Even Astral Configurations

Leah Wrenn Berman
Department of Mathematics and Computer Science
Ursinus College, Collegeville, PA, USA
lberman@ursinus.edu

Submitted: May 30, 2003; Accepted: May 28, 2004; Published: Jun 11, 2004.
MR Subject Classifications: 51A20, 52C35

Abstract

A configuration (p_q, n_k) is a collection of p points and n straight lines in the Euclidean plane so that every point has q straight lines passing through it and every line has k points lying on it. A configuration is *astral* if it has precisely $\lfloor \frac{q+1}{2} \rfloor$ symmetry classes (transitivity classes) of lines and $\lfloor \frac{k+1}{2} \rfloor$ symmetry classes of points. An *even* astral configuration is an astral configuration configuration where q and k are both even. This paper completes the classification of all even astral configurations.

1 Introduction

A combinatorial configuration (p_q, n_k) is a collection of p "points" and n collections of points, called "lines", so that each "point" is contained in q of the "lines" and each "line" contains k of the "points". Combinatorial configurations have been studied since the mid-1800s (see, e.g., [5]). Much of the study of configurations, both in the past (see [5]) and recently ([4]), has focused on the question of enumerating all combinatorial configurations and determining whether the combinatorial configurations have any geometric realization (e.g., [13]). However, even when it has been determined that combinatorial configurations do have a geometric realization, little investigation has been done as to how 'nice' such a realization can be. For example, the Pappus configuration, a $(9_3, 9_3)$ configuration (usually denoted simply as (9_3)), admits geometric realizations that have no nontrivial Euclidean symmetries, as well as realizations with quite a lot of symmetry (see Figure 2).

There are a few papers that focus on geometrically realizable configurations, as opposed to (or in addition to) combinatorial configurations; for example, see [6], [8], [9], and [5]. In [1], a particular kind of highly symmetric (n_4) configurations, called *astral* configurations, were classified; this paper will classify (p_{2s}, q_{2t}) astral configurations.

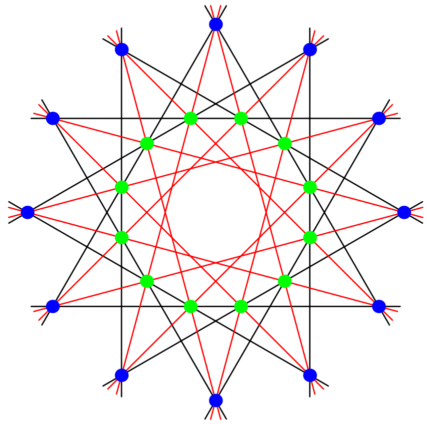

Figure 1: An astral configuration with 24 points and 24 lines, with 4 points on each line and 4 lines through each point.

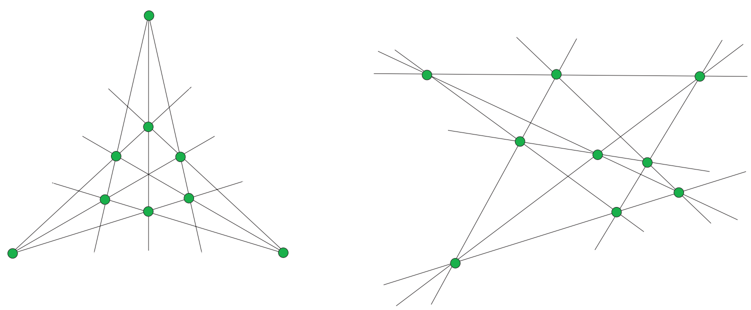

Figure 2: Two embeddings of the Pappus configuration, one with nontrivial geometric symmetries and one without.

2 Definitions and preliminary results

A (p_q, n_k) configuration is a collection of p points and n straight lines, in the Euclidean plane, with the condition that every point has q lines passing through it and every line has k points lying on it. Such a configuration is *astral* if the set of Euclidean isometries of the plane that map the configuration to itself partitions the lines into $\lfloor (q+1)/2 \rfloor$ symmetry classes and the points into $\lfloor (k+1)/2 \rfloor$ symmetry classes. This is the least number of symmetry classes (i.e., the most symmetry) that a configuration can have. To see this, note that if a straight line in the plane has k points on it, at most two of the points can be in the same symmetry class (see Figure 3), so the configuration must have at least $\lfloor (k+1)/2 \rfloor$ symmetry classes of points, and similarly with the lines, since two lines can intersect only at a single point. (Note that the symmetry classes being considered are precisely the transitivity classes of the points or lines under the appropriate rotations and reflections of the plane.)

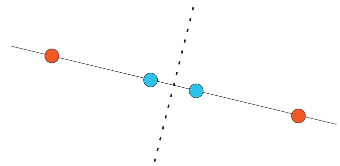

Figure 3: At most two points can be in the same symmetry class

Note that by counting incidences, $pq = nk$, so if $p = n$ then $q = k$. An (n_k, n_k) configuration is denoted (n_k). For example, Figure 1 shows a configuration with 24 points and 24 lines, with each point incident to four lines and each line incident with four points. Moreover, it has precisely two symmetry classes of points and two symmetry classes of lines. Hence, it is an astral configuration (24_4). In general, diagrams in this paper will distinguish the symmetry classes by color. In Figure 1, the colors used for the two symmetry classes of points are green and blue, and the colors used for the two symmetry classes of lines are red and black.

Often, one is interested only in the number of points on a line and the number of lines through a point, rather than in how many points and lines there are in the configuration. A (p_q, n_k) configuration is called a configuration of *class* $[q, k]$, or, usually, a $[q, k]$ *configuration*, when we are only interested in indicating the number of points on each line and the number of lines passing through each point, rather than in the total number of points and lines. An astral configuration of class $[q, k]$ is called *even* if both q and k are even; otherwise, the configuration is called *odd*.

In an astral configuration with q lines incident with each point, where q is odd, there is one symmetry class of lines, called the *special* symmetry class of lines, with exactly one of its members incident with each point, while in all the other symmetry classes of lines, there are exactly two lines incident with each point. Similarly, in an astral $[q, k]$ configuration with k odd, the *special* symmetry class of points is the symmetry class of points with exactly one point in this class incident with each line. It follows from the definitions of astral and even that in an even astral configuration, no symmetry classes are special. Astral configurations come in two varieties. An astral $[q, k]$ configuration of *type 1* satisfies the condition that each of its symmetry classes of points forms the vertices of a regular polygon, all of which are concentric; such a configuration is denoted $[q, k]^1$. In an astral *type 2* configuration, there is some symmetry class of points which does not form the vertices of a regular polygon; astral type 2 configurations are denoted $[q, k]^2$. The configuration in Figure 1 is a $[4, 4]^1$ configuration, while Figure 4 shows a $[4, 4]^2$ configuration.

The *size* of a type 1 configuration is the cardinality of the largest symmetry class of points that form the vertices of a regular polygon.

One method of constructing type 1 astral configurations is to consider one of the symmetry classes of points as the vertices of a regular polygon; in a type 1 configuration,

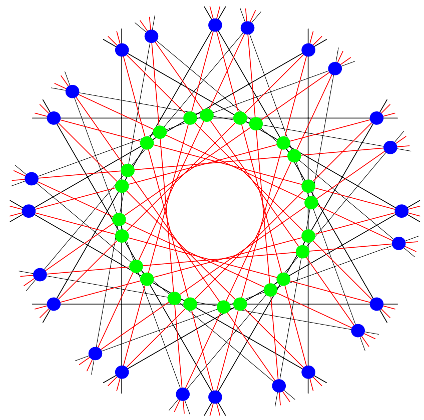

Figure 4: A $[4,4]^2$ astral configuration.

the lines will be diagonals of the polygon. Given a diagonal of a regular polygon, its *span* is the (smaller, usually) number of sides of the polygon intercepted by the diagonal.

Lemma 2.1. *If no astral* $[2s, 2t]$ *configuration exists, then no astral* $[2(s+x), 2(t+y)]$ *configuration exists either, where* $x, y = 0, 1, 2, \ldots$.

Proof. Suppose there exists an astral $[2(s+x), 2(t+y)]$ configuration. Remove all but s symmetry classes of lines and all but t symmetry classes of points from the $[2(s+x), 2(t+y)]$ configuration. The resulting configuration is a $[2s, 2t]$ configuration. □

2.1 Multiples of a configuration

Given a type 1 astral configuration of size m with the symmetries of a regular m-gon, then additional type 1 configurations may be formed by adding $r-1$ equally-spaced copies of the original configuration—i.e., the new configuration will have the j^{th} copy rotated by $\frac{2j\pi}{mr}$ radians. This new configuration is called an *r-multiple*, or, more simply, a *multiple* of the original configuration; Figure 5 shows an example. Note that any $[2s, 2t]^1$ astral configuration of size m will have the symmetries of a regular m-gon.

In addition, taking two copies of a size m type 1 configuration, rotating one through any angle α which is not an integer multiple of $\frac{\pi}{m}$, and placing it concentrically on the first one yields a type 2 astral configuration; that such a configuration is astral is shown in Lemma 2.2. The type 2 configurations produced from this process are called *ordinary* type 2 configurations; other type 2 configurations are called *extraordinary*. With this terminology, the configuration in Figure 4 is an ordinary $[4,4]^2$ configuration formed from two copies of the configuration in Figure 1.

Lemma 2.2. *Ordinary* $[q, k]^2$ *configurations are astral.*

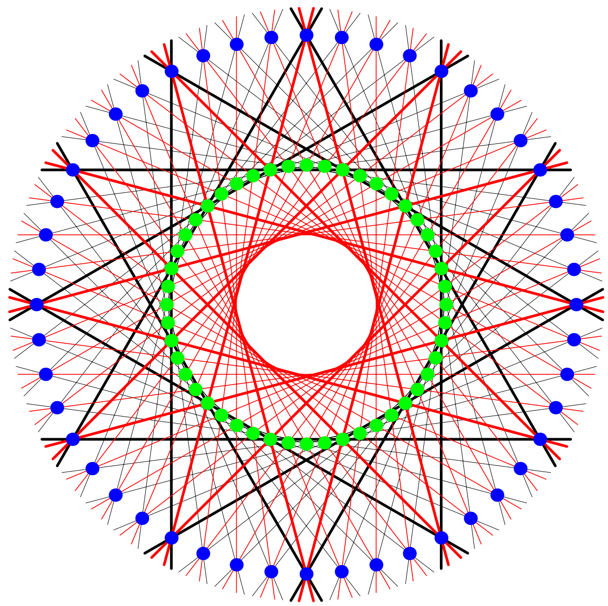

Figure 5: A (96_4) configuration, formed from four evenly spaced multiples of the $[4,4]$ configuration shown in Figure 1; one copy is shown with thicker lines.

Proof. The ordinary configuration (the 'main configuration') is constructed from two smaller $[q,k]^1$ configurations, called the subconfigurations. Suppose that the two subconfigurations are colored red and black and that each subconfiguration is of size m. The symmetries of the main configuration consist of rotations by multiples of $\frac{2\pi}{m}$ and reflections through the mirrors that are at an angle halfway between corresponding points of the red and black configurations. Any point in a symmetry class in a subconfiguration can be rotated onto any other point in the same symmetry class of the same subconfiguration. Reflection through a mirror sends black points to red points of the corresponding symmetry class, so any point in a symmetry class of a subconfiguration may be mapped to any other point in that symmetry class or in the corresponding symmetry class of the other subconfiguration. Similarly, for the lines of the configuration, rotation maps any line in a subconfiguration's symmetry class to any other line in that class, and reflection maps black lines to red lines. □

2.2 Diametral points

If the vertices of an m-gon are consecutively labelled $v_0, \ldots, v_{m-1}$, a diagonal has *span* c if it connects vertices v_i and v_{i+c}, where indices are taken modulo m and in general, $2 \le c \le m/2$. In Figure 1, the red lines may be viewed as diagonals of the dodecagon of span 4 and the blue lines as diagonals of span 5. Given a regular polygon and a diagonal of span c, label the intersection points of the diagonal with other span c diagonals as

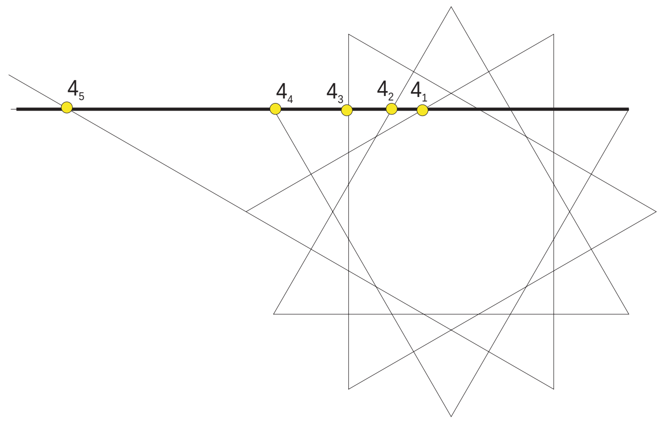

Figure 6: Examples of the symbols c_d; in this case $c = 4$.

$c_1, c_2, \ldots, c_{\lfloor \frac{m}{2} \rfloor}$, counted from the midpoint of the diagonal and travelling in one direction, say, to the left. Note that considering the set of points with symbol c_i, if $i > c$, the point is outside the polygon, for $i = c$ the point is a vertex of the polygon, and if $i < c$ the point is interior to the polygon; see Figure 6. Also, the point with symbol c_{-d} is the d-th intersection point along the span c diagonal counted to the right of the the midpoint.

A line is *diametral* with respect to a regular convex m-gon if it passes through the center of the m-gon and one of the vertices of the polygon. Note that if m is even, diametral lines correspond to the ordinary notion of diameters of a regular polygon, i.e, they pass through two vertices and the center of the polygon and are lines of span $\frac{m}{2}$. A line in a type 1 configuration is *diametral* if it is diametral for the underlying regular polygon formed by the ring of vertices which are farthest from the center of the configuration. A line in a configuration is *semidiametral* if it passes through the center of the m-gon and lies halfway between two diametral lines. A point is diametral if it lies on a diametral line, and a point is *semidiametral* if it lies on a semidiametral line.

Lemma 2.3. *Choose a span c diagonal of a regular, convex m-gon, and label the intersection points of the diagonal with other span c diagonals as $c_1, c_2, \ldots, c_c, \ldots, c_{\lfloor \frac{m}{2} \rfloor}$. If m is even, the intersection points c_i which are diametral are precisely those for which the parity of c and i is the same, and the other intersection points are semidiametral. If m is odd, all points c_i are diametral.*

Proof. Note that the geometric object produced by taking all span c diagonals of an m-gon has the dihedral symmetry group of an m-gon. Without loss of generality, we may assume that the m-gon is centered at the origin in $\mathbb{R}^2$ and that one vertex is located at the point $(1, 0)$. In this case, the lines of reflective symmetry (mirrors) are those that pass through the origin and have an angle of $\frac{q\pi}{m}$ for $q = 0, 1, 2, \ldots, m-1$. Every intersection point c_i lies on one of the lines of reflective symmetry of the figure.

Case 1: m is even.

If q is also even, the corresponding mirrors are diametral lines, while if q is odd, the mirrors are semidiametral lines; thus, the intersection points alternate between lying on a diametral line and not lying on a diametral line. Finally, if c is even, the midpoint of a span c diagonal lies on a diameter, while if c is odd, it does not.

Case 2: m is odd.

If m is odd, all the lines of reflective symmetry (mirrors) are diametral lines as defined above. Every point c_i lies on one of the mirrors, so all the points c_i are diametral.

□

2.3 Polars

In the study of combinatorial configurations and of (geometric) configurations in the projective plane, if a $[q, k]$ configuration exists, then a $[k, q]$ configuration exists as well, by duality. One may view the projective plane as the extended Euclidean plane, i.e., the Euclidean plane with the line at infinity appended, and define a configuration to be astral if isometries of the Euclidean plane that send points at infinity to points at infinity partition the points and lines (including those that may be at infinity) into the required number of symmetry classes. Given an astral $[q, k]$ configuration in the extended Euclidean plane, a new astral $[k, q]$ configuration may be constructed by taking the polar of the configuration with respect to a circle that passes through one of the symmetry classes of finite points. The resulting configuration is astral in the ordinary Euclidean plane as long as the original configuration contained no lines passing through the center of the configuration. In particular, since an even astral configuration must have two lines from each symmetry class passing through each point, no members of a symmetry class of lines are diametral lines, so the polar of an astral $[2s, 2t]$ configuration is an astral $[2t, 2s]$ configuration.

2.4 Type 2 distributions of points

In a type 2 configuration, there is some symmetry class of points which does not form the vertices of a regular polygon. The only other possible arrangement is that they are dispersed 'long-short' equally around the circle (see Figure 7), since a finite set of points either has only rotational symmetry or it has dihedral symmetry. This second distribution is called a *type 2 distribution of points*. Note that this forces the number of points, say $n = 2m$, in the symmetry class to be even. If every other point is considered to be colored red, with the others black, the m red points are the vertices of a regular polygon, as are the m black points, and the red points are formed by rotating the black points through an arbitrary angle which is not an integer multiple of π/m, since rotation by any multiple of π/m would yield equally-spaced points.

Lemma 2.4. *Given a type 2 distribution of* $2m$ *points in a* $[2s, 2t]$ *configuration with*

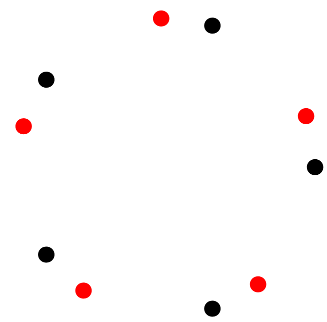

Figure 7: A type 2 distribution of points

every other point colored black or red as above, lines in a symmetry class must connect points of the same color.

Proof. In a $[2s, 2t]$ configuration, every symmetry class of lines has the property that two lines in the class are incident with each point.

Choose a symmetry class, and suppose that the lines of that symmetry class connect black vertices to red vertices. For convenience, assume that the type 2 distribution of points is distributed on the unit circle, centered at $(0, 0)$ in $\mathbb{R}^2$. Label the points of the type 2 distribution as $v_{0b}, v_{0r}, v_{1b}, v_{1r}, \ldots, v_{(m-1)b}, v_{(m-1)r}$, where points with subscript b are colored black and those with subscript r are colored red. Assume that v_{0b} is the point $(1, 0)$. Since the black points are evenly spaced,

$$v_{ib} = \left(\cos\left(\frac{2\pi i}{m}\right), \sin\left(\frac{2\pi i}{m}\right)\right).$$

In a type 2 distribution of points, the red points are obtained by rotating the black points about the origin through an angle α where α is not an integer multiple of $\frac{\pi}{m}$. If R_α is rotation by α about the origin,

$$v_{ir} = R_\alpha(v_{ib}) = \left(\cos\left(\frac{2\pi i}{m} + \alpha\right), \sin\left(\frac{2\pi i}{m} + \alpha\right)\right).$$

Consider point $v_{0b} = (1, 0)$. Suppose that one of the lines of the symmetry class passes through point v_{0b} and point v_{ir}. Since there are two lines from the symmetry class incident with every point, in particular, there are two lines from the symmetry class incident with the point v_{0b}. That is, there is a line in the symmetry class which passes through v_{0b} and some other red vertex v_{jr}. Moreover, symmetry conditions imply that the reflection through the horizontal axis (i.e., the mirror passing through $(0, 0)$ and v_{0b}) must map the line $\langle v_{0b}, v_{ir}\rangle$ to the line $\langle v_{0b}, v_{jr}\rangle$.

Since the reflection of v_{ir} over the horizontal axis is the point

$$\left(\cos\left(\frac{2\pi i}{m} + \alpha\right), -\sin\left(\frac{2\pi i}{m} + \alpha\right)\right),$$

it follows that

$$\left(\cos\left(\frac{2\pi i}{m}+\alpha\right), -\sin\left(\frac{2\pi i}{m}+\alpha\right)\right) = \left(\cos\left(\frac{2\pi j}{m}+\alpha\right), \sin\left(\frac{2\pi j}{m}+\alpha\right)\right)$$

for some j, and hence

$$-\left(\frac{2\pi i}{m}+\alpha\right) = \frac{2\pi j}{m}+\alpha$$

so that $\alpha = -\frac{\pi}{m}(i+j)$. This is a contradiction, since it was assumed that α is not an integer multiple of $\frac{\pi}{m}$. $\square$

3 $[2s, 2]$ and $[2, 2t]$ configurations

Note that the situation for $[2s, 2]$ and $[2, 2t]$ astral configurations is quite different from that of $[2s, 2t]$ configurations where $s, t \geq 2$. For example, as will be shown below,(p_{2s}, n_2) configurations exist whenever p greater than $2s$, while if $s, t \geq 2$, (p_{2s}, n_{2t}) configurations may possibly exist only if p is divisible by 12. Thus, the treatment of $[2s, 2]$ and $[2, 2t]$ configurations is separate from the other cases.

3.1 $[2, 2]$ configurations

A $[2, 2]$ configuration, i.e., a (n_2) configration, has 2 points on each line and two lines through each point. A type 1 astral (n_2) configuration has a single symmetry class of points and a single symmetry class of lines, and so may be viewed as a regular p-gon (including the star polygons). If the lines of the configuration are viewed as diagonals of span a, then the configuration may be denoted by $n\#a$. Thus:

Theorem 3.1. *Type 1 (n_2) configurations exist for all integers $n \geq 3$.*

Proposition 3.2. *All $[2, 2]^2$ configurations are ordinary.*

Proof. The single symmetry class of points in a $[2, 2]^2$ configuration is a type 2 distribution. If the points of the type 2 distribution are colored red and black as before, Lemma 2.4 implies that the single symmetry class of lines must connect black points to black points and red points to red points. Thus, the collection of black points and their connecting lines forms a $[2, 2]^1$ subconfiguration, as does the collection of red points and their connecting lines, so the $[2, 2]^2$ configration is ordinary. $\square$

Theorem 3.3. *Type 2 (n_2) configurations exist for all even integers $n \geq 6$.*

3.2 $[2s, 2]$ configurations

A $[2s, 2]$ astral configuration has $2s$ lines through each point, forming s symmetry classes. Type 1 configurations may be denoted $n\#a^1,\ a^2, \ldots,\ a^s$, where each of the symmetry classes of lines is formed from diagonals of a regular n-gon of span a^i (with the superscript merely for indexing purposes, to distinguish a line of span a^i from a line of span a with intersection point i, denoted a_i).

Theorem 3.4. *Astral $[2s, 2]^1$ configurations exist whenever $\frac{p}{2} > s$.*

Proof. For example, one way to construct such a configuration is $p\#1, 2, \ldots s$. □

An example is shown in Figure 8, where $p = 11$ and $s = 3$.

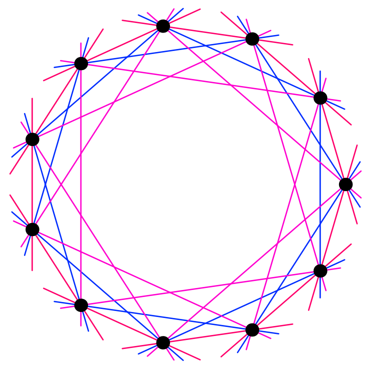

Figure 8: An $(11_6, 33_2)$ configuration, with symbol $11\#1, 2, 3$.

Theorem 3.5. *All astral $[2s, 2]^2$ configurations are ordinary.*

Proof. Note that it follows from Lemma 2.4 that each symmetry class of lines connects black points to black points and red points to red points. Thus, the subset consisting of all black points and their connecting lines forms an astral $[2s, 2]^1$ configuration, so astral $[2s, 2]^2$ configurations must be formed from two concentric copies of a $[2s, 2]^1$ configuration with one rotated arbitrarily with respect to the other. □

Theorem 3.6. *Astral $[2s, 2]^2$ configurations exist for all even integers $p > 2s$.*

3.3 $[2, 2t]$ configurations

Note that the polar of a $[2s, 2]$ configuration is a $[2, 2s]$ configuration. For completeness and for notation, the following results are presented.

A $[2, 2t]$ astral configuration has a single symmetry class of lines and t symmetry classes of points, which lie on concentric circles. Since each symmetry class of points has the same cardinality, either all symmetry classes of points form the vertices of regular polygons or none of them do. In the latter case, as has been discussed previously, the points in a single symmetry class must be distributed as in Figure 7.

In the case of an astral $[2, 2t]^1$ configuration, the various symmetry classes of points fall on intersection points of the single span of diagonals: these may be labelled $n\#a_{b_1}, a_{b_2}, \ldots, a_{b_t}$. Figure 9 is an example of a $[2, 6]$ configuration denoted $10\#3_1, 3_2, 3_3$ or, more compactly, $10\#3_1, 3_2$, where the outside vertices with label 3_3 are understood to be part of the configuration. In general, an astral $[2, 2t]^1$ configuration may constructed whenever $n > 2t$ and $p = nt$; one way to do this is $n\#a_1, a_2, \ldots, a_t$, where a is any line of span at least t.

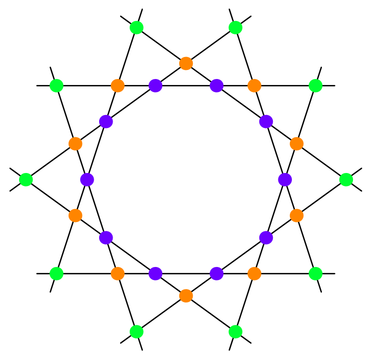

Figure 9: An astral type 1 configuration $(30_2, 10_6)$

4 Astral $[4, 4]$ configurations

Astral configurations of class $[4, 4]$ — that is, astral (n_4) configurations — have been characterized completely, beginning in [8] and finishing in [1]. For clarity in the subsequent discussion, I will summarize the main results.

Following the notation in [8], a $[4, 4]^1$ configuration, where the vertices of each symmetry class of points forms an m-gon, will be notated as $m\#a_b\, c_d$, where m is the number of vertices of the outside m-gon and a and c are the spans of diagonals of the m-gon corresponding to lines of the configuration. Note the difference in symbols from those in the previous section. Since a $[4, 4]$ configuration must have four lines passing through each point, b and d must be chosen so that a_b and c_d are the same point of the configuration.

Theorem 4.1. *All* $[4, 4]^1$ *configurations are listed in the following: there are two infinite families,* $(6k)\#(3k - j)_{3k-2j}\, (2k)_j$ *for* $j = 1, \ldots, 2k - 1, k > 1, j \neq k$ *and* $j \neq \frac{3k}{2}$*, and* $(6k)\#(3k - 2j)_j\, (3k - j)_{2k}$*, for* $k > 1, j = 1, \ldots, k - 1$*. There are 27 connected sporadic configurations, with* $m = 30, 42,$ *and 60, listed in Table 1, where a configuration is sporadic* *if it is not a member of one of the infinite families. Finally, there are multiples of the sporadic configurations.*

$m = 30$

$30\#4_1\ 7_6$	$30\#6_1\ 7_4$	$30\#6_1\ 11_{10}$
$30\#6_2\ 8_6$	$30\#7_2\ 12_{11}$	$30\#8_1\ 13_{12}$
$30\#10_1\ 11_6$	$30\#10_6\ 12_{10}$	$30\#10_7\ 13_{12}$
$30\#11_2\ 12_7$	$30\#11_6\ 14_{13}$	$30\#12_1\ 13_8$
$30\#12_4\ 14_{12}$	$30\#12_7\ 13_{10}$	$30\#13_6\ 14_{11}$

$m = 42$

$42\#6_1\ 13_{12}$	$42\#11_6\ 18_{17}$	$42\#12_1\ 13_6$
$42\#12_5\ 19_{18}$	$42\#17_6\ 18_{11}$	$42\#18_5\ 19_{12}$

$m = 60$

$60\#9_2\ 22_{21}$	$60\#12_5\ 25_{24}$	$60\#14_3\ 27_{26}$
$60\#21_2\ 22_9$	$60\#24_5\ 25_{12}$	$60\#26_3\ 27_{14}$

Table 1: The sporadic astral $[4, 4]^1$ configurations

In addition, $[4, 4]^2$ configurations were classified in the following (slightly restated from [1]):

Theorem 4.2. *All type 2 (n_4) configurations are ordinary.*

The proof of Theorem 4.1 was the main content of [1].

5 Configurations of class $[2s, 2t]$, for $s, t \geq 2$

5.1 Some general results for even configurations

Lemma 5.1. *Every $[2s, 4]^2$ configuration is ordinary.*

Proof. The proof proceeds by induction on the number of symmetry classes of lines through each point. The base case, that all type 2 (n_4) configurations are ordinary, has been shown in Theorem 4.2.

Suppose there exists an astral $[2s, 4]^2$ configuration, where $s \geq 3$. Removing one of the symmetry classes of lines yields an astral $[2s - 2, 4]^2$ configuration. By induction, this is an ordinary astral $[2s - 2, 4]^2$ configuration, made up of two copies of an astral $[2s - 2, 4]^1$ configuration. For convenience, suppose that one of the subconfigurations is colored red and the other is colored black and that both subconfigurations are of size m. Note that the red configuration is the black configuration rotated through an angle α, where $\alpha \neq \frac{t\pi}{m}$ for any integer t.

By Lemma 2.4, the lines of the symmetry class which was removed connect black vertices to black vertices and red vertices to red vertices. Hence, the black configuration with no lines removed is an astral $[2s, 4]^1$ configuration, as is the red configuration, so the original astral $[2s, 4]^2$ configuration is ordinary. □

Corollary 5.2. *Every* $[4, 2t]^2$ *configuration is ordinary.*

Proof. This follows from Lemma 5.1 and polarity. □

Corollary 5.3. *Every* $[2s, 2t]^2$ *configuration is ordinary.*

Proof. This follows by induction on the number of lines passing through a point; note that $s \geq 2$. Corollary 5.2 proves the base case. Given an extraordinary $[2s, 2t]^2$ configuration for $s > 2$, removal of one of the symmetry classes of lines yields a $[2s-2, 2t]^2$ configuration which must be ordinary by the induction hypothesis. Color one of the $[2s-2, 2t]^2$ component configurations red and the other black; again, the additional class of lines must be added connecting black vertices to black vertices and red vertices to red vertices. The argument that these additions only produce ordinary type 2 configurations proceeds identically as in Lemma 5.1, substituting the ordinary $[2s-2, 2t]^2$ configurations for the $[2s-2, 4]^2$ configurations used in the proposition. □

6 Astral $[6, 4]$ configurations

An astral $[6, 4]^1$ configuration has three symmetry classes of lines and two symmetry classes of points. Consider the outer symmetry class of points to be the vertices of a regular m-gon; then the three classes of lines may be viewed as diagonals of that m-gon. The diagonals must have some spans associated with them, say a, c, and z. The remaining symmetry class of points is interior to the m-gon by construction. Following the notation introduced for astral (n_4) configurations, an astral $[6, 4]^1$ configuration may be denoted $m\#a_b\, c_d\, z_w$, where a_b, c_d, and z_w represent the same point of the configuration.

If one of the symmetry classes of lines is ignored, the result is a configuration with four points on every line, four lines through every point, two symmetry classes of lines, and two symmetry classes of points: that is, an astral $[4, 4]$ configuration. Therefore, every astral $[6, 4]$ configuration must be made up of three astral $[4, 4]$ configurations, one corresponding to each way a symmetry class of lines can be ignored.

Proposition 6.1. *The only astral* $[6, 4]^1$ *configurations are the following:* $30\#8_1\ 10_7\ 13_{12}$, $30\#6_1\ 7_4\ 11_{10}$, $30\#11_2\ 12_7\ 13_{10}$, $30\#9_3\ 10_6\ 12_{10}$, $30\#10_1\ 11_6\ 14_{13}$, *and multiples of these.*

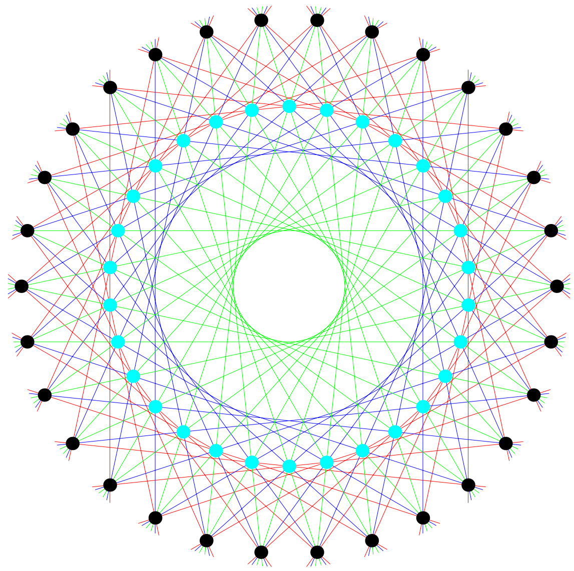

Figure 10: The astral $[6,4]^1$ configuration with symbol $30\#8_1\,10_7\,13_{12}$

Proof. Three astral (n_4) configurations will combine into an astral $[6,4]^1$ configuration only if they are type 1 astral (n_4) configurations with the following relationships between the symbols:

$$\{m\#a_b\,c_d,\ m\#a_b\,z_w,\ m\#c_d\,z_w\}.$$

To see this, note if only the span a and c lines are considered, then they form an astral (n_4) configuration with symbol $m\#a_b\,c_d$. On the other hand, using the z-span and a-span lines forms a configuration $m\#a_b\,z_w$. Since the $[6,4]$ configuration has only two symmetry classes of points, it follows that a_b, c_d and z_w all represent the same point (class) of the configuration. Thus, $m\#c_d\,z_w$ must also be a configuration.

Recall from Theorem 4.1 that there are two infinite families of astral (n_4) configurations: family 1, consisting of configurations $(6k)\#(3k-j)_{3k-2j}\,(2k)_j$ for $k>1, j=1,\ldots,2k-1, j\neq k$ and $j\neq\frac{3k}{2}$, and family 2, consisting of configurations $(6k)\#(3k-2j)_j\,(3k-j)_{2k}$, for $k>1, j=1,\ldots,k-1$; and there are 27 sporadics plus multiples of those sporadics.

Case 1: Inspection of the list of sporadics shows that there is no triple consisting entirely of sporadics.

Case 2: Two (n_4) configurations come from a single infinite family. By the previous remarks, they must look either like (1) $m\#a_b c_d$ and $m\#a_b z_w$ or (2) $m\#a_b c_d$ and $m\#z_w a_b$, in order — i.e., they agree in a span or they agree in the different span. In family 1, (1) implies there exists $j_1\neq j_2$ with $3k-j_1=3k-j_2$, since $a=a$, forcing $j_1=j_2$, a contradiction; in family 2, (1) implies a similar contradiction. In family 1, (2) implies there exists a j such that $3k-j=2k$, since $a=a$, which implies that $j=k$, which is not

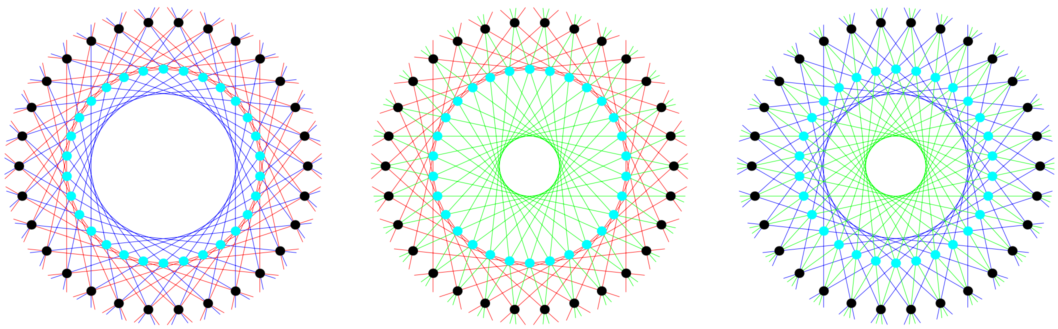

Figure 11: The three $[4,4]$ configurations which together form the $[6,4]^1$ configuration in Figure 10; from left to right, they are $30\#10_7\,13_{12}$, $30\#8_1\,10_7$, and $30\#8_1\,13_{12}$.

a valid choice of j for family 1. A similar contradiction is reached with family 2 using (2) by observing that since $b = b$, $j = 2k$, which is an invalid choice for family 2.

Therefore, any triple which forms an astral $[6,4]^1$ configuration must contain at least one of the sporadic (n_4) configurations or their multiples, and at least one configuration which is not sporadic.

Suppose a triple contains a q-multiple of a sporadic configuration—i.e., it consists of q copies of one of the sporadic (n_4) configurations listed in Table 1. Then $m = 30q$, $42q$, or $60q$. Hence, for the infinite families, $m = 30q$, $42q$, or $60q$ as well, but since $m = 6k$, it follows that $k = 5q, 7q$, or $10q$.

Case 3: A triple contains one (n_4) configuration from each infinite family and one sporadic configuration. If the (n_4) configurations are $m\#a_b\,c_d$ from family 1 and $m\#a_b\,z_w$ from family 2, then easy algebra as in case 1 shows that using j_1 with family 1 and j_2 with family 2 gives that $j_1 = 2j_2$ and $j_1 = \frac{6k}{5}$. Since the triple must contain a sporadic configuration or its multiple, using $k = 5q$ yields the pair of configurations $\{(30q)\#(9q)_{(3q)}\,(10q)_{(6q)},\ (30q)\#(9q)_{(3q)}\,(12q)_{(10q)}\}$ which may be joined with the q-multiple of the sporadic configuration $30\#10_6\,12_{10}$ to yield the astral $[6,4]$ configuration $(30q)\#(9q)_{(3q)}(10q)_{(6q)}(12q)_{(10q)}$. Using $k = 10q$ yields a multiple of the $[6,4]$ configuration found using $k = 5q$, and no configuration exists using $k = 7q$.

Case 4: a triple contains two sporadic configurations. It suffices to determine what pairs of sporadic configurations exist of the form $\{m\#a_b\,c_d, m\#a_b\,z_w\}$ where the order of the pairs a_b and z_w is now irrelevant. Inspection yields the following pairs for the $m = 30$ sporadics: $\{30\#8_1 13_{12},\ 30\#10_7 13_{12}\}$; $\{30\#6_1 7_4,\ 30\#6_1 11_{10}\}$; $\{30\#12_7 13_{10},\ 30\#11_2 12_7\}$; $\{30\#10_1\,11_6,\ 30\#11_6\,14_{13}\}$. No pairs were found using the $m = 42$ sporadics or the combination of the $m = 60$ sporadics and twice the $m = 30$ sporadics (except those already found). Each of these pairs combines with a member of one of the infinite families to form the needed triple. These, together with the triple found in case 3, form the astral $[6,4]$ configurations listed in the theorem. □

The type 1 astral $(60_6, 90_4)$ configuration $30\#8_1\,10_7\,13_{12}$ has been exhibited in Figure 10. The other four type 1 $(60_6, 90_4)$ astral configurations are listed in Figures 12 and 13.

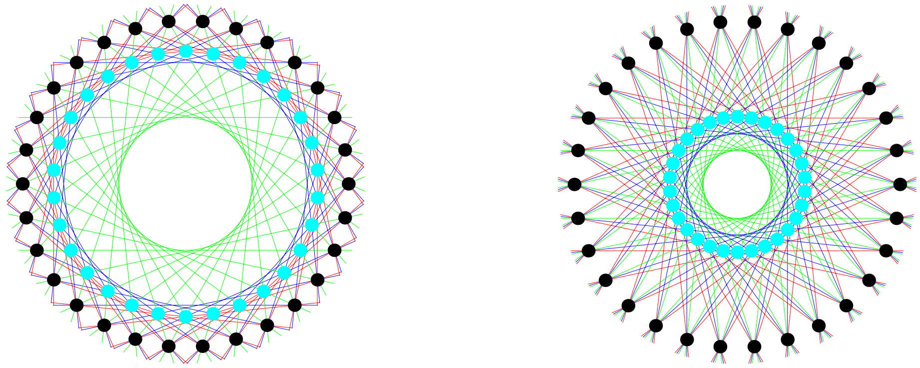

Figure 12: Left: The astral $[6,4]^1$ configuration $30\#6_1\,7_4\,11_{10}$. Right: The astral $[6,4]^1$ configuration $30\#11_2\,12_7\,13_{10}$.

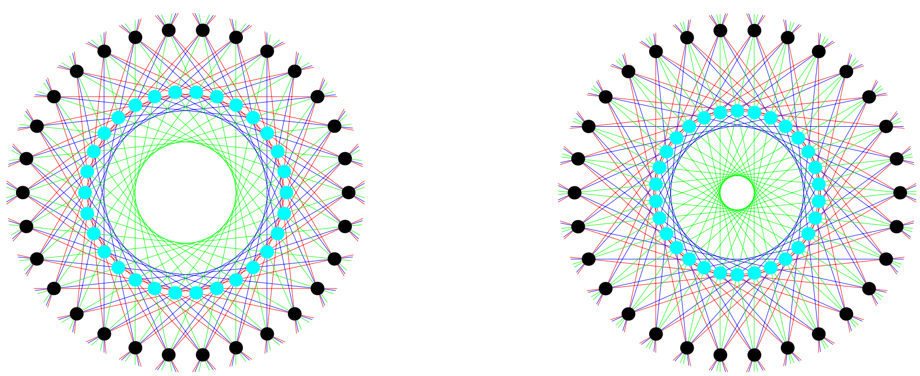

Figure 13: Left: The astral $[6,4]^1$ configuration $30\#9_3\,10_6\,12_{10}$. Right: The astral $[6,4]^1$ configuration $30\#10_1\,11_6\,14_{13}$.

Theorem 6.2. *These are all the astral* $[6,4]$ *configurations: the type 1 configurations* $30\#8_1\,10_7\,13_{12}$, $30\#6_1\,7_4\,11_{10}$, $30\#11_2\,12_7\,13_{10}$, $30\#9_3\,10_6\,12_{10}$, $30\#10_1\,11_6\,14_{13}$, *multiples of these, and ordinary type 2 configurations formed from the already-listed configurations.*

Proof. The type 1 configurations were determined in Proposition 6.1 and the fact that all $[6,4]^2$ configurations are ordinary is a consequence of Lemma 5.1. □

7 Astral $[4,6]$ configurations

None of the astral $[6,4]^1$ configurations listed above contains a diameter, so their polars through a circle concentric with the configuration are astral $[4,6]^1$ configurations. No astral $[4,6]^1$ configuration may contain a diameter by reasons of symmetry, since each point must have four lines passing through it, two from each symmetry class, and only one diameter can pass through a given point. It follows that there are no other possible astral $[4,6]^1$ configurations. As before, all $[4,6]^2$ configurations are ordinary.

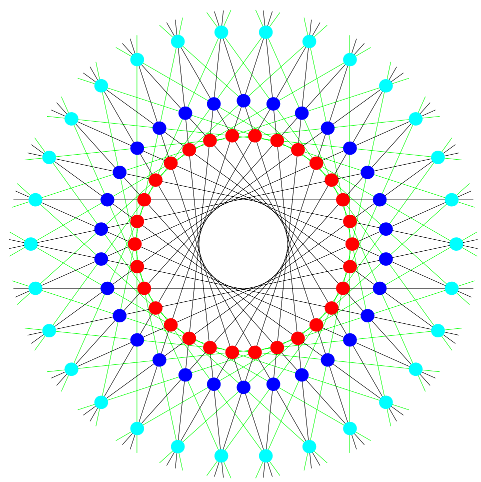

Figure 14: The astral $[4,6]^1$ configuration $30\#(10_2\,13_{11})(10_7\,13_{12})$.

In [7], Branko Grünbaum showed using elementary trigonometry that the dual of an astral (n_4) configuration with symbol $m\#a_b\,c_d$ is the configuration with symbol $m\#d_b\,c_a$ (assuming, without loss of generality, that $c > a$). Using this information and the facts that three (n_4) configurations combine to form the astral $[6,4]^1$ configuration and that every astral $[4,6]^1$ configuration is the dual of an astral $[6,4]^1$ configuration, one can easily devise symbols associated with the astral $[4,6]^1$ configurations. An astral $[4,6]^1$ configuration formed from three astral (n_4) configurations $m\#a_b\,c_d$, $m\#a_e\,c_f$, $m\#b_d\,e_f$ will be denoted $m\#(a_b\,c_d)(a_e\,c_f)$, where the symbols enclosed in parentheses represent one of the intersection points of the a and c diagonals.

The astral $[4,6]$ configurations are the following: $30\#(12_1\,13_8)(12_7\,13_{10})$, $30\#(10_4\,11_7)(10_1\,11_6)$, $30\#(10_2\,13_{11})(10_7\,13_{12})$, $30\#(10_3\,12_9)(10_6\,12_{10})$, $30\#(13_1\,14_{10})(13_6\,14_{11})$, and their multiples, plus ordinary type 2 configurations formed from these (by Lemma 5.1). They are listed here in the order which corresponds to that of the list of $[6,4]$ configurations to which they are polar. The configuration $30\#(10_2\,13_{11})(10_7\,13_{12})$ is shown in Figure 14.

8 Astral (n_6) configurations

An astral (n_6) configuration has three symmetry classes of points and three symmetry classes of lines with six lines through every point (two of each class) and six points on every line (also two of each class). Removing a class of points leaves an astral $[6,4]$ configuration, and removing a class of lines leaves an astral $[4,6]$ configuration.

Theorem 8.1. *No astral (n_6) configurations exist.*

Proof. Suppose there exists a type 1 astral (n_6) configuration. Choose one class of lines to remove, say of span z; the remainder is an astral $[4,6]^1$ configuration, say $m\#(a_b c_d)(a_e c_f)$, which is formed from the three (n_4) configurations $m\#a_b\, c_d$, $m\#a_e\, c_f$, and $m\#b_e\, d_f$. The span z lines which were removed must pass through precisely the points already identified, which have symbols $a_b = c_d$ and $a_e = c_f$. That is, there must be some intersection points w and x so that $a_b = c_d = z_w$ and $a_e = c_f = z_x$. This information yields two $[6,4]$ configurations which must exist if the type 1 astral (n_6) configuration exists: they are $\{m\#a_b\, c_d,\ m\#a_b\, z_w,\ m\#c_d\, z_w\} = m\#a_b\, c_d\, z_w$ and $\{m\#a_e\, c_f,\ m\#a_e\, z_x,\ m\#c_f\, z_x\} = m\#a_e\, c_f\, z_x$. (The configuration $m\#b_d\, e_f\, w_x$ also must exist; however, it suffices to consider only the first two configurations.)

That is, for a type 1 astral (n_6) configuration to exist, it must be possible to find two astral $[6,4]^1$ configurations with the same set of three spans a, c, z but different intersection points. But the spans of possible $[6,4]^1$ configurations are $\{8q, 10q, 13q\}$, $\{6q, 7q, 11q\}$, $\{11q, 12q, 13q\}$, $\{9q, 10q, 12q\}$, and $\{10q, 11q, 14q\}$ for any positive number q (obtained by taking q equally spaced copies of the corresponding astral $[6,4]$ configuration with $m = 30$). Since the ratios of the spans are different, these do not combine as necessary to form an astral (n_6) configuration of type 1. Hence, there are no astral $[6,6]^1$ configurations.

Corollary 5.3 says that if any type 2 (n_6) configurations exist, they must be formed from two type 1 configurations. Since no type 1 configurations exist, it follows that no type 2 configurations exist either. □

Corollary 8.2. *No astral configurations $[2s, 2t]$ exist where s and $t \geq 3$.*

Proof. Combine Theorem 8.1 with Lemma 2.1. □

9 Astral $[q,k]$ configurations for q or $k \geq 8$

In [11], Poonen and Rubinstein prove that it is impossible to have eight or more diagonals of a regular polygon meeting at a point other than the center. Since the lines of an astral type 1 configuration of size m may be viewed as diagonals of a regular m-gon, it immediately follows that there are no astral $[q,8]^1$ configurations. The polar of a $[q,8]^1$ configuration is a $[8,q]^1$ configuration. Since in an $[8,q]$ configuration there are an even number of lines passing through each point, diameters cannot be lines of the $[8,q]^1$ configuration. Hence any $[8,q]^1$ configuration must occur as the polar of an astral $[q,8]^1$ configuration, so there are no $[8,q]^1$ configurations either. Lemma 5.3 implies that there are no $[q,8]^2$ or $[8,q]^2$ configurations for even $q \geq 8$ and in particular, there are no $[4,8]$ and $[8,4]$ configurations.

Theorem 9.1. *For $s \geq 2$ and $t \geq 4$, the configurations $[2s, 2t]$ and $[2t, 2s]$ do not exist, either of type 1 or of type 2.*

Proof. This follows immediately from the preceeding discussion and Lemma 2.1. □

10 A summary of known results about odd astral configurations

There are several results known about the classification of odd astral configurations. For completeness, they are summarized without proof here. They will be discussed more thoroughly in a subsequent paper.

10.1 Astral $[2s, 2t+1]^1$ and $[2t+1, 2s]^1$ configurations

Lemma 10.1. *If an astral $[2s, 2t+1]^1$ configuration exists, then an astral $[2s, 2t+2]^1$ configuration must also exist. Hence, if no astral $[2s, 2t+2]^1$ configuration exists, then no astral $[2s, 2t+1]^1$ configuration exists, either.*

Corollary 10.2. *If an astral $[2t+1, 2s]^1$ configuration exists which does not use diameters, then an astral $[2t+2, 2s]^1$ configuration must also exist. Hence, if no astral $[2t+2, 2s]^1$ configuration exists, then if an astral $[2t+1, 2s]^1$ configuration exists, it must be constructed by adding diameters to a $[2t, 2s]^1$ astral configuration.*

10.2 Astral $[4, 5]^1$ and $[5, 4]^1$ configurations

By $m\#(a_b\ c_d)(a_e\ c_f)^*$ denote the $[4, 5]^1$ configuration which has vertices with symbols $(a_a)_i = (c_c)_i$, $(a_b)_i = (c_d)_i$ for all i and $(a_e)_i = (c_f)_i$ for $i = 0, 2, 4, \ldots, m-2$ (so that every other vertex in the a_e ring is used).

Theorem 10.3. *The only astral $[4, 5]^1$ configurations are $(30q)\#((10q)_{(6q)}\ (12q)_{(10q)})((10q)_{(3q)}\ (12q)_{(9q)})^*$, where q is odd.*

Note that the polar of $(30q)\#((10q)_{(6q)}\ (12q)_{(10q)})((10q)_{(3q)}\ (12q)_{(9q)})^*$ is an astral $[5, 4]^1$ configuration. However, many more $[5, 4]^1$ configurations may be formed by adding diameters appropriately:

Theorem 10.4. *Diameters may be added to the following astral (n_4) configurations to yield astral $[5, 4]^1$ configurations:*

1. $(2t) \cdot m\#a_b\ c_d$ *for any astral configuration* $m\#a_b\ c_d$*;*

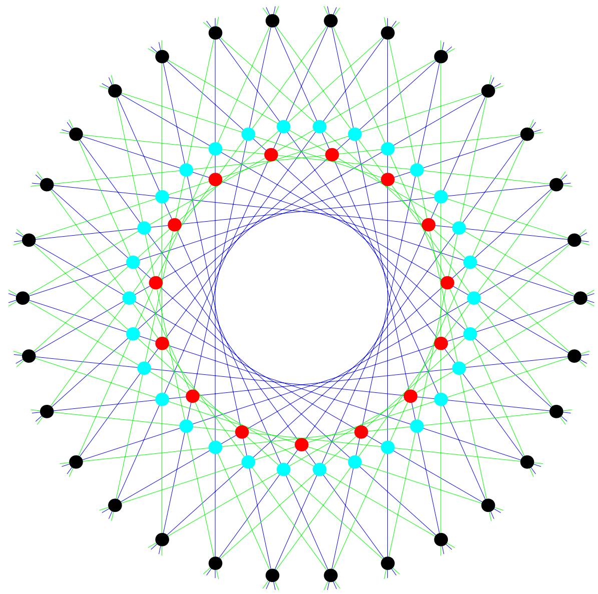

Figure 15: The astral $[4,5]^1$ configuration $30\#(10_6\,12_{10})(10_3\,12_9)^*$.

2. *$(6k)\#(3k-j)_{(3k-2j)}\,2k_j$, if j is even and k is odd (the case where j and k are both even is listed in (1));*

3. *$(6k)\#(3k-2j)_j\,3k-j_{2k}$, if j and k are both odd (the case where j and k are both even is listed in (1));*

4. *$q\cdot 30\#6_2\,8_6$, $q\cdot 30\#10_6\,12_{10}$, $q\cdot 30\#12_4\,14_{12}$ for any (odd) q (the case q is even is listed in (1)).*

10.3 Mixed configurations

There is another way to construct astral $[4,2t+1]^2$ configurations. Given an astral $[4,2t]^1$ configuration of size m, with the two symmetry classes of lines of span a and span c, there are many points of intersection of a single span a diagonal with a single span c diagonal (i.e., not an intersection point that participates in a 4-diagonal intersection); these points will be called *embryonic*. Choose one of them, called $\mathbf{x}$; it is not on a mirror of symmetry of the configuration. To see this, note that each point of intersection of a span a diagonal with another span a diagonal lies on one of the lines of symmetry of the $[4,2t]^1$ configuration, which has the symmetries of a regular m-gon. For the chosen point of intersection to lie on a line of symmetry, it would also have to be part of an a-a intersection, and symmetry would force it to be a 4-diagonal intersection point. But it was chosen to be the intersection of precisely two diagonals, a span a-diagonal and a span c-diagonal.

Assume the $[4, 2t]^1$ configuration is centered at the origin with one of its vertices located at the point $(1, 0)$. Let the angle formed by the ray $\langle(0, 0), \mathbf{x}\rangle$ and horizontal be called α. Take another copy of the $[4, 2t]^1$ configuration and rotate it through 2α (about the origin); color the original configuration black and the rotated configuration red. This yields a configuration with four diagonals passing through point $\mathbf{x}$: the black a and c diagonals that passed through $\mathbf{x}$ originally and the red a and c diagonals from the rotated configuration. If all of the points $\mathbf{x}$ formed in the same manner are taken as points of the configuration as well, the result is an astral $[4, 2t + 1]^1$ configuration. A configuration constructed in this fashion will be called a *mixed* configuration. Figure 16 shows a mixed $[4, 5]^2$ configuration.

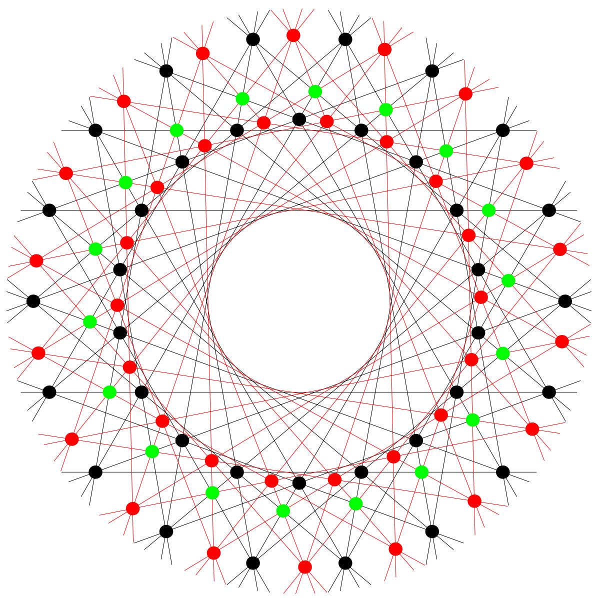

Figure 16: An astral $[4, 5]^2$ configuration mixed from two $18\#6_2\,7_5$ configurations, using the third a-c intersection. Note that this configuration violates the usual coloring convention, where elements in the same symmetry classes are colored the same color; in this configuration, one of the $[4, 4]$ subconfigurations is colored red and the other is colored black, and the green points are formed from the corresponding embryonic points.

Lemma 10.5. *Mixed* $[4, 2t + 1]^2$ *configurations are astral.*

Lemma 10.6. *The only astral* $[2s, 2t + 1]^2$ *configurations are ordinary and mixed.*

10.4 Astral $[6, 5]$ and $[5, 6]$ configurations

Theorem 10.7. *There are no astral* $[6, 5]^1$ *configurations and no ordinary* $[6, 5]^2$ *configurations.*

Theorem 10.8. *The only mixed astral* $[6,5]^2$ *configurations whose special class of points lie closer to the center of the configuration than one of the non-special classes of points are formed from the astral* $[6,4]^1$ *configurations* $30\#8_1\, 10_7\, 13_{12}$ *and* $30\#10_1\, 11_6\, 14_{13}$.

Theorem 10.9. *Polars of mixed astral* $[6,5]^2$ *configurations are astral* $[5,6]^2$ *configurations. The only astral* $[5,6]^1$ *configurations are those formed by adding diameters to any even multiple of a* $[4,6]$ *configuration.*

10.5 Astral $[7,4]$ and $[4,7]$ configurations

Theorem 10.10. *The only astral* $[7,4]^1$ *configurations are those formed by adding diameters to any even multiple of a* $[6,4]$ *configuration and to any multiple of the configuration* $30\#9_3\, 10_6\, 12_{10}$.

There are no astral $[4,7]^1$ configurations. Starting with a $[4,6]^1$ configuration, it is easy to construct mixed $[4,7]^2$ configurations, using the same embryonic points and construction methods which were used to construct mixed $[4,5]^2$ configurations. The polars of these configurations will be $[7,4]^2$ configurations.

10.6 $[5,5]$, $[5,7]$, $[7,5]$ configurations

Proposition 10.11. *There are no astral* $[5,5]^1$ *configurations.*

Proposition 10.12. *There are no astral* $[5,5]^2$ *configurations mixed from two* $[5,4]^1$ *configurations.*

Conjecture 1. *There are no astral* $[5,5]$ *configurations.*

Conjecture 2. *There are no astral* $[7,5]$ *and* $[5,7]$ *configurations.*

11 Conclusions and Open Questions

For $[2,2s]$ and $[2t,2]$ astral configurations, it is easy to construct configurations, but they are rather uninteresting. Astral $[4,4]$ configurations are more constrained, but there are still a variety of configurations. As things get more constrained, with the $[6,4]$ and $[4,6]$ configurations, it is very hard to construct astral configurations, so much so that there are really only five kinds of each configuration. If any additional constraints are added, as in the $[4,8]$ or $[6,6]$ configurations, then no configurations exist.

The situation with odd configurations is much more complicated. Type 1, ordinary type 2, and extraordinary type 2 $[q,k]$ astral configurations all may exist, depending on the choices for q and k. As discussed in section 10, a partial classification of odd astral $[q,k]$ configurations exists for $q,k \geq 4$ (also see [2]; odd astral $[q,k]$ configurations for

$q, k \geq 4$ are classified except for classes $[5, 5]$, $[5, 7]$ and $[7, 5]$). Behavior of astral (n_3) configurations is even stranger: it is known, for example, that there exist families of astral (n_3) configurations with several discrete parameters and *one continuous parameter!* This is far removed from the case of even astral configurations, where to a choice of discrete parameters (e.g., m, a, b, c, d to form a $[4, 4]$ configuration $m\#a_b\, c_d$) there exists at most one corresponding configuration. Some $[3, k]$ and $[q, 3]$ configurations and results are known for $q, k \geq 3$, but a general theory does not yet exist.

References

[1] Berman, L.W. A characterization of astral (n_4) configurations. *Discrete and Computational Geometry.* **26** 4 (2001), 603 – 612.

[2] Berman, L.W. Astral Configurations. Ph.D. Thesis, University of Washington. (2002)

[3] Bol, G. Beantwoording van prijsvraag no. 17. *Nieuw Archief voor Wiskunde* **18** (1936), 14 – 66.

[4] Betten, A., Brinkmann, G., Pisanski, T. Counting symmetric configurations v_3. *Discrete and Applied Mathematics* **99** (2000), 331 – 338.

[5] Gropp, H. Configurations and their realization. *Discrete Mathematics* **174** (1997), 137 – 151.

[6] Grünbaum, B. Astral (n_k) configurations. *Geombinatorics* **3** (1993), 32 – 37.

[7] Grünbaum, B. *Configurations*. Unpublished manuscript/class notes. 1999.

[8] Grünbaum, B. Astral (n_4) configurations. *Geombinatorics* **9** (2000), 127 – 134.

[9] Grünbaum, B. Which (n_4) configurations exist? *Geombinatorics* **9** (2000), 164 – 169.

[10] Grünbaum, B. and J. F. Rigby. The real configuration (21_4). *J. London Math. Soc (2).* **41** (1990), 336 – 346.

[11] Poonen, B. and M. Rubinstein. The number of intersection points made by the diagonals of a regular polygon. *SIAM J. Discrete Math.* **11** (1998), 135 – 156.

[12] Rigby, J.F. Multiple intersections of diagonals of regular polygons. *Geometriae Dedicata.* **9** (1980), 207–238.

[13] Sturmfels, B. and N. White. All 11_3 and 12_3 configurations are rational. *Aequationes Mathematicae* **39** (1990) 254 – 260.