Regular character tables of symmetric groups

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Abstract

We generalize a well-known result on the determinant of the character tables of finite symmetric groups.

It is a well-known fact that if X_n is the character table of the symmetric group S_n, then the absolute value of the determinant of X_n equals a_n, which is defined as the product of all parts of all partitions of n. It also equals b_n, which is defined as the product of all factorials of all multiplicities of parts in partitions of n. Proofs of this may be found in [6], [5]. We sketch a proof below.

In this brief note we present generalizations of this to certain submatrices of X_n (called regular/singular character tables). We get such character tables for each choice of an integer $\ell \geq 2$. This is a perhaps slightly surprising consequence of results in [4]. The above result is obtained when we choose $\ell \geq n$.

If μ is a partition of n we write $\mu \vdash n$ and then z_μ denotes the order of the centralizer of an element of (conjugacy) type μ in S_n. Suppose $\mu = (1^{m_1}, 2^{m_2}, ...)$, is written in exponential notation. Then we may factor $z_\mu = a_\mu b_\mu$, where

$$a_\mu = \prod_{i \geq 1} i^{m_i}, \quad b_\mu = \prod_{i \geq 1} m_i!$$

We define

$$a_n = \prod_{\mu \vdash n} a_\mu, \quad b_n = \prod_{\mu \vdash n} b_\mu.$$

Proposition 1: *We have that* $|\det(X_n)| = a_n = b_n$.

Proof: (See also [6].) By column orthogonality for the irreducible characters of S_n, $X_n^t X_n$ is a diagonal matrix with the integers $z_\mu, \mu \vdash n$ on the diagonal. It follows that in the above notation $\det(X_n)^2 = \prod_{\mu\vdash n} z_\mu = a_n b_n$. By [2], Corollary 6.5 we have $|\det(X_n)| = a_n$. The result follows.

Another proof of the fact that $a_n = b_n$ for all n may be found in [3].

We choose an integer $\ell \geq 2$, which is fixed from now on. Several concepts below, like regular, singular, defect etc. refer to the integer ℓ.

A partition is called *regular* if no part is repeated ℓ or more times and is called *class regular,* if no part is divisible by ℓ. A partition which is not regular (class regular) is called *singular (class singular).* We let $p(n)$ be the number of partitions of n. The number $p^*(n)$ of regular partitions of n equals the number of class regular partitions of n and then $p'(n) = p(n) - p^*(n)$ is the number of (class)singular partitions of n. The irreducible characters and the conjugacy classes of X_n are labelled canonically by the partitions of n. An irreducible character is called *regular (singular),* if the partition labelling it is regular (singular). A conjugacy class is called *regular (singular),* if the partition labelling it is class regular (class singular). The *regular character table* X_n^{reg} contains the values of regular characters on regular classes and the *singular character table* X_n^{sing} is defined analogously.

Let

$$a_n^{\mathrm{creg}} = \prod_{\mu \text{ class regular}} a_\mu, \quad b_n^{\mathrm{creg}} = \prod_{\mu \text{ class regular}} b_\mu$$

and define a_n^{csing} and b_n^{csing} correspondingly such that a_n and b_n are factored into a “regular” and a “singular” component, $a_n = a_n^{\mathrm{creg}}\ a_n^{\mathrm{csing}}$, $b_n = b_n^{\mathrm{creg}}\ b_n^{\mathrm{csing}}$.

Our main results are:

Theorem 2: *The regular character table satisfies:* $|\det(X_n^{\mathrm{reg}})| = a_n^{\mathrm{creg}}$.

Theorem 3: *The singular character table satisfies:* $|\det(X_n^{\mathrm{sing}})| = b_n^{\mathrm{csing}}$.

Remark: In the case where $\ell = p$ is a prime number, we have that the absolute value of the determinant of the *Brauer character table* of S_n in characteristic p is also a_n^{creg}.

When $\mu \vdash n$, say $\mu = (i^{m_i(\mu)})$ we define the *defect* of μ by

$$d_\mu = \sum_{i,j\geq 1} \left\lfloor \frac{m_i(\mu)}{\ell^j} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ means “integral part of.”

We start the proof of Theorems 2 and 3 with a key result which may be of independent interest. It generalizes the identity $a_n = b_n$ above and is obtained by modifying an idea implicit in [6], see also [7], Exercise 26, p.48 and p.59. An unpublished note of John Graham communicated to the author by Gordon James has been useful. The case where ℓ is a prime is implicit in [5], where proofs are based on modular representation theory.

Theorem 4: *We have that* $b_n^{\mathrm{creg}}/a_n^{\mathrm{creg}} = \ell^{c_n}$, *where*

$$c_n = \sum_{\mu \text{ class regular}} d_\mu.$$

Proof: Consider the set $\mathcal{T}$ of triples

$$\mathcal{T} = \{(\mu, i, j) | \mu \text{ class regular}, i, j \geq 1, m_i(\mu) \geq j\}.$$

We claim that

$$a_n^{\text{creg}} = \prod_{(\mu,i,j)\in\mathcal{T}} i, \quad b_n^{\text{creg}} = \prod_{(\mu,i,j)\in\mathcal{T}} j.$$

Indeed, for a fixed class regular μ and a fixed non-zero block $i^{m_i(\mu)}$ in μ, the elements $(\mu, i, 1)$, $(\mu, i, 2), \cdots, (\mu, i, m_i(\mu))$ are precisely the ones in $\mathcal{T}$ starting with μ and i. These elements give a contribution $i^{m_i(\mu)}$ to a_n^{creg} and a contribution $m_i(\mu)!$ to b_n^{creg}.

We define an involution ι on $\mathcal{T}$ as follows. If $(\mu, i, j) \in \mathcal{T}$ then ℓ does not divide i, since μ is class regular. Also note that μ contains at least j parts equal to i. Write $j = \ell^v j'$, where v is a non-negative integer and $\ell \nmid j'$. We refer then to j' as the ℓ'-part of j. Let $\mu_{(i,j)}$ be obtained from μ by replacing j parts equal to i in μ by $\ell^v i$ parts equal to j'. Then $\iota(\mu, i, j)$ is defined as $(\mu_{(i,j)}, j', \ell^v i)$, an element of $\mathcal{T}$. It is easily checked that ι^2 is the identity.

This shows that

$$a_n^{\text{creg}} = \prod_{(\mu,i,j)\in\mathcal{T}} i = \prod_{(\mu,i,j)\in\mathcal{T}} j',$$

where as above j' is the ℓ'-part of j. Thus $b_n^{\text{creg}}/a_n^{\text{creg}} = \ell^c$, where c is the sum of the exponents of the powers ℓ^v of ℓ, occuring as factors in the integers of the product $\prod_{\mu \ \mathit{class}, i\geq 1} m_i(\mu)!$. If m is a positive integer, then there are $\lfloor m/\ell \rfloor$ numbers among $1, \cdots, m$ which are divisible by ℓ, $\lfloor m/\ell^2 \rfloor$ numbers divisible by ℓ^2, etc., giving a total exponent $\sum_{j\geq 1} \lfloor m/\ell^j \rfloor$ of ℓ in $m!$. Applying this fact to each $m_i(\mu)$, we get our result.

Let χ_λ denote the irreducible character of S_n, labelled by the partition $\lambda \vdash n$, and χ_λ^0 the restriction of χ_λ to the regular classes of S_n. In [4], Section 4, it was shown that there exist integers $d_{\lambda\rho}$ such that for each irreducible character χ_λ we have

$$\chi_\lambda^0 = \sum_{\rho \ \text{regular}} d_{\lambda\rho} \chi_\rho^0. \qquad (1)$$

It follows from (1) that for any λ the character

$$\psi_\lambda = \chi_\lambda - \sum_{\rho \ \text{regular}} d_{\lambda\rho} \chi_\rho \qquad (2)$$

vanishes on all regular classes.

Proof of Theorem 2: The matrix form of (1) above may be stated as

$$Y_n = D_n X_n^{\text{reg}},$$

where Y_n is the $p(n) \times p^*(n)$-submatrix of X_n containing the values of all irreducible characters on regular classes, and $D_n = (d_{\lambda\rho})$ is the "decomposition matrix". Consider the corresponding "Cartan matrix" $C_n = (D_n)^t D_n$. (For an explanation of the terms decomposition matrix and Cartan matrix we refer to [4].)

Column orthogonality shows that

$$(Y_n)^t Y_n = (X_n^{\text{reg}})^t C_n X_n^{\text{reg}} = \Delta(z_\mu).$$

Here Δ is a diagonal matrix. Taking determinants we see that

$$\det(X_n^{\text{reg}})^2 \det(C_n) = \prod_{\mu \text{ class regular}} z_\mu = a_n^{\text{creg}} b_n^{\text{creg}} \qquad (3).$$

By Proposition 6.11 in [4] (see also [1], Theorem 3.3) we have that $\det(C_n) = \ell^{c_n}$. It follows then from Theorem 4 that

$$\det(C_n) = b_n^{\text{creg}} / a_n^{\text{creg}}. \qquad (4)$$

From (3) and (4) we conclude $|\det(X_n^{\text{reg}})| = a_n^{\text{creg}}$, which proves the theorem.

Proof of Theorem 3: We assume that the rows and columns of X_n are ordered such that the regular characters and classes are the first. Then the submatrix consisting of the intersection of the first $p^*(n)$ rows and the first $p^*(n)$ columns in X_n is exactly X_n^{reg}. In fact X_n has a block form

$$X_n = \begin{bmatrix} X_n^{\text{reg}} & A_n \\ B_n & X_n^{\text{sing}} \end{bmatrix}.$$

We do some row operations on X_n to get a new matrix $\bar{X}_n$ as follows: For each *singular* partition λ' and each *regular* partition ρ, subtract $d_{\lambda'\rho}$ times the row labelled by ρ from the row labelled by λ'. Thus in $\bar{X}_n$ the row labelled by the singular partition λ' contains the values of the character $\psi_{\lambda'}$ on all conjugacy classes. Since $\psi_{\lambda'}$ vanishes on regular classes $\bar{X}_n$ looks like this:

$$\bar{X}_n = \begin{bmatrix} X_n^{\text{reg}} & A_n \\ 0 & Q_n \end{bmatrix}$$

for a suitable square $p'(n)$-matrix Q_n. We have then $\det(X_n) = \det(\bar{X}_n) = \det(X_n^{\text{reg}})\det(Q_n)$, whence by Theorem 2

$$\det(Q_n) = a_n^{\text{csing}}. \qquad (5)$$

We now have that if λ', λ'' are singular partitions, then since $\psi_{\lambda'}$ vanishes on regular classes

$$< \psi_{\lambda'}, \chi_{\lambda''} >= \sum_{\mu} \frac{1}{z_\mu} \psi_{\lambda'}(x_\mu)\chi_{\lambda''}(x_\mu) = \sum_{\mu' \text{ class singular}} \frac{1}{z_{\mu'}} \psi_{\lambda'}(x_{\mu'})\chi_{\lambda''}(x_{\mu'}).$$

Here x_μ is an element in the conjugacy class labelled by μ. On the other hand by (2) $< \psi_{\lambda'}, \chi_{\lambda''} >= \delta_{\lambda'\lambda''}$. Translating these equations in terms of matrices

$$Q_n \Delta(\frac{1}{z_{\mu'}})(X_n^{\text{sing}})^t = E.$$

Here again Δ is a diagonal matrix and E is a $p'(n)$-square identity matrix. Taking determinants

$$\det(X_n^{\text{sing}})\ \det(Q_n) = \prod_{\mu'\ \text{class singular}} z_{\mu'} = a_n^{\text{csing}}\ b_n^{\text{csing}} \qquad (6)$$

Now Theorem 3 follows from (5) and (6).

It should be remarked that Theorems 2 and 3 also hold, if we replace the irreducible characters χ_λ by the Young characters η_λ.

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