unique* path of length at most two, which is equivalent to A satisfying the matrix equation

$$A^2 + A - (\Delta - 1)I = J \tag{2}$$

where I and J are the identity and the all-one matrices of dimension n, respectively. Since eigenvalues of A and J must satisfy the same polynomial equation and the spectrum of J is $n^1 0^{n-1}$, it follows that A has just two eigenvalues r, s distinct from Δ and these satisfy the equation $x^2 + x - (\Delta - 1) = 0$, that is, $r, s = (-1 \pm \sqrt{4\Delta - 3})/2$. For multiplicities a, b of r, s one obviously has $a + b = \Delta^2$, and the trace of A gives $\mathrm{tr}(A) = 1 \times \Delta + a \times r + b \times s = 0$. If r and s are irrational, then $a = b$ and $\Delta = 2$. If r, s are rational, then they must be integral; also, then $4\Delta - 3 = m^2$, for some positive integer m. Letting $b = \Delta^2 - a$ and then substituting $\Delta = (m^2 + 3)/4$ into the trace equation yields, after simplification, a polynomial equation for m of the fifth degree (with parameter a) with leading coefficient 1 and absolute term equal to 15. It follows that m must be a divisor of 15 and hence $m \in \{1, 3, 5, 15\}$, that is, $\Delta \in \{1, 3, 7, 57\}$. Excluding $\Delta = 1$ for trivial reasons, the solutions 3 and 7 have unique realisation, namely, the Petersen graphs (which is easy to prove by hand) and the Hoffman-Singleton graph (which is harder to prove and we refer to [205] for details).

The known Moore graphs are all vertex-transitive; in addition, both the Petersen and the Hoffman-Singleton graph are known to be non-Cayley graphs. No example of a Moore graph of degree $\Delta = 57$ and diameter $D = 2$ has been found yet; all one can say is that its existence is 'arithmetically feasible'. In contrast to the other Moore graphs, however, it was shown by Higman in the 1960's in one of his lectures for graduate students that Moore graph(s) of degree 57 and diameter 2 cannot be vertex-transitive, cf. [81].

Investigation into the structure of the automorphism group(s) of the missing Moore graphs(s) of degree 57 and diameter 2 was initiated by Makhnev and Paduchikh [247] and rather deep results were obtained later by Mačaj and Širáň [246] by a mixture of spectral techniques and rational group characters. Severe restrictions were proved in [246] on both the abstract group structure (in terms of possible Sylow subgroups, for example) and on the permutation action of the automorphism group of any such 'missing' Moore graph(s). In particular, it follows from [246] that the order of the automorphism group any such Moore graph (of diameter 2, degree 57 and hence with $M_{57,2} = 3250$ vertices) is bounded above by 375 if the order is odd, and by 110 if the order is even; in both cases the bound is surprisingly small compared to the order of the graph.

It turns out that no Moore graphs exist for the parameters $\Delta \geqslant 3$ and $D \geqslant 3$. This was shown by Damerell [114] by way of an application of his theory of distance-regular graphs to the classification of Moore graphs. An independent proof of this result was also given by Bannai and Ito [23]. Another independent proof of a partial result of the nonexistence (degree at most 100 and diameter at most 100) was given by Plesnik [286].

Main results concerning Moore graphs can therefore be summed up as follows. Moore graphs for diameter $D = 1$ and degree $\Delta \geqslant 1$ are the complete graphs $K_{\Delta+1}$. For diameter $D = 2$, Moore graphs are the cycle C_5 for degree $\Delta = 2$, the Petersen graph (see Fig. 1) for degree $\Delta = 3$, and the Hoffman-Singleton graph (see Fig. 2, drawn by Slamin

[319]) for degree $\Delta = 7$. Finally, for diameter $D \geqslant 3$ and degree $\Delta = 2$, Moore graphs are the cycles on $2D + 1$ vertices C_{2D+1}.

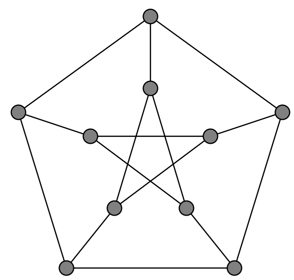


Figure 1: Petersen graph.

Between the time of the publication of the Hoffman-Singleton paper (1960) and the time of the publication of the results by Bannai-Ito and Damerell (both in 1973), there were several related partial results published. For example, Friedman [169] showed that there are no Moore graphs for parameters (Δ, D), where $\Delta = 3, 4, 5, 6, 8$ and $3 < D \leqslant 300$, except, possibly, for the pair $(5, 7)$. He also showed that there are no Moore graphs with parameters $(3, D)$, when $D \geqslant 3$ and $2D + 1$ is prime. Bosák [66] proved the nonexistence of Moore graphs of degree 3 and diameter D, $3 \leqslant D \leqslant 8$. For another contribution to nonexistence proofs, see also Plesník [286]. A combinatorial proof that the Moore graph with $\Delta = 7$ and $D = 2$ is unique was given by James [212]. Fan and Schwenk [147] gave a simpler proof of the same. As an aside, a connection of Moore graphs with design theory was found by Benson and Losey [42], by embedding the Hoffman-Singleton graph in the projective plane $PG(2, 5^2)$.

A problem closely related to the degree/diameter problem is the so-called degree/girth problem, which can be posed as follows:

> *Degree/girth problem*: Given natural numbers $d \geqslant 2$ and $g \geqslant 3$, find the smallest possible number of vertices in a regular graph of degree d and girth g.

A regular graph of degree Δ, girth g and minimum possible order is called a *cage*. Tutte [334] was the first to study (Δ, g)-cages. However, researchers became really interested in this class of graphs when Erdős and Sachs [137] proved that a (Δ, g)-cage exists for all $\Delta \geqslant 2$ and $g \geqslant 3$. At present only a few (Δ, g)-cages are known. The state-of-the-art in the study of cages can be found in the dynamic survey by Exoo and Jajcay [142]. See also the earlier survey by Wong [344].

The Moore bound represents not only an upper bound on the number $n_{\Delta,D}$ of vertices of a graph of maximum degree Δ and diameter D, but it is also a lower bound on the

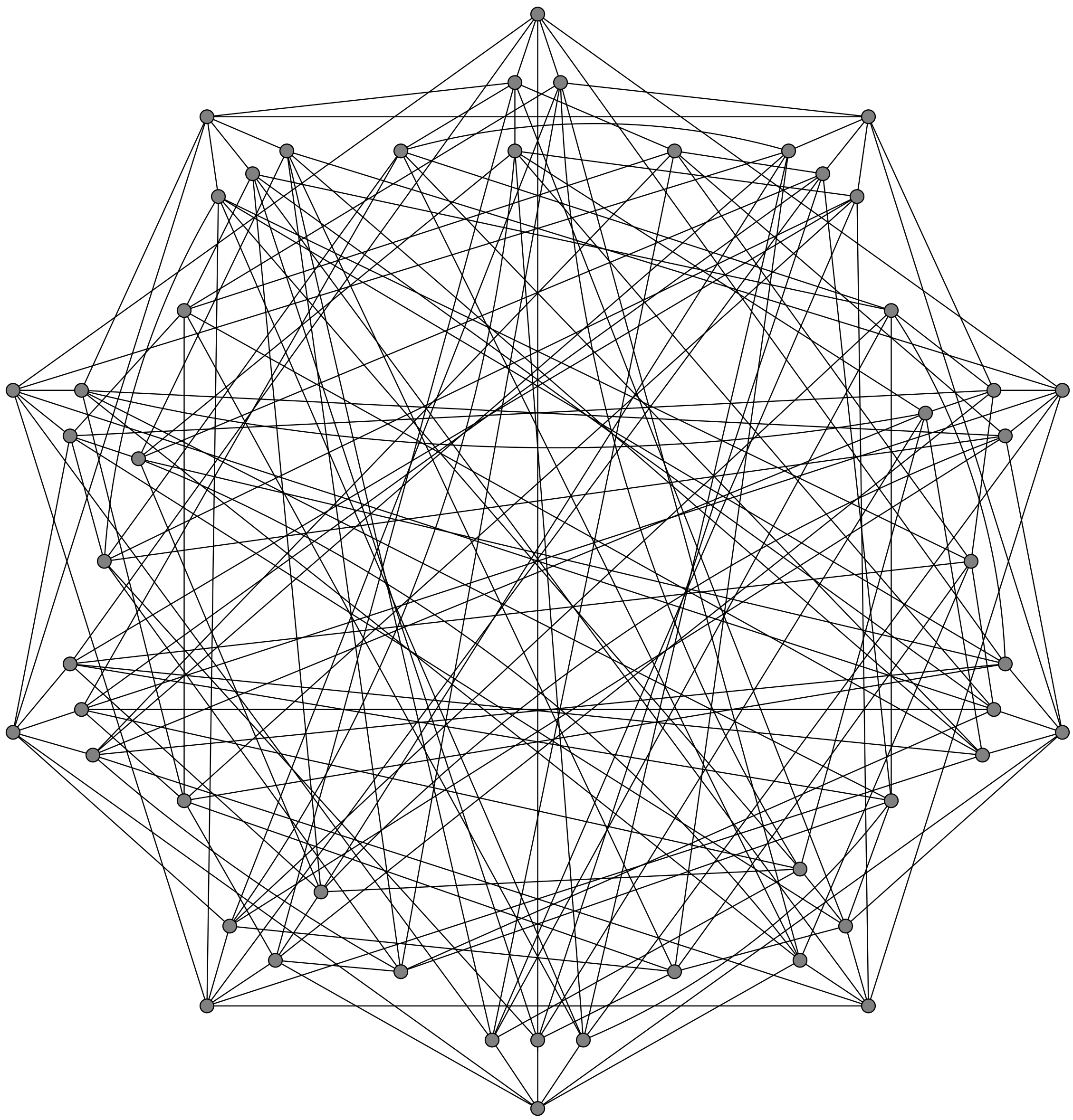


Figure 2: Hoffman-Singleton graph.

number of vertices of a regular graph of degree Δ and girth $2D+1$ [58]. A (Δ, g)-cage of order $M_{\Delta,D}$ is therefore a Moore graph if $g = 2D+1$.

In this connection, Singleton [317] proved that a graph of diameter D and girth $2D+1$ is necessarily regular.

Several other areas of research in graph theory turn out to be related or inspired by the theory of Moore graphs; examples include antipodal graphs, Moore geometries, and Moore groups.

A graph is *antipodal* if for each vertex x there exists a vertex z such that $d(x,y)+d(y,z) = d(x,z)$, for all vertices y of the graph. Sabidussi [297] showed that Moore graphs of diameter 2 and degree 3, 7, or possibly 57 are 'antipodal quotients' of certain extremal antipodal graphs of odd diameter.

Fuglister [170, 171] and Bose and Dowling [70] investigated finite Moore geometries which are a generalisation of Moore graphs; other contribution to this area include Damerell and Georgiacodis [116], Damerell [115], and Roos and van Zanten [293, 294]. Finally, Fried and Smith [167] defined a Moore group and proved results that limit the possible degrees that Moore groups of fixed rank can have, by reducing the problem to the study of Moore graphs.

We conclude this section by a summary of the known results about Moore graphs in Table 1 below.

Maximum Degree Δ	*Diameter* D	*Moore Graphs*
$\geqslant 2$	1	Complete graphs $K_{\Delta+1}$
2	$\geqslant 2$	Cycles C_{2D+1}
3	2	Petersen graph
7	2	Hoffman-Singleton graph
57	2	?

Table 1: Moore graphs.

3.2 Graphs of order close to Moore bound

Since Moore graphs exist only in a small number of cases, the study of the existence of large graphs of given diameter and maximum degree focuses on graphs whose order is 'close' to the Moore bound, that is, graphs of order $M_{\Delta,D} - \delta$, for δ small. The parameter δ is called the *defect*, and the most usual understanding of a 'small defect' is that $\delta \leqslant \Delta$.

For convenience, by a (Δ, D)-*graph* we will understand any graph of maximum degree Δ

and of diameter at most D; if such a graph has order $M_{\Delta,D} - \delta$ then it will be referred to as a (Δ, D)-*graph of defect* δ or a $(\Delta, D, -\delta)$-*graph*.

Erdős, Fajtlowitcz and Hoffman [135] proved that, apart from the cycle C_4, there are no graphs of degree Δ, diameter 2 and defect 1, that is, of order one less than the Moore bound; for a related result, see Fajtlowicz [146].

This was subsequently generalized by Bannai and Ito [24], and also by Kurosawa and Tsujii [230], to all diameters. Hence, for all $\Delta \geqslant 3$, there are no (Δ, D)-graphs of defect 1, and for $\Delta = 2$ the only such graphs are the cycles C_{2D}. It follows that, for $\Delta \geqslant 3$, we have $n_{\Delta,D} \leqslant M_{\Delta,D} - 2$.

Let us now discuss the case of defect $\delta = 2$. Clearly, if $\Delta = 2$ then the $(\Delta, D, -2)$-graphs are the cycles C_{2D-1}. For $\Delta \geqslant 3$, only five $(\Delta, D, -2)$-graphs are known at present: Two $(3, 2, -2)$-graphs of order 8, a $(4, 2, -2)$-graph of order 15, a $(5, 2, -2)$-graph of order 24, and a $(3, 3, -2)$-graph of order 20.

The last three of these graphs were found by Elspas [134] and are known to be unique; in Bermond, Delorme and Farhi [46], the $(3, 3)$-graph was constructed as a certain product of a 5-cycle with the field of order four. All these graphs are depicted in Figure 3.

Considering $\Delta = 3$, $D \geqslant 2$, for $(3, D, -2)$-graphs Jørgensen [217] proved that there are exactly two non-isomorphic $(3, 2, -2)$-graphs, and that there is a unique $(3, 3, -2)$-graph. Furthermore, he proved that these are the only $(3, D, -2)$-graphs for $D \geqslant 2$.

The uniqueness of the $(4, 2, -2)$-graph was proved by Broersma and Jagers [75], while the uniqueness of the $(5, 2, -2)$-graph was proved by Nguyen and Miller [275].

For diameter 2 and defect 2, Miller, Nguyen and Pineda-Villavicencio [260, 262, 261] proved the following results. When $\Delta \geqslant 6$ is even, if $\Delta \not\equiv 1 \pmod 3$ then there is no $(\Delta, 2, -2)$-graph. When $\Delta \geqslant 7$ is odd, if $\Delta \neq l^2 + l + 3$ and $\Delta \neq l^2 + l - 1$, for each positive integer l, then there is no $(\Delta, 2, -2)$-graph.

Miller, Nguyen and Pineda-Villavicencio [260] conjectured that for $\Delta \geqslant 6$ there are no graphs of maximum degree Δ, diameter 2 and defect 2. Further support for the conjecture was provided by Conde and Gimbert in [105], where the authors proved that there are no $(\Delta, 2, -2)$-graphs for $5 < \Delta < 50$.

Considering the case of $\Delta = 4$, in an earlier work, Stanton, Seah and Cowan [327] proved that a graph with maximum degree 4 and diameter $D \geqslant 4$ cannot have defect 2. Subsequently, Miller and Simanjuntak [267] provided several structural properties of $(4, D, -2)$-graphs. A full proof of the non-existence of $(4, D, -2)$-graphs for $D \geqslant 3$ was obtained by Feria-Purón, Miller and Pineda-Villavicencio [153]. Structural properties of $(5, D)$-graphs

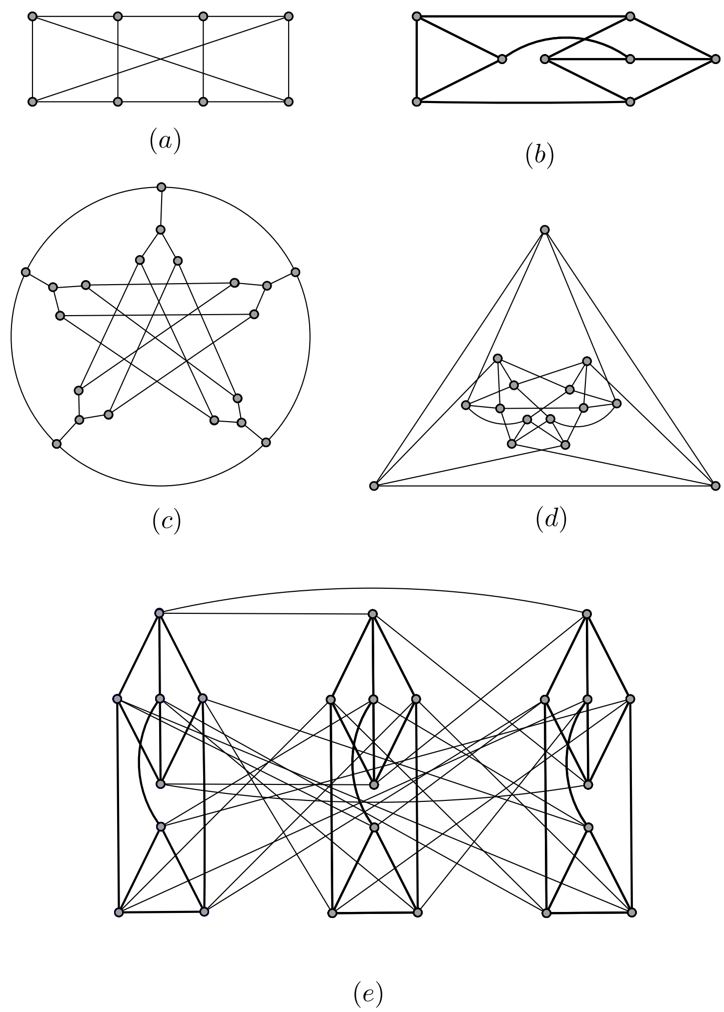


Figure 3: Known graphs of defect 2. (a) and (b) the $(3, 2, -2)$-graphs, (c) the unique $(3, 3, -2)$-graph, (d) the unique $(4, 2, -2)$-graph and (e) the unique $(5, 2, -2)$ graph (note that this graph is formed by 3 copies of the graph (b)).

of defect 2, $D \geqslant 2$, were obtained by Pineda-Villavicencio and Miller [285].

The paper [153] proved not only the nonexistence of $(4, D, -2)$-graphs for $D \geqslant 3$ but also the nonexistence of (Δ, D)-graphs of defect 2 for all even $\Delta \geqslant 4$ and $D \geqslant 4$; this outcome, together with a proof on the non-existence of $(4, 3)$-graphs of defect 2 completed the catalogue of $(4, D)$-graphs of defect δ with $D \geqslant 2$ and $0 \leqslant \delta \leqslant 2$. Such a catalogue is only the second census of (Δ, D)-graphs of defect 2 known at present, the first being the one of $(3, D)$-graphs of defect δ with $D \geqslant 2$ and $0 \leqslant \delta \leqslant 2$ [217].

The paper [153] also includes some necessary conditions for the existence of (Δ, D)-graphs of defect 2 with odd $\Delta \geqslant 5$ and $D \geqslant 4$, and the non-existence of (Δ, D)-graphs of defect 2 with odd $\Delta \geqslant 5$ and $D \geqslant 5$ such that $\Delta \equiv 0, 2 \pmod{D}$. Based on these results, the authors have conjectured that there are no (Δ, D)-graphs of defect 2 with $\Delta \geqslant 4$ and $D \geqslant 4$. Moreover, these results (together with [24]) imply that $n_{4,2} = 15$, $n_{5,2} = 24$, and $n_{3,3} = 20$.

In [274], Nguyen and Miller proved some structural properties of graphs of diameter 2 and maximal repeats, that is, graphs with the property that there exists a vertex with unique paths of lengths at most the diameter to all the other vertices except one vertex to which the number of walks of length at most the diameter is equal exactly to the defect plus 1. Furthermore, in [276], they considered graphs with diameter 2 and defect 3. They proved that such graphs must contain a certain induced subgraph, which in turn leads to the proof that, for degree 6 and diameter 2, the largest order of a vertex-transitive graph is 32.

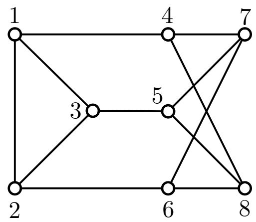
Before leaving the topic of graphs of defect 2, we discuss a particular subclass of the graphs of defect 2. For a graph Γ of degree Δ with adjacency matrix A, we define the polynomials $G_{\Delta,m}(x)$ for $x \in \mathbb{R}$:

$$\begin{cases} G_{\Delta,0}(x) = 1 \\ G_{\Delta,1}(x) = x + 1 \\ G_{\Delta,m+1}(x) = xG_{\Delta,m}(x) - (\Delta - 1)G_{\Delta,m-1}(x) \text{ for } m \geqslant 1 \end{cases} \tag{3}$$

It is known that the entry $(G_{d,m}(A))_{\alpha,\beta}$ counts the number of paths of length at most m joining the vertices α and β in Γ; see [205, 24, 316].

Regular graphs with degree Δ and defect δ satisfy the equation $G_{\Delta,D}(A) = J_n + B$, where A denotes the adjacency matrix of the graph in question, n its order, J_n the $n \times n$ matrix whose entries are all 1's and B a matrix with the row and column sums equal to δ.

For Moore graphs (graphs with $\delta = 0$), the matrix B is the null matrix. For graphs with defect or excess 1, B can be considered as the adjacency matrix of a matching with n



$$A = \begin{pmatrix} 0&1&1&1&0&0&0&0\\ 1&0&1&0&0&1&0&0\\ 1&1&0&0&1&0&0&0\\ 1&0&0&0&0&0&1&1\\ 0&0&1&0&0&0&1&1\\ 0&1&0&0&0&0&1&1\\ 0&0&0&1&1&1&0&0\\ 0&0&0&1&1&1&0&0 \end{pmatrix} \qquad B = \begin{pmatrix} 0&1&1&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&1&0&1&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&2\\ 0&0&0&0&0&0&2&0 \end{pmatrix}$$

Figure 4: Labelling of a $(3, 2, -2)$-graph that produces the desired structure of the corresponding defect matrix B.

vertices (see [24]), while for graphs with defect or excess 2, B can be assumed to be the adjacency matrix of a union of vertex-disjoint cycles (see [126]). Figure 4 shows a $(3, 2)$-graph of defect 2 labelled in such a way that the matrix B displays the aforementioned structure.

Graphs with defect 2 having the adjacency matrix of a cycle of order n as the matrix B are called *graphs with cyclic defect*. Hoffman asked whether the Möbius ladder on 8 vertices is the only graph of diameter 2 and cyclic defect. A positive answer was given by Fajtlowicz [146]. What about such graphs of cyclic defect having a larger diameter? Combining the results of Delorme and Pineda-Villavicencio [126] and Miller [257], it turns out that the Möbius ladder is the only such graph for any diameter.

Little is known about graphs with defects larger than two. One of the few works in this direction is by Nguyen and Miller [276], in which the authors provided structural properties of graphs with diameter 2 and defect 3. Another work is by Miller and Pineda-Villavicencio [265] who gave the complete catalogue of connected graphs of maximum degree 3 and defect at most 4. This is the first such catalogue for a particular degree, defect 4 and any diameter.

Some further upper bounds on the maximum number of vertices for graphs which are not Moore were given by Smyth [323]. See also Buskens and Stanton [79], Buskens, Rogers and Stanton [80], and Cerf, Cowan, Mullin and Stanton [89, 90] for work related to (small) graphs of order close to Moore bound; see McKay and Stanton [251] for an early survey of generalized Moore graphs.

In Sections 3.1 and 3.2, we have seen that for certain pairs (Δ, D) there exist graphs of order close to (and in some cases equal to) the Moore bound. The situation for pairs (Δ, D) not discussed above is largely unknown. In this connection, Bermond and Bollobás [43] asked the following interesting question:

Is it true that for each positive integer c there exist Δ and D such that the order of the largest graph of maximum degree Δ and diameter D is at most $M_{\Delta,D} - c$?

3.3 Constructions of large graphs

Another way to study graphs close to the Moore bound is by constructing large graphs in order to find improvements in the lower bound on the maximum possible order of graphs for given D and Δ. This has been done in various ways, and often by considering particular classes of graphs, such as vertex-transitive and Cayley graphs (which will be discussed in more detail in the forthcoming sections).

The pioneers in the construction of large graphs were Elspas and Green [134]. In 1964 Elspas constructed, among others, a $(4, 2, -2)$-graph and a $(5, 2, -2)$-graph, while he credited Green with producing a $(3, 3, -2)$-graph.

In the quest for large graphs many ingenious techniques have been used, for instance, the star product [45], the voltage assignment technique [29, 71, 72] and graph compounding [47]. Researchers have also made use of computers to find large graphs [129, 139, 242]. An overview of these constructions is given in this section.

In a general sense we could divide all the known constructions into two wide categories: general and *ad hoc* constructions. As general methods, below we present the constructions of the de Bruijn graphs and Kautz graphs, while as *ad hoc* constructions, we give some of the most relevant constructions for small diameters.

3.3.1 General overview

The *undirected de Bruijn graph* of type (t, k) has vertex set V formed by all sequences of length k, the entries of which are taken from a fixed alphabet consisting of t distinct letters. In the graph, two vertices $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ are joined by an edge if either $a_i = b_{i+1}$ for $1 \leqslant i \leqslant k-1$, or if $a_{i+1} = b_i$, for $1 \leqslant i \leqslant k-1$. Obviously, the undirected deBruijn graph of type (t, k) has order t^k, degree $\Delta = 2t$ and diameter $D = k$. These graphs therefore give, for any Δ and D, the lower bound

$$n_{\Delta,D} \geqslant \left(\frac{\Delta}{2}\right)^D .$$

Kautz graphs of type (t, k) [222] are easily derived from de Bruijn graphs of type (t, k). Given a de Bruijn graph of type (t, k), a Kautz graph of type (t, k) is obtained by deleting words with two consecutive identical letters in the de Bruijn graph. The Kautz graph is therefore an induced subgraph of the de Bruijn graph, and if $t \geqslant 3$ and $k \geqslant 3$, it has order $t(t-1)^{k-1}$, diameter k and maximum degree $2t-2$. Thus, for any D and even Δ, such graphs improve the bound given de Bruijn graphs, and consequently, we have

$$N_{\Delta,D} \geqslant \left(\frac{\Delta}{2}\right)^D + \left(\frac{\Delta}{2}\right)^{D-1}$$

Independently, ignoring directions in the digraph construction of Baskoro and Miller [32] produces graphs of even maximum degree Δ and diameter at most D; the order of these graphs is also

$$\left(\frac{\Delta}{2}\right)^D + \left(\frac{\Delta}{2}\right)^{D-1}.$$

A substantial progress was achieved by Canale and Gómez [84] by exhibiting, for an infinite set of values of Δ, families of graphs showing that

$$n_{\Delta,D} \geqslant \left(\frac{\Delta}{1.6}\right)^D$$

for all $D \geqslant 3$; the constant in the denominator can be replaced by 1.57 for D congruent with -1, 0, or 1 (mod 6).

For completeness, we mention several related results. Certain extensions of deBruijn graphs were studied by Canale and Gómez [85]. An adaptation of the digraph construction of Imase and Itoh [208] also gives (Δ, D)-graphs of order at least $\lceil \frac{\Delta}{2} \rceil^D$. The list of general lower bounds also includes constructions by Elspas [134], Friedman [168], Korn [228], Akers [5] and Arden and Lee [13], all giving (Δ, D)-graphs of order

$$f(\Delta)(\Delta-1)^{\lceil \frac{D}{2} \rceil} + g(\Delta),$$

where f and g depend on Δ but not on D.

Much better results have been obtained for small values of D. By far the best result is furnished by Brown's construction [76], with the help of finite projective geometries. Let q be a prime power and let F be the Galois field of order q. A *projective point* is any collection of $q-1$ triples of the form (ta, tb, tc), where $t \in F$ and (a, b, c) is a non-zero vector in F^3; any non-zero triple in this set is a *representative* of the point. Let $\mathcal{P}$ be the set of all such points; it is easy to see that

$$|\mathcal{P}| = q^2 + q + 1.$$

Let G be the graph with vertex set $\mathcal{P}$, where two vertices are adjacent if the corresponding projective points have orthogonal representatives. Since any two non-orthogonal representatives are orthogonal to some non-zero element of F^3, the graph G has diameter 2. In general, G need not be regular but its maximum degree is always $\Delta = q + 1$. Therefore, for each Δ such that $\Delta - 1$ is a prime power, we have [76]

$$n_{\Delta,2} \geqslant \Delta^2 - \Delta + 1. \tag{4}$$

As observed by Erdős, Fajtlowicz and Hoffman [135], and by Delorme [120], this bound can be improved to

$$n_{\Delta,2} \geqslant \Delta^2 - \Delta + 2 \tag{5}$$

if $\Delta - 1$ is a power of 2. In order to obtain a lower bound for the remaining values of Δ one may use the result of [20] about gaps between primes, stating that for any sufficiently large x there is a prime p such that $x - x^{0.525} \leqslant p \leqslant x$. Using this fact, it was shown in [318] that Brown graphs can be modified to give, for all sufficiently large Δ, the inequality

$$n_{\Delta,2} \geqslant \Delta^2 - 2\Delta^{1.525} \tag{6}$$

The algebraic structure of Brown graphs and their automorphism groups have been investigated in great detail by Bachratý and Širáň [19].

For larger diameter, it seems more reasonable to focus on asymptotic behaviour of $n_{\Delta,D}$ for fixed D while $\Delta \to \infty$. Delorme [119] introduced the parameter

$$\mu_D = \lim\ \inf_{\Delta\to\infty} \frac{n_{\Delta,D}}{\Delta^D}.$$

Trivially, $\mu_D \leqslant 1$ for all D, and $\mu_1 = 1$; the bound (6) shows that $\mu_2 = 1$ as well. Further results of Delorme [118] imply that μ_D is equal to 1 also for $D = 3$ and $D = 5$.

The above facts can be seen as an evidence in favour of an earlier conjecture of Bollobás [62] that, for each $\varepsilon > 0$, it should be the case that

$$n_{\Delta,D} > (1 - \varepsilon)\Delta^D$$

if Δ and D are sufficiently large.

The values of μ_D for other diameters D are unknown. For example, for diameter 4 we only know that $\mu_4 \geqslant 1/4$; see Delorme [120] for more information.

3.3.2 Star product and compounding

A number of sophisticated constructions arose in the quest for large graphs of given degree and diameter. We comment in some detail on two that seem to be most important: the star product of Bermond, Delorme and Farhi [45, 46] and the compounding of graphs introduced by Bermond, Delorme and Quisquater [47].

The concept of the *star product* of two graphs H and K was introduced by Bermond, Delorme and Farhi [45] as follows. Fix an arbitrary orientation of all edges of H and let $\vec{E}$ be the corresponding set of the fixed darts of H. For each dart $uv \in \vec{E}$, let ϕ_{uv} be a bijection on the set $V(K)$. Then the vertex set of the star product $H*K$ is $V(H)\times V(K)$, and a vertex (u,k) is joined in $H*K$ to a vertex (v,l) if and only if either $u = v$ and kl is an edge of K, or if $uv \in \vec{E}$ and $l = \phi_{uv}(k)$.

Loosely speaking, the star product of H and K can be formed by taking $|V(H)|$ copies of K, whereby two copies of K 'represented' by vertices $u, v \in V(H)$ are interconnected by a perfect matching (that depends on the bijection ϕ_{uv}), whenever $uv \in \vec{E}$.

With the help of the star product, Bermond, Delorme and Farhi [45, 46] described several families of large (Δ, D)-graphs for various values of Δ and D. An inspection of their examples reveals, however, that in *all* instances they actually used a special case of the star product that we describe next.

Let Γ be a group and let S be a symmetric unit-free generating set S, meaning that $S^{-1} = S$ and $1_\Gamma \neq S$. The *Cayley graph* $C(\Gamma, S)$ is the graph with vertex set Γ, two vertices a, b being adjacent if $a^{-1}b \in S$. In the above definition of the $*$-product $H * K$, take now $K = C(\Gamma, S)$ and $\phi_{uv}(k) = g_{uv}\psi_{uv}(k)$, where g_{uv} is an arbitrary element of Γ and ψ_{uv} is an automorphism of Γ. In [45, 46], the authors used this special version of the $*$-product mainly with Cayley graphs of cyclic groups and of the additive groups of finite fields.

We now briefly comment on compounding. Roughly speaking, compounding of two graphs G and H is obtained by taking $|V(H)|$ copies of G, indexed by the vertices of H, and joining two copies G_u, G_v of G by a single edge (or a pair of edges) whenever uv is an edge of H. Depending on particular positions of edges between copies of the graph G, one may obtain various large graphs of given degree and diameter.

This method tends to give good results, especially in *ad hoc* combinations with other methods. For instance, Fiol and Fábrega [159], and Gómez [180], considered compounding combined with graphs on alphabets, where vertices are words over a certain alphabet and adjacency is defined by various relations between words. Large graphs of diameter 6, obtained by methods in this category, were given by Gómez [181]. Other related results were produced by Fiol, Yebra and Fábrega [166], and by Gómez and Fiol [185].

Several other *ad hoc* methods have been designed in connection with searching for large (Δ, D)-graphs for relatively small values of Δ and D. As most of these methods are based on graphs related in one way or another to algebraic structures (mostly groups), we will discuss them in more detail in the next subsection.

Here we mention a method by Gómez, Pelayo and Balbuena [191] that produces large graphs of diameter six by replacing some vertices of a Moore bipartite graph of diameter six with graphs K_h which are joined to each other and to the rest of the graph using a special graph of diameter two. The degree of the constructed graph remains the same as the degree of the original graph. In an extension to this work, Gómez and Miller [188] presented two new generalizations of two large compound graphs.

By means of the compounding of complete graphs into a bipartite Moore graph of diameter 6, Gómez, Miller, Pérez-Rosés, and Pineda-Villavicencio [284] obtained a family of large graphs of the same diameter. For maximum degrees $\Delta = 5, 6, 9, 12$ and 14, the members of this family constitute the current largest known graphs of diameter 6.

3.3.3 Graph lifting

Graph lifting has been well known in algebraic and topological graph theory for decades [192]. It is well suited for producing large (Δ, D)-graphs since a number of other construction methods can be reduced to lifting.

In order to describe the graph lifting construction, we will think of (undirected) edges as being formed by pairs of oppositely directed *darts*; if e is a dart then e^{-1} will denote its reverse. The set $D(G)$ of all darts of G then satisfies $|D(G)| = 2|E(G)|$.

For a finite group Γ, a mapping $\alpha : \ D(G) \to \Gamma$ will be called a *voltage assignment* if $\alpha(e^{-1}) = (\alpha(e))^{-1}$, for any dart $e \in D(G)$. The pair (G, α) determines the *lift* G^α of G. The vertex set and the dart set of the lift are $V(G^\alpha) = V(G) \times \Gamma$ and $D(G^\alpha) = D(G) \times \Gamma$, In the lift, (e, g) is a dart from the vertex (u, g) to the vertex (v, h) if and only if e is a dart from u to v in the *base graph* G and, at the same time, $h = g\alpha(e)$. The lift is an undirected graph because the darts (e, g) and $(e^{-1}, g\alpha(e))$ are mutually reverse and form an undirected edge of G^α.

Figure 5 shows an example of a base graph with ordinary voltages in the group $\mathcal{Z}_5 \times \mathcal{Z}_5$ which lifts to the Hoffman-Singleton graph, displayed in Fig. 2; the function $p(i)$ in Fig. 5 can be any quadratic polynomial over $\mathcal{Z}_5$ in the variable i, as follows from [306].

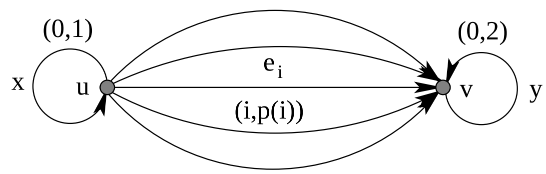


Figure 5: The base graph for the Hoffman-Singleton graph.

It is known [192] that a graph H is a lift (of a smaller graph) if and only if the automorphism group of H contains a non-trivial subgroup acting freely on the vertex set of H. This condition is in fact satisfied for most of the current largest examples of (Δ, D)-graphs, and hence most of these can be described as lifts.

The latest examples of reformulating an existing construction in terms of lifts are the largest known (Δ, D)-graphs for the pairs (3,7), (3,8), (4,4), (5,3), (5,5), (6,3), (6,4), (7,3), (14,3), and (16,2), initially obtained by Exoo [139] by computer search. Having mentioned computers, we note that the diameter of the lift can be conveniently expressed in terms of voltages on walks of the base graph [71]; besides its theoretical importance, this fact can be used to design efficient diameter-checking algorithms.

The cases when the base graphs are bouquets of loops (possibly with semi-edges, i.e., 'dangling' non-loop edges incident with just one vertex) are of particular importance, since their lifts are Cayley graphs. A more colloquial but equivalent definition of a Cayley graph was given in the previous subsection. While Cayley graphs are always lifts of single-vertex graphs, in many instances quite complex Cayley graphs (such as the Cayley graphs of certain semidirect products of Abelian groups considered in [129]) can actually be described as ordinary lifts of smaller Cayley graphs, with voltages in Abelian (mostly cyclic) groups; see [72].

For more general base graphs, there exist convenient sufficient conditions [72] for a lift to be a vertex transitive (or a Cayley) graph, which can be successfully used to produce large vertex transitive (Δ, D)-graphs by lifts. Results for vertex-transitive and Cayley graphs will be surveyed in Subsections 3.4.1 and 3.4.2. We conclude this subsection with a remark relating lifts of graphs with the $*$-product $G * H$: If H is a Cayley graph and if the group values on the edges of G are taken in the Cayley group of H then $G * H$ is just a lift of G.

3.3.4 Tables of large graphs

Needless to say that in many cases the largest currently known (Δ, D)-graphs have been found with the assistance of computers. It is clear that computation of diameter is much easier in the case of graphs that admit a lot of symmetries; here of particular advantage are vertex-transitive graphs which we will discuss in the next subsection. At this point we note that Toueg and Steiglitz [333] present a local search algorithm for the design of small diameter networks, for both directed and undirected graphs. The resulting graphs tend to have small diameter and small average shortest distance.

It is clear that discovering large graphs through computer search involves dealing with large search spaces. Therefore, techniques are needed to reduce such search spaces, allowing that some heuristic methods for combinatorial searches could be used later.

Before 2006 the most encouraging results had been obtained by Dinneen and Hafner [129], who used computer search and clever techniques to reduce the search space. Their large graphs were Cayley graphs of semidirect products of cyclic groups and other types of groups.

In order to obtain large (Δ, D)-graphs, authors are currently relaxing the symmetry conditions, and are therefore considering a wider spectrum of graphs. In this area Exoo [139] obtained a family of large graphs by seeking graphs whose order is a small integral multiple of the size of the respective automorphism group.

At present the table of the largest known graphs (see [239]) shows that most of the entries have been obtained by Loz and Širáň [242] and Loz and Pineda-Villavicencio [241]. These are two examples of searches for large graphs that have not been restricted to Cayley graphs; only some of the obtained graphs are Cayley.

The paper [241] gives a complete overview of the state-of-the-art methodology that can be used to construct large graphs of bounded degree and small diameter.

Descriptions of many new constructions, often accompanied by a new corresponding table of the largest known values of (Δ, D)-graphs, are published frequently. These include constructions by Alegre, Fiol and Yebra [8], Bar-Yehuda and Etzion [26], Bermond, Delorme and Farhi [45, 46], Bermond, Delorme and Quisquater [48, 49, 51, 50], Campbell *et al.* [83], Carlsson, Cruthirds, Sexton and Wright [87], Chudnovsky, Chudnovsky and Denneau [93], Chung [94], Comellas and Gómez [103], Delorme [118, 120], Delorme and Farhi [122], Dinneen and Hafner [129], Doty [131], Gómez, Fiol and Serra [186], Gómez, Fiol and Yebra [187], Hafner [200], Memmi and Raillard [252], Smyth [322], and Storwick [328].

Table 2 shows a summary of current largest known graphs for degree $\Delta \leqslant 16$ and diameter $D \leqslant 10$. These graphs provide the best current lower bounds on the order of graphs for given values of degree and diameter. This table can be found on the website

`http://maite71.upc.es/grup_de_grafs/grafs/taula_delta_d.html`

which is updated regularly by Francesc Comellas. A latex file of this table can be obtained upon request from Charles Delorme at email "cd@lri.fr".

More recently, in 2009 Eyal Loz, Hebert Pérez-Rosés and Guillermo Pineda-Villavicencio created CombinatoricsWiki

`http://combinatoricswiki.org`

a project aiming to present the latest concepts, results, conjectures and references in various topics of Combinatorics. At present CombinatoricsWiki contains material about various topics, including the degree/diameter problem.

Before going on to consider some restricted versions of the degree/diameter problem, we shall now summarise all the known exact solutions of the degree/diameter problem.

Due to the complete graphs and cycles of odd length we have $n_{\Delta,1} = \Delta + 1$ and $n_{2,D} = 2D + 1$. There are only a few exact values of $n_{\Delta,D}$ known for $\Delta, D \geqslant 2$. We have the values of $n_{\Delta,2}$ involving Moore graphs, $n_{3,2} = 10$, $n_{7,2} = 50$. We also have the values of $n_{3,3} = 20$, $n_{4,2} = 15$ and $n_{5,2} = 24$. The latest two known values of $n_{\Delta,D}$ were obtained as follows.

For maximum degree 3 and diameter 4 the Moore bound is $M_{3,4} = 46$. Jørgensen [216] and [217] proved that there is no $(3, 4)$-graph of defect 2 or 4, and Buset [78] showed that there is no $(3, 4)$-graph of defect 6. Therefore, the two known non-isomorphic $(3, 4)$-graphs of defect 8 constructed by Doty [131] and by von Conta [110] are maximal, and thus, $n_{3,4} = 38$.

D \ Δ	2	3	4	5	6	7	8	9	10
3	**10**	**20**	**38**	**70**	**132**	**196**	**336**	**600**	**1 250**
4	**15**	**41**	**98**	**364**	**740**	**1 320**	**3 243**	**7 575**	**17 703**
5	**24**	**72**	**212**	**624**	**2 772**	**5 516**	**17 030**	**57 840**	**187 056**
6	**32**	**111**	**390**	**1 404**	**7 917**	**19 383**	**76 461**	**331 387**	**1 253 615**
7	**50**	**168**	**672**	**2 756**	**11 988**	**52 768**	**249 660**	**1 223 050**	**6 007 230**
8	**57**	**253**	**1100**	**5 060**	**39 672**	**131 137**	**734 820**	**4 243 100**	**24 897 161**
9	**74**	**585**	**1 550**	**8 268**	**75 893**	**279 616**	**1 686 600**	**12 123 288**	**65 866 350**
10	**91**	**650**	**2 286**	**13 140**	**134 690**	**583 083**	**4 393 452**	**27 997 191**	**201 038 922**
11	**104**	**715**	**3 200**	**19 500**	**156 864**	**1 001 268**	**7 442 328**	**72 933 102**	**600 380 000**
12	**133**	**786**	**4 680**	**29 470**	**359 772**	**1 999 500**	**15 924 326**	**158 158 875**	**1 506 252 500**
13	**162**	**851**	**6 560**	**40 260**	**531 440**	**3 322 080**	**29 927 790**	**249 155 760**	**3 077 200 700**
14	**183**	**916**	**8 200**	**57 837**	**816 294**	**6 200 460**	**55 913 932**	**600 123 780**	**7 041 746 081**
15	**187**	**1 215**	**11 712**	**76 518**	**1 417 248**	**8 599 986**	**90 001 236**	**1 171 998 164**	**10 012 349 898**
16	**198**	**1 600**	**14 640**	**132 496**	**1 771 560**	**14 882 658**	**140 559 416**	**2 025 125 476**	**12 951 451 931**

Table 2: The order of the largest known graphs of maximum degree Δ and diameter D, CombinatoricsWiki, accessed on 10 March 2013.

For maximum degree 6 and diameter 2 the Moore bound is $M_{6,2} = 37$. By computer generation, Molodtsov [270] showed that $n_{6,2} = M_{6,2} - 5 = 32$, and presented the 6 non-isomorphic graphs of degree 6 and diameter 2.

3.4 Restricted versions of the degree/diameter problem

The study of large graphs of given degree and diameter has often been restricted to special classes of graphs. The most obvious candidates here are vertex-transitive and Cayley graphs, suitable because of their quick computer generation as well as from the point of view of diameter checking. Other special classes, for which the degree/diameter problem has been considered, include bipartite graphs and graphs embeddable in a fixed surface (most notably, planar graphs).

3.4.1 Vertex-transitive graphs

Let $vt_{\Delta,D}$ be the largest order of a vertex-transitive (Δ, D)-graph. As mentioned in Subsection 3.1, if a Moore graph of degree 57 and diameter 2 does exist, then it cannot be vertex-transitive [81]. Although vertex-transitivity is a rather restrictive property, until recently there was no better general upper bound on $vt_{\Delta,D}$ than the bounds listed in the previous sections. The situation has improved by the latest results of Jajcay, Mačaj and Širáň [213] by which for any fixed $\Delta \geqslant 3$ and $c \geqslant 2$ there exists a set S of natural numbers of positive density such that $vt_{\Delta,D} \leqslant M_{\Delta,D} - c$, for all $D \in S$. This represents at least a partial progress towards answering the question of Bermond and Bollobás appearing at the end of Subsection 3.2 at least for vertex-transitive graphs.

For $\Delta = 3$, the values of $vt_{3,D}$ have been determined for $D \leqslant 8$ by Potočnik, Spiga and Verret [288] from their census of cubic vertex-transitive graphs of order up to 1280; the same work implies lower bounds on $vt_{3,D}$ for $9 \leqslant D \leqslant 12$. With the exception of $D = 6$, 9 and 10, the values and bounds on $vt_{3,D}$ from [288] are obtained by Cayley graphs. In this connection it is worth mentioning that Conder [109] generated all cubic arc-transitive graphs on up to 2048 vertices. With the exception of the graphs in the censuses of [288] and [109], to our knowledge so far no further study has been carried out in the study of vertex- and arc-transitive cubic graphs of a given diameter.

As regards lower bounds, a number of the existing examples of large (Δ, D)-graphs are vertex-transitive; many of them actually are Cayley graphs and they will be discussed in the next subsection. In this subsection we will therefore focus on vertex-transitive non-Cayley graphs, or on vertex-transitive graphs for which possibilities of a Cayley representation have not yet been investigated.

For diameter 2, the best result here seems to be the one of McKay, Miller and Širáň [249] who showed that

$$vt_{\Delta,2} \geqslant \frac{8}{9}\left(\Delta + \frac{1}{2}\right)^2 \tag{7}$$

for all degrees of the form $\Delta = (3q-1)/2$, where q is a prime power congruent to 1 (mod 4). The graphs that prove the inequality (7) are quite remarkable: They are all vertex-transitive but non-Cayley; the graph corresponding to the value $q = 5$ turns out to be the Hoffman-Singleton graph, and for $q = 9$, the corresponding $(13, 2)$-graph has order 162, just 8 off the Moore bound $M_{13,2} = 170$.

The result of [249] was further extended by Širáň, Šiagiová and Ždímalová [318] by showing that $vt_{\Delta,} \geqslant 2q^2$ for all odd $\Delta \geqslant 5$, where q is the largest prime power such that $q \equiv 5 \mod 8$ and $(\Delta+1)/2 < q < (2\Delta+1)/3$.

The construction of McKay-Miller-Širáň graphs [249] relies on a suitable lift of the complete bipartite graph $K_{q,q}$. A simplified version (in the form of a lift of a dipole with q edges and $(q-1)/4$ loops at each of its two vertices) was presented by Šiagiová [306], based on her results about compositions of regular coverings [305, 308]. In this connection it is interesting to mention another result of Šiagiová [307], who showed that, among all regular lifts of a dipole of degree Δ, the maximum order of a lift of diameter 2 is, for sufficiently large Δ, bounded above by

$$(4(10+\sqrt{2})/49)\Delta^2 \simeq .93\Delta^2.$$

This compares well with the Moore bound $M_{\Delta,2} = \Delta^2 + 1$, and is larger than the bound from (7), which is approximately $.89\Delta^2$.

It is also worth noting that the graphs of McKay-Miller-Širáň are very rich in symmetries; their full automorphism groups were determined by Hafner [201], using ideas related to combinatorial geometry.

The results of [249] and [72] strongly suggest that computer search over lifts of small graphs, using various voltage assignments, may lead to further new examples of highly symmetric large graphs of given diameter and degree.

For larger diameters the record-holders are the graphs of Faber-Moore-Chen type, studied in detail by Macbeth, Šiagiová, Širáň and Vetrík [245]. These graphs are obtained from the digraphs of Faber, Moore and Chen [144] simply by ignoring directions and suppressing parallel edges resulting from (directed) cycles of length 2. Up to a set of isolated exceptions these graphs are vertex-transitive but not Cayley, by [245], and furnish the bound $vt_{\Delta,D} \geqslant ((\Delta+3)/2)!/((\Delta+3)/2-D)!$ for all pairs (Δ, D) in the range $3 \leqslant D \leqslant 0.3\Delta$; we have used the factor of 0.3 to simplify the result and refer to [245] for the exact formulation. A slight strengthening (but not in asymptotic terms) of this bound can be obtained from the work of Gómez [178] who improved the original construction of [144].

Large vertex-transitive graphs of given degree and diameter can also be obtained from the digraphs constructed by Comellas and Fiol in [101] as described in Subsection 4.5.1 of this survey, again by ignoring directions. There appears to be no good way of comparing the constructions of Faber-Moore-Chen and Comellas-Fiol in general, for reasons discussed in detail in [245]. However, for a few infinite classes where a comparison can be done it turns out [245] that the Faber-Moore-Chen type graphs are larger than the Comellas-Fiol type graphs. A possible merger of the two constructions for digraphs was studied by Gómez [179], but it appears to give no better bounds for graphs by suppressing edge directions.

3.4.2 Cayley graphs

Let $C_{\Delta,D}$ denote the largest order of a Cayley graph of degree Δ and diameter D. Obviously, $C_{\Delta,D} \leqslant vt_{\Delta,D}$ in general, with strict inequality for some pairs (Δ, D), such as $(3,2)$ and $(7,2)$ due to the existence of the Petersen graph and the Hoffman-Singleton graphs which are vertex-transitive but non-Cayley, and $(3,6)$ due to [288]. Since the work of Jajcay, Mačaj and Širáň [213] mentioned at the beginning of Subsection 3.4.1 applies also to Cayley graphs, it follows that for any fixed $\Delta \geqslant 3$ and $c \geqslant 2$ there exists a set S of natural numbers of positive density such that $C_{\Delta,D} \leqslant M_{\Delta,D} - c$ for all $D \in S$. For the remaining pairs (Δ, D) with $\Delta \geqslant 3$ we currently do not have a better upper bound on $C_{\Delta,D}$ than $M_{\Delta,D} - 2$.

We proceed by surveying lower bounds on $C_{\Delta,D}$. The best available result for diameter 2, obtained by Šiagiová and Širáň [312], states that for any degree Δ from the set $\mathcal{D} = \{2^{2m+\mu} + (2+\delta)2^{m+1} - 6,\ m \geqslant 1,\ \mu \in \{0,1\}\}$ one has $C_{\Delta,2} > \Delta^2 - 6\sqrt{2}\Delta^{3/2}$. This means that, at least for degrees $\Delta \in \mathcal{D}$, the Moore bound $M_{\Delta,2}$ can be approached asymptotically by Cayley graphs. Since the set $\mathcal{D}$ is rather sparse, it does not allow for good approximations by adding generators. With the help of results on gaps between primes mentioned in Subsection 3.3.1, Širáň, Šiagiová and Ždímalová [318] improved an earlier result of the first two authors [311] by showing that $C_{\Delta,2} > (1/2)\Delta^2 - 1.39\Delta^{1.525}$ for every sufficiently large Δ. This is much weaker than the result of [312] but applies to *all* sufficiently large degrees.

For larger diameters the best currently known general results were obtained by Macbeth, Šiagiová, Širáň and Vetrík [245] which, in a simplified form, says that $C_{\Delta,D} \geqslant D((\Delta - 3)/3)^D$ for $\Delta \geqslant 5$ and $D \geqslant 3$, and by Macbeth, Šiagiová and Širáň [244] who established the bound $C_{\Delta,D} \geqslant D((\Delta-2)/3)^D - D$ for $\Delta, D \geqslant 3$ using just Cayley graphs of metacyclic groups. Using the same type of groups as in [245] but with somewhat different generating sets, Vetrík [335] showed that $C_{\Delta,3} \geqslant (3/16)(\Delta - 3)^3$, $C_{\Delta,4} \geqslant 32((\Delta - 8)/5)^4$ and $C_{\Delta,5} \geqslant 25((\Delta - 7)/4)^5$ for all $\Delta \geqslant 10$.

An important stream of research in Cayley graphs, one that is closely related to the degree/diameter problem, is bounding the diameter of a Cayley graph in terms of a logarithm of the order of the group. The relation relies on the fact that, for $k \geqslant 3$ and $d \geqslant 2$, we have $M_{\Delta,D} < \Delta^D$, and therefore also $n < \Delta^D$ for $n = n_{\Delta,D}$. It follows that, for the diameter of a graph of order n, we always have

$$D > b \times \log n, \quad \text{where} \quad b = 1/\log \Delta.$$

Although taking logarithms results in a substantial loss of precision, it is still reasonable to ask about *upper* bounds on the diameter D in terms of the logarithm of the largest order a (Δ, D)-graph; as indicated earlier, this has been considered primarily for Cayley graphs.

From a result of Babai and Erdős [14], it follows that there exists a constant c, such that, for any finite group G, there exists a set of $t \leqslant c \log |G|$ generators, such that the associated Cayley graph has diameter at most t. This settles the general question about an upper bound on D, at least in terms of a constant multiple of the logarithm of $C_{\Delta,D}$, the largest order of a Cayley graph of degree Δ and diameter D. Further refinements have been obtained for special classes of groups, with emphasis on reducing the size of the generating sets (and hence reducing the degree).

Babai, Kantor and Lubotzky [16] gave an elementary and constructive proof of the fact that every nonabelian finite simple group G contains a set of at most *seven* generators for which the diameter of the associated Cayley graph is at most $c \log |G|$, for an absolute constant c. For projective special linear groups $G = PSL(m, q)$, this was improved by Kantor [220] by showing that, for each $m \geqslant 10$, there is a *trivalent* Cayley graph for G of diameter at most $c \log |G|$.

For an *arbitrary* transitive subgroup G of the symmetric group of degree r and any symmetric generating set of G, Babai and Seress [18] proved that the diameter of the corresponding Cayley graph is at most

$$\exp((r \ln r)^{1/2}(1 + o(1))).$$

Note that this bound is quite far from

$$\log |G| = \log (r!) \approx cr \log r;$$

however, the strength of the statement is in that it is valid for arbitrary groups and generating sets. An earlier result by the same authors [17] states that if G is either the symmetric or the alternating group of degree r, then, for an *arbitrary* symmetric generating set, the corresponding Cayley graph of G has diameter not exceeding

$$\exp((r - \ln r)^{1/2}(1 + o(1))),$$

which is better than the previous bound (however, for more special groups). By probabilistic arguments, Babai and Hetyei [15] showed that, for almost every pair of random permutations (p_1, p_2) from the symmetric group of degree r, the diameter of the Cayley graph of the group $G = \langle p_1, p_2 \rangle$ with generating set $S = \{p_1^{\pm 1}, p_2^{\pm 1}\}$ is less than $\exp((\frac{1}{2} + o(1))(\ln r)^2)$. Since such a group almost surely (for $r \to \infty$) contains the alternating group of degree r, this result (at least in a probabilistic sense) is substantially stronger than the previous two bounds. Nevertheless, it is still far from the conjectured [17] upper bound r^c for the diameter of *any* Cayley graph of the symmetric group of degree r, for an absolute constant c. Another remarkable and still unresolved conjecture of Babai [18] states that there is an absolute constant c such that for any group G the diameter of any Cayley graph for G is bounded above by $(\log|G|)^c$.

In terms of computer generation, note that roughly one half of the values in Table 2 have been obtained from Cayley graphs. A description of computer-assisted constructions of large (Δ, D)-graphs (for manageable values of Δ and D) from Cayley graphs of semidirect products of (mostly cyclic) groups can be found in Hafner [200]. Later, Branković *et al.* [72] showed that the constructions of [200] can be obtained as lifts of smaller Cayley graphs with voltage assignments in smaller, mostly cyclic, groups. Researchers who have contributed in the quest for large Cayley graphs of given degree and diameter also include Campbell [82], and Akers and Krishnamurthy [6].

To conclude on a more general note, it is worth mentioning that Lakshmivarahan, Jwo and Dhall [231] produced a survey of Cayley graph network designs. Apart from the usual properties of order, degree and diameter, they also consider shortest path distance, vertex-transitivity, arc-transitivity and several forms of distance transitivity. The survey emphasises algebraic features, such as cosets, conjugacy classes, and automorphism actions, in the determination of some topological properties of over 18 types of networks. Cayley graphs have been used numerous times as a tool for the design and analysis of interconnection networks, for an example see Schibell and Stafford [301].

3.4.3 Abelian Cayley graphs

Further restrictions on the classes of groups yield better upper bounds. We discuss here in more detail the Cayley graphs of *abelian* groups. Let $AC_{\Delta,D}$ denote the largest order of a Cayley graph of an abelian group of degree Δ and diameter D. Inequalities for such graphs are often stated in terms of the number of generators of the reduced generating set rather than the degree. Given a Cayley graph $C(\Gamma, S)$, the reduced generating set is a subset S' of S such that, for each $s \in S$, exactly one of s, s^{-1} appears in S'. If the reduced generating set has d elements then the degree of the Cayley graph is equal to $\Delta = 2d - d'$, where d' is the number of generators of order two in S.

Investigations of large abelian Cayley graphs of given size of reduced generating set and given diameter can be based on the following simple but ingenious idea (see [132] for genesis and background). Any finite abelian group Γ with a symmetric generating set S and a reduced generating set $S' = \{g_1, \ldots, g_d\}$ of size d is a quotient group of the free abelian d-generator group $\mathcal{Z}^d$ by the subgroup N (of finite index), that is, the kernel of the natural homomorphism $\mathcal{Z}^d \to \Gamma$, which sends the unit vector $\mathbf{e}_i \in \mathcal{Z}^d$ onto g_i. For any given D, define

$$W_{d,D} = \{(x_1, \ldots, x_d) \in \mathcal{Z}^d;\ |x_1| + \ldots + |x_d| \leqslant D\}.$$

Then the Cayley graph $C(\Gamma, S)$ has diameter at most D if and only if $W_{d,D} + N = \mathcal{Z}^d$.

D / Δ	2	3	4	5	6	7	8	9	10
3	**8**	**14**	**24**	**60**	**72**	**168**	**300**	**506**	**820**
4	**13**	**30**	**84**	**216**	**513**	**1 155**	**3 080**	**7 550**	**17 608**
5	**18**	**60**	**210**	**546**	**1 640**	**5 500**	**16 965**	**57 840**	**187 056**
6	**32**	**108**	**375**	**1 395**	**5 115**	**19 383**	**76 461**	**331 387**	**1 253 615**
7	**36**	**168**	**672**	**2 756**	**11 988**	**52 768**	**249 660**	**1 223 050**	**6 007 230**
8	**48**	**253**	**1100**	**5 060**	**23 991**	**131 137**	**734 820**	**4 243 100**	**24 897 161**
9	**60**	**294**	**1 550**	**8 200**	**45 612**	**279 616**	**1 686 600**	**12 123 288**	**65 866 350**
10	**72**	**406**	**2 286**	**13 140**	**81 235**	**583 083**	**4 393 452**	**27 997 191**	**201 038 922**
11	**84**	**486**	**2 860**	**19 500**	**139 446**	**1 001 268**	**7 442 328**	**72 933 102**	**500 605 110**
12	**96**	**605**	**3 775**	**29 470**	**229 087**	**1 999 500**	**15 924 326**	**158 158 875**	**1 225 374 192**
13	**112**	**680**	**4 788**	**40 260**	**346 126**	**3 322 080**	**29 927 790**	**233 660 788**	**2 129 329 324**
14	**128**	**873**	**6 510**	**57 837**	**530 448**	**5 600 532**	**50 128 239**	**579 328 377**	**7 041 746 081**
15	**144**	**972**	**7 956**	**76 518**	**787 116**	**8 599 986**	**88 256 520**	**1 005 263 436**	**10 012 349 898**
16	**155**	**1 155**	**9 576**	**100 650**	**1 125 264**	**12 500 082**	**135 340 551**	**1 995 790 371**	**12 951 451 931**

Table 3: The order of the largest known Cayley graphs of maximum degree Δ and diameter D, CombinatoricsWiki, accessed on 10 March 2013.

This has two immediate consequences. Firstly, $|W_{d,D}|$ is an upper bound on $AC_{2d,D}$. Secondly, if N is a subgroup of $\mathcal{Z}^d$ of finite index with the property $W_{d,D} + N = \mathcal{Z}^d$ then N determines a d-dimensional lattice that induces 'shifts' of the set $W_{d,D}$ which completely cover the elements of $\mathcal{Z}^d$; the index $[\mathcal{Z}^d : N] = |\Gamma|$ (which is a lower bound on $AC_{2d,D}$) is also equal to the absolute value of the determinant of the d-dimensional matrix formed by the d generating vectors of N. The search for bounds on $AC_{2d,D}$ can therefore be reduced to interesting problems in combinatorial geometry [132].

An exact formula for $|W_{d,D}|$ (which, as we know, is automatically an upper bound on $AC_{2d,D}$) was given, for example, by Stanton and Cowan [326]. A general lower bound on $AC_{2d,D}$, based on a thorough investigation of lattice coverings discussed above, was obtained by Dougherty and Faber [132]. We state both bounds in the following form: There exists a constant c (not depending on d and D), such that for any fixed $d \geqslant 2$ and all D,

$$\frac{c \times 2^d}{d! d (\ln d)^{1+\log_2 e}} D^d + O(D^{d-1}) \leqslant AC_{2d,D} \leqslant \sum_{i=0}^{d} 2^i \binom{d}{i} \binom{D}{i} \tag{8}$$

Note that the upper bound can be considered to be the *abelian Cayley Moore bound* for abelian groups with d-element reduced generating sets. It differs from the Moore bound $M_{2d,D}$ rather dramatically; if the number of generators d is fixed and $D \to \infty$ then the right hand side of (8) has the form

$$2^d D^d / d! + O(D^{d-1}).$$

Exact values of $AC_{2d,D}$ are difficult to determine. With the help of lattice tilings, Dougherty and Faber (and many other authors, also using different methods – see [132]) showed that, for $d = 2$, there actually exist 'abelian Cayley Moore graphs', that is,

$$AC_{4,D} = |W_{2,D}| = 2D^2 + 2D + 1;$$

the analysis here is facilitated by a nice shape of the set $W_{2,D} \subset \mathcal{Z}^2$. For $d = 3$, the same type of analysis [132] gives

$$AC_{6,D} \geqslant (32D^3 + 48D^2)/27 + f(D),$$

where $f(D)$ is a linear function that depends on the residue class of D mod 3; the abelian Cayley Moore bound here is

$$AC_{6,D} \leqslant |W_{3,D}| = (4D^3 + 6D^2 + 8D + 3)/3.$$

A table of exact values of $AC_{6,D}$ for $D \leqslant 14$ is included in [132].

It should be noted that the method of lattice-induced shifts of the sets $W_{d,D}$ tends to be manageable for small values of d while $D \to \infty$, as can be seen from (8). At the other end of the spectrum, for diameter $D = 2$, a folklore result says that

$$AC_{\Delta,2} \geqslant \lfloor \frac{\Delta + 2}{2} \rfloor \lceil \frac{\Delta + 2}{2} \rceil \tag{9}$$

This can be obtained from a Cayley graph for the product of cyclic groups $Z_{\lfloor(\Delta+2)/2\rfloor} \times Z_{\lceil(\Delta+2)/2\rceil}$, with the generating set consisting of all pairs (x_1, x_2), in which exactly one of x_1, x_2 is equal to 0. This, however, is far from the upper bound $AC_{\Delta,2} \leqslant \Delta^2/2 + \Delta + 1$ that can be obtained by simple counting. An unstated folklore conjecture asserts that one should have $AC_{\Delta,2} \approx \Delta^2/2$ for all Δ; computational evidence due to McKay [248] has confirmed this for all $\Delta \leqslant 16$.

The current best general result in this direction is the inequality

$$AC_{\Delta,2} > (3/8)\Delta^2 - 1.45D^{1.525}$$

for all sufficiently large Δ, proved by Širáň, Šiagiová and Ždímalová [318] by way of extending an earlier result of Macbeth, Šiagiová and Širáň [244] stating that $AC_{\Delta,2} \geqslant (3/8)(\Delta^2 - 4)$ for all Δ of the form $4q - 2$, where q is an odd prime power. Another result of [244] in this direction is the inequality $AC_{\Delta,2} \geqslant (\Delta + 1)^2/3$ for $\Delta = 3q - 1$ and q an odd prime power. Since the last two results refer to different degrees, one cannot compare them by just looking at the multiplicative constants at the Δ^2 term, especially for relatively small values of Δ.

A completely different type of a bound on $AC_{\Delta,D}$ for $D \leqslant \Delta$ was found by Garcia and Peyrat [172] with the help of covering codes; for Δ large enough and for $4 \leqslant D \leqslant \Delta$ they showed that

$$AC_{\Delta,D} \geqslant \frac{\Delta^{D-2.17}}{21D!}$$

Using similar groups as in [244], Vetrík [336] proved a bound for diameter 3 of the form $AC_{\Delta,3} \geqslant (9/128)(\Delta + 3)^2(\Delta - 5)$ for $\Delta = 8q - 3$ and q a prime power; trick developed in [318] in conjunction with the prime-gap result of [20] again allow to conclude that $AC_{\Delta,3} \geqslant (9/128)\Delta^3 + O(D^{2.525}$ for all sufficiently large Δ.

We conclude by mentioning the work done to find the largest order $CC_{\Delta,D}$ of a Cayley graph of a *cyclic* group with degree Δ and diameter D; obviously, $CC_{\Delta,D} \leqslant AC_{\Delta,D}$. This is of special interest for diameter 2, as the expectation is that $CC_{\Delta,D} \approx \Delta^2/2$ for all sufficiently large Δ.

Macbeth, Šiagiová and Širáň [244] proved that $CC_{\Delta,2} \geqslant (9/25)(\Delta+3)(\Delta-2)$ for all $\Delta = 5p-3$, where p is a prime such that $p \equiv 2 \bmod 3$, and also that $CC_{\Delta,2} \geqslant \Delta^2/3 + O(\Delta^{3/2})$ for $\Delta = 3p + O(\sqrt{p})$ for the same set of primes. Using the same type of groups but with modified generating sets, Vetrík [336] showed that $CC_{\Delta,2} \geqslant (13/36)(\Delta + 2)(\Delta - 4)$ for $\Delta = 6p - 2$, where p is a prime such that $p \neq 13$ and $p \not\equiv 1 \bmod 13$. We reiterate that since these results refer to different degree sets, comparison by just looking at the leading terms may be misleading. No results for large Cayley graphs of cyclic groups for diameters larger than 2 have been known at the time of writing this survey.

We note that inequalities which include assumptions of the form $\Delta = f(q)$ where q is a prime or a prime power satisfying some extra congruence relations cannot be converted to inequalities valid for all Δ using results of [20] on gaps between primes, since no such 'gap results' appear to be available for primes subject to congruence restrictions.

3.4.4 Bipartite graphs

As in the case of the degree/diameter problem for general graphs, research activities concerning the degree/diameter problem for bipartite graphs include both proofs of the non-existence or otherwise of bipartite graphs of order close to the bipartite Moore bound, and constructions of large bipartite graphs.

The 'bipartite Moore bound', that is, the maximum number $B_{\Delta,D}$ of vertices in a bipartite graph of maximum degree Δ and diameter at most D, was given by Biggs [58]:

$$B_{2,D} = 2D \quad \text{and} \quad B_{\Delta,D} = \frac{2(\Delta-1)^D - 1}{\Delta - 2} \quad \text{if } \Delta > 2.$$

Note that the bipartite Moore bound represents not only an upper bound on the number of vertices of a bipartite graph of maximum degree Δ and diameter D but it is also a lower bound on the number of vertices of a regular graph G of degree Δ and girth $g = 2D$ [58]. A (Δ, g)-cage of order $B_{\Delta,D}$ is therefore a bipartite Moore graph if $g = 2D$.

Bipartite graphs satisfying the equality are called *bipartite Moore graphs*. For degrees 1 or 2, bipartite Moore graphs are K_2 and the $2D$-cycles, respectively. When $\Delta \geqslant 3$ the possibility of the existence of bipartite Moore graphs was settled by Feit and Higman [148] in 1964 and, independently, by Singleton [316] in 1966. They proved that such graphs exist only if the diameter is 2, 3, 4 or 6.

For $D = 2$ and each $\Delta \geqslant 3$ the bipartite Moore graphs of degree Δ are the complete bipartite graphs of degree Δ. For $D = 3, 4, 6$ bipartite Moore graphs of degree Δ have been constructed only when $\Delta - 1$ is a prime power [41]. Furthermore, Singleton [316] proved that the existence of a bipartite Moore graph of diameter 3 is equivalent to the existence of a projective plane of order $\Delta - 1$.

On the other hand, for $D = 3$, there are values of Δ with no bipartite Moore graphs. The question of whether or not bipartite Moore graphs of diameter 3, 4 or 6 exist for other values of Δ remains open, and represents one of the most famous problems in combinatorics.

In view of the scarcity of bipartite Moore graphs, we next turn our attention to bipartite graphs of small defect. Note that if the defect is less than

$$1 + (\Delta - 1) + \cdots + (\Delta - 1)^{D-2}$$

then the graph must be regular. Since the required graph is to be also bipartite, it follows that in such a case the defect cannot be odd.

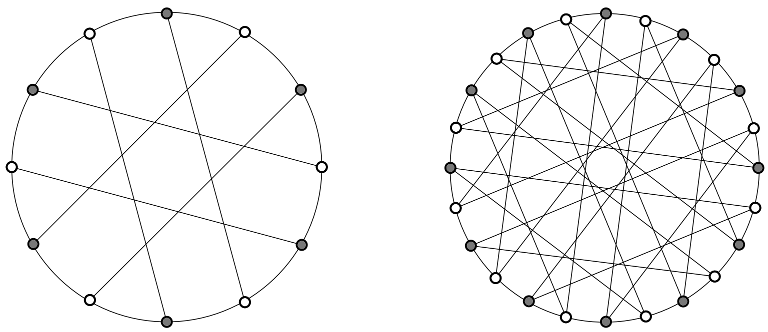


Figure 6: (a) the unique bipartite $(3, 3, -2)$-graph and (b) the unique bipartite $(4, 3, -2)$-graph.

Concerning bipartite graphs of defect 2, it was proved by Delorme, Jørgensen, Miller, and Pineda-Villavicencio [124] that the known bipartite $(3, 3)$-graph and bipartite $(4, 3)$-graph are unique. Moreover, they gave several necessary conditions for the existence of bipartite $(\Delta, 3)$-graphs of defect 2, including the fact that either Δ or $\Delta - 2$ must be a perfect square. The story of defect 2 was almost completed by Delorme, Jørgensen, Miller, and Pineda-Villavicencio [125], and Pineda-Villavicencio [283], which proved the nonexistence of such graphs for $\Delta \geqslant 3$ and $D \geqslant 4$. Thus, bipartite graphs of defect 2 can possibly exist only for diameter 3. Figure 3.4.4 shows the known bipartite graphs of defect 2.

The paper [283] went on to suggest the following conjecture: For a given Δ_0 and $\delta_0 = f(\Delta_0)$ there exists a constant $D_1 \geqslant D_0$ such that regular bipartite $(\Delta_0, D, -\delta_0)$-graphs with $D \geqslant D_1$ do not exist.

Bipartite graphs of defect 4 were studied by Feria-Purón and Pineda-Villavicencio [154] and by Feria-Purón, Miller, and Pineda-Villavicencio [152]. In [154] the authors provided the following results: (1) several necessary conditions for the existence of bipartite (Δ, D)-graphs of defect 4, (2) the complete catalogue of bipartite $(3, D)$-graphs with $D \geqslant 2$ and $0 \leqslant \delta \leqslant 4$, (3) the complete catalogue of bipartite $(\Delta, D, -\epsilon)$-graphs with $\Delta \geqslant 2$, $5 \leqslant D \leqslant 187$ ($D \neq 6$) and $0 \leqslant \epsilon \leqslant 4$, (4) and a proof of the non-existence of all bipartite $(\Delta, D, -4)$-graphs with $\Delta \geqslant 3$ and odd $D \geqslant 5$.

Then, the follow-up paper [152] proved the optimality of the bipartite graph of degree 7, diameter 3 and defect 6 found by Hafner and Loz independently, and the non-existence of bipartite graphs of degree $\Delta \geqslant 3$, diameter $D \geqslant 5$ and defect 4. Thus, bipartite graphs of defect 4 exist only for diameters 3 and 4. Figure 3.4.4 shows all the known bipartite graphs of defect 4.

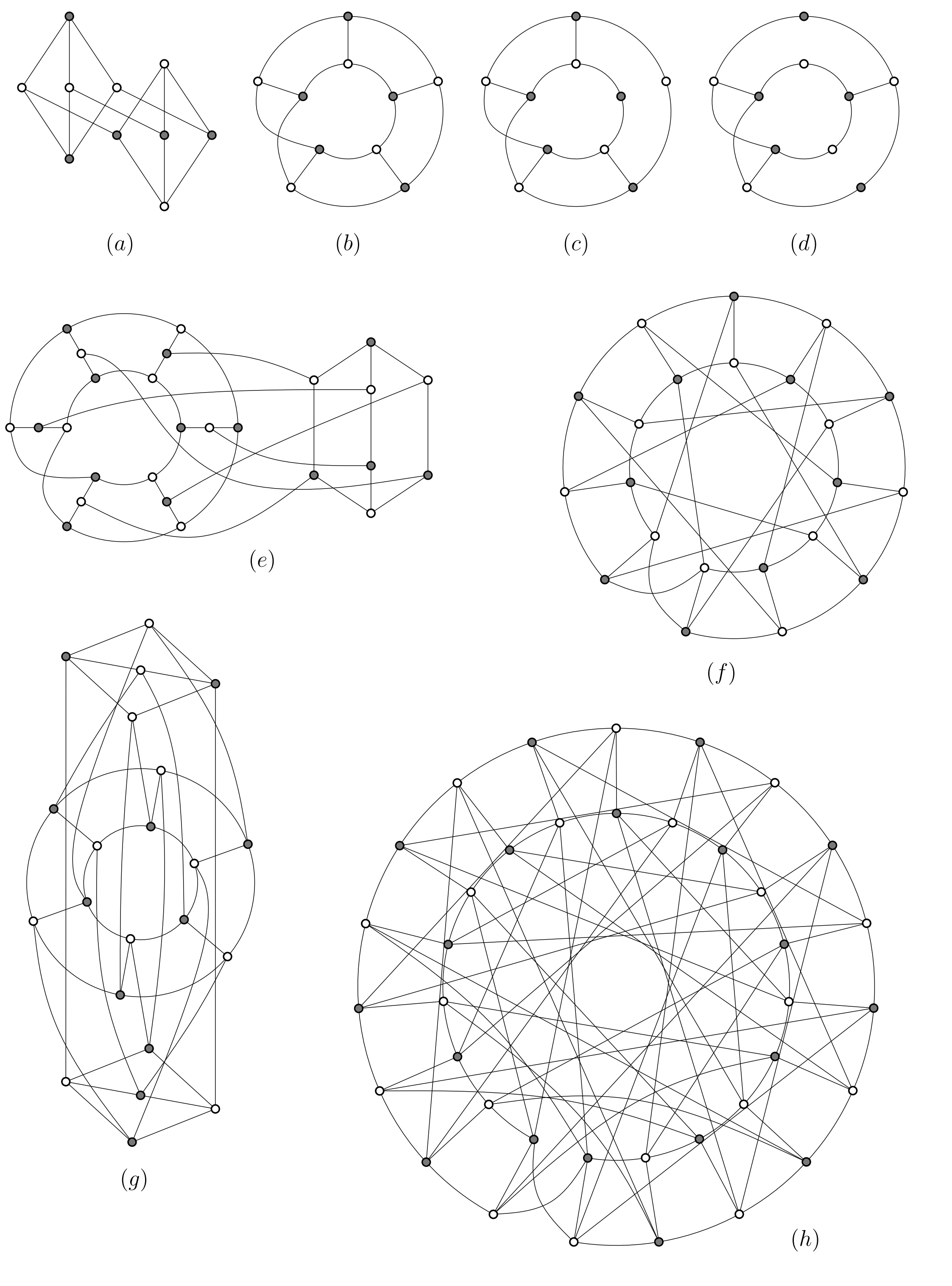


Figure 7: $(a)-(d)$ all the bipartite $(3,3)$-graphs of defect 4, (e) the unique bipartite $(3,4)$-graph of defect 4, $(f)-(g)$ all the bipartite $(4,3)$-graphs of defect 4, and (g) the only known bipartite $(5,3)$-graph of defect 4.

At present there are only a few exact values of $b_{\Delta,D}$ known. In particular, considering bipartite Moore graphs for $D = 2$ and any $\Delta \geqslant 2$, we have $b_{\Delta,2} = B_{\Delta,2}$. For $D = 3, 4$ and 6, whenever $\Delta - 1$ is a prime power, the value of $b_{\Delta,D}$ equals $B_{\Delta,D}$. Additionally, we know that $b_{3,5} = B_{3,5} - 6$ and $b_{7,3} = B_{7,3} - 6$. The value of $b_{3,5}$ corresponds to an optimal bipartite graph of degree 3, diameter 5 and order 56 found by Bond and Delorme [64], whose optimality was proved by Jørgensen [216]. A bipartite graph of order $b_{7,3} = 80$ was independently constructed by Hafner and Loz, and its optimality was settled by Feria-Purón, Miller, and Pineda-Villavicencio [152].

The non-existence of bipartite Moore graphs of diameter 3, and degrees 7 and 11, respectively, follows from the non-existence of projective planes of orders 6 and 10. The non-existence of a projective plane of order 6 follows from Bruck, Ryser and Chowla's theorem on symmetric designs [296], while the non-existence of a projective plane of order 10 was proved by Lam *et al.* [232] using computer search.

3.4.5 Constructions of large bipartite graphs

In the past many constructions of large bipartite graphs were proposed by Delorme and his collaborators.

Bond and Delorme [64, 65] gave new constructions of large bipartite graphs with given degree and diameter, using their concept of a partial Cayley graph. Other constructions of large bipartite graphs were found by Delorme [118, 119], using bipartite versions of operations described earlier, most notably, $*$-product and compounding.

In the same papers, Delorme also studied the asymptotic behaviour of the problem by introducing the parameter

$$\beta_D = \lim\ \inf_{\Delta\to\infty} \frac{b_{\Delta,D}}{2\Delta^{D-1}}$$

where $b_{\Delta,D}$ is the largest order of a bipartite (Δ, D)-graph. Comparing this with the bipartite Moore bound, we see that $\beta_D \leqslant 1$ for all D; so far, it is only known [118, 119] that equality holds for $D = 2, 3, 4$ and 6.

So far, specific constructions for certain degrees or girths have produced the best results; see, for instance, [12, 140, 142]. For cubic graphs of girths greater than 8, Biggs [60] gave an account of the degree/girth problem. Additionally, computer search has played a significant role in this area; see, for example, [74, 250, 253, 142].

Loz, Pérez-Rosés and Pineda-Villavicencio [240] adapted the voltage-assignment-based method developed by Loz and Širáň [242] for general graphs to the case of bipartite graphs; this resulted in updating more than one third of the table of the largest known bipartite graphs.

D / Δ	3	4	5	6	7	8	9	10
3	**14**	**30**	**56**	**126**	**168**	**256**	**506**	**800**
4	**26**	**80**	**160**	**728**	**840**	**2 184**	**4 970**	**11 748**
5	**42**	**170**	**336**	**2 730**	**3 110**	**9 234**	**27 936**	**90 068**
6	**62**	**312**	**684**	**7 812**	**8 310**	**29 790**	**117 360**	**452 032**
7	**80**	**346**	**1 134**	**8 992**	**23 436**	**80 940**	**400 160**	**1 987 380**
8	**114**	**800**	**1 710**	**39 216**	**40 586**	**201 480**	**1 091 232**	**6 927 210**
9	**146**	**1 170**	**2 496**	**74 898**	**117 648**	**449 480**	**2 961 536**	**20 017 260**
10	**182**	**1 640**	**4 000**	**132 860**	**224 694**	**1 176 480**	**7 057 400**	**50 331 156**
11	**190**	**1 734**	**5 850**	**142 464**	**398 580**	**2 246 940**	**15 200 448**	**130 592 1354**
12	**266**	**2 928**	**8 200**	**354 312**	**664 300**	**4 650 100**	**30 001 152**	**300 383 050**
13	**270**	**3 064**	**11 480**	**374 452**	**1 062 936**	**5 314 680**	**50 990 610**	**617 330 936**
14	**366**	**4 760**	**14 760**	**804 468**	**1 771 560**	**14 172 480**	**95 087 738**	**1 213 477 190**
15	**370**	**4 946**	**20 496**	**842 048**	**2 480 184**	**14 172 480**	**168 016 334**	**2 300 326 510**
16	**394**	**5 134**	**27 300**	**884 062**	**4 022 340**	**36 201 060**	**288 939 118**	**4 119 507 330**

Table 4: The order of the largest known bipartite graphs of maximum degree Δ and diameter D, CombinatoricsWiki, accessed on 10 March 2013.

3.4.6 Graphs on surfaces

Let $\mathcal{S}$ be an arbitrary connected, closed surface (orientable or not) and let $n_{\Delta,D}(\mathcal{S})$ be the largest order of a graph of maximum degree at most Δ and diameter at most k, embeddable in $\mathcal{S}$. Let $\mathcal{S}_0$ be a sphere.

The planar (or, equivalently, spherical) version of the degree/diameter problem was considered by several authors. Seyffarth [303] proved that such graphs with maximum degree Δ and diameter two have no more than $3/2\Delta + 1$ vertices and showed that such graphs do exist.

Hell and Seyffarth [204] have shown that, for diameter 2 and $\Delta \geqslant 8$, we have

$$n_{\Delta,2}(\mathcal{S}_0) = \lfloor \frac{3}{2}\Delta \rfloor + 1.$$

For $\Delta \leqslant 7$, the exact values of $n_{\Delta,2}(\mathcal{S}_0)$ were determined by Yang, Lin and Dai [349].

Subsequently, Fellows, Hell and Seyffarth [149] established upper and lower bounds for planar graphs of diameter 3 and any maximum degree Δ as

$$\lfloor \frac{9}{2}\Delta \rfloor - 3 \leqslant n_{\Delta,3}(\mathcal{S}_0) \leqslant 8\Delta + 12.$$

The case $\Delta = 3$ was also considered by Tishchenko [331]. For planar graphs with general diameter D and with $\Delta \geqslant 4$, the authors in [149] (see also [150]) apply a special case of a theorem of Lipton and Tarjan [236], to show that

$$n_{\Delta,D}(\mathcal{S}_0) \leqslant (6D + 3)(2\Delta^{\lfloor \frac{D}{2} \rfloor} + 1) \ .$$

An interesting generalisation of the result of [204] to arbitrary surfaces was obtained by Knor and Širáň [226]. Let $\mathcal{S}$ be an arbitrary surface (orientable or not) other than the sphere, and let $\Delta_{\mathcal{S}} = 2^8(2 - \chi(\mathcal{S}))^2 + 2$. Then, for diameter $D = 2$ and any maximum degree $\Delta \geqslant \Delta_{\mathcal{S}}$,

$$n_{\Delta,2}(\mathcal{S}) = n_{\Delta,2}(\mathcal{S}_0) = \lfloor \frac{3}{2}\Delta \rfloor + 1.$$

The striking fact here is that this bound is, for $\Delta \geqslant \Delta_{\mathcal{S}}$, independent of the surface $\mathcal{S}$ and is the same as for the plane! The bound can therefore be considered to be the *surface Moore bound* for $\Delta \geqslant \Delta_{\mathcal{S}}$.

In [226], it is also shown that, for all $\Delta \geqslant \Delta_{\mathcal{S}}$, there exist triangulations of $\mathcal{S}$ of diameter 2, maximum degree Δ, and order $\lfloor (3/2)\Delta \rfloor + 1$; moreover, these 'surface Moore graphs' are not unique. The largest order of graphs of diameter 2 and degree at most 6 on surfaces with Euler characteristic $\geqslant 0$ was determined by Tishchenko [332].

When the diameter is even, Tishchenko completely solved the degree/diameter problem for planar graphs, providing the exact value of $p_{\Delta,D}$ for large Δ. Indeed,

$$p_{\Delta,2d} = \frac{3\Delta(\Delta-1)^d - 1}{2(\Delta-2)}.$$

Tables compiling the largest known planar graphs are maintained in CombinatoricsWiki [239] and by Preen [289].

Šiagiová and Simanjuntak [310] considered bounds on the order of graphs of arbitrary maximum degree $\Delta \geqslant 3$ and arbitrary diameter D, embeddable in a general surface of Euler genus ε. Setting $c_{\mathcal{S},D} = (6D(\varepsilon+1)+3)$, their result can be stated in the form

$$\frac{\Delta((\Delta-1)^{\lfloor \frac{D}{2} \rfloor} - 2)}{\Delta-2} < n_{\Delta,D}(\mathcal{S}) \leqslant c_{\mathcal{S},D}\frac{\Delta((\Delta-1)^{\lfloor \frac{D}{2} \rfloor} - 2)}{\Delta-2}.$$

In view of these bounds, the authors of [310] raise the natural question of the existence and the value of the limit of $n_{\Delta,D}(\mathcal{S})/\Delta^{\lfloor D/2 \rfloor}$ as $\Delta \to \infty$.

The degree/diameter problem has also been studied for graphs embedded on a torus, that is *toroidal graphs*. Note all planar graphs are a proper subclass of toroidal graphs. A table compiling the largest known toroidal graphs is maintained by Preen [290].

3.5 Related topics

The relationships between parameters such as order, diameter, minimum degree and maximum degree have been considered by Chung [95]. She reviews the status of a number of interrelated problems on diameters of graphs, including:

- degree/diameter problem
- order/degree problem
- given n, D, D', s, determine the minimum number of edges in a graph on n vertices of diameter D having the property that after removing any s or fewer edges the remaining graph has diameter at most D'
- as above but with a constraint on the maximum degree
- for a given graph, find the optimum way to add t edges so that the resulting graph has minimum diameter
- for a given graph, find the optimum way to add t vertex disjoint edges to reduce the diameter as much as possible.

A variety of interrelated diameter problems are discussed by Chung in [94], including determining extremal graphs of bounded degrees and small diameters, finding orientations for undirected or mixed graphs to minimise diameters, investigating diameter bounds for networks with possible node and link failures, and algorithmic aspects of determining the diameters of graphs.

In her study of properties of eigenvalues of the adjacency matrix of a graph, Chung [96] proved that the second largest eigenvalue (in absolute value) λ is related to the diameter D by means of the inequality

$$D \leqslant \lceil \log(n-1)/\log(\Delta/\lambda) \rceil.$$

Bermond and Bollobás [43] studied the following extremal problem: Given integers n, D, D', Δ, k and l, determine or estimate the minimum number of edges in a graph G of order n and with the following properties: (i) G has maximum degree at most Δ, (ii) the diameter of G is at most D, (iii) if G' is obtained from G by suppressing any k of the vertices or any l of the edges, the diameter of G' is at most D'.

Bollobás [62] considered another extremal problem on diameters: given diameter and maximum degree, find the minimum number of edges.

Gómez and Escudero [184] investigated constructions of graphs with a given diameter D and a given maximum degree Δ and having a large number of vertices, whose edges can be well coloured by exactly p colours. They include a table of such digraphs for $D \leqslant 10$ and $p \leqslant 16$.

The two additional parameters that have been considered most systematically in relation with the degree/diameter problem are *girth* (= length of the shortest cycle) and *connectivity*; we consider them in separate subsections, after taking a short detour in the next subsection to consider how we can get close to the ideal of Moore graphs.

3.5.1 Approximating Moore graphs

Since Moore graphs exist for only a few combinations of the degree and diameter values, we are interested in studying the existence of large graphs which are in some way 'close' to Moore graphs. Given that we are dealing with three parameters, namely, order, degree and diameter, in order to get close to Moore graphs, we may consider relaxing each of these parameters in turn. An initial discussion on the relaxation of the three parameters was presented by Miller and Pineda-Villavicencio in [264], and is described next.

Relaxing the order: Here we look for graphs of given maximum degree Δ and diameter D, whose order is $M_{\Delta,D} - \delta$. As mentioned earlier, the parameter δ is called the *defect* and such a graph is called a $(\Delta, D, -\delta)$*-graph*.

Relaxing the order corresponds to the degree/diameter problem. This is the direction that has traditionally been considered when trying to approximate the idea of Moore graphs.

Relaxing the degree: As the maximum degree is a global measure of the degrees of the vertices of a graph, we could choose a finer measure, the degree sequence. This approach could be dealt with in several ways. A graph could be considered to be close to a Moore graph if it has $M_{\Delta,D}$ vertices, diameter D and if, for example,

(i) the number δ of vertices of degree $\Delta + 1$ is the smallest possible, while the rest of the vertices all have degree at most Δ, or

(ii) there is one vertex of degree $\Delta + \delta$, δ as small as possible, while the rest of the vertices all have degree at most Δ, or

(iii) the average degree of a vertex is $\Delta + \delta$, δ as small as possible.

Relaxing the diameter: As the diameter is a global measure of the distances between the vertices of a graph, we could choose a finer measure, the eccentricity. Relaxing the diameter could mean, for example, that a graph is close to a Moore graph if it has $M_{\Delta,D}$ vertices, maximum degree Δ and if

(i) the number δ of vertices with eccentricity equal to $D + 1$ is the smallest possible, while the remaining vertices all have eccentricity at most D, or

(ii) the average eccentricity of a vertex is $D + \delta$, δ as small as possible.

A relaxation of the diameter was first considered by Knor [224] in the context of digraphs. In that paper Knor studied the so-called radially Moore digraphs and showed that for every diameter and every maximum out-degree there exists a corresponding radially (or radial) Moore digraph.

Interestingly, the undirected version of this problem has proved to be much more difficult and it is not clear whether or not there exists an undirected radial Moore graph for all possible values of degree and diameter. A *radial Moore graph* (or, a *radially Moore graphs* G, has radius $D = rad(G)$, diameter at most $D + 1$ degree Δ and $M_{\Delta,D}$ vertices. Capdevila, Conde, Exoo, Gimbert and López [86] consider the existence of radial Moore graphs and they note that such graphs exist for radius 2 and any degree. They also consider some natural measures of how well a radial Moore graph approximates a Moore graph.

Using de Bruijn graph, Exoo, Gimbert, López, and Gómez [141] gave a nice construction of radial Moore graphs of diameter 3 for all degrees greater than or equal to 22. This, together with earlier constructions by Knor [225], ensures that radial Moore graphs of diameter 3 exist for every degree. The only other radial Moore graphs are known for the values of (Δ, D) = (3,4), (3,5), (4,4) and (5,4). These graphs were obtained using the algorithm described in [86] and their computer representations are available at http://cs.indstate.edu/ge/COMBIN/RADMOORE

For all the remaining values of degree and radius the question of the existence or otherwise of radial Moore graphs is totally open.

3.5.2 Girth

Biggs [59] studied the number of vertices of a regular graph whose girth and degree are given. If the degree is $D \geqslant 3$ and girth $g = 2r+1$, $r \geqslant 2$, then there is a simple lower bound

$$n_0(g, D) = 1 + \frac{D}{D-2}((D-1)^r - 1)$$

for the number of vertices. It has been proved by Bannai and Ito [23], and by Damerell [114], that the bound can be attained only when $g = 5$ and $D = 3, 7$ or 57. For related work, see also Biggs and Ito [61].

On the other hand, attempts to find general constructions for graphs with given girth and degree have yielded only much larger graphs than the lower bound. Bollobás [62] gives an overview of the problem and presents open questions regarding the behaviour of the number of excess vertices $n - n_0(g, D)$, where n is the smallest possible order. Cubic (that is, trivalent) graphs of a given girth and with the smallest possible number of vertices have been known as *cages*. For a survey article about cages, we recommend Wong [344]; for later results, the interested reader should consult Exoo [140] and Exoo and Jajcay [142].

Using matrix theory, Bannai and Itoh [24] proved that there do not exist any regular graphs with excess 1 and girth $2r+1 \geqslant 5$, and that, for $r \geqslant 3$, there are no antipodal regular graphs with diameter $r+1$ and girth $2r+1$.

Dutton and Brigham [133] gave upper bounds for the maximum number of edges e possible in a graph depending upon its order n, girth g (and sometimes minimum degree δ).

3.5.3 Connectivity

Chung, Delorme and Solé [97] define the k-diameter of a graph G as the largest pairwise minimum distance of a set of k vertices in G, i.e., the best possible distance of a code of size k in G. They study a function $N(k, \Delta, D)$, the largest size of a graph of degree at most Δ and k-diameter D, and give constructions of large graphs with given degree and k-diameter. They also give upper bounds for the eigenvalues, and new lower bounds on spectral multiplicity.

A parameter which is believed to be particularly important in networks is the reliability of the network: it is desirable that if some stations (resp., branches) are unable to work, the message can still be always transmitted. This corresponds to the connectivity (resp., edge-connectivity) of the associated graph. It is well known that the connectivity is less than or equal to the edge-connectivity, which is less than or equal to the minimum degree of the graph.

Seidman [302] gives an upper bound for the diameter of a connected graph in terms of its number of vertices, minimum degree and connectivity. Earlier results in this direction were also obtained by Watkins [341] and Kramer [229].

Bauer, Boesch, Suffel and Tindell [40] introduced the notion of super-λ graphs for the study of network reliability. A graph is *super*-λ if every edge cut of minimum size is an edge cut isolating a vertex. Soneoka [324] surveyed sufficient conditions for connectivity or edge-connectivity to be equal to the minimum degree. Additionally, the author proved a sufficient condition for super-λ in terms of the diameter D, order n, minimum degree δ and maximum degree Δ. He proves that a graph is super-λ if

$$n \geqslant \delta(((\Delta-1)^{D-1}-1)/(\Delta-2)+1)+(\Delta-1)^{D-1}.$$

The bounds are best possible for graphs with diameter 2,3,4 and 6.

Fiol [155] considers the relation between connectivity (resp., superconnectivity) and other parameters of a graph G, namely, its order n, minimum degree, maximum degree, diameter, and girth.

Using the same parameters, Balbuena, Carmona, Fábrega and Fiol [21] show that the connectivity, as well as arc-connectivity, of a bipartite graph is maximum possible, provided that n is large enough.

Quaife [291] gives an overview and some new results concerning the optimisation problem of the order of a graph given maximum degree, diameter and another parameter μ which expresses a redundancy. An undirected finite graph G is a (Δ, D, μ)-graph if, for each pair of distinct vertices of G, there exist at least μ edge-disjoint paths joining these vertices, each path consisting of k or fewer edges. The original (Δ, D) problem is then the $(\Delta, D, 1)$ problem.

Other papers relating the order of a graph, its maximum degree and diameter (and possibly other parameters) with the connectivity of a graph, include the studies by Fiol [155, 156], Fiol, Fábrega and Escudero [158], Bermond, Homobono and Peyrat [55], [56].

We conclude the first part of this survey by a new subsection on a recent generalisation of the degree/diameter problem.

3.6 Degree/diameter problem inside a host architecture

An interesting new direction in the degree/diameter problem was recently initiated by Dekker, Perez-Roses, Pineda-Villavicencio and Watters [117]. They proposed the following problem.

- *MaxDDBS Problem*: Given a connected undirected host graph G, an upper bound Δ for the maximum degree, and an upper bound D for the diameter, find the largest connected subgraph S with maximum degree $\leqslant \Delta$ and diameter $\leqslant D$.

MaxDDBS is a natural generalization of the degree/diameter problem and which can be seen as MaxDDBS when G is the complete graph K_n for sufficiently large n. After introducing the problem in [117], the authors went on to discuss various practical applications and, since it is computationally hard, give a heuristic approximation algorithm to solve it.

Regarding its computational complexity, MaxDDBS is known to be NP-hard, since it contains other well-known NP-hard problems as subproblems. In fact, restricting the search to only one constraint (either on the degree or the diameter), is enough to ensure NP-hardness [215]. The Largest Degree-Bounded Subgraph Problem is NP-hard as long as we insist that the subgraph be connected, but can be solved in polynomial time otherwise (Problem GT26 of [173]). On the other hand, the Maximum Diameter-Bounded Subgraph becomes the Maximum Clique for $D = 1$, which was one of Karp's original 21 NP-hard problems [221]. MaxDDBS also turns out not to be in Apx, the class of NP-hard optimization problems for which there is a polynomial-time algorithm with a constant approximation ratio.

A case of special interest is when the host graph G is a common parallel architecture, such as the mesh, the hypercube, the butterfly, or the cube-connected cycles. If there are any constraints on communication time between two arbitrary processors, then MaxDDBS corresponds to the largest subnetwork that can be allocated to perform the computation. The case of the mesh and the hypercube as host graphs were already treated in [117], where some bounds were found for the order of MaxDDBS in a k-dimensional mesh. The problem was treated in more detail when the host graph is a mesh in [263] and the bounds given in [117] were refined for the order of the largest subgraph in arbitrary $k \geqslant 1$. For the particular cases $k = 3, \Delta = 4$ and $k = 2, \Delta = 3$ the authors give constructions that result in larger lower bounds.

In general, the initial results obtained in this degree/diameter subproblem tend to result in upper and lower bounds that are much closer than is the case in the general degree/diameter problem.

4 Part 2: Directed graphs

4.1 Moore digraphs

As in the case of undirected graphs, there is a natural upper bound $n_{d,k}$ on the order of directed graphs (digraphs), given maximum out-degree d and diameter k. For any given vertex v of a digraph G, we can count the number of vertices at a particular distance from that vertex. Let n_i, for $0 \leqslant i \leqslant k$, be the number of vertices at distance i from v. Then $n_i \leqslant d^i$, for $0 \leqslant i \leqslant k$, and consequently,

$$\begin{aligned} n_{d,k} = \sum_{i=0}^{k} n_i &\leqslant 1 + d + d^2 + \cdots + d^k \\ &= \begin{cases} \frac{d^{k+1}-1}{d-1} & \text{if } d > 1 \\ k+1 & \text{if } d = 1 \end{cases} \end{aligned} \tag{10}$$

The right-hand side of (10), denoted by $M_{d,k}$, is called the *Moore bound* for digraphs. If the equality sign holds in (10) then the digraph is called a *Moore digraph*.

It is well known that Moore digraphs exist only in the trivial cases when $d = 1$ (directed cycles of length $k+1$, C_{k+1}, for any $k \geqslant 1$) or $k = 1$ (complete digraphs of order $d+1$, K_{d+1}, for any $d \geqslant 1$). This was first proved by Plesník and Znám in 1974 [287] and later independently by Bridges and Toueg who presented in 1980 a short and very elegant proof [73].

Because of the importance of the result we include an outline of the proof of the rarity of Moore digraphs, due to Bridges and Toueg.

Let A be the adjacency matrix of a digraph of degree d, diameter k and order $n = M_{d,k} = 1 + d + \cdots + d^k$. In such a digraph, any two distinct vertices are connected by a *unique* directed path of length at most k (note that this means that there are no directed cycles of length less than $k+1$ in the digraph), which is equivalent to A satisfying the matrix equation

$$I + A + \cdots + A^k = J \tag{11}$$

where I and J are the identity and the all-one matrices of dimension n, respectively. Since eigenvalues of A and J must satisfy the same polynomial equation and the spectrum of J is $n^1 0^{n-1}$, it follows that A has eigenvalue d corresponding to the eigenvalue n of J, and some of the roots of

$$1 + x + \cdots + x^k = 0$$

corresponding to the eigenvalue 0 of J, that is, $\frac{x^{k+1}-1}{x-1} = 0$.

Since $x \neq 0$, we have $x^{k+1} - 1 = 0$, and the solutions are complex numbers which come in conjugate pairs, $\lambda_i, \overline{\lambda_i}$, and $\overline{\lambda_i} = \lambda_i^k$.

Since there are no directed cycles of length less than $k+1$ then the trace of A, $\mathrm{tr}(A^j) = 0$ for $1 \leqslant j \leqslant k$.

Therefore we have $\mathrm{tr}(A) = d + \Sigma_{i=1}^{n-1}\lambda_i = 0$ and $\mathrm{tr}(A^k) = d^k + \Sigma_{i=1}^{n-1}\lambda_i^k = 0$.

Then $-d = \Sigma_{i=1}^{n-1}\lambda_i = \Sigma_{i=1}^{n-1}\overline{\lambda_i} = \Sigma_{i=1}^{n-1}\lambda_i^k = -d^k$.

This is possible only when $d = 1$ or $k = 1$ (the directed cycle of length $k+1$ or the complete digraph on $k+1$ vertices, respectively). Therefore, when $d > 1$ and $k > 1$ there are no Moore digraphs.

Throughout, a digraph of maximum out-degree d and diameter k will be referred to as *(d,k)-digraph*. Since there are no Moore (d,k)-digraphs for $d \geqslant 2$ and $k \geqslant 2$, the study of the existence of large digraphs focuses on (d,k)-digraphs whose order is close to the Moore bound, that is, digraphs of order $n = M_{d,k} - \delta$, where the *defect* δ is as small as possible.

4.2 Digraphs of order close to Moore bound

We start this section with a survey of the existence of digraphs of order one less than the Moore bound, that is, with (d,k)-digraphs of defect one; such digraphs are alternatively called *almost Moore digraphs*.

For the diameter $k = 2$, line digraphs of complete digraphs are examples of almost Moore digraphs for any $d \geqslant 2$, showing that $n_{d,2} = M_{d,2} - 1$. Interestingly, for out-degree $d = 2$, there are exactly three non-isomorphic diregular digraphs of order $M_{2,2} - 1$: the line digraph of K_3 plus two other digraphs (cf [258]), see Fig. 8. However, for maximum out-degree $d \geqslant 3$, Gimbert [174, 176] completely settled the classification problem for diameter 2 when he proved that line digraphs of complete digraphs are the only almost Moore digraphs.

Conde, Gimbert, Gonzalez, Miret and Moreno [108] used concepts and techniques from algebraic number theory combined with spectral techniques to prove that almost Moore digraphs do not exist for any degree when the diameter is 3.

Subsequently, using a similar approach, Conde, Gimbert, Gonzalez, Miret and Moreno [107] were able to also prove that almost Moore digraphs of diameter 4 also do not exist. Their proof relies on the irreducibility of a certain polynomial. Furthermore, they have provided a very general result for the nonexistence of all almost Moore digraphs of diameter at least 3 whenever the corresponding polynomial is irreducible [106].

On the other hand, focusing on small out-degree instead of diameter, Miller and Fris [258] proved that there are no almost Moore digraphs of maximum out-degree 2, for any $k \geqslant 3$.

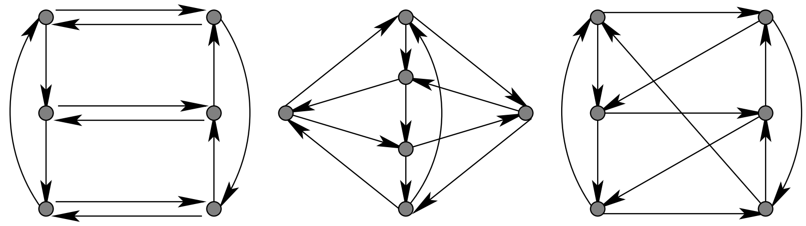


Figure 8: Three non-isomorphic diregular digraphs of order $M_{2,2} - 1$.

Moreover, Baskoro, Miller, Širáň and Sutton [38] showed that there are no almost Moore digraphs of maximum out-degree 3 and any diameter greater than or equal to 3.

The question of whether or not the equality can hold in $n_{d,k} \leqslant M_{d,k} - 1$, for $d \geqslant 4$ and $k \geqslant 5$, is open in general. A number of structural and non-existence results concerning almost Moore digraphs (the (d, k)-digraphs of defect one) are based on the following concept. If G is an almost Moore digraph then for each vertex $v \in V(G)$ there exists exactly one vertex, denoted by $r(v)$ and called the *repeat* of v, such that there are exactly two $v \to r(v)$ walks of length at most k. If S is a set (resp. multiset) of vertices then $r(S)$ is the set or a multiset of all the repeats of all the elements of S. We denote by $N^+(u)$ the set (or multiset) of the out-neighbours of a vertex u, and we denote by $N^-(u)$ the set (or multiset) of the in-neighbours of u.

If the almost Moore digraph G is diregular then the map r that assigns to each vertex $v \in V(G)$ its repeat $r(v)$ is an automorphism of G. This follows from the *Neighbourhood Lemma* of Baskoro, Miller, Plesník and Znám [35] which asserts that $N^+(r(v)) = r(N^+(v))$ and $N^-(r(v)) = r(N^-(v))$ for any vertex v of a diregular almost Moore digraph. Moreover, the permutation matrix P associated with the automorphism r (viewed as a permutation on the vertex set of the digraph) satisfies the equation

$$I + A + A^2 + \cdots + A^k = J + P,$$

where A is the adjacency matrix of G and J denotes the $n \times n$ matrix of all 1's.

The rest of the results mentioned in this section have been proved with the help of repeats (often combined with other techniques, most notably, matrix methods).

Miller and Fris [258] proved that there are no almost Moore digraphs for $d = 2$ and $k \geqslant 3$.

Baskoro, Miller, Plesník and Znám [35] gave a necessary divisibility condition for the existence of (diregular) almost Moore digraphs of degree 3, namely that if a diregular almost Moore digraph of degree 3 and diameter $k \geqslant 3$ exists then $k + 1$ divides $\frac{9}{2}(3^k - 1)$. Using this condition they deduce that such digraphs do not exist for infinitely many values of the diameter (if k is odd or if 27 divides $k + 1$).

Baskoro, Miller, Plesník and Znám [34] considered diregular almost Moore digraphs of diameter 2. Using the eigenvalues of adjacency matrices, they give several necessary conditions for the existence of such digraphs. For degree 3, they prove that there is no such digraph other than the line digraph of the complete digraph K_4 (a Kautz digraph).

For diregular digraphs, Baskoro, Miller and Plesník [36] gave various properties of repeats and structural results involving repeats and especially selfrepeats (vertices for which $r(v) = v$. These culminate in the theorem stating that for $d \geqslant 3$, $k \geqslant 3$, an almost Moore digraph contains either no selfrepeats or exactly k selfrepeats, that is, an almost Moore digraph contains at most one C_k.

In [37] Baskoro, Miller and Plesník gave further necessary conditions for the existence of almost Moore digraphs. They consider the cycle structure of the permutation r (repeat) and find that certain induced subdigraphs in a diregular almost Moore digraph are either cycles or, more interestingly, smaller almost Moore digraphs. For $k = 2$ and degree $2 \leqslant d \leqslant 12$ they show that if there is a C_2 then every vertex lies on a C_2 (that is, all vertices are selfrepeats or none is).

Baskoro, Miller and Širáň [39] studied almost Moore digraphs of degree 3 and found that such a digraph cannot be a Cayley digraph of an abelian group.

Gimbert [175] dealt with the problem of (h, k)-digraphs, where there is a unique directed walk of length at least h and at most k between any two vertices of the digraph and found that such digraphs exist only when $h = k$ and $h = k - 1$ if $d \geqslant 2$. In the cases of $d = 2$ or $k = 2$, it is shown, using algebraic techniques, that the line digraph $L(K_{d+1})$ of the complete digraph K_{d+1} is the only $(1, 2)$-digraph of degree d, that is, the only digraph whose adjacency matrix A satisfies the equation $A + A^2 = J$. As a consequence, there does not exist any other almost Moore digraph of diameter $k = 2$ with all selfrepeat vertices apart from the Kautz digraph.

Gimbert [174] used the characteristic polynomial of an almost Moore digraph to obtain some new necessary conditions for the cycle structure of the automorphism r of such a digraph. In particular, he applied the results to the cases of diameters 2 and 3 and proved that there is exactly one almost Moore digraph for $d = 4$ and $k = 2$, the line digraph of K_5.

Inspired by the technique of Bridges and Toueg [73], Baskoro, Miller, Plesník and Znám [35] used matrix theory (the eigenvalues of the adjacency matrix) to prove that there is no diregular almost Moore digraph of degree $\geqslant 2$, diameter $k \geqslant 3$ and with every vertex a selfrepeat, that is, every vertex on a directed cycle C_k. Note that Bosák [67] already studied diregular digraphs satisfying the more general matrix equation

$$A^a + A^{a+1} + \cdots + A^b = J,\ a \leqslant b,$$

and he proved that for $d > 1$ such digraphs exist only if either $b = a$ (de Bruijn digraphs [77]) or $b = a + 1$ (Kautz digraphs [222, 223]). Thus the result of [35] is only the case $a = 1$ and $b = k \geqslant 3$ but we mention it here because the proof is much simpler than Bosák's proof.

Cholily, Baskoro and Uttunggadewa [92] gave some conditions for the existence of almost Moore digraphs containing selfrepeat. The smallest positive integer p such that the composition $r^p(u) = u$ is called the *order* of u. Baskoro, Cholily and Miller [30, 31] investigated the number of vertex orders present in an almost Moore digraphs containing selfrepeat. An exact formula for the number of all vertex orders in a graph is given, based on the vertex orders of the outneighbours of any selfrepeat vertex.

There is a strong evidence that for diameter greater than 2, there are no almost Moore digraphs. Teska, Kuzel and Miller [330] provided further structural conditions for digraphs of defect 1 in the case when the digraph contains selfrepeats. Based on these conditions they were able to establish the nonexistence of such digraphs for many combinations of d and k values.

In 2011 Cholily [91] published a proof of the nonexistence of almost Moore digraphs of diameter greater than 2. However, the proof assumes the validity of a very strong conjecture which is of similar degree of difficulty as the conjecture of the nonexistence.

Although the nonexistence of almost Moore digraphs has been proved for many values of d and k, we do not have an unconditional general proof. The conditional proof due to Conde et al. [106] assumes the validity of the so-called *cyclotomic conjecture*, due to Gimbert [174].

Conjecture 1 *[174] Let $n > 2$ and $k > 1$ be integers. Then*

(i) If k is even then $F_{n,k}(x)$ is reducible in $\mathbb{Q}[x]$ if and only if $n \mid (k+2)$, in which case $F_{n,k}(x)$ has just two factors.

(ii) If k is odd then $F_{n,k}(x)$ is reducible in $\mathbb{Q}[x]$ if and only if n is even and $n \mid 2(k+2)$, in which case $F_{n,k}(x)$ has just two factors.

The case $k = 2$ of the cyclotomic conjecture was proved by Lenstra Jr. and Poonen [233] and the cases $k = 3$ and $k = 4$ by Conde et al. in [108] and [107], respectively.

In [106] the authors established that proving the cyclotomic conjecture implies the nonexistence of almost Moore digraphs.

The study of digraphs of defect 2 has so far concentrated on digraphs of maximum out-degree $d = 2$. In the case of diameter $k = 2$, it was shown by Miller [255] that there are exactly five non-isomorphic diregular digraphs of defect 2. In [255], Miller proved the non-existence of digraphs of defect two for out-degree 2 and diameter $k \geqslant 3$ by deriving a necessary condition, namely, that $k+1$ must divide $2(2^{k+1}-3)$, the number of arcs in the digraph of defect 2. Interestingly, this condition excludes many values of k. For example, for $3 \leqslant k \leqslant 10^7$ there are only two values ($k = 274485$ and $k = 5035921$) for which the divisibility condition holds. Consequently, for all but these two values of k, $3 \leqslant k \leqslant 10^7$, it has been known for some time that digraphs of defect 2 do not exist for out-degree $d = 2$. Miller and Širáň [268] improved this result by showing that digraphs of defect 2 do not exist for out-degree $d = 2$ and all $k \geqslant 3$.

For the remaining values of $k \geqslant 2$ and $d \geqslant 3$, the question of whether digraphs of defect 2 exist or not remains completely open; see Miller *et al.* [259, 268]. Our current knowledge of the upper bound on the order of digraphs of out-degree d and diameter k is summarised in Table 5.

Diameter k	*Degree d*	*Upper Bound of order* $n_{d,k}$
$k = 1$	all $d \geqslant 1$	$M_{d,1}$
$k = 2$	$d = 1$	$M_{1,2}$
	all $d \geqslant 2$	$M_{d,2} - 1$
$k = 3, 4$	$d = 1$	$M_{1,k}$
	$d = 2$	$M_{2,k} - 3$
	all $d \geqslant 3$	$M_{d,k} - 2$
$k \geqslant 5$	$d = 1$	$M_{1,k}$
	$d = 2$	$M_{2,k} - 3$
	$d = 3$	$M_{3,k} - 2$
	all $d \geqslant 4$	$M_{d,k} - 1$

Table 5: Upper bounds on the order of digraphs of degree d and diameter k.

4.3 Diregularity of digraphs close to Moore bound

We shall next consider the question of diregularity of digraphs, given maximum out-degree d and diameter k. To get a more complete picture, we make a short detour and briefly consider the much simpler issue of the regularity of undirected graphs.

For undirected graphs, if there is a vertex of degree less than Δ then the order of the graph cannot be more than

$$
\begin{aligned}
n_{\Delta,D} = \sum_{i=0}^{D} n_i &\leqslant 1 + (\Delta-1) + (\Delta-1)(\Delta-1) + \cdots + (\Delta-1)(\Delta-1)^{D-1} \\
&= 1 + (\Delta-1)(1 + (\Delta-1) + \cdots + (\Delta-1)^{D-1}) \\
&= \begin{cases} 1 + (\Delta-1)\ \frac{(\Delta-1)^D-1}{\Delta-2} = M_{\Delta,D} - \frac{(\Delta-1)^D-1}{\Delta-2} & \text{if } \Delta > 2 \\ D+1 = M_{2,D} - D & \text{if } \Delta = 2 \end{cases}
\end{aligned} \tag{12}
$$

Obviously, it follows that graphs with the number of vertices 'close' to the Moore bound cannot have any vertex of degree less than Δ, that is, the graphs are necessarily regular, end of story. However, for directed graphs the situation is much more interesting.

The only strongly connected digraph of out-degree $d = 1$ is the directed cycle C_{k+1}. For $d > 1$, if the maximum out-degree is d and if there is a vertex of out-degree less than d then we have

$$
\begin{aligned}
n_{d,k} = \sum_{i=0}^{k} n_i &\leqslant 1 + (d-1) + (d-1)d + \cdots + (d-1)d^{k-1} \\
&= (1 + d + d^2 + \cdots + d^k) - (1 + d + d^2 + \cdots + d^{k-1}) \\
&= M_{d,k} - M_{d,k-1}.
\end{aligned} \tag{13}
$$

Therefore, a digraph of maximum out-degree $d \geqslant 2$, diameter k and order $n = M_{d,k} - \delta$ must be out-regular if $\delta < M_{d,k-1}$. However, establishing the regularity or otherwise of the in-degree of digraphs (given maximum out-degree) is not so straightforward. Indeed, there exist digraphs of out-degree d and diameter k, whose order is just two or three less than the Moore bound and in which not all vertices have the same in-degree. These graphs are out-regular but not in-regular.

For example, when $d = 2$, $k = 2$, $n = 5$ (that is, defect 2), there are 9 non-isomorphic digraphs. Of these, 5 are diregular (see Fig. 9) and 4 are non-diregular (see Fig. 10).

It is interesting to note that there are more diregular digraphs than non-diregular ones for the parameters $n = 5$, $d = 2$, $k = 2$, while for the next larger digraphs of defect 2, namely, when $n = 11$, $d = 3$, $k = 2$, the situation is quite different: there are at least four non-isomorphic non-diregular digraphs [319] but only one diregular digraph [27] (see Figs. 11 and 12.

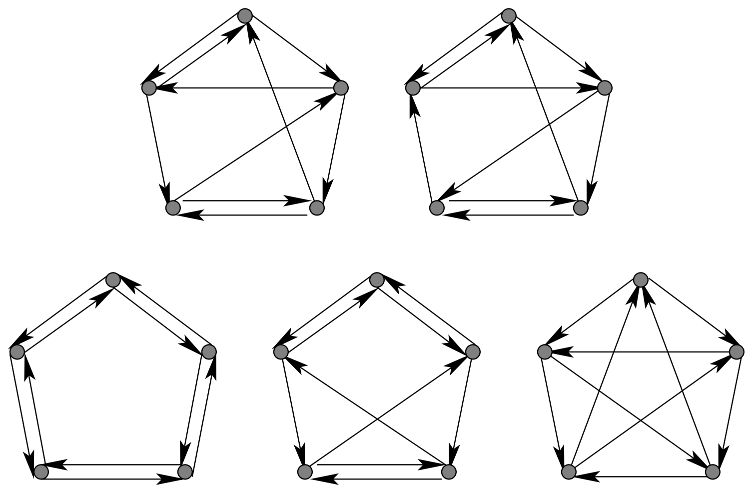


Figure 9: Five non-isomorphic diregular digraphs of order $M_{2,2} - 2$.

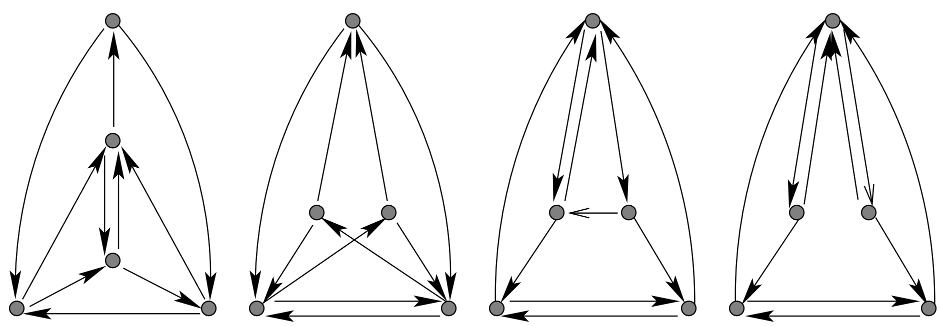


Figure 10: Four non-isomorphic non-diregular digraphs of order $M_{2,2} - 2$.

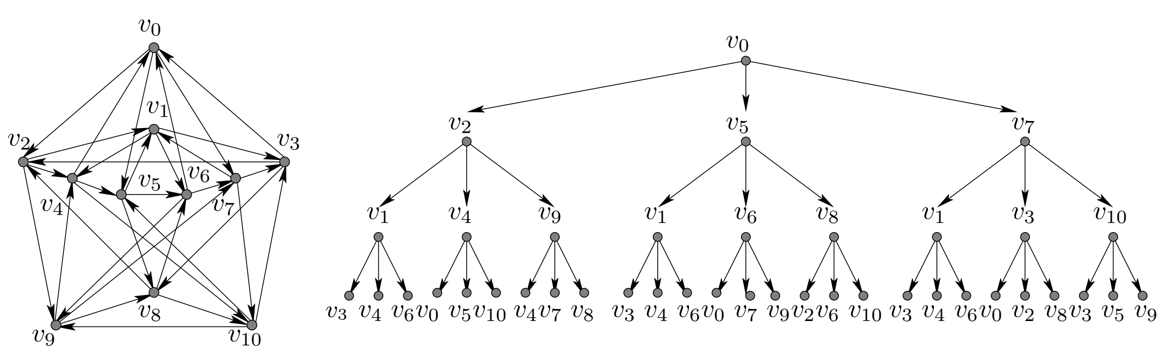


Figure 11: The unique diregular digraph of order $M_{3,2} - 2$.

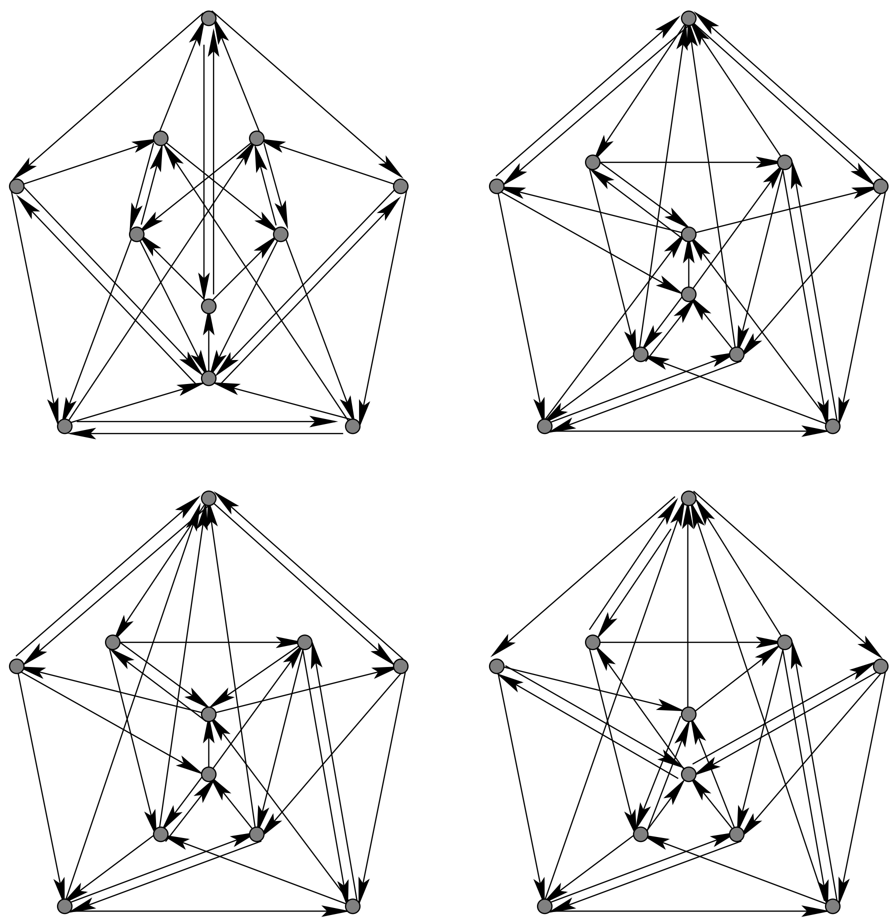


Figure 12: Four non-isomorphic non-diregular digraphs of order $M_{3,2} - 2$.

Miller, Gimbert, Širáň and Slamin [259] proved that every almost Moore digraph is diregular. Miller and Slamin [269] proved that every digraph of defect 2, maximum out-degree 2 and diameter $k \geqslant 3$ is diregular. Slamin, Baskoro and Miller [320] studied diregularity of digraphs of defect 2 and maximum out-degree 3. Miller and Slamin conjecture that all defect 2 digraphs of maximum out-degree $d \geqslant 2$ are diregular, provided $k \geqslant 3$.

Dafik, Miller, Iliopolous and Ryjacek [111] proved that all almost Moore digraphs are almost diregular (a digraph of defect δ is *almost in-regular* if the number of vertices of in-degree less than d is at most δ and the number of vertices of in-degree greater than d is also at most δ; *almost out-regularity* is defined as expected; a digraph is *almost regular* if it is both almost in-regular and almost out-regular). For maximum out-degree 3 they proved that apart from the diregular case there is only one other possible in-degree sequence in an almost Moore digraph of diameter greater than 2.

The question of diregularity or otherwise of digraphs with defect greater than 2 is completely open.

4.4 Constructions of large digraphs

The best lower bound on the order of digraphs of maximum out-degree d and diameter k is as follows. For maximum out-degree $d \geqslant 2$ and diameter $k \geqslant 4$,

$$n_{d,k} \geqslant 25 \times 2^{k-4}. \tag{14}$$

This lower bound is obtained from the *Alegre digraph* [7] which is a diregular digraph of degree 2, diameter 4 and order 25 (see Fig. 13), and from its iterated line digraphs. For the remaining values of maximum out-degree and diameter, a general lower bound is

$$n_{d,k} \geqslant d^k + d^{k-1}. \tag{15}$$

This bound is obtained from *Kautz digraphs*, that is, the diregular digraphs of degree d, diameter k and order $d^k + d^{k-1}$ [222]. Kautz digraphs, although defined in the literature in various ways, are just iterated line digraphs of complete digraphs (as an example, see the Kautz digraph on 24 vertices of degree 2 and diameter 4, in Fig. 14). Line digraph iterations were also studied by Fiol, Alegre and Yebra [157]; for a nice partial line digraph technique, see Fiol and Lladó [164]. For an example of work related to Kautz and de Bruijn digraphs, we refer to Barth and Heydemann [25]. A new family of digraphs that includes both de Bruijn and Kautz digraphs was studied by Llado, Villar and Fiol [237].

In [208, 209], Imase and Itoh considered the minimum diameter problem and the lower bound for diameter k, given the number of nodes n and the in- and out-degree of each node being d or less. From the Moore bound, they obtained

$$k \geqslant \lceil \log_d(n(d-1)+1) \rceil - 1 = l(n,d),$$

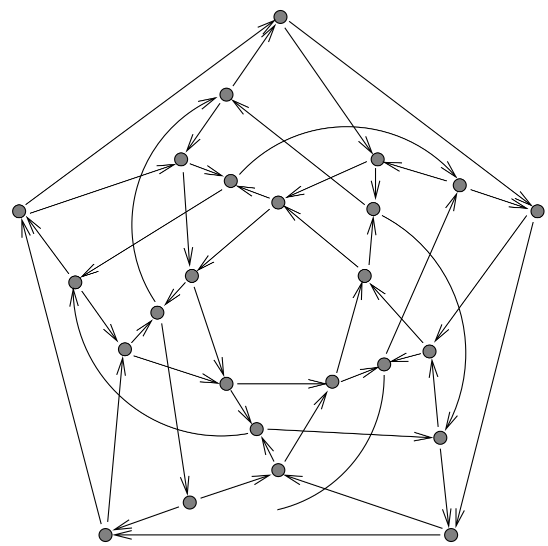


Figure 13: Alegre digraph.

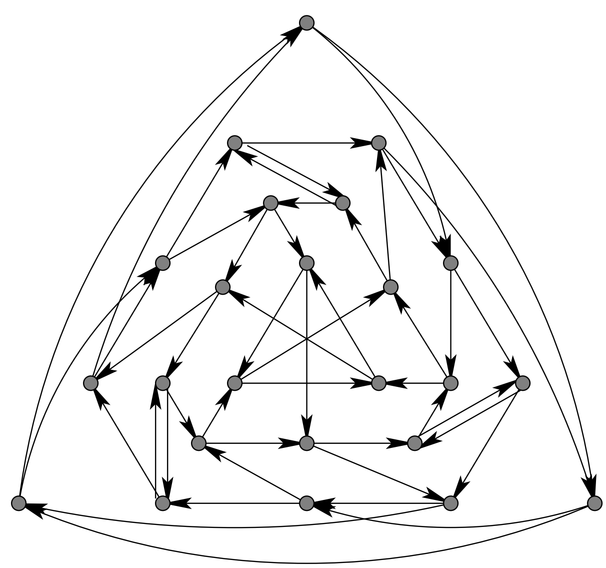


Figure 14: Kautz digraph.

where $1 < d \leqslant n-1$ and $\lceil x \rceil$ denotes the minimum integer not smaller than x. In [208], they gave the following construction of a (d,k)-digraph of order n with diameter $k = \lceil \log_d n \rceil$. For any n and d ($1 < d \leqslant n-1$), the vertex set of the digraph is $\{0,1,\ldots,n-1\}$, and there is an arc from i to j if and only if $j \equiv id + q \pmod n$, $q = 0,1,\ldots,d-1$. The diameter of the digraph is either equal to $l(n,d)$ or is at most one more. If $\frac{d^k-1}{d-1} \leqslant n \leqslant d^k$ or if $n = d^{k-b}(d^b+1)$, b odd, $b \leqslant m$, then the construction achieves diameter equal to the lower bound $l(n,d)$.

The construction by Imase and Itoh [209] was improved by Baskoro and Miller [32] who produced a construction for digraphs of $d^{k-b}(d^b+1)$ vertices for any b (note that the construction in [209] worked only for b odd). The procedure makes use of de Bruijn digraphs [77]. For other constructions based on adjacency defined by congruence relations, we refer to Opatrný [279, 280], and to Gómez, Padró and Perennes [190].

Examples of large digraphs of given degree and diameter have also been constructed by heuristic search, see e.g., Allwright [9].

The lifting method described in Subsection 3.3.2 is suitable for constructing digraphs as well as undirected graphs. In fact, most of the concepts introduced in Subsection 3.3.2 apply to digraphs with no or just minor changes. Let G be a *base digraph* with arc set $A(G)$ and let Γ be a finite group. This time, a *voltage assignment* on G in Γ is *any* mapping $\alpha: A(G) \to \Gamma$; no extra condition on voltages is needed because edge directions are a part of the description of the digraph G. The definition of the lift G^α is formally the same as in Subsection 3.3.2, and the lift is automatically a digraph.

For an example, we refer to Fig. 15 that shows how the Alegre digraph can be obtained as a lift of a base digraph of order 5, endowed with voltages in the group $\mathcal{Z}_5$.

Lifts of graphs implicitly appear in a number of constructions of large (d,k)-digraphs. To our knowledge, the first to explicitly use lifts of graphs were Annexstein, Baumslag and Rosenberg [10] in connection with their group action graphs. Such graphs were then later studied by Espona and Serra [138] to produce large Cayley (d,k)-digraphs based on the so-called de Bruijn networks. We recall that, given a group Γ and an arbitrary generating sequence Y of elements $y_1, y_2, \ldots, y_d$ of Γ, the Cayley digraph $C(\Gamma, Y)$ has vertex set Γ, and for each $g \in \Gamma$ and each $y_i \in Y$, there is a directed edge from g to gy_i.

We point out that the role of lifts in the context of digraphs is similar to the situation we have encountered in undirected graphs, and the reasons are essentially the same. To name a few of the advantages of lifts, the diameter of the lifted digraph can be expressed in terms of voltages on walks of the base digraph [29], which can be used to design efficient diameter-checking algorithms. Further, if a digraph contains a non-trivial group of automorphisms acting freely on its vertex set, then the digraph is a lift of a smaller digraph. This remark,

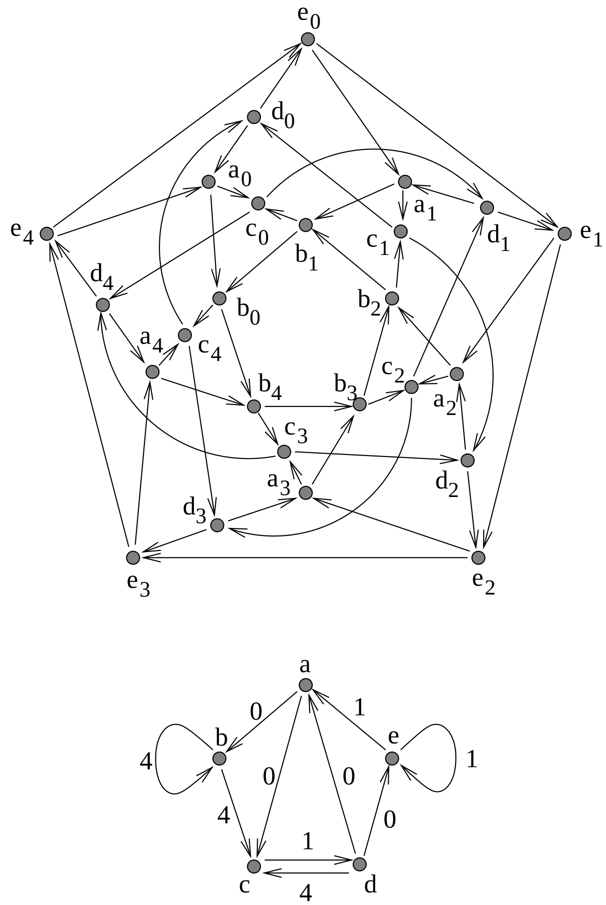


Figure 15: A base graph G with voltage assignment in $\mathcal{Z}_5$ and its lift, the Alegre digraph.

which directly follows from [192], applies to most currently known largest examples of (d, k)-digraphs, and so most of them can be described as lifts. Likewise, all constructions where incidence is defined by linear congruences are, in fact, lifting constructions.

Any Cayley digraph $C(\Gamma, S)$ is a lift of a single-vertex digraph (with $|S|$ directed loops carrying voltages from the generating set S). Additionally, quite complex Cayley digraphs that have appeared in the directed version of the degree/diameter problem (such as the ones of certain semidirect products of Abelian groups considered in [129]) can be described as ordinary lifts of smaller Cayley digraphs, with voltages in Abelian (mostly cyclic) groups; see [72].

As regards transitivity, convenient sufficient conditions can be extracted from [72] for a lift to be a vertex transitive (or a Cayley) digraph, which is suitable for producing large vertex transitive (d, k)-digraphs by lifts.

The theoretical background for lifts in the study of large (d, k)-digraphs can be found in Baskoro *et al.* [29] and Branković *et al.* [72]. Some of the results of [29] were in particular cases strengthened by Zlatoš [355], who proved several upper bounds on the diameter of the lift in terms of some properties of the base digraph and the voltage group. A number of his results give significantly improved upper bounds when the digraph is a Cayley digraph and the voltage group is abelian.

Table 6 gives a summary of the current largest known digraphs for maximum out-degree $d \leqslant 13$ and diameter $k \leqslant 11$.

4.5 Restricted versions of the degree/diameter problem

Unlike the undirected case, the restrictions of the degree/diameter problem for digraphs that have been considered in the literature are mostly connected with vertex transitivity. Issues such as biparticity and connectivity have received less attention so far.

4.5.1 Vertex-transitive digraphs

Let $vt_{d,k}$ be the largest order of a vertex-transitive digraph of maximum out-degree d and diameter k. Obviously, we have $vt_{d,k} = M_{d,k}$ if $d = 1$ or if $k = 1$. Moreover, as line digraphs of complete digraphs are vertex-transitive, we also have $vt_{d,2} = M_{d,2} - 1$, for all $d \geqslant 2$. Apart from this, there do not seem to be any general upper bounds on $vt_{d,k}$. Constructions that yield lower bounds on $vt_{d,k}$ rely mostly on coset graphs or on certain compositions.

Let Γ be a finite group, let Λ be a subgroup of Γ, and let X be a set of distinct Λ-coset representatives, such that Γ is generated by $\Lambda \cup X$, $X \cap \Lambda = \emptyset$, and $\Lambda X \Lambda \subseteq X\Lambda$. The *Cayley coset digraph* $Cos(\Gamma, \Lambda, X)$ has vertex set $\{g\Lambda;\ g \in \Gamma\}$, and there is an arc from $g\Lambda$ to $h\Lambda$ if $h\Lambda = gx\Lambda$, for some $x \in X$. It is an easy exercise to prove that Cayley coset graphs are well defined, $|X|$-diregular, connected, and vertex-transitive.

For a prominent example, let $\Gamma = S_{d+1}$ be the symmetric group acting on the set $[d+1] = \{1, 2, \dots, d, d+1\}$, and let Λ_k be the subgroup of Γ that pointwise fixes the subset $[k] = \{1, 2, \dots, k\}$, for some k, $2 \leqslant k \leqslant d$. Further, for $2 \leqslant i \leqslant d+1$ let ξ_i be the cyclic permutation $(i \dots 21)$, and let $X = \{\xi_i;\ 2 \leqslant i \leqslant d+1\}$. It can be checked that the above conditions on Γ, Λ, and X are satisfied; the Cayley coset digraph $Cos(S_{d+1}, \Lambda_k, X)$ is known as a *cycle prefix digraph* (see Faber, Moore and Chen [144], and also Comellas and Fiol [101]). The cycle prefix digraphs $Cos(S_{d+1}, \Lambda_k, X)$ are (d, k)-digraphs of order $(d+1)!/(d+1-k)!$ and they yield most of the entries of the lower triangular part in the table of largest known vertex-transitive (d, k)-digraphs (see end of this subsection). In particular,

$$vt_{d,k} \geqslant (d+1)!/(d+1-k)! \text{ if } d \geqslant k \geqslant 3.$$

Moderate improvements of the above lower bound can be obtained by removing certain adjacencies in the cycle prefix digraphs; for details we refer to [101].

d \ k	2	3	4	5	6	7	8	9	10	11
2	Ka **6**	Ka **12**	Al **25**	$L(Al)$ **50**	$L^2(Al)$ **100**	$L^3(Al)$ **200**	$L^4(Al)$ **400**	$L^5(Al)$ **800**	$L^6(Al)$ **1 600**	$L^7(Al)$ **3 200**
3	Ka **12**	Ka **36**	Ka **108**	Ka **324**	Ka **972**	Ka **2 916**	Ka **8 748**	Ka **26 244**	Ka **78 732**	Ka **236 196**
4	Ka **20**	Ka **80**	Ka **320**	Ka **1 280**	Ka **5 120**	Ka **20 480**	Ka **81 920**	Ka **327 680**	Ka **1 310 720**	Ka **5 242 880**
5	Ka **30**	Ka **150**	Ka **750**	Ka **3 750**	Ka **18 750**	Ka **93 750**	Ka **468 750**	Ka **2 343 750**	Ka **11 718 750**	Ka **58 593 750**
6	Ka **42**	Ka **252**	Ka **1 512**	Ka **9 072**	Ka **54 432**	Ka **326 592**	Ka **1 959 552**	Ka **11 757 312**	Ka **70 543 872**	Ka **423 263 232**
7	Ka **56**	Ka **392**	Ka **2 744**	Ka **19 208**	Ka **134 456**	Ka **941 192**	Ka **6 588 344**	Ka **46 118 408**	Ka **322 828 856**	Ka **2 259 801 992**
8	Ka **72**	Ka **576**	Ka **4 608**	Ka **36 864**	Ka **294 912**	Ka **2 359 296**	Ka **18 874 368**	Ka **150 994 944**	Ka **1 207 959 552**	Ka **9 663 676 416**
9	Ka **90**	Ka **810**	Ka **7 290**	Ka **65 610**	Ka **590 490**	Ka **5 314 410**	Ka **47 829 690**	Ka **430 467 210**	Ka **3 874 204 890**	Ka **34 867 844 010**
10	Ka **110**	Ka **1 100**	Ka **11 000**	Ka **110 000**	Ka **1 100 000**	Ka **11 000 000**	Ka **110 000 000**	Ka **1 100 000 000**	Ka **11 000 000 000**	Ka **110 000 000 000**
11	Ka **132**	Ka **1 452**	Ka **15 972**	Ka **175 692**	Ka **1 932 612**	Ka **21 258 732**	Ka **233 846 052**	Ka **2 572 306 572**	Ka **28 295 372 292**	Ka **311 249 095 212**
12	Ka **156**	Ka **1 872**	Ka **22 464**	Ka **269 568**	Ka **3 234 816**	Ka **38 817 792**	Ka **465 813 504**	Ka **5 589 762 048**	Ka **67 077 144 576**	Ka **804 925 734 912**
13	Ka **182**	Ka **2 366**	Ka **30 758**	Ka **399 854**	Ka **5 198 102**	Ka **67 575 326**	Ka **878 479 238**	Ka **11 420 230 094**	Ka **148 462 991 222**	Ka **1 930 018 885 886**

Table 6: The order of the largest known digraphs of maximum out-degree d and diameter k.

We now give an example of a composition method [101]. We say that a digraph is k-*reachable* if for every pair of its vertices u, v there exists a directed path from u to v of length exactly k. For example, the Kautz digraphs of diameter k are $(k+1)$-reachable, and the cycle prefix (d,k)-digraphs are k-reachable for all $k \geqslant 3$. Now, let G be a digraph with vertex set V. Let $n \geqslant 2$ and $t \geqslant 1$ be integers. Form a new digraph $G_{n,t}$ on the vertex set $\mathcal{Z}_{tn} \times V^n$, with adjacencies given by

$$(i, v_0, \ldots, v_i, \ldots, v_{n-1}) \to (i+j, v_0, \ldots, w_i, \ldots, v_{n-1}),$$

where $j \in \{1, b\}$, for some $b \in \mathcal{Z}_{tn}$, all indices on the vertices of G taken mod n, and w_i adjacent from v_i in G. This construction was introduced by Comellas and Fiol [101] who also proved the following result: If G is a vertex-transitive d-diregular k-reachable digraph then $G_{n,t}$ is also a vertex-transitive digraph, diregular of degree $2d$, of order $nt|V|^n$, and of diameter at most $kn + \ell$, where ℓ is the diameter of the Cayley digraph $C(\mathcal{Z}_{tn}, \{1, b\})$. The number b is then chosen to minimise ℓ. If j is restricted to assume the value 1 only, then the result is a vertex-transitive (d, k')-digraph of order $nt|V|^n$ and of diameter at most $(k+t)n - 1$. Both constructions yield certain record examples of vertex-transitive digraphs of diameter between 7 and 11; we refer to [101] for details.

The current largest known orders of vertex-transitive digraphs for maximum out-degree $d \leqslant 13$ and diameter $k \leqslant 11$ can be found on the website

`http://maite71.upc.es/grup_de_grafs/grafs/taula_delta`

which is updated regularly by Francesc Comellas. For an earlier version of the table, see Comellas and Fiol [101].

See also the CombinatoricsWiki website [239]. Note that all the currently largest known values for $d \geqslant 5$ and $k \geqslant 5$ are due to Gomez [178].

4.5.2 Cayley digraphs

Let $C_{d,k}$ and $AC_{d,k}$ be the largest order of a Cayley digraph and a Cayley digraph of an abelian group, respectively, of out-degree d and diameter k. Very little is known about $C_{d,k}$ in general. Clearly, $C_{d,1} = M_{d,1}$, and for $k \geqslant 3$ we know only that $C_{d,k} \leqslant M_{d,k}$ but we can say a little more in the case when $k = 2$. As we know from [174], for $d \geqslant 3$, we have $n_{d,2} = M_{d,2} - 1$ and the unique digraph of out-degree d and diameter 2 is the line digraph of the complete digraph on $d+1$ vertices.

As in the undirected case, the study of large abelian Cayley digraphs of a given out-degree (equal to the number of elements in the generating set) and given diameter can be based on a combination of group-theoretic and geometric ideas, whose genesis and background have been explained in [132].

The starting point is again the fact that any finite abelian group Γ with an arbitrary (not necessarily symmetric) generating set $Y = \{y_1, \ldots, y_d\}$ of size d is a quotient group of the free abelian d-generator group $\mathcal{Z}^d$ by a subgroup N (of finite index) that is the kernel of the natural homomorphism $\mathcal{Z}^d \to \Gamma$ which sends the unit vector $\mathbf{e}_i \in \mathcal{Z}^d$ onto y_i.

Since this time we are discussing directed graphs, and therefore in our Cayley digraphs we cannot use an inverse of a generator (unless it belongs to Y), in our quotient group we are allowed to use only linear combinations of the vectors $\mathbf{e}_i$ with *non-negative* integer coefficients. Therefore, for any given diameter k, define

$$W'_{d,k} = \{(x_1, \dots, x_d) \in \mathcal{Z}^d;\ x_i \geqslant 0,\ x_1 + \dots + x_d \leqslant k\}.$$

Then the Cayley digraph $C(\Gamma, Y)$ has diameter at most k if and only if $W'_{d,k} + N = \mathcal{Z}^d$. This allows us to conclude that $|W'_{d,k}|$ is an upper bound on $AC_{d,k}$.

The geometric connection lies again in the fact that any subgroup N of $\mathcal{Z}^d$ of finite index, with the property $W'_{d,k} + N = \mathcal{Z}^d$, determines a d-dimensional lattice that induces 'shifts' of the set $W'_{d,k}$ so that they completely cover the elements of $\mathcal{Z}^d$. We also recall that the index $[\mathcal{Z}^d : N] = |\Gamma|$ (which gives a lower bound on $AC_{d,k}$) is equal to the absolute value of the determinant of the d-dimensional matrix formed by the d generating vectors of N. This reduces the search for bounds on $AC_{d,k}$ to interesting and deep problems in combinatorial geometry (cf. [132]).

Unlike the undirected case, an exact formula for $|W'_{d,k}|$ (which, as we know, is automatically an upper bound on $AC_{d,k}$) is a matter of easy counting and it forms the right hand side of (16) below. A lower bound is much harder to obtain, and we present here the one given in Dougherty and Faber [132], based on a deep study of lattice coverings. We give both bounds as follows: There exists a constant c (not depending on d and k) such that for any fixed $d \geqslant 2$ and all k,

$$\frac{c}{d!d(\ln d)^{1+\log_2 e}} k^d + O(k^{d-1}) \leqslant AC_{d,k} \leqslant \binom{k+d}{d}. \tag{16}$$

Note that the upper bound can be considered to be the *abelian Cayley directed Moore bound* for abelian groups with d-element generating sets. Once again, this differs from the Moore bound $M_{d,k}$ rather dramatically; if the number of generators d is fixed and $k \to \infty$ then the right hand side of (16) has the form $k^d/d! + O(k^{d-1})$.

It should not come as a surprise that the exact values of $AC_{d,k}$ are difficult to determine. With the help of lattice tilings, Dougherty and Faber [132] (and others, mainly Wong and Coopersmith [343]) showed that

$$AC_{2,k} = |W'_{2,k}| = \lfloor (k+2)^2/3 \rfloor.$$

For $d = 3$ and $k \geqslant 8$, similar methods (see [132] for details and more references) yield the bounds

$$0.084k^3 + O(k^2) \leqslant AC_{3,k} \leqslant 3(k+3)^3/25.$$

A table of the current best values of $AC_{3,k}$ for $k \leqslant 30$ appears in [132] as well.

Large Cayley digraphs can also be obtained by lifting [29, 72, 355]. As a representative example, we briefly summarise the work of Espona and Serra [138]. Let G be a connected diregular digraph of out-degree d and let $\mathcal{F} = \{F_1, F_2, \ldots, F_d\}$ be a factorization of G into directed factors F_i of in- and out-degree 1 (that is, each F_i is a union of directed cycles, covering all vertices of G). Each factor F_i then defines, in a natural way, a permutation ϕ_i of the vertex set of G, where $\phi_i(v)$ is the vertex adjacent from v in the factor F_i. Let Γ be the permutation group generated by the permutations $\phi_1, \ldots, \phi_d$ and let $X = \{\phi_1, \phi_2, \ldots, \phi_d\}$. Then the Cayley digraph $C(\Gamma, X)$ is a lift of the original digraph G. We note that this procedure can easily be translated into the language of voltage assignments on G.

It was pointed out in [138] that interesting large (d, k)-digraphs (such as butterfly digraphs) can be obtained by the above construction applied to various factorizations of the de Bruijn digraphs. For more constructions of large Cayley digraphs of given degree and diameter, see [4].

Since bounds on the diameter of a Cayley digraph in terms of a logarithm of the order of the group are essentially the same as in the undirected case (Subsection 3.4.2), we do not discuss them here.

4.5.3 Digraphs on surfaces

The planar version of the degree/diameter problem for digraphs was considered by Simanjuntak and Miller [314]. They showed that a planar digraph of diameter 2 and maximum out-degree $d \geqslant 41$ cannot have more than $2d$ vertices and that this bound is the best possible. They conjecture that the same bound holds also for $d \leqslant 40$. The planar version of the degree/diameter problem for $k > 2$ is totally open. Unlike in the undirected case, directed graphs embeddable on a fixed surface, other than the sphere, have not been considered from the point of view of the degree/diameter problem.

4.6 Related topics

In the directed case it seems that less attention has been paid to topics closely related to the ones presented in the previous sections. We therefore only consider separately approximating Moore digraphs and the connectivity issue, while other miscellaneous contributions are summed up in Subsection 4.6.3.

4.6.1 Approximating Moore digraphs

Since Moore digraphs exist for only a few combinations of the degree and diameter values, we are interested in studying the existence of large digraphs which are in some way 'close' to Moore digraphs. Given that we are dealing with three parameters, namely, order, maximum out-degree and diameter, in order to get close to Moore digraphs, we may consider relaxing each of these parameters in turn.

Relaxed Moore digraphs were first considered by Tang, Miller and Lin in [329].

Relaxing the order means looking for digraphs of given maximum out-degree d and diameter k, whose order is $M_{d,k} - \delta$. The parameter δ is called the *defect*. Such a digraph is called a $(d,k) - digraph\ of\ defect\ \delta$ or a $(d,k,-\delta)$-*digraph*. This is the direction that has traditionally been considered when trying to approximate the idea of Moore digraphs; this is the degree/diameter problem.

Next we consider relaxing the degree. As the maximum degree of a graph is a global measure of the out-degrees of the vertices of a graph, we could choose a finer measure, the in- and out-degree sequences. This approach could be dealt with in several ways. For example, a digraph could be considered to be close to a Moore digraph if it has $M_{d,k}$ vertices, diameter k and if

(i) the number of vertices of out-degree $d+1$ is the smallest possible, while the rest of the vertices all have out-degree at most d, or

(ii) there is one vertex of out-degree $d+\delta$, δ as small as possible, while the rest of the vertices all have degree at most d, or

(iii) the average degree of a vertex is $d+\delta$, δ as small as possible.

We have not found any study which has considered such a relaxation of the maximum degree for graphs close to the ideal of a Moore digraph.

Next we consider relaxing the diameter. As the diameter is a global measure of the distances between the vertices of a digraph, we could choose a finer measure, the in- and out-eccentricity. Relaxing the diameter could mean, for example, that a digraph is close to a Moore digraph if it has $M_{d,k}$ vertices, maximum out-degree d and if

(i) the number of vertices with out-eccentricity equal to $k+1$ is the smallest possible, while the remaining vertices all have out-eccentricity at most k, or

(ii) the average out-eccentricity of a vertex is $k+\delta$, δ as small as possible.

A relaxation of the diameter was first considered by Knor [224]. In that paper Knor studied the so-called radially Moore digraphs. Knor defined *radially Moore digraphs* (also called *radial Moore digraphs*) as regular digraphs with radius k, diameter $\leqslant s+1$ and the maximum possible number of nodes. He showed that, for every k and d, there exists a regular radially Moore digraph of degree d and radius k. He also gave an upper bound on the number of central nodes in a radially Moore digraph of degree 2.

In [329] the authors built a Hybrid Simulated Annealing and Genetic Algorithm (HSAGA) for the construction of digraphs which are in some sense 'close' to being Moore digraphs.

4.6.2 Connectivity

Homobono and Peyrat [206] considered the connectivity of the digraphs proposed by Imase and Itoh. They proved that, provided the diameter is greater than 4, the connectivity of these digraphs is d if $n = k(d+1)$ and $\gcd(n, d) > 1$; and $d-1$ otherwise. Imase, Soneoka and Itoh earlier proved that the connectivity is greater than or equal to $d-1$ if the graph's diameter is greater than 4. Homobono and Peyrat's paper improves upon this result.

Imase, Soneoka and Okada [210] considered the relation between the diameter k and the edge (resp., vertex) connectivity of digraphs. They found that diameter minimisation results in maximising the connectivity and that all proposed small diameter digraphs have a node connectivity either $d-1$ or d.

A digraph is super-λ if every edge cut of minimum size is an edge cut isolating a vertex. Soneoka [325] proved a sufficient condition for super-λ in terms of the diameter k, order n, minimum out-degree δ and maximum out-degree d. He proves that a digraph is super-λ if

$$n \geqslant \delta((d^{k-1}-1)/(d-1)+1)+d^{k-1}.$$

The bounds are the best possible for digraphs with diameter 2 or 3. The sufficient conditions are satisfied by many well-known digraphs, including the de Bruijn and Kautz digraphs.

Fiol [155] considers the relation between connectivity (resp., superconnectivity) and other parameters of a digraph G, namely, its order n, minimum out-degree, maximum out-degree, diameter, and a new parameter related to the number of short walks in G. Maximally connected and superconnected iterated line digraphs are characterised.

Fiol and Yebra [165] showed that the Moore-like bound for strongly connected bipartite digraphs $G = (V_1 \cup V_2, A)$, $d > 1$,

$$|V_1| + |V_2| \leqslant 2(d^{k+1}-1)/(d^2-1) \quad \text{for } k \text{ odd};$$

$$|V_1| + |V_2| \leqslant 2(d^{k+1}-d)/(d^2-1) \quad \text{for } k \text{ even}$$

is attainable only when $k = 2$,3 or 4. The interested reader can find out more about bipartite and almost bipartite Moore digraphs in studies by Fiol, Gimbert, Gómez and Wu [161], and Fiol and Gimbert [160]; for multipartite version, see Fiol, Gimbert and Miller [162].

Balbuena, Carmona, Fábrega and Fiol [21] showed that the connectivity, as well as arc-connectivity, of a bipartite digraph is the maximum possible, provided that n is large enough.

Other papers relating the order of a digraph, its maximum out-degree and diameter (and possibly other parameters) with the connectivity and/or super-connectivity of a digraph include the studies by Fábrega and Fiol [145], Fiol [156] and Xu [347].

Along with connectivity, modified concepts of diameter were considered, such as the k-diameter of k-connected graphs studied by Xu and Xu [348], or the conditional diameter in superconnected digraphs looked at by Balbuena, Fàbrega, Marcote and Pelayo [22].

4.6.3 Other related problems

Analogously to the generalisation of Moore graphs proposed by Cerf, Cowan Mullin and Stanton [89], Sampels [299] proposed a generalisation of Moore digraphs and in the same paper he constructed vertex-symmetric generalised Moore graphs. Fiol, Lladó and Villar [163] considered the order/degree problem for digraphs; they constructed a family of digraphs with the smallest possible diameter, given order and maximum out-degree.

Aider [1] studied bipartite digraphs with maximum in- and out-degree d (> 1) and diameter k. He showed that the order of such a digraph is at most

$$2\frac{d^{k+1}-1}{d^2-1} \qquad \text{if } k \text{ is odd;}$$

$$2\frac{d^{k+1}-d}{d^2-1} \qquad \text{if } k \text{ is even.}$$

The author then finds some pairs d and k, for which there exist bipartite digraphs of the given order ('bipartite Moore digraphs') and some pairs for which there are no such bipartite digraphs. Additionally, a variety of properties of such digraphs are established.

Gómez, Morillo and Padró [189] consider (d, k, k', s)-digraphs (digraphs with maximum out-degree d and diameter k such that, after the deletion of any s of its vertices, the resulting digraph has diameter at most k'). The authors' goal is to find such bipartite digraphs with order as large as possible. They give new families of digraphs satisfying a Menger-type condition, namely, between any pair of non-adjacent vertices there are $s+1$ internally disjoint paths of length at most k', and they obtain new families of bipartite (d, k, k', s)-digraphs with order very close to the upper bound.

Munoz and Gómez [273] continued this research and obtained new families of asymptotically optimal (d, k, k', s)-digraphs.

Morillo, Fiol and Fábrega [271], Morillo, Fiol and Yebra [272], Comellas, Morillo and Fiol [104] used plane tessellations to construct families of bipartite digraphs of degree two and with maximum order, minimum diameter, and minimum mean distance, defined by $\bar{k} = \sum_{i,j \in V} d_{ij}/n^2$. The last parameter was also studied earlier by Wong [342], who considered a subclass of digraphs in which the number of nodes is n and diameter is k; he showed that the minimum values of diameter and the average distance are both of the order of $d^d\sqrt{n}$.

Unilaterally connected digraphs were studied from the perspective of the degree/diameter problem by Gómez, Canale and Munoz [182, 183].

In the next subsection we consider a generalization of the degree/diameter problem which will subsume both undirected and directed versions of the problem.

4.7 Partially directed graphs

In many real-world networks, a mixture of both unidirectional and bidirectional connections may exist (e.g., the World Wide Web network, where pages are nodes and hyperlinks describe the connections). For such networks, mixed graphs provide a perfect modeling framework.

While it would appear that the undirected and the directed versions of the degree/diameter problem are sufficiently different from each other - indeed so much so that we have decided to organise this survey into two separate parts - there is nevertheless a way of treating both undirected and directed versions together in a unified manner.

Recall that the Moore bound for a directed graph of maximum out-degree d and diameter k is $M^*_{d,k} = 1 + d + d^2 + \cdots + d^k$.

Similarly, the Moore bound for an undirected graph of maximum degree d and diameter k is $M_{d,k} = 1 + d + d(d-1) + \cdots + d(d-1)^{k-1}$.

Nguyen and Miller [277] showed that both undirected and directed Moore graphs are special cases of "mixed" (or "partially directed") Moore graphs.

Partially directed Moore graphs (also called *mixed Moore graphs*) were introduced and investigated by Bosák [68]. Bosák [68] defines a partially directed Moore graph as a simple, finite and homogeneous (each vertex is an endpoint of r undirected (two-way) edges and is an origin and a terminal of z directed (one-way) edges, where r and z are independent of the choice of a vertex) graph satisfying the condition: There exists exactly one trail (an edge can be used only once and wrong way is not allowed) from any vertex u to any vertex v of length at most the diameter. Bosák found divisibility conditions concerning the distribution of undirected and directed edges in mixed Moore graphs of diameter 2, and he produced some examples of mixed Moore graphs.

This line of research was continued by Nguyen, Miller and Gimbert [278] who proved the equivalence of mixed tied graphs and mixed Moore graphs. A *proper* mixed graph contains at least one edge and at least one arc. It is shown that all proper mixed tied graphs must be totally regular.

With the exception of the graph of order $n = 18$ (see Figure 16), all the other known proper mixed Moore graphs of diameter 2, constructed by Bosák, can be considered isomorphic to Kautz digraphs (see [222] and [223]) of the same degree and order. Indeed, they are the Kautz digraphs $Ka(d, 2)$ with all digons considered as undirected edges. The following conjecture was proposed by Bosák [68] (Z_n denotes directed cycle on n vertices).

Conjecture 2 [68] *Let d and k be integers, $d \geqslant 1$, $k \geqslant 3$. A finite graph G is a mixed Moore graph of degree d and diameter k if and only if either $d = 1$ and G is Z_{k+1}, or $d = 2$ and G is C_{2k+1}.*

Nguyen, Miller and Gimbert [278] presented a combinatorial proof that Conjecture 2 holds. They also proved that all the known mixed Moore graphs of diameter 2 are unique.

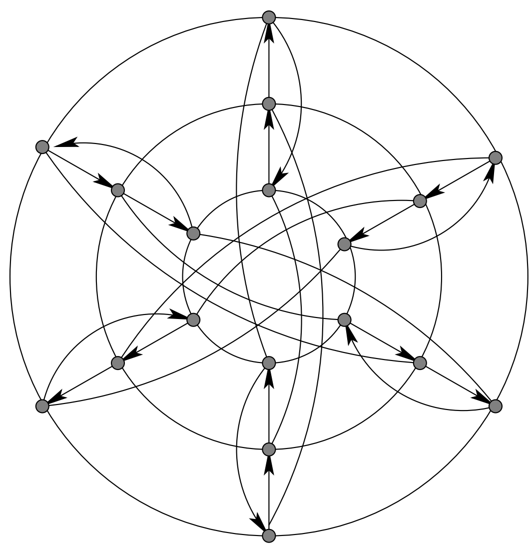


Figure 16: The Bosák graph.

Let G be a mixed Moore graph of order n, degree d and diameter $k \geqslant 2$. Then G is a loopless totally regular graph. If z is the directed degree of G and r is the undirected degree of G then

$$n = M_{z,r,k} = 1 + (z + r) + z(z + r) + r(z + r - 1) + \cdots + z(z + r)^{k-1} + r(z + r - 1)^{k-1}.$$

The right hand side is the *Moore bound for mixed graphs*, or *mixed Moore bound*. A mixed graph of maximum (undirected) degree r, maximum out-degree z, diameter k and order $M_{z,r,k}$ is called a *mixed Moore graph*.

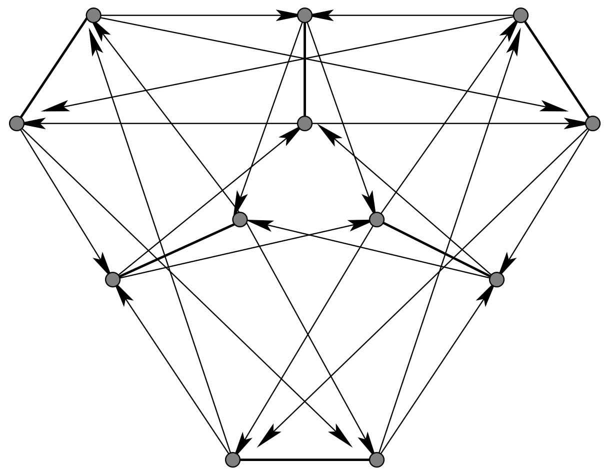


Figure 17: The proper mixed Moore graph $Ka(3, 2)$ of order 12.

Note that $M_{z,r,k} = M_{d,k}$ when $z = 0$ and $M_{z,r,k} = M^*_{d,k}$ when $r = 0$ $(d = r + z)$.

Table 7 lists all the values of $n \leqslant 100$ with the corresponding possible values of r and z such that a mixed Moore graph either exists or is not known not to exist. Clearly, unlike in the case of directed Moore graphs which are all known, or the case of undirected Moore graphs which are all known with the exception when the degree is 57 and diameter 2, there are still many values of r and z, for which the existence of a mixed Moore graph of diameter 2 has not been settled.

The most recent new result is due to Jørgensen [218] who has consructed a new mixed Moore graph of diameter 2, undirected degree 3 and out-degree 7; this graph contains 108 vertices.

n	d	z	r	*existence*	*uniqueness*
3	1	1	0	Z_3	$\surd$
5	2	0	2	C_5	$\surd$
6	2	1	1	$Ka(2,2)$	$\surd$
10	3	0	3	Petersen graph	$\surd$
12	3	2	1	$Ka(3,2)$ (Figure 7)	$\surd$
18	4	1	3	Bosák graph	$\surd$
20	4	3	1	$Ka(4,2)$	$\surd$
30	5	4	1	$Ka(5,2)$	$\surd$
40	6	3	3	unknown	unknown
42	6	5	1	$Ka(6,2)$	$\surd$
50	7	0	7	Hoffman-Singleton graph	$\surd$
54	7	4	3	unknown	unknown
56	7	6	1	$Ka(7,2)$	$\surd$
72	8	7	1	$Ka(8,2)$	$\surd$
84	9	2	7	unknown	unknown
88	9	6	3	unknown	unknown
90	9	8	1	$Ka(9,2)$	$\surd$
104	10	3	7	unknown	unknown
108	10	7	3	Jorgensen graph	unknown
110	10	9	1	$Ka(10,2)$	$\surd$
126	11	4	7	unknown	unknown
130	11	8	3	unknown	unknown
132	11	10	1	$Ka(11,2)$	$\surd$
150	12	5	7	unknown	unknown
154	12	9	3	unknown	unknown
156	12	11	1	$Ka(12,2)$	$\surd$
176	13	6	7	unknown	unknown
180	13	10	3	unknown	unknown
182	13	12	1	$Ka(13,2)$	$\surd$
198	14	1	13	unknown	unknown

Table 7: Feasible values of the parameters for proper mixed Moore graphs of order up to 200.

5 Conclusion

In this survey we have presented results and research directions concerning the degree/diameter problem.

Here we give a list of some of the open problems in this area.

1. Does there exist a Moore graph of diameter 2 and degree 57?
2. At present we have only non-diregular examples of a digraph with $n = 49$, $d = 2$ and $k = 5$. Does there exist a diregular version of a digraph with the same parameters n, d, k?
3. Is $n_{d,k}$ monotonic in d and/or in k?
4. Find graphs (resp., digraphs) which have larger number of vertices than the currently largest known graphs (resp., digraphs).
5. Investigate the various restricted forms of the degree/diameter problem, for example, for vertex-transitive graphs, Cayley graphs, bipartite graphs.
6. Answer the question of Bermond and Bollobás (end of Section 3.2), which asks if, for each integer $c > 0$, there exist Δ and D, such that $n_{\Delta,D} \leqslant M_{\Delta,D} - c$.
7. Prove or disprove the conjecture of Bollobás (Subsection 3.3.1), stating that for each $\varepsilon > 0$, $n_{\Delta,D} > (1-\varepsilon)\Delta^D$, for sufficiently large Δ and D.
8. Is it true that $n_{\Delta,D} = vt_{\Delta,D}$, for infinitely many pairs of $\Delta \geqslant 3$ and $D \geqslant 2$?
9. Does there exist a radially Moore undirected graph for every diameter and degree?
10. Is there a mixed Moore graph of order 40, diameter 2, and such that each vertex is incident with 3 undirected edges and each vertex is the starting point of 3 directed arcs?
11. More generally, classify all proper mixed Moore graphs.
12. Prove the diregularity or otherwise of digraphs close to Moore bound for defect greater than one.
13. Prove or disprove the following generalisation of the result of Knor and Širáň from Subsection 3.4.6: For each surface $\mathcal{S}$ and for each $D \geqslant 2$, there exist a constant $\Delta_{\mathcal{S}}$ such that for each $\Delta \geqslant \Delta_{\mathcal{S}}$, we have $n_{\Delta,D}(\mathcal{S}) = n_{\Delta,D}(\mathcal{S}_0)$.
14. Motivated by the result of Šiagiová and Simanjuntak (Subsection 3.4.6), investigate the existence of the limit of $n_{\Delta,D}(\mathcal{S})/\Delta^{\lfloor D/2 \rfloor}$ as $\Delta \to \infty$.

15. Are there any bipartite graphs of defect 2 when the diameter is 3?

16. Classify graphs of defect 2 and diameter 2 or 3.

17. For $D \geqslant 4$ are there any graphs with defect 2, diameter D and odd degree?

18. Prove or disprove the nonexistence of all almost Moore digraphs of degree $\geqslant 3$ and diameter $\geqslant 5$.

19. Consider the degree/diameter problem as a subgraph within a given host architecture.

In conclusion, we would like to comment briefly on the relationships between the three parameters that have featured heavily in this survey; namely, order, degree and diameter. Throughout this survey, we have considered the degree/diameter problem, that is, maximising the order of a graph, resp., digraph. However, considering the three parameters of a graph: order, degree and diameter, there are two additional extremal problems that arise if we optimise in turn each one of the parameters while holding the other two parameters fixed, namely,

- *Order/degree problem*: Given natural numbers n and Δ, find the smallest possible diameter $D_{n,\Delta}$ in a graph of order n and maximum degree Δ.

- *Order/diameter problem*: Given natural numbers n and D, find the smallest possible maximum degree $\Delta_{n,D}$ in a graph of order n and diameter D.

The statements of the directed version of the problems differ only in that 'degree' is replaced by 'out-degree.

For both undirected and directed cases, most of the attention has been given to the degree/diameter problem, some attention has been received by the order/degree problem but the order/diameter problem has been largely overlooked so far. For more details concerning the three problems and their relationships, see [254, 269, 256].

Although we tried to include all references to the degree/diameter problem and other research related to the Moore bounds, it is quite likely that we have accidentally or out of ignorance left out some references that should have been included. We apologise for any such oversights. Fortunately, this is a dynamic survey and we will be updating it periodically. We will very much appreciate finding out about any omissions, as well as new results in the degree/diameter problem and related topics.

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References

[1] M. Aider, Réseaux d'interconnexion bipartis orientés, *Rev. Maghrébine Math.* **1** (1) (1992) 79–92.

[2] N. Alon, S. Hoory and N. Linial, The Moore bound for irregular graphs, *Graphs and Combinatorics* **18** (2002) 53–57.

[3] F. Aguiló-Gost, New dense families of triple loop networks. *Discrete Math.* **197-198** (1999) 15–27.

[4] F. Aguiló, M.A. Fiol and C. Garcia, Triple loop networks with small transmission delay, *Discrete Mathematics* **167/168** (1997) 3–16.

[5] S.B. Akers, On the construction of (d, k) graphs, *IEEE Trans. Electronic Computers* (Short Notes) **EC-14** (1965) 448.

[6] S.B. Akers and B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, *IEEE Transactions on Computers* **38** (1989) 555–566.

[7] I. Alegre, personal communication to M. Miller via K. Culik II and I. Fris, 1984.

[8] I. Alegre, M.A. Fiol and J.L.A. Yebra, Some large graphs with given degree and diameter, *J. Graph Theory* **10** (1986) 219–224.

[9] J. Allwright, New (Δ, D)-graphs discovered by heuristic search, *Discrete Applied Mathematics* **37/38** (1992) 3–8.

[10] F. Annexstein, M. Baumslag, A. L. Rosenberg, Group action graphs and parallel architectures, *SIAM J. Comput.* **19** (1990) 544–569.

[11] R.P. Anstee, An Analogue of Group Divisible Designs for Moore Graphs, *Journal of Combinatorial Theory B* **30** (1981) 11–20.

[12] G. Araujo, D. González, J. J. Montellano-Ballesteros, O. Serra, On upper bounds and connectivity of cages, *The Australasian Journal of Combinatorics* **38** (2007) 221–228.

[13] B. W. Arden and H. Lee, Analysis of chordal ring network, *IEEE Trans. Comput.* **30** (4) (1981) 291–295.

[14] L. Babai, P. Erdős, Representation of group elements as short products, *"Theory and practice of combinatorics"*, North Holland Math. Stud. 60, North Holland, Amsterdam (1982) 27–30.

[15] L. Babai, G. L. Hetyei, On the diameter of random Cayley graphs of the symmetric group, *Combin. Probab. Comput.* **1** (3) (1992) 201–208.

[16] L. Babai, W. M. Kantor, A. Lubotzky, Small-diameter Cayley graphs for finite simple groups, *European J. Combin.* **10** (6) (1989) 507–522.

[17] L. Babai, A. Seress, On the diameter of Cayley graphs of the symmetric group, *J, Combinat. Theory Ser A* **49** (1) (1988) 175–179.

[18] L. Babai, A. Seress, On the diameter of permutation groups, *European J. Combinat.* **13** (4) (1992) 231–243.

[19] M. Bachratý and J. Širáň, Brown graphs revisited, submitted.

[20] R. C. Baker, G. Harman, J. Pintz, The difference between consecutive primes II, *Proceedings of the London Mathematical Society. Third Series*, **83** (3) (2001) 532–562.

[21] M.C. Balbuena, A. Carmona, J. Fábrega and M.A. Fiol, On the order and size of s-geodetic digraphs with given connectivity, *Discrete Mathematics* **174** (1997) 19–27.

[22] M.C. Balbuena, J. Fàbrega, X. Marcote, I. Pelayo, Superconnected digraphs and graphs with small conditional diameters, *Networks* **39** (3) (2002) 153-160.

[23] E. Bannai and T. Ito, On finite Moore graphs, *J. Fac. Sci. Tokyo Univ.* **20** (1973) 191–208.

[24] E. Bannai and T. Ito, Regular graphs with excess one, *Discrete Mathematics* **37** (1981) 147–158.

[25] D. Barth and M.-C. Heydemann, A new digraph composition with applications to de Bruijn and generalised de Bruijn digraphs, *Discrete Applied Mathematics* **77** (1997) 99–118.

[26] R. Bar-Yehuda and T. Etzion, Connections between two cycles – a new design of dense processor interconnection networks, *Discrete Applied Mathematics* **37/38** (1992) 29–43.

[27] E.T. Baskoro, *Optimal Directed Networks*, Master Thesis, UNE, Armidale, Australia (1991).

[28] E.T. Baskoro and Amrullah, Almost Moore digraphs of degree 4 without selfrepeats, *Journal of the Indonesian Mathematical Society (MIHMI)*, **8** (1) (2002) 51 - 60.

[29] E.T. Baskoro, L. Branković, M. Miller, J. Plesník, J. Ryan and J. Širáň, Large digraphs with small diameter: A voltage assignment approach, *Journal of Combinatorial Mathematics and Combinatorial Computing* **24** (1997) 161–176.

[30] E.T. Baskoro, Y.M. Cholily and M. Miller,Structure of repeat cycles in almost Moore digraph with selfrepeats and diameter 3, *Bulletin of ICA* **46** (2006) 99-109.

[31] E.T. Baskoro, Y.M. Cholily and M. Miller, Enumeration of vertex orders of almost Moore digraphs with selfrepeats, *Discrete Mathematics* **308** (1) (2008) 123-128.

[32] E.T. Baskoro and M. Miller, On the construction of networks with minimum diameter, *Australian Computer Science Communications* **C 15** (1993) 739–743.

[33] E.T. Baskoro and M. Miller, A procedure for constructing a minimum-diameter digraph, *Bulletin of ICA* **17** (1996) 8–14.

[34] E.T. Baskoro, M. Miller, J. Plesník and Š. Znám, Regular digraphs of diameter 2 and maximum order, *the Australasian Journal of Combinatorics* **9** (1994) 291–306. See also Errata, **13** (1995).

[35] E.T. Baskoro, M. Miller, J. Plesník and Š. Znám, Digraphs of degree 3 and order close to Moore bound, *J. Graph Theory* **20** (1995) 339–349.

[36] E.T. Baskoro, M. Miller and J. Plesník, On the structure of digraphs with order close to the Moore bound, *Graphs and Combinatorics* **14** (2) (1998) 109–119.

[37] E.T. Baskoro, M. Miller and J. Plesník, Further results on almost Moore digraphs, *Ars Combinatoria* **56** (2000) 43–63.

[38] E.T. Baskoro, M. Miller, J. Širáň and M. Sutton, Complete characterization of almost Moore digraphs of degree three, *Journal of Graph Theory* **48** (2) (2005) 112-126.

[39] E.T. Baskoro, M. Miller and J. Širáň, A remark on almost Moore digraphs of degree three, *Acta Math. Univ. Comenianae* **LXVI** (2) (1997) 285–291.

[40] D. Bauer, F.T. Boesch, C. Suffel and R.Van Slyke, On the validity of a reduction of reliable networks design to a graph extremal problem, *IEEE Trans. on Circuits and Systems* **CAS-34** (1987) 1579–1581.

[41] C.T. Benson, Minimal regular graphs of girth eight and twelve, *Canad. J. Math* **18** (1966) 1091–1094.

[42] C.T. Benson and N.E. Losey, On a graph of Hoffman and Singleton, *J. Combinatorial Theory Ser. B* **11** (1971) 67–79.

[43] J.-C. Bermond and B. Bollobás, The diameter of graphs: A survey, *Congressus Numerantium* **32** (1981) 3–27.

[44] J.-C. Bermond, F. Comellas, D.F. Hsu, Distributed loop computer networks: a survey *J. Parallel and Distributed Computing* **24** (1995) 2–10.

[45] J.-C. Bermond, C. Delorme and G. Farhi, Large graphs with given degree and diameter III, *Annuals of Discrete Mathematics* **13** (1982) 23–32.

[46] J.-C. Bermond, C. Delorme and G. Farhi, Large graphs with given degree and diameter II, *Journal of Combinatorial Theory* (B) **36** (1984) 32–48.

[47] J.-C. Bermond, C. Delorme and J.-J. Quisquater, Grands graphes non dirigés de degré et diamètre fixés, *Ann. Discrete Math.* **17** (1982) 65–73.

[48] J.-C. Bermond, C. Delorme and J.-J. Quisquater, Tables of large graphs with given degree and diameter, *Information Processing Letters* **15** (1982) 10–13.

[49] J.-C. Bermond, C. Delorme and J.-J. Quisquater, Grands graphes de degré et diametre fixés, *Combinatorial Mathematics* (Marseille-Luminy, 1981) 65–73; *North-Holland Math. Stud.* **75** North-Holland, Amsterdam-New York (1983).

[50] J.-C. Bermond, C. Delorme and J.-J. Quisquater, Tables of large (Δ, D)-graphs, *Discrete Applied Mathematics* **37/38** (1992) 575–577.

[51] J.-C. Bermond, C. Delorme and J.-J. Quisquater, Strategies for interconnection networks: some methods from graph theory, *Journal of Parallel and Distributed Computing* **3** (1986) 433–449.

[52] J.-C. Bermond, F.O. Ergincan, Bus interconnection networks, Discrete Appl. Math. **68** (1996) 1–15.

[53] J.-C. Bermond, P. Hell, On even factorizations and the chromatic index of the Kautz and de Bruijn digraphs, *J. Graph Theory* **17** (1993) 647–655.

[54] J.-C. Bermond, P. Hell, J.-J. Quisquater, Construction of large packet radio networks, *Parallel Processing Letters* **2** (1992) 3–12.

[55] J.-C. Bermond, N. Homobono and C. Peyrat, Large fault-tolerant interconnection networks, *Graphs and Combinatorics* **5** (1989) 107–123.

[56] J.-C. Bermond, N. Homobono and C. Peyrat, Connectivity of Kautz networks, *Discrete Mathematics* **114** (1–3) (1993) 51–62.

[57] J.-C. Bermond, C. Peyrat, De Bruijn and Kautz networks, a competitior for the hypercube? *Proc. 1st European Workshop on Hypercubes and Distributed Computers*, Rennes (1989) 279–293.

[58] N.I. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Second Edition, Great Britain (1993).

[59] N.I. Biggs, Girth, valency and excess, *Linear Algebra Appl.* **31** (1980) 55–59.

[60] N. L. Biggs, Constructions for cubic graphs with large girth, *The Electronic Journal of Combinatorics* **5** (1998) 1–25.

[61] N.I. Biggs and T. Ito, Graphs with even girth and small excess, *Math. Proc. Cambridge Phil. Soc.* **88** (1) (1980) 1–10.

[62] B. Bollobás, *Extremal Graph Theory*, Academic Press, New York (1978).

[63] B. Bollobás, *Random Graphs*, volume 73 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 2nd edition (2001).

[64] J. Bond and C. Delorme, New large bipartite graphs with given degree and diameter, *Ars Combinatoria* **25C** (1988) 123–132.

[65] J. Bond and C. Delorme, A note on partial Cayley graphs, *Discrete Mathematics* **114** (1–3) (1993) 63–74.

[66] J. Bosák, Cubic Moore graphs, *Mat. Časopis Slovenskej Akadémie Vied* **20** (1970) 72–80.

[67] J. Bosák, Directed graphs and matrix equations, *Math. Slovaca* **28** (2) (1978) 189–202.

[68] J. Bosák, Partially directed Moore graphs, *Math. Slovaca* **29** (2) (1979) 181–196.

[69] J. Bosák, A. Kotzig and Š. Znám, Strongly geodetic graphs, *Journal of Combinatorial Theory* **5** (1968) 170–176.

[70] R.C. Bose and T.A. Dowling, A generalization of Moore graphs of diameter two, *J. Comb. Th.* **11** (1971) 213–226.

[71] L. Branković, M. Miller, J. Plesník, J. Ryan, J. Širáň, Large graphs with small degree and diameter: A voltage assignment approach, *Australas. J. Combin.* **18** (1998) 65–76.

[72] L. Branković, M. Miller, J. Plesník, J. Ryan, J. Širáň, A note on constructing large Cayley graphs of given degree and diameter by voltage assignments, *Electronic J. Combin.* **5** (1) (1998) paper R9.

[73] W.G. Bridges and S. Toueg, On the impossibility of directed Moore graphs, *J. Combinatorial Theory B* **29** (1980) 339–341.

[74] G. Brinkmann, B. D. McKay, C. Saager, The smallest cubic graphs of girth 9, *Combinatorics Probability and Computing* **4** (1995) 317–330.

[75] H. J. Broersma, A. A. Jagers, The unique 4-regular graphs on 14 and 15 vertices with diameter 2, *Ars Combinatoria* **25** (C) (1988) 55–62.

[76] W.G. Brown, On graphs that do not contain a Thompsen graph, *Canad. Math. Bull.* **9** (1966) 281–285.

[77] N.G. de Bruijn, A combinatorial problem, *Koninkl. Nederl. Acad. Wetensch. Proc. Ser* A **49** (1946) 758–764.

[78] D. Buset, Maximal cubic graphs with diameter 4, *Discrete Applied Mathematics* **101** (1-3) (2000) 53–61.

[79] R.W. Buskens and R.G. Stanton, Generalized Moore graphs on twelve and thirteen vertices, *Ars Combinat.* **23B** (1987) 87–132.

[80] R.W. Buskens, M.J. Rogers and R.G. Stanton, A census of tetravalent generalized Moore networks, *Congressus Numerantium* **52** (1986) 255–296.

[81] P.J. Cameron, Automorphism groups of graphs, in: *Selected Topics in Graph Theory* **2** (1983) Acad. Press.

[82] L. Campbell, Dense group networks, *Discrete Applied Mathematics* **37/38** (1992) 65–71.**IEEE Transactions on Computers 41** (2) (1992) 218–220.

[83] L. Campbell, G.E. Carlsson, M.J. Dinneen, V. Faber, M.R. Fellows, M.A. Langston, J.W. Moore, A.P. Mullhaupt, H.B. Sexton, Small diameter symmetric networks from linear groups,

[84] E.A. Canale and J. Gómez, Asymptotically large (Δ, D)-graphs, *Discrete Applied Math.* **152** (1-3) (2005) 89–108.

[85] E.A. Canale and J. Gómez, Superfluous edges and exponential expansions of de-Bruijn and Kautz digraphs, *Discrete Applied Math.* **138** (2004), 303-331.

[86] C. Capdevila, J. Conde, G. Exoo, J. Gimbert, N. Lopez, Ranking Measures for Radially Moore Graphs, *Networks* **56** (4) (2010) 255–262.

[87] G.E. Carlsson, J. Cruthirds, H. Sexton and C. Wright, Interconnection networks based on generalization of cube-connected cycles, *IEEE Trans. Comput.* **C-34** (8) (1985) 769–772.

[88] J.D.P Carracedo, J.A.M. Pereda, A.G. Marcos, Optomization method in a WDM static routed optical network, *Optica Pura y Aplicada* **42** (4)(2009) 245–250.

[89] V.G. Cerf, D.D. Cowan, R.C. Mullin and R.G. Stanton, Computer networks and generalized Moore graphs, *Congr. Num.* **9** (1973) 379–398.

[90] V.G. Cerf, D.D. Cowan, R.C. Mullin and R.G. Stanton, Trivalent generalized Moore networks on sixteen nodes, *Utilitas Math.* **6** (1974) 259–283.

[91] Y.M. Cholily, On a conjecture on the existence of almost Moore digraphs, *Adv. and Applications in Discrete Math.* **8**(1) (2011) 57–64.

[92] Y.M. Cholily, E.T. Baskoro and S. Uttunggadewa, Some conditions for the existence of (d, k)-digraphs containing selfrepeat, *LNCS* **3330** (2005) 55–61.

[93] D.V. Chudnovsky, G.V. Chudnovsky and M.M. Denneau, Regular graphs with small diameter as models for interconnection networks, *Proc. 3rd Internat. Conf. on Supercomputing ICS'88*, Boston (1988) 232–239.

[94] F.R.K. Chung, Diameters of communication networks, in: *AMS Short Course on the Mathematics of Information Processing* **34**, Amer. Math. Soc., Providence, RI (1986) 1–18.

[95] F.R.K. Chung, Diameters of graphs: old problems and new results, *Congr. Numer.* **60** (1987) 295–317.

[96] F.R.K. Chung, Diameters and eigenvalues, *Journal of American Mathematical Society* **2** (2) (1989) 187–196.

[97] F.R.K. Chung, C. Delorme and P. Solé, Multidiameters and multiplicities, *European J. Combin.* **20** (1999) 629–640.

[98] Chung, F.R.K.; Faber, V.; Manteuffel, Thomas A. An upper bound on the diameter of a graph from eigenvalues associated with its Laplacian. *SIAM J. Discrete Math.* **7** (1994) No.3, 443–457.

[99] F. Comellas, The degree/diameter problem for graphs, *http://www-mat.upc.es/grup.de.grafs/table.g.html.*

[100] F. Comellas and M.A. Fiol, Using simulated annealing to design interconnection networks, *Technical Report* DMAT-05-0290, Universitat Politécnica de Catalunya (1990).

[101] F. Comellas and M.A. Fiol, Vertex symmetric digraphs with small diameter, *Discrete Applied Mathematics* **58** (1995) 1–12.

[102] F. Comellas, M.A. Fiol and J. Gómez, On large vertex symmetric 2-reachable digraphs, *Parallel Processing Letters* **4** (1994) 379–384.

[103] F. Comellas and J. Gómez, New large graphs with given degree and diameter, *Graph Theory, Combinatorics and Algorithms* **1,2**: Proc. 7th Quadrennial Int'l Conf. on the theory and Appl. of Graphs, Kalamazoo (MI, USA) (1992), edited by Y. Alavi and A. Schwenk (1995) 221–233.

[104] F. Comellas, P. Morillo and M.A. Fiol, Diameter and mean distance of bipartite digraphs related to plane tessellations, *Ars Combin.* **29** B (1990) 245–254.

[105] J. Conde, J. Gimbert, On the existence of graphs of diameter two and defect two. *Discrete Mathematics* **309** (10) (2009) 3166–3172.

[106] J. Conde, J. Gimbert, J. González, M. Miller, J.M. Miret, On the nonexistence of almost Moore digraphs, preprint.

[107] J. Conde, J. Gimbert, J. González, J.M. Miret, R. Moreno, On the nonexistence of almost Moore digraphs of diameter four, *The Electronic Journal of Combinatorics* **20** (2013) P75.

[108] J. Conde, J. Gimbert, J. González, J.M. Miret, R. Moreno, Nonexistence of almost Moore digraphs of diameter three, *The Electronic Journal of Combinatorics* **15** (2008) R87 1–14.

[109] M. Conder, Trivalent (cubic) symmetric graphs on up to 2048 vertices, http://www.math.auckland.ac.nz/conder/symmcubic2048list.txt (2006).

[110] C. Von Conta, Torus and other networks as communication networks with up to some hundred points, *IEEE Transactions on Computers* **32** (7) (1983) 657–666.

[111] Dafik, M. Miller, C. Iliopoulos, Z. Ryjacek, On diregularity of digraphs of defect at most two, *Journal of Combinatorial Mathematics and Combinatorial Computing* **71** (2009) 21-38.

[112] S. Damaj, T. Goubier, F. Blanc, B. Pottier, A heuristic (delta,D) digraph to interpolate between Hypercube and de Bruijn topologies for future on Chip Interconnection Networks, *2009 International Conference on Parallel Processing Workshops* (2009) 492–498.

[113] S. Damaj, T. Goubier, B. Pottier, SWH: A Small-World Heuristic for Future Network, *IEEE PACRIM'09* (2009), 234–238.

[114] R.M. Damerell, On Moore graphs, *Proc. Cambridge Phil. Soc.* **74** (1973) 227–236.

[115] R.M. Damerell, On Moore geometries II, *Math. Proc. Camb. Phil. Soc.* **90** (1981) 33–40.

[116] R.M. Damerell and M.A. Georgiacodis, On Moore geometries I, *J. London Math. Soc.* **23** (2) (1981) 1–9.

[117] A. Dekker, H. Perez-Roses, G. Pineda-Villavicencio and P. Watters, The maximum degree & diameter-bounded subgraph and its applications, *Journal of Mathematical Modelling and Algorithms* **11** (3) (2012) 249-268,

[118] C. Delorme, Grands graphes de degré et diamètre donnés, *European J. Combin.* **6** (1985) 291–302.

[119] C. Delorme, Large bipartite graphs with given degree and diameter, *J. Graph Theory* **8** (1985) 325–334.

[120] C. Delorme, Examples of products giving large graphs with given degree and diameter, *Discrete Applied Mathematics* **37/38** (1992) 157–167.

[121] C. Delorme, private communication (2008).

[122] C. Delorme and G. Farhi, Large graphs with given degree and diameter-part I, *IEEE Transactions on Computers* **C-33** (1984) 857–860.

[123] C. Delorme and J. Gómez, Some new large compound graphs, *European Journal of Combinatorics* **23** (5) (2002) 539–547.

[124] C. Delorme, L. K. Jørgensen, M. Miller, and G.Pineda-Villavicencio, On bipartite graphs of diameter 3 and defect 2, *Journal of Graph Theory* **61** (4) (2009) 271–288.

[125] C. Delorme, L. K. Jørgensen, M. Miller, and G. Pineda-Villavicencio, On bipartite graphs of defect 2, *European Journal of Combinatorics* **30** (4) (2009) 798–808.

[126] C. Delorme and G. Pineda-Villavicencio, On graphs with cyclic defect or excess. *The Electronic Journal of Combinatorics* **17** (1) (2010) R143.

[127] C. Delorme and J.J. Quisquater, Some new large compound graphs, *Internal Report* **317**, LRI, December 1986.

[128] M.J. Dinneen, *Algebraic methods for efficient network constructions*, Master's Thesis, Department of Computer Science, University of Victoria, Canada (1991).

[129] M.J. Dinneen and P.R. Hafner, New results for the degree/diameter problem, *Networks* **24** (1994) 359–367.

[130] A.C. Doring, Analysis of Network Topologies and Fault-Tolerant Routing Algorithms using Binary Decision Diagrams, *2010 IEEE International Symposium on Parallel & Distributed Processing, Workshops (IPDPSW)* (2010) 1–5.

[131] K.W. Doty, New designs for dense processor interconnection networks, *IEEE on Transactions on Computers* **C-33** (5) (1984) 447–450.

[132] R. Dougherty and V. Faber, The degree-diameter problem for several varieties of Cayley graphs, I: The Abelian case, *SIAM J. Discrete Math.* **17** (3) (2004) 478–519.

[133] R.D. Dutton and R.C. Brigham, Edges in graphs with large girth, *Graphs and Combinatorics* **7** (1991) 315–321.

[134] B. Elspas, Topological constraints on interconnection-limited logic, *Proc. IEEE Fifth Symposium on Switching Circuit Theory and Logical Design*, IEEE **S–164** (1964) 133-147.

[135] P. Erdős, S. Fajtlowicz and A.J. Hoffman, Maximum degree in graphs of diameter 2, *Networks* **10** (1980) 87–90.

[136] P. Erdős, A. Rényi, and V. T. Sós, On a problem of graph theory, *Studia Scientiarum Mathematicarum Hungarica. A Quarterly of the Hungarian Academy of Sciences* **1** (1966) 215–235.

[137] P. Erdős and H. Sachs, Regülare graphen gegebener taillenweite mitminimaler knotenzahl, *Wiss. Z. Uni. Halle (Math. Nat.)* **12** (1963) 251–257.

[138] M. Espona and O. Serra, Cayley digraphs based on the networks, *SIAM J. Discrete Math.* **11** (2) (1998) 305–317.

[139] G. Exoo, A Family of Graphs and the Degree/Diameter Problem, *J. Graph Theory* **37** (2) (2001) 118–124.

[140] G. Exoo, Voltage graphs, group presentations and cages, *Electron. J. Combinat.* **11** (2004) Note 2, 1–7.

[141] G. Exoo, J. Gimbert, N. Lopez, J. Gomez, Radial Moore graphs of radius three, *Discrete Applied Matehmatics* **160** (2012), 1507-1512.

[142] G. Exoo and R. Jajcay, Dynamic cage survey, *The Electronic Journal of Combinatorics* DS16 (2011) 1–54.

[143] V. Faber and J.W. Moore, High-degree low-diameter interconnection networks with vertex symmetry: the directed case. *Technical Report* LA-UR-88-1051, Los Alamos National Laboratory, Los Alamos, New Mexico (1988).

[144] V. Faber, J.W. Moore and W.Y.C Chen, Cycle prefix digraphs for symmetric interconnection networks, *Networks* **23** (1993) 641–649.

[145] J. Fàbrega and M.A. Fiol, Maximally connected digraphs, *Journal of Graph Theory* **13** (1989) 657-668.

[146] S. Fajtlowicz, Graphs of diameter 2 with cyclic defect, *Colloquium Mathematicum* **51** (1987) 103–106.

[147] C. Fan and A.J. Schwenk, Structure of the Hoffman-Singleton Graph, *Congressus Numerantium* **94** (1993) 3–8.

[148] W. Feit and G. Higman, The nonexistence of certain generalized polygons, *Journal of Algebra* **1** (1964) 114–131.

[149] M. Fellows, P. Hell and K. Seyffarth, Large planar graphs with given diameter and maximum degree, *Discrete Appl. Mathematics* **61** (1995) 133–153.

[150] M. Fellows, P. Hell and K. Seyffarth, Constructions of large planar networks with given degree and diameter, *Networks* **32** (1998) 275–281.

[151] M.R. Fellows, M.A. Langston, J.W. Moore, A.P. Mullhaupt and H.B. Sexton, Small diameter symmetric networks from linear groups, *IEEE Transactions on Computers* **41** (2) (1992) 218–220.

[152] R. Feria-Puron, M. Miller and G. Pineda-Villavicencio, On large bipartite graphs of diameter 3, Discrete Mathematics **313** (4) (2013) 381–390.

[153] R. Feria-Purón, M. Miller, and G. Pineda-Villavicencio, On graphs of defect at most 2, *Discrete Applied Mathematics* **159** (13) (2011) 1331–1344.

[154] R. Feria-Purón and G. Pineda-Villavicencio, On bipartite graphs of defect at most 4, *Discrete Applied Mathematics* **160** (2012) 140–154.

[155] M.A. Fiol, Connectivity and superconnectivity of large graphs and digraphs, Twelfth British Combinatorial Conference (Norwich, 1989), *Ars Combinatoria* **29B** (1990) 5–16.

[156] M.A. Fiol, The connectivity of large digraphs and graphs, *J. Graph Theory* **17** (1993) 31–45.

[157] M.A. Fiol, I. Alegre and J.L.A. Yebra, Line digraph iterations and the (d,k) problem for directed Graphs, *IEEE Transactions on Computers* **C-33** (1984) 400–403.

[158] M.A. Fiol, J. Fàbrega and M. Escudero, Short paths and connectivity in graphs and digraphs, *Ars Combinatoria* **29B** (1990) 17–31.

[159] M.A. Fiol and J. Fàbrega, Some compound graphs, *Stochastica* **7** (2) (1983) 137–143.

[160] M.A. Fiol and J. Gimbert, On almost Moore bipartite digraphs with odd diameter, *Congr. Numer.* **152** (2001), 169-187.

[161] M.A. Fiol, J. Gimbert, J. Gómez and Y. Wu, On Moore bipartite digraphs, *J. Graph Theory* **43** (3) (2003) 171-181.

[162] M.A. Fiol, J. Gimbert and M. Miller, Multipartite Moore digraphs, *Linear Algebra and its Applications* **419** (2006) 234–250.

[163] M.A. Fiol, A.S. Lladó and J.L. Villar, Digraphs on alphabets and the (d,N) digraph problem, *Ars Combinatoria* **25C** (1988) 105–122.

[164] M.A. Fiol and A.S. Lladó, The partial line digraph technique in the design of large interconnection networks, *IEEE Transactions on Computers* **41** (1992) 848–857.

[165] M.A. Fiol and J.L.A. Yebra, Dense bipartite digraphs, *J. Graph Theory* **14** (1990) 687–700.

[166] M. A. Fiol, J. L. A. Yebra, J. Fabrega, Sequence graphs and interconnection networks, *Ars Combinatoria* **16A** (1983) 7–13.

[167] M. Fried and J.H. Smith, Primitive groups, Moore graphs, and rational curves, *Michigan Math. J.* **19** (1972) 341–346.

[168] H. Friedman, A design for (d,k) Graphs, *IEEE Trans. Electronic Computers* **EC-15** (1996) 253–254.

[169] H. Friedman, On the impossibility of certain Moore graphs, *J. Combinatorial Theory Ser. B* **10** (1971) 245–252.

[170] F.J. Fuglister, On finite Moore geometries, *J. Comb. Th.* A **23** (1977) 187–197.

[171] F.J. Fuglister, The nonexistence of Moore geometries of diameter 4, *Discrete Mathematics* **45** (1983) 229238.

[172] C. Garcia and C. Peyrat, Large Cayley graphs on an abelian group, *Discrete Appl. Mathematics* **75** (2) (1997) 125–133.

[173] M. Garey and D. Johnson. *Computers and Intractability. A Guide to the Theory of NP-completeness*. Freeman and Co., 1979.

[174] J. Gimbert, On the existence of (d,k)-digraphs, 16th British Combinatorial Conference (London, 1997), *Discrete Mathematics* **197/198** (1999) 375–391.

[175] J. Gimbert, On digraphs with unique walks of closed lengths between vertices, *Australasian Journal of Combinatorics* **20** (1999) 77–90.

[176] J. Gimbert, Enumeration of almost Moore digraphs of diameter two, *Discrete Math.* **231** (2001) 177–190.

[177] C. Godsil and G. Royle, *Algebraic Graph Theory*, Volume 207 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2001.

[178] J. Gómez, Large vertex-symmetric digraphs, *Networks* **50** (4) (2007) 241–250.

[179] J. Gómez, On large vertex-symmetric digraphs, *Discrete Math.* **309** (6) (2009) 1213–1221.

[180] J. Gómez, Generalized compound graphs, *Ars Combinatoria* **29B** (1990) 33–53.

[181] J. Gómez, On large $(\Delta, 6)$-graphs, *Networks* **46** (2005) 82–87.

[182] J. Gómez, E.A. Canale and X. Munoz, On the unilateral (Δ, D^*)-problem, *Networks* **34** (2000), 164-171.

[183] J. Gómez, E.A. Canale and X. Munoz, Unilaterally connected large digraphs and generalized cycles, *Networks* **42** (2000), 181-188.

[184] J. Gómez and M. Escudero, Optimal edge coloring of large graphs, *Networks* **34** (1999) 61–65.

[185] J. Gómez and M.A. Fiol, Dense compound graphs, *Ars Combinatoria* **20A** (1985) 211–237.

[186] J. Gómez, M.A. Fiol and O. Serra, On large (Δ, D)-graphs, *Discrete Mathematics* **114** (1993) 219–235.

[187] J. Gómez, M.A. Fiol and J.L.A. Yebra, Graphs on alphabets as models for large interconnection networks, *Discrete Applied Mathematics* **37/38** (1992) 227–243.

[188] J. Gómez and M. Miller, Two new families of compound graphs, *Networks* **47** (3) (2006) 140-146.

[189] J. Gómez, P. Morillo and C. Padró, Large (d, D, D', s)-bipartite digraphs, *Discrete Applied Mathematics* **59** (2) (1995) 103–114.

[190] J. Gómez, C. Padró and S. Perennes, Large generalized cycles, *Discrete Applied Mathematics* **89** (1998) 107–123.

[191] J. Gómez, I. Pelayo and C. Balbuena, New large graphs with given degree and diameter six, *Networks* **34** (2) (1999) 154–161.

[192] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley, New York (1987).

[193] D. Guo, T. Chen, D. Li, M Li, Y. Liu, G. Chen, Expansible and Cost-Effective Network Structures for Data Centers Using Dual-Port Servers, *IEEE Transactions on Computers* (2012) 1–12.

[194] D. Guo, J. Wu, H. Chen, X. Luo, Moore: An Extendable Peer-to-Peer Network Based on Incomplete Kautz Digraph with Constant Degree, *26th IEEE INFOCOM 2007 Proceedings* (2007) 821–829.

[195] D. Guo, Y. Liu, X. Li, BAKE: A Balanced Kautz Tree Structure for Peer-to-Peer Networks, *IEEE INFOCOM 2008 Proceedings* (2008) 351–355.

[196] D. Guo, H. Chen, Y. He, H. Jin, C. Chen, H. Chen, Z. Shu, G. Huang, KCube: A novel architecture for interconnection networks, *Information Processing Letters* **110** (2010) 821–825.

[197] D. Guo, J. Wu, Y. Liu, H. Jin, H. Chen, T. Chen, Quasi-Kautz Digraphs for Peer-to-Peer Networks, *IEEE Transactions on Parallel and Distributed Systems* **22** (2011) 1042–1055.

[198] D. Guo, Y. Liu, H. Jin, Z. Liu, W. Zhang, H. Liu, Theory and network application of balanced kautz tree structures, *ACM Transactions on Internet Technology (TOIT)* **12** (1) (2012).

[199] W. H. Haemers, *Eigenvalue technique in design and graph theory*, Mathematical Centre Tracts **121**, Matematisch Zentrum, Amsterdam (1980).

[200] P. R. Hafner, Large Cayley graphs and digraphs with small degree and diameter, *Computational Algebra and Number Theory* (W. Bosma and van der Poorten, eds.) Kluwer, Amsterdam (1995) 291–302.

[201] P. R. Hafner, On the graphs of McKay-Miller-Širáň, *Proc. 8th Intl. Colloquium on Structural Information and Communication Complexity (SIROCCO 8), Proceedings in Informatics* **11** (F. Comellas, J. Fabrega, P. Fraigniaud, eds.), Carleton Scientific (2001) 209–215.

[202] P. R. Hafner, The Hoffman-Singleton graph and its automorphisms, *J. Algebraic Combin.* **18** (2003) 7–12.

[203] P. R. Hafner, Geometric realisation of the automorphism group of the graphs of McKay-Miller-Širáň, *J. Combin. Theory* Ser. B **90** (2) (2004) 223–232.

[204] P. Hell and K. Seyffarth, Largest planar graphs of diameter two and fixed maximum degree, *Discrete Mathematics* **111** (1993) 313–332.

[205] A.J. Hoffman and R.R. Singleton, On Moore graphs with diameter 2 and 3, *IBM J. Res. Develop.* **4** (1960) 497–504.

[206] N. Homobono and C. Peyrat, Connectivity of Imase and Itoh digraphs, *IEEE TRansactions on Computers* **37** (1988) 1459–1461.

[207] M. N. Huxley, On the difference between consecutive primes, *Invent. Math.* **15** (1972) 164–170.

[208] M. Imase and M. Itoh, Design to minimize diameter on building-block network, *IEEE Trans. on Computers* **C-30** (1981) 439–442.

[209] M. Imase and M. Itoh, A design for directed graphs with minimum diameter, *IEEE Trans. on Computers* **C-32** (1983) 782–784.

[210] M. Imase, T. Soneoka and K. Okada, Connectivity of regular directed graphs with small diameter, *IEEE Transactions on Computers* **C-34** (1985) 267–273.

[211] H. Iswadi and E.T. Baskoro, On (4,2)-digraphs containing a cycle of length 2, *Bull. Malaysian Math. Sci. Soc.* (2) **23** (1) (2000) 79–91.

[212] L.O. James, A combinatorial proof that the Moore (7,2) graph is unique, *Utilitas Math.* **5** (1974) 79–84.

[213] R. Jajcay, M. Mačaj and J. Širáň, The vertex-transitive Moore bound, in preparation.

[214] J. Jaros, V. Dvorak, An Evolutionary Design Technique for Collective Communications on Optimal Diameter-Degree Networks, *Proceedings of ACM GECCO'08* (2008) 1539–1546.

[215] D. Johnson, The NP-completeness column: An ongoing guide, *J. Algorithms* 6 (1985) 145–159.

[216] L. K. Jørgensen, Nonexistence of certain cubic graphs with small diameters, *Discrete Mathematics* **114** (1993) 265–273.

[217] L.K. Jørgensen, Diameters of cubic graphs, *Discrete Appl. Mathematics* **37/38** (1992) 347–351.

[218] L.K. Jørgensen, personal communication, 2012.

[219] S. Jung-hyun, L. HyeongOk, J. Moon-suk, Petersen-Torus Network for Multicomputer Systems, *IEEE 4th Conference on Networked Computing and Advanced Information Management* (2008) 567–571.

[220] W. M. Kantor, Some large trivalent graphs having small diameters, *Discrete Applied Math.* **37/38** (1992) 353–357.

[221] R. M. Karp. Reducibility Among Combinatorial Problems, *In R. E. Miller, J. W. Thatcher (eds), Complexity of Computer Computations* (1972) 85–103.

[222] W.H. Kautz, Bounds on directed (d, k) graphs, *Theory of cellular logic networks and machines*, AFCRL-68-0668 Final Report (1968) 20–28.

[223] W.H. Kautz, Design of optimal interconnection networks for multiprocessors, *Architecture and Design of Digital Computers*, NATO Advanced Summer Institute (1969) 249–272.

[224] M. Knor, A note on radially Moore digraphs, *IEEE Trans. Comput.* **45** (1996) 381–383.

[225] M. Knor, Small radial Moore graphs of radius 3, in *Extended Abstracts of IWONT 2010*, Barcelona (2010) 159–162.

[226] M. Knor and J. Širáň, Extremal graphs of diameter two and given maximum degree, embeddable in a fixed surface, *Journal of Graph Theory* **24** (1997) 1–8.

[227] M. Knor, private communication (2007).

[228] I. Korn, On (d, k) graphs, *IEEE Trans. Electronic Computers* **EC-16** (1967) 90.

[229] F. Kramer, Relations between the diameter of a graph and the degrees of the nodes of a graph (Romanian), *Rev. Anal. Numer. Teoria Aproximaţiei* **1** (1972) 125–131.

[230] K. Kurosawa and S. Tsujii, Considerations on diameter of communication networks, *Electronics and Communications in Japan* **64A** (4) (1981) 37–45.

[231] S. Lakshmivarahan, J.S. Jwo and S.K. Dhall, Symmetry in interconnection networks based on Cayley graphs of permutation groups: A survey, *Parallel Computing* **19** (1993) 361–407.

[232] C. W. H. Lam, L. Thiel, and S. Swiercz, The nonexistence of finite projective planes of order 10, *Canadian Journal of Mathematics* **41** (1989) 1117–1123.

[233] H.W. Lenstra Jr. and B. Poonen, *Personal communication* via Joan Gimbert (1998).

[234] D. Li, X. Lu, J. Wu, FISSIONE: A Scalable Constant Degree and Low Congestion DHT Scheme Based on Kautz Graphs, *Proceedings of IEEE INFOCOM* (2005).

[235] H. Li, Z. Shen, A Kautz-based Real-Time and Energy-Efficient Wireless Sensor and Actuator Network, *2012 IEEE 32nd International Conference on Distributed Computing Systems (ICDCS)* (2012) 62–71.

[236] R. Lipton and R. E. Tarjan, A separator theorem for planar graphs, *SIAM J. Appl. Math.* **36** (2) (1979) 177–189.

[237] A.S. Llado, J. Villar, M.A. Fiol, A new family of digraphs om alphabets, *COMBINATORICS '88, Proceedings of the International Conference on Incidence Geometries and Combinatorial Structures*, A. Barlotti et al. (Eds.), Mediterranean Press (1991) 181–193.

[238] N. López, *Contribución al estudio de excentricidades en grafos dirigidos*, PhD thesis, Department of mathematics, University of Lleida, Spain, (2008).

[239] E. Loz, H. Pérez-Rosés, and G. Pineda-Villavicencio, combinatoricswiki.org.

[240] E. Loz, H. Pérez-Rosés, and G. Pineda-Villavicencio, Properties of finite metabelian groups for algebraic constructions, (2012) preprint.

[241] E. Loz and G. Pineda-Villavicencio, New benchmarks for large-scale networks with given maximum degree and diameter, *The Computer Journal* **53** (7) (2010) 1092–1105.

[242] E. Loz and J. Širáň, New record graphs in the degree-diameter problem, *The Australasian Journal of Combinatorics* **41** (2008) 63–80.

[243] A. Lubotzky and R. Meshulam, A Moore bound for simplicial complexes, *Bull. London Math. Soc.* (2007) 1–6.

[244] H. Macbeth, J. Šiagiová, and J. Širáň, Cayley graphs of given degree and diameter for cyclic, abelian, and metacyclic groups, *Discrete Mathematics* **312** (1) (2012) 94 – 99.

[245] H. Macbeth, J. Šiagiová, J. Širáň and T. Vetrík, Large Cayley graphs and vertex-transitive non-Cayley graphs of given degree and diameter, *J. Graph Theory* **64** (2010) 87–98.

[246] M. Mačaj and J. Širáň, Search for properties of the missing Moore graph, *Linear Algebra and its Applications* **432** (2010) 2381–2398.

[247] A.A. Makhnev and D.V. Paduchikh, Automorphisms of Aschbacher graphs, *Algebra Logic* **40** (2) (2001) 69–74.

[248] B. D. McKay, personal communication, 2006.

[249] B.D. McKay, M. Miller and J. Širáň, A note on large graphs of diameter two and given maximum degree, *J. Combin. Theory Ser. B* **74** (1) (1998) 110–118.

[250] B. D. McKay, W. Myrvold, and J. Nadon, Fast backtracking principles applied to find new cages, In *Proccedings of 9th Annual ACM-SIAM Symposium on Discrete Algorithms*, San Francisco (1998) 188–191.

[251] B. D. McKay and R.G. Stanton, The current status of the generalized Moore graph problem, *Combinatorial Math.* **VI** (1978) 21–31.

[252] G. Memmi, and Y. Raillard, Some new results about the (d, k) graph problem, *IEEE Trans. on Computers* **C-31** (1982) 8.

[253] M. Meringer, Fast generation of regular graphs and construction of cages, *Journal of Graph Theory* (1999) **30** 137–146.

[254] M. Miller, *Extremal Digraphs*, M.A. Thesis, Department of Mathematics, Statistics and Computing Science, University of New England, Australia (1986).

[255] M. Miller, Digraph covering and its application to two optimization problems for digraphs, *Australasian Journal of Combinatorics* **3** (1991) 151–164.

[256] M. Miller, Some open problems in optimal digraphs, *Tatra Mountains Mathematical Publications* **9** (1996) 89–96.

[257] M. Miller, Nonexistence of graphs with cyclic defect, *The Electronic Journal of Combinatorics* **18** (1) (2011) Paper 71, 1–5.

[258] M. Miller and I. Fris, Maximum order digraphs for diameter 2 or degree 2, *Pullman volume of Graphs and Matrices*, *Lecture Notes in Pure and Applied Mathematics* **139** (1992) 269–278.

[259] M. Miller, J. Gimbert, J. Širáň and Slamin, Almost Moore digraphs are diregular, *Discrete Mathematics* **218** (2000) 265–270.

[260] M. Miller, M. Nguyen, and G. Pineda-Villavicencio, On the nonexistence of graphs of diameter 2 and defect 2, *Journal of Combinatorial Mathematics and Combinatorial Computing* **71** (2009) 5–20.

[261] M. Miller, M. H. Nguyen, and G. Pineda-Villavicencio, On the non-existence of odd degree graphs with diameter 2 and defect 2, in L. Brankovic, Y. Q. Lin, and W. F. Smyth, editors, *Proceedings of IWOCA 2007, the 18th International Workshop on Combinatorial Algorithms*, IWOCA Proceedings, Lake Macquarie, NSW, Australia, College Publications (2007) 143–150.

[262] M. Miller, M. H. Nguyen, and G. Pineda-Villavicencio, On the non-existence of even degree graphs with diameter 2 and defect 2, in J. Harland and P. Manyem,

editors, *Fourteenth Computing: The Australasian Theory Symposium (CATS 2008)*, Wollongong, NSW, Australia, *CRPIT* **77** (2008) 93–95.

[263] M. Miller, H. Perez-Roses, J. Ryan, The maximum degree&diameter-bounded subgraph in the mesh, *Discrete Applied Mathematics* **160** (2012) 1782-1790.

[264] M. Miller and G. Pineda-Villavicencio, On large graphs with given degree and diameter, in J. Ryan, P. Manyem, K. Sugeng, and M. Miller, editors, *Proceedings of AWOCA 2005, the 16th Australasian Workshop on Combinatorial Algorithms*, Ballarat, Australia, (2005) 239–247.

[265] M. Miller and G. Pineda-Villavicencio, Complete catalogue of graphs of maximum degree 3 and defect at most 4, *Discrete Applied Mathematics* **157** (13) (2009) 2983–2996.

[266] M. Miller, M.H. Nguyen and R. Simanjuntak, Repeat structure in regular graphs and digraphs, *Proceedings of the 2nd European Conference on Combinatorics, Graph Theory and Applications (EUROCOMB2003), Prague* (2003) 265–270.

[267] M. Miller and R. Simanjuntak, Graphs of order two less than the Moore bound, *Discrete Mathematics* **308** (13) (2008) 2810–2821.

[268] M. Miller and J. Širáň, Digraphs of degree two which miss the Moore bound by two, *Discrete Mathematics* **226** (2001) 269–274.

[269] M. Miller and Slamin, On the monotonicity of minimum diameter with respect to order and maximum out-degree, *Proceedings of COCOON 2000, Lecture Notes in Computer Science* **1858** (D.-Z. Du, P. Eades, V. Estivill-Castro, X. Lin (eds.)) (2000) 192–201.

[270] S. G. Molodtsov, Largest Graphs of Diameter 2 and Maximum Degree 6, *General Theory of Information Transfer and Combinatorics*, **4123** *Lecture Notes in Computer Science* (2006) 853–857.

[271] P. Morillo, M.A. Fiol and J. Fábrega, The diameter of directed graphs associated to plane tessellations, *Ars Combin.* **20** A (1985) 17–27.

[272] P. Morillo, M.A. Fiol and J.L.A. Yebra, A new family of graphs associated with plane tessellations, *Ars Combin.* **20** A (1985) 163–169.

[273] X. Munoz and J. Gómez, Asymptotically optimal (Δ, D', s)-digraphs, *Ars Combin.* **49** (1998) 97–111.

[274] M.H. Nguyen and M. Miller, Structural properties of graphs of diameter 2 with maximal repeats, *Discrete Mathematics* **308** (2008) 2337-2341.

[275] M. H. Nguyen and M. Miller, Structural properties of graphs of diameter 2 with defect 2, *AKCE International Journal of Graphs and Combinatorics*, **7** (1) (2010) 29–43.

[276] M. H. Nguyen and M. Miller, Structural properties of graphs of diameter 2 with defect 3, preprint.

[277] M.H. Nguyen and M. Miller, Moore bound for mixed networks, *Discrete Mathematics* **308** (2008) 5499–5503.

[278] M.H. Nguyen, M. Miller and J. Gimbert, On mixed Moore graphs, *Discrete Mathematics* **307** (2007) 964–970.

[279] J. Opatrný, D. Sotteau, Linear congruential graphs, *Graph theory, combinatorics, algorithms, and applications* (San Francisco, CA, 1989), SIAM, Philadelphia (1991), 404–415.

[280] J. Opatrný, D. Sotteau, N. Srinivasan, K. Thulasiraman, DCC linear congruential graphs: a new class of interconnection networks. *IEEE Trans. Comput.* **45** (1996) No. 2, 156–164.

[281] Z. Peng, Z. Duan, J.-J. Qi, Y. Cao, E. Lv, HP2P: A Hybrid Hierarchical P2P Network, *IEEE Proceedings on the 1st International Conference on the Digital Society (ICDS'07)* (2007).

[282] S. Perennes, A proof of Jean de Rumeur's Conjecture *Discrete Appl. Math.* **74** (1997) 295–299.

[283] G. Pineda-Villavicencio, Non-existence of bipartite graphs of diameter at least 4 and defect 2, *Journal of Algebraic Combinatorics* (2011) **34** 163–182.

[284] G. Pineda-Villavicencio, J. Gómez, M. Miller, and H. Pérez-Rosés, New largest known graphs of diameter 6, *Networks* **53** (4) (2009) 315–328.

[285] G. Pineda-Villavicencio and M. Miller, On graphs of maximum degree 5, diameter D and defect 2, In L. Matyska, A. Kučera, T. Vojnar, Z. Kotásek, D. Antoš, and O. Krajíček, editors, *Proceedings of MEMICS 2006, Second Doctoral Workshop on Mathematical and Engineering Methods inComputer Science*, Mikulov, Czech Republic (2006) 182–189.

[286] J. Plesník, One method for proving the impossibility of certain Moore graphs, *Discrete Mathematics* **8** (1974) 363–376.

[287] J. Plesník and Š. Znám, Strongly geodetic directed graphs, *Acta F. R. N. Univ. Comen. - Mathematica* **XXIX** (1974) 29–34.

[288] P. Potočnik, P. Spiga and G. Verret, Cubic vertex-transitive graphs on up to 1280 vertices, *Journal of Symbolic Computations***50** (2013) 465–477.

[289] J. Preen, The degree/diameter problem for planar graphs, `http://faculty.capebretonu.ca/jpreen/degdiam.html`, accessed on 14 Dec 2011.

[290] J. Preen, The degree/diameter problem for toroidal graphs. `http://faculty.capebretonu.ca/jpreen/torvaldiam.html`, accessed on 14 Dec 2011.

[291] H. J. Quaife, On (d, k, μ) graphs, *IEEE Trans. Computers* **C-18** (1969) 270–272.

[292] J. J. Quisquater, Chordal rings, communicated February 27 1998.

[293] C. Roos and A.J. van Zanten, On the existence of certain generalized Moore geometries, Part I, *Discrete Mathematics* **51** (1984) 179–190.

[294] C. Roos and A.J. van Zanten, On the existence of certain generalized Moore geometries, Part II, *Discrete Mathematics* **51** (1984) 277–282.

[295] R.R. Rout, K. Kant, Reduction of Look up Latency in Location-aware Clusters of Large-scale P2P Overlay Networks, *TENCON 2008 IEEE Proceedings* (2008).

[296] H. J. Ryser, The existence of symmetric block designs. *Journal of Combinatorial Theory, Series A* **32** (1) (1982) 103–105.

[297] G. Sabidussi, Graphs without dead ends, *Europ. J. Combinatorics* **17** (1996) 69–87.

[298] M. Sampels, Vertex-symmetric generalized Moore graphs, *Discrete Applied Mathematics* **138** (2004) 195–202.

[299] M. Sampels, On Generalized Moore Digraphs, *Discrete Applied Mathematics* **138** (2004) 195–202.

[300] M. Sampels and S. Schöf, Massively parallel architectures for parallel discrete event simulation, *Proceedings of the 8th European Simulation Symposium (ESS'96)* **2** (1996) 374–378.

[301] S.T. Schibell and R.M. Stafford, Processor interconnection networks from Cayley graphs, *Discrete Applied Mathematics* **40** (1992) 333–357.

[302] S.B. Seidman, Network structure and minimum diameter, *Social Networks* **5** (3) (1983) 269–287.

[303] K. Seyffarth, Maximal Planar Graphs of Diameter Two, *Journal of Graph Theory* **13** (5) (1989) 619–648.

[304] J. Šiagiová, Minimum diameter lifts of complete graphs, *Proc. Conf. 60th Anniversary of the Fac. Civil Eng.*, Slovak University of Technology, Mathematics and Physics, Bratislava (1998) 229–233.

[305] J. Šiagiová, Composition of regular coverings of graphs, *J. Electrical Engg.* **50** (10) (1999) 75–77.

[306] J. Šiagiová, A note on the McKay-Miller-Širáň graphs, *Journal of Combinatorial Theory B* **81** (2001) 205–208.

[307] J. Šiagiová, A Moore-like bound for graphs of diameter 2 and given degree obtained as regular lifts of small graphs, *Acta Math. Univ. Comen.* **LXXI** (2) (2002) 157–161.

[308] J. Šiagiová, Composition of regular coverings of graphs and voltage assignments, *Australasian J. Combin.* **28** (2003) 131–136.

[309] J. Šiagiová, Permutation voltage assignments and constructions of large (Δ, D)-graphs, preprint.

[310] J. Šiagiová and R. Simanjuntak, A Note on a Moore Bound for Graphs Embedded in Surfaces, *Acta Math. Univ. Comen.* **LXXIII** (1) (2004) 115–118.

[311] J. Šiagiová and J. Širáň, A note on large Cayley graphs of diameter two and given degree, *Discrete Math.* **35** (2005) 379–382.

[312] J. Šiagiová and J. Širáň, Approaching the Moore bound for diameter two by Cayley graphs, *Journal of Combinatorial Theory, Series B* 102(2) (2012) 470–473.

[313] R. Simanjuntak and E.T. Baskoro, The uniqueness of almost Moore digraphs with degree 4 and diameter 2, *Proceeding ITB* **32** (1) (2000) 7–11.

[314] R. Simanjuntak and M. Miller, Largest planar digraphs of diameter two, *Proceedings of AWOCA'02*, Frazer Island, Australia, July 2002, 43–52.

[315] S. Singh, Challenges: Wide-Area wireless NETworks (WANETs), *ACM Proceedings of MobiCom08* (2008) 38–45.

[316] R. C. Singleton, On minimal graphs of maximum even girth, *Journal of Combinatorial Theory* **1** (3) (1966) 306–332.

[317] R. Singleton, There is no irregular Moore graph, *Amer. Math. Monthly* **75** (1968) 42–43.

[318] J. Širáň, J. Šiagiová and M. Ždímalová, Large graphs of diameter two, preprint.

[319] Slamin, *Diregularity of Digraphs Close to Moore Bound*, PhD Thesis, Department of Computer Science and Software Engineering, The University of Newcastle (2001).

[320] Slamin, E. T. Baskoro and M. Miller, Diregularity of digraphs of out-degree three and order two less than Moore bound, *Proceedings of AWOCA'01*, Bandung, Indonesia (2001) 167–173.

[321] Slamin and M. Miller, Diregularity of digraphs close to Moore bound, *Prosiding Konperensi Nasional Matematika X, MIHMI* **6** (5) (2000) 185–192.

[322] W.F. Smyth, The construction of large graphs of given maximum degree and diameter, *Congressus Numerantium* **51** (1986) 277–285.

[323] W.F. Smyth, An upper bound on the order of graphs of given maximum degree and diameter, *Congressus Numerantium* **57** (1987) 201–211.

[324] J. Soares, Maximum diameter of regular digraphs, *Journal of Graph Theory* **16** (5) (1992) 437–450.

[325] T. Soneoka, Super edge-connectivity of dense digraphs and graphs, *Discrete Appl. Mathematics* **37/38** (1992) 511–523.

[326] R. Stanton and D. Cowan, Note on a "square" functional equation, *SIAM Review* **12** (1970) 277–279.

[327] R.G. Stanton, S.T.E. Seah and D.D. Cowan, Nonexistence of an extremal graph of a certain type, *J. Austral. Math. Soc.* (Series A) **30** (1980) 55–64.

[328] R.M. Storwick, Improved Construction Techniques for (d, k) Graphs, *IEEE Trans. on Computers* (1970) 1214–1216.

[329] J. Tang, M. Miller, Y. Lin, HSAGA and its application for the construction of near-Moore digraphs, *J. Discrete Algorithms* **6** (1) (2008) 73-84.

[330] J. Teska, R. Kuzel, M. Miller, Divisibility conditions in almost Moore digraphs with selfrepeats, *Electronic Notes in Discrete Mathematics* **24** (2006) 161-163.

[331] S.A. Tishchenko, The maximum size of a planar graph ($\Delta = 3$, $D = 3$), *Fundam. Prikl, Math.* **7** (1) (2001) 159-171.

[332] S.A. Tishchenko, The maximum order of a graph of diameter 2 with a fixed Euler characteristic, *Fundam. Prikl, Math.* **7** (4) (2001) 1203-1225.

[333] S. Toueg and K. Steiglitz, The design of small-diameter network by local search, *IEEE Trans. on Computers* **C-28** (1979) 537–542.

[334] W. T. Tutte, A family of cubical graphs, *Proceedings of the Cambridge Philosophical Society*, **43** (1947) 459–474.

[335] T. Vetrík, Cayley graphs of given degree and diameter 3, 4 and 5, *Discrete Math.*, to appear.

[336] T. Vetrík, Abelian Cayley graphs of given degree and diameter 2 and 3, preprint (2012).

[337] T. Vetrík, Large Cayley digraphs of given degree and diameter, *Discrete Mathematics* **312** (2012) 472 - 475.

[338] T. Vetrík, R. Simanjuntak, E. Baskoro, Large bipartite Cayley graphs of given degree and diameter, *Discrete Mathematics* **311** (4) (2011) 324–326.

[339] T. Vetrík, A note on bipartite Cayley digraphs of diameter 3, *Mathematics, Geometry and their Applications* **24** (2009) 27–28.

[340] T. Vetrík, An upper bound for graphs of diameter 3 and given degree obtained as Abelian lifts of dipoles, *Discussiones Mathematicae Graph Theory* **28** (2008) 91–96.

[341] M.E. Watkins, Connectivity of transitive graphs, *Journal of Combinatorial Theory* **8** (1970) 23–29.

[342] C.K. Wong, A combinatorial problem related to multimodule memory organization, *J. Ass. Comput. Mach.* **21** (1974) 392–402.

[343] C. K. Wong, D. Coppersmith, A combinatorial problem related to multimodule memory organizations, *J. ACM* **21** (1974) 392–402.

[344] P.K. Wong, Cages - a survey, *J. Graph Theory* **6** (1) (1982) 1–22.

[345] W. Xiao, B. Parhami, Further mathematical properties of Cayley digraphs applied to hexagonal and honeycomb meshes, *Discrete Applied Mathematics* **155** (2007) 1752–1760.

[346] W. Xiao, B. Parhami, Structural properties of Cayley digraphs with applications to mesh and pruned torus interconnection networks, *Journal of Computer and System Sciences* **73** (2007) 1232–1239.

[347] J.M. Xu, An inequality relating the order, maximum degree, diameter and connectivity of a strongly connected digraph. *Acta Math. Appl. Sinica (English Ser.)* **8** (2) (1992) 144–152.

[348] J.M. Xu and K. Xu, On k-diameter of k-connected graphs, *Appl. Math. J. Chinese Univ. Ser. B* **16** (3) (2001) 231-236.

[349] Y. Yang and J. Lin, Largest planar graphs and largest maximal planar graphs of diameter two, *J. Comput. Appl. Math.* **144** (1–2) (2002) 349-358.

[350] Y. Yang, J. Lin, and Y. Dai, Largest planar graphs and largest maximal planar graphs of diameter two, *Journal of Computational and Applied Mathematics* **144** (1–2) (2002) 349–358.

[351] J.L.A. Yebra, M.A. Fiol and J. Fábrega, Semiregular bipartite Moore graphs, *Ars Combin.* **16** (1983) 131–139.

[352] A.J. van Zanten, A short proof of a theorem of Bannai and Ito, *Discrete Mathematics* **78** (1989) 321–325.

[353] S. Zhou, A note on the degree-diameter problem for arc-transitive graphs, *Bull. of the ICA*, to appear.

[354] S. Zhou, A class of arc-transitive Cayley graphs as models for interconnection networks, *SIAM J. Discrete Mathematics* **23** (2009) 694-714.

[355] A. Zlatoš, The diameter of lifted digraphs, *Australasian J. Combin.* **19** (1999) 73–82.

[356] Š. Znám, On existence and regularity of graphs with certain properties, *Discrete Mathematics* **16** (1976) 83–90.