Sign-graded posets, unimodality of W-polynomials and the Charney-Davis Conjecture

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Submitted: Jul 6, 2004; Accepted: Nov 6, 2004; Published: Nov 22, 2004
Mathematics Subject Classifications: 06A07, 05E99, 13F55

Dedicated to Richard Stanley on the occasion of his 60th birthday

Abstract

We generalize the notion of graded posets to what we call sign-graded (labeled) posets. We prove that the W-polynomial of a sign-graded poset is symmetric and unimodal. This extends a recent result of Reiner and Welker who proved it for graded posets by associating a simplicial polytopal sphere to each graded poset. By proving that the W-polynomials of sign-graded posets has the right sign at -1, we are able to prove the Charney-Davis Conjecture for these spheres (whenever they are flag).

1 Introduction and preliminaries

Recently Reiner and Welker [10] proved that the W-polynomial of a graded poset (partially ordered set) P has unimodal coefficients. They proved this by associating to P a simplicial polytopal sphere, $\Delta_{eq}(P)$, whose h-polynomial is the W-polynomial of P, and invoking the g-theorem for simplicial polytopes (see [15, 16]). Whenever this sphere is flag, i.e., its minimal non-faces all have cardinality two, they noted that the Neggers-Stanley Conjecture implies the Charney-Davis Conjecture for $\Delta_{eq}(P)$. In this paper we give a different proof of the unimodality of W-polynomials of graded posets, and we also prove the Charney-Davis Conjecture for $\Delta_{eq}(P)$ (whenever it is flag). We prove it by studying a family of labeled posets, which we call sign-graded posets, of which the class of graded naturally labeled posets is a sub-class.

*Part of this work was financed by the EC's IHRP Programme, within the Research Training Network "Algebraic Combinatorics in Europe", grant HPRN-CT-2001-00272, while the author was at Universitá di Roma "Tor Vergata", Rome, Italy.

In this paper all posets will be finite and non-empty. For undefined terminology on posets we refer the reader to [13]. We denote the cardinality of a poset P with the letter p. Let P be a poset and let $\omega : P \to \{1, 2, \dots, p\}$ be a bijection. The pair (P, ω) is called a *labeled poset*. If ω is order-preserving then (P, ω) is said to be *naturally labeled*. A (P, ω)-*partition* is a map $\sigma : P \to \{1, 2, 3, \dots\}$ such that

- σ is order reversing, that is, if $x \leq y$ then $\sigma(x) \geq \sigma(y)$,
- if $x < y$ and $\omega(x) > \omega(y)$ then $\sigma(x) > \sigma(y)$.

The theory of (P, ω)-partitions was developed by Stanley in [14]. The number of (P, ω)-partitions σ with largest part at most n is a polynomial of degree p in n called the *order polynomial* of (P, ω) and is denoted $\Omega(P, \omega; n)$. The W-polynomial of (P, ω) is defined by

$$\sum_{n \geq 0} \Omega(P, \omega; n+1) t^n = \frac{W(P, \omega; t)}{(1-t)^{p+1}}. \tag{1.1}$$

The set, $\mathcal{L}(P, \omega)$, of permutations $\omega(x_1), \omega(x_2), \dots, \omega(x_p)$ where $x_1, x_2, \dots, x_p$ is a linear extension of P is called the *Jordan-Hölder set* of (P, ω). A *descent* in a permutation $\pi = \pi_1 \pi_2 \cdots \pi_p$ is an index $1 \leq i \leq p-1$ such that $\pi_i > \pi_{i+1}$. The number of descents in π is denoted $\mathrm{des}(\pi)$. A fundamental result in the theory of (P, ω)-partitions, see [14], is that the W-polynomial can be written as

$$W(P, \omega; t) = \sum_{\pi \in \mathcal{L}(P, \omega)} t^{\mathrm{des}(\pi)}.$$

The Neggers-Stanley Conjecture is the following:

Conjecture 1.1 (Neggers-Stanley). *Let (P, ω) be a labeled poset. Then $W(P, \omega; t)$ has real zeros only.*

This was first conjectured by Neggers [8] in 1978 for natural labelings and by Stanley in 1986 for arbitrary labelings. The conjecture has been proved for some special cases, see [1, 2, 10, 17] for the state of the art. If a polynomial has only real non-positive zeros then its coefficients form a unimodal sequence. For the W-polynomials of graded posets unimodality was first proved by Gasharov [7] whenever the rank is at most 2, and as mentioned by Reiner and Welker [10] for all graded posets.

For the relevant definitions concerning the topology behind the Charney-Davis Conjecture we refer the reader to [3, 10, 16].

Conjecture 1.2 (Charney-Davis, [3]). *Let Δ be a flag simplicial homology $(d-1)$-sphere, where d is even. Then the h-vector, $h(\Delta, t)$, of Δ satisfies*

$$(-1)^{d/2} h(\Delta, -1) \geq 0.$$

Recall that the nth *Eulerian polynomial*, $A_n(x)$, is the W-polynomial of an anti-chain of n elements. The Eulerian polynomials can be written as

$$A_n(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} a_{n,i} x^i (1+x)^{n-1-2i},$$

where $a_{n,i}$ is a nonnegative integer for all i, see [5, 11]. From this expansion we see immediately that $A_n(x)$ is symmetric and that the coefficients in the standard basis are unimodal. It also follows that $(-1)^{(n-1)/2} A_n(-1) \geq 0$.

We will in Section 2 define a class of labeled poset whose members we call sign-graded posets. This class includes the class of naturally labeled graded posets. In Section 4 we show that the W-polynomial of a sign-graded poset (P, ω) of rank r can be expanded, just as the Eulerian polynomial, as

$$W(P, \omega; t) = \sum_{i=0}^{\lfloor (p-r-1)/2 \rfloor} a_i(P, \omega) t^i (1+t)^{p-r-1-2i}, \tag{1.2}$$

where $a_i(P, \omega)$ are nonnegative integers. Hence, symmetry and unimodality follow, and $W(P, \omega; t)$ has the right sign at -1. Consequently, whenever the associated sphere $\Delta_{eq}(P)$ of a graded poset P is flag the Charney-Davis Conjecture holds for $\Delta_{eq}(P)$. We also note that all symmetric polynomials with non-positive zeros only, admit an expansion such as (1.2). Hence, that $W(P, \omega; t)$ has such an expansion can be seen as further evidence for the Neggers-Stanley Conjecture. After the completion of the first version of this paper we were informed that S. Gal [6] has conjectured that if Δ is flag simplicial homology $(d-1)$-sphere, then its h-vector admits an expansion

$$h(\Delta, t) = \sum_{i=0}^{\lfloor d/2 \rfloor} a_i(\Delta) t^i (1+t)^{d-2i},$$

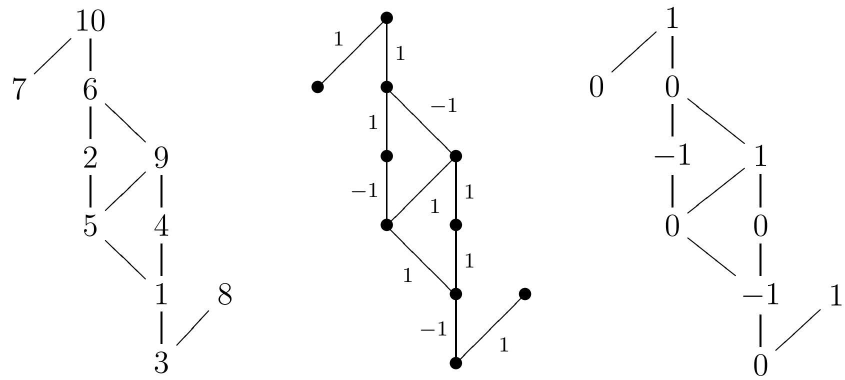
where $a_i(\Delta)$ are nonnegative integers. This would imply the Charney-Davis conjecture and (1.2) can be seen as further evidence for Gal's conjecture.

In [9] the Charney-Davis quantity of a graded naturally labeled poset (P, ω) of rank r was defined to be $(-1)^{(p-1-r)/2} W(P, \omega; -1)$. In Section 5 we give a combinatorial interpretation of the Charney-Davis quantity as counting certain reverse alternating permutations. Finally in Section 7 we characterize sign-graded posets in terms of properties of order polynomials.

2 Sign-graded posets

Recall that a poset P is *graded* if all maximal chains in P have the same length. If P is graded one may associate a *rank function* $\rho : P \to \mathbb{N}$ by letting $\rho(x)$ be the length of any saturated chain from a minimal element to x. The *rank* of a graded poset P is defined

Figure 1: A sign-graded poset, its two labelings and the corresponding rank function.



as the length of any maximal chain in P. In this section we will generalize the notion of graded posets to labeled posets.

Let (P, ω) be a labeled poset. An element y *covers* x, written $x \prec y$, if $x < y$ and $x < z < y$ for no $z \in P$. Let $E = E(P) = \{(x, y) \in P \times P : x \prec y\}$ be the covering relations of P. We associate a labeling $\epsilon : E \to \{-1, 1\}$ of the covering relations defined by

$$\epsilon(x, y) = \begin{cases} 1 \text{ if } & \omega(x) < \omega(y), \\ -1 \text{ if } & \omega(x) > \omega(y). \end{cases}$$

If two labelings ω and λ of P give rise to the same labeling of $E(P)$ then it is easy to see that the set of (P, ω)-partitions and the set of (P, λ)-partitions are the same. In what follows we will often refer to ϵ as the labeling and write (P, ϵ).

Definition 2.1. Let (P, ω) be a labeled poset and let ϵ be the corresponding labeling of $E(P)$. We say that (P, ω) is *sign-graded*, and that P is ϵ-*graded* (and ω-*graded*) if for every maximal chain $x_0 \prec x_1 \prec \cdots \prec x_n$ the sum

$$\sum_{i=1}^{n} \epsilon(x_{i-1}, x_i)$$

is the same. The common value of the above sum is called the *rank* of (P, ω) and is denoted $r(\epsilon)$.

We say that the poset P is ϵ-*consistent* (and ω-consistent) if for every $y \in P$ the principal order ideal $\Lambda_y = \{x \in P : x \leq y\}$ is ϵ_y-graded, where ϵ_y is ϵ restricted to $E(\Lambda_y)$. The *rank function* $\rho : P \to \mathbb{Z}$ of an ϵ-consistent poset P is defined by $\rho(x) = r(\epsilon_x)$. Hence, an ϵ-consistent poset P is ϵ-graded if and only if ρ is constant on the set of maximal elements.

See Fig. 1 for an example of a sign-graded poset. Note that if ϵ is identically equal to 1, i.e., if (P, ω) is naturally labeled, then a sign-graded poset with respect to ϵ is just a graded poset. Note also that if P is ϵ-graded then P is also $-\epsilon$-graded, where $-\epsilon$ is defined by $(-\epsilon)(x,y) = -\epsilon(x,y)$. Up to a shift, the order polynomial of a sign-graded labeled poset only depends on the underlying poset:

Theorem 2.2. *Let P be ϵ-graded and μ-graded. Then*

$$\Omega(P,\epsilon;t-\frac{r(\epsilon)}{2}) = \Omega(P,\mu;t-\frac{r(\mu)}{2}).$$

Proof. Let ρ_ϵ and ρ_μ denote the rank functions of (P,ϵ) and (P,μ) respectively, and let $\mathcal{A}(\epsilon)$ denote the set of (P,ϵ)-partitions. Define a function $\xi : \mathcal{A}(\epsilon) \to \mathbb{Q}^P$ by $\xi\sigma(x) = \sigma(x) + \Delta(x)$, where

$$\Delta(x) = \frac{r(\epsilon)-\rho_\epsilon(x)}{2} - \frac{r(\mu)-\rho_\mu(x)}{2}.$$

Table 1:

$\epsilon(x,y)$	$\mu(x,y)$	σ	Δ	$\xi\sigma$
1	1	$\sigma(x) \geq \sigma(y)$	$\Delta(x) = \Delta(y)$	$\xi\sigma(x) \geq \xi\sigma(y)$
1	-1	$\sigma(x) \geq \sigma(y)$	$\Delta(x) = \Delta(y)+1$	$\xi\sigma(x) > \xi\sigma(y)$
-1	1	$\sigma(x) > \sigma(y)$	$\Delta(x) = \Delta(y)-1$	$\xi\sigma(x) \geq \xi\sigma(y)$
-1	-1	$\sigma(x) > \sigma(y)$	$\Delta(x) = \Delta(y)$	$\xi\sigma(x) > \xi\sigma(y)$

The four possible combinations of labelings of a covering-relation $(x,y) \in E$ are given in Table 1.

According to the table $\xi\sigma$ is a (P,μ)-partition provided that $\xi\sigma(x) > 0$ for all $x \in P$. But $\xi\sigma$ is order-reversing so it attains its minima on maximal elements and if z is a maximal element we have $\xi\sigma(z) = \sigma(z)$. Hence $\xi : \mathcal{A}(\epsilon) \to \mathcal{A}(\mu)$. By symmetry we also have a map $\eta : \mathcal{A}(\mu) \to \mathcal{A}(\epsilon)$ defined by

$$\eta\sigma(x) = \sigma(x) + \frac{r(\mu)-\rho_\mu(x)}{2} - \frac{r(\epsilon)-\rho_\epsilon(x)}{2}.$$

Hence, $\eta = \xi^{-1}$ and ξ is a bijection.

Since σ and $\xi\sigma$ are order-reversing they attain their maxima on minimal elements. But if z is a minimal element then $\xi\sigma(z) = \sigma(z) + \frac{r(\epsilon)-r(\mu)}{2}$, which gives

$$\Omega(P,\mu;n) = \Omega(P,\epsilon;n+\frac{r(\mu)-r(\epsilon)}{2}),$$

for all nonnegative integers n and the theorem follows. □

Theorem 2.3. *Let P be ϵ-graded. Then*

$$\Omega(P,\epsilon;t) = (-1)^p\Omega(P,\epsilon;-t-r(\epsilon)).$$

Proof. We have the following reciprocity for order polynomials, see [14]:

$$\Omega(P,-\epsilon;t) = (-1)^p \Omega(P,\epsilon;-t). \tag{2.1}$$

Note that $r(-\epsilon) = -r(\epsilon)$, so by Theorem 2.2 we have:

$$\Omega(P,-\epsilon;t) = \Omega(P,\epsilon;t - r(\epsilon)),$$

which, combined with (2.1), gives the desired result. □

Corollary 2.4. *Let P be an ϵ-graded poset. Then $W(P,\epsilon;t)$ is symmetric with center of symmetry $(p - r(\epsilon) - 1)/2$. If P is also μ-graded then*

$$W(P,\mu;t) = t^{(r(\epsilon)-r(\mu))/2} W(P,\epsilon;t).$$

Proof. Suppose that $W(P,\epsilon;t) = \sum_{i\geq 0} w_i(P,\epsilon)t^i$. From (1.1) it follows that $\Omega(P,\epsilon;t) = \sum_{i\geq 0} w_i(P,\epsilon)\binom{t+p-1-i}{p}$. Let $r = r(\epsilon)$. Theorem 2.3 gives:

$$\begin{aligned}
\Omega(P,\epsilon;t) &= \sum_{i\geq 0} w_i(P,\epsilon)(-1)^p \binom{-t-r+p-1-i}{p} \\
&= \sum_{i\geq 0} w_i(P,\epsilon)\binom{t+r+i}{p} \\
&= \sum_{i\geq 0} w_{p-r-1-i}(P,\epsilon)\binom{t+p-1-i}{p},
\end{aligned}$$

so $w_i(P,\epsilon) = w_{p-r-1-i}(P,\epsilon)$ for all i, and the symmetry follows. The relationship between the W-polynomials of (P,ϵ) and (P,μ) follows from Theorem 2.2 and the expansion of order-polynomials in the basis $\binom{t+p-1-i}{p}$. □

We say that a poset P is *parity graded* if the size of all maximal chains in P have the same parity. Also, a poset is P is *parity consistent* if for all $x \in P$ the order ideal Λ_x is parity graded. These classes of posets were studied in [12] in a different context. The following theorem tells us that the class of sign-graded posets is considerably greater than the class of graded posets.

Theorem 2.5. *Let P be a poset. Then*

- *there exists a labeling $\epsilon : E \to \{-1,1\}$ such that P is ϵ-consistent if and only if P is parity consistent,*
- *there exists a labeling $\epsilon : E \to \{-1,1\}$ such that P is ϵ-graded if and only if P is parity graded.*

Moreover, the labeling ϵ can be chosen so that the corresponding rank function has values in $\{0,1\}$.

Proof. It suffices to prove the equivalence regarding parity graded posets. It is clear that if P is ϵ-graded then P is parity graded.

Let P be parity graded. Then, for any $x \in P$, all saturated chains from a minimal element to x have the same length modulo 2. Hence, we may define a labeling $\epsilon : E(P) \to \{-1, 1\}$ by $\epsilon(x, y) = (-1)^{\ell(x)}$, where $\ell(x)$ is the length of any saturated chain starting at a minimal element and ending at x. It follows that P is ϵ-graded and that its rank function has values in $\{0, 1\}$. □

We say that $\omega : P \to \{1, 2, \dots, p\}$ is *canonical* if (P, ω) has a rank-function ρ with values in $\{0, 1\}$, and $\rho(x) < \rho(y)$ implies $\omega(x) < \omega(y)$. By Theorem 2.5 we know that P admits a canonical labeling if P is ϵ-consistent for some ϵ.

3 The Jordan-Hölder set of an ϵ-consistent poset

Let P be ω-consistent. We may assume that $\omega(x) < \omega(y)$ whenever $\rho(x) < \rho(y)$. This is because any labeling ω' of P for which $\rho(x) < \rho(y)$ implies $\omega'(x) < \omega'(y)$ will give rise to the same labeling of $E(P)$ as (P, ω).

Suppose that $x, y \in P$ are incomparable and that $\rho(y) = \rho(x) + 1$. Then the Jordan-Hölder set of (P, ω) can be partitioned into two sets: One where in all permutations $\omega(x)$ comes before $\omega(y)$ and one where $\omega(y)$ always comes before $\omega(x)$. This means that $\mathcal{L}(P, \omega)$ is the disjoint union

$$\mathcal{L}(P, \omega) = \mathcal{L}(P', \omega) \sqcup \mathcal{L}(P'', \omega), \tag{3.1}$$

where P' is the transitive closure of $E \cup \{x \prec y\}$, and P'' is the transitive closure of $E \cup \{y \prec x\}$.

Lemma 3.1. *With definitions as above P' and P'' are ω-consistent with the same rank-function as (P, ω).*

Proof. Let $c : z_0 \prec z_1 \prec \cdots \prec z_k = z$ be a saturated chain in P'', where z_0 is a minimal element of P''. Of course z_0 is also a minimal element of P. We have to prove that

$$\rho(z) = \sum_{i=0}^{k-1} \epsilon''(z_i, z_{i+1}),$$

where ϵ'' is the labeling of $E(P'')$ and ρ is the rank-function of (P, ω).

All covering relations in P'', except $y \prec x$, are also covering relations in P. If y and x do not appear in c, then c is a saturated chain in P and there is nothing to prove. Otherwise

$$c : y_0 \prec \cdots \prec y_i = y \prec x = x_{i+1} \prec x_{i+2} \prec \cdots \prec x_k = z.$$

Note that if $s_0 \prec s_1 \prec \cdots \prec s_\ell$ is any saturated chain in P then $\sum_{i=0}^{\ell-1} \epsilon(s_i, s_{i+1}) = \rho(s_\ell) - \rho(s_0)$. Since $y_0 \prec \cdots \prec y_i = y$ and $x = x_{i+1} \prec x_{i+2} \prec \cdots \prec x_k = z$ are saturated chains in P we have

$$\begin{aligned}\sum_{i=0}^{k-1}\epsilon''(z_i,z_{i+1}) &= \rho(y)+\epsilon''(y,x)+\rho(z)-\rho(x)\\ &= \rho(y)-1-\rho(x)+\rho(z)\\ &= \rho(z),\end{aligned}$$

as was to be proved. The statement for (P',ω) follows similarly. □

We say that a ω-consistent poset P is *saturated* if for all $x,y\in P$ we have that x and y are comparable whenever $|\rho(y)-\rho(x)|=1$. Let P and Q be posets on the same set. Then Q *extends* P if $x<_Q y$ whenever $x<_P y$.

Theorem 3.2. *Let P be a ω-consistent poset. Then the Jordan-Hölder set of (P,ω) is uniquely decomposed as the disjoint union*

$$\mathcal{L}(P,\omega)=\bigsqcup_Q \mathcal{L}(Q,\omega),$$

where the union is over all saturated ω-consistent posets Q that extend P and have the same rank-function as (P,ω).

Proof. That the union exhausts $\mathcal{L}(P,\omega)$ follows from (3.1) and Lemma 3.1. Let Q_1 and Q_2 be two different saturated ω-consistent posets that extend P and have the same rank-function as (P,ω). We may assume that Q_2 does not extend Q_1. Then there exists a covering relation $x\prec y$ in Q_1 such that $x\not< y$ in Q_2. Since $|\rho(x)-\rho(y)|=1$ we must have $y<x$ in Q_2. Thus $\omega(x)$ precedes $\omega(y)$ in any permutation in $\mathcal{L}(Q_1,\omega)$, and $\omega(y)$ precedes $\omega(x)$ in any permutation in $\mathcal{L}(Q_2,\omega)$. Hence, the union is disjoint and unique. □

We need two operations on labeled posets: Let (P,ϵ) and (Q,μ) be two labeled posets. The *ordinal sum*, $P\oplus Q$, of P and Q is the poset with the disjoint union of P and Q as underlying set and with partial order defined by $x\le y$ if $x\le_P y$ or $x\le_Q y$, or $x\in P, y\in Q$. Define two labelings of $E(P\oplus Q)$ by

$$\begin{aligned}(\epsilon\oplus_1\mu)(x,y) &= \epsilon(x,y) \text{ if } (x,y)\in E(P),\\ (\epsilon\oplus_1\mu)(x,y) &= \mu(x,y) \text{ if } (x,y)\in E(Q) \text{ and}\\ (\epsilon\oplus_1\mu)(x,y) &= 1 \text{ otherwise.}\\ (\epsilon\oplus_{-1}\mu)(x,y) &= \epsilon(x,y) \text{ if } (x,y)\in E(P),\\ (\epsilon\oplus_{-1}\mu)(x,y) &= \mu(x,y) \text{ if } (x,y)\in E(Q) \text{ and}\\ (\epsilon\oplus_{-1}\mu)(x,y) &= -1 \text{ otherwise.}\end{aligned}$$

With a slight abuse of notation we write $P\oplus_{\pm1}Q$ when the labelings of P and Q are understood from the context. Note that ordinal sums are associative, i.e., $(P\oplus_{\pm1}Q)\oplus_{\pm1} R = P\oplus_{\pm1}(Q\oplus_{\pm1}R)$, and preserve the property of being sign-graded. The following result is easily obtained by combinatorial reasoning, see [2, 17]:

Proposition 3.3. *Let (P, ω) and (Q, ν) be two labeled posets. Then*

$$W(P \oplus Q, \omega \oplus_1 \nu; t) = W(P, \omega; t) W(Q, \nu; t)$$

and

$$W(P \oplus Q, \omega \oplus_{-1} \nu; t) = t W(P, \omega; t) W(Q, \nu; t).$$

Proposition 3.4. *Suppose that (P, ω) is a saturated canonically labeled ω-consistent poset. Then (P, ω) is the direct sum*

$$(P, \omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,$$

where the A_is are anti-chains.

Proof. Let $\pi \in \mathcal{L}(P, \omega)$. Then we may write π as $\pi = w_0 w_1 \cdots w_k$ where the w_is are maximal words with respect to the property: If a and b are letters of w_i then $\rho(\omega^{-1}(a)) = \rho(\omega^{-1}(b))$. Hence $\pi \in \mathcal{L}(Q, \omega)$ where

$$(Q, \omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,$$

and A_i is the anti-chain consisting of the elements $\omega^{-1}(a)$, where a is a letter of w_i (A_i is an anti-chain, since if $x < y$ where $x, y \in A_i$ there would be a letter in π between $\omega(x)$ and $\omega(y)$ whose rank was different than that of x, y). Now, (Q, ω) is saturated so $P = Q$. $\square$

Note that the argument in the above proof also can be used to give a simpler proof of Theorem 3.2 when ω is canonical.

4 The W-polynomial of a sign-graded poset

The space S^d of symmetric polynomials in $\mathbb{R}[t]$ with center of symmetry $d/2$ has a basis

$$B_d = \{t^i (1+t)^{d-2i}\}_{i=0}^{\lfloor d/2 \rfloor}.$$

If $h \in S^d$ has nonnegative coefficients in this basis it follows immediately that the coefficients of h in the standard basis are unimodal. Let S^d_+ be the nonnegative span of B_d. Thus S^d_+ is a cone. Another property of S^d_+ is that if $h \in S^d_+$ then it has the correct sign at -1 i.e.,

$$(-1)^{d/2} h(-1) \geq 0.$$

Lemma 4.1. *Let $c, d \in \mathbb{N}$. Then*

$$\begin{aligned} S^c S^d &\subset S^{c+d} \\ S^c_+ S^d_+ &\subset S^{c+d}_+. \end{aligned}$$

Suppose further that $h \in S^d$ has positive leading coefficient and that all zeros of h are real and non-positive. Then $h \in S^d_+$.

Proof. The inclusions are obvious. Since $t \in S_+^2$ and $(1+t) \in S_+^1$ we may assume that none of them divides h. But then we may collect the zeros of h in pairs $\{\theta, \theta^{-1}\}$. Let $A_\theta = -\theta - \theta^{-1}$. Then

$$h = C \prod_{\theta < -1} (t^2 + A_\theta t + 1),$$

where $C > 0$. Since $A_\theta > 2$ we have

$$t^2 + A_\theta t + 1 = (t+1)^2 + (A_\theta - 2)t \in S_+^2,$$

and the lemma follows. □

We can now prove our main theorem.

Theorem 4.2. *Suppose that (P, ω) is a sign-graded poset of rank r. Then $W(P, \omega; t) \in S_+^{p-r-1}$.*

Proof. By Corollary 2.4 and Lemma 2.5 we may assume that (P, ω) is canonically labeled. If Q extends P then the maximal elements of Q are also maximal elements of P. By Theorem 3.2 we know that

$$W(P, \omega; t) = \sum_Q W(Q, \omega; t),$$

where (Q, ω) is saturated and sign-graded with the same rank function and rank as (P, ω). The W-polynomials of anti-chains are the Eulerian polynomials, which have real nonnegative zeros only. By Propositions 3.3 and 3.4 the polynomial $W(Q, \omega; t)$ has only real non-positive zeros so by Lemma 4.1 and Corollary 2.4 we have $W(Q, \omega; t) \in S_+^{p-r-1}$. The theorem now follows since S_+^{p-r-1} is a cone. □

Corollary 4.3. *Let (P, ω) be sign-graded of rank r. Then $W(P, \omega; t)$ is symmetric and its coefficients are unimodal. Moreover, $W(P, \omega; t)$ has the correct sign at -1, i.e.,*

$$(-1)^{(p-1-r)/2} W(P, \omega; -1) \geq 0.$$

Corollary 4.4. *Let P be a graded poset. Suppose that $\Delta_{eq}(P)$ is flag. Then the Charney-Davis Conjecture holds for $\Delta_{eq}(P)$.*

Theorem 4.5. *Suppose that P is an ω-consistent poset and that $|\rho(x) - \rho(y)| \leq 1$ for all maximal elements $x, y \in P$. Then $W(P, \omega; t)$ has unimodal coefficients.*

Proof. Suppose that the ranks of maximal elements are contained in $\{r, r+1\}$. If Q is any saturated poset that extends P and has the same rank function as (P, ω) then Q is ω-graded of rank r or $r+1$. By Theorems 3.2 and 4.2 we know that

$$W(P, \omega; t) = \sum_Q W(Q, \omega; t),$$

where $W(Q, \omega; t)$ is symmetric and unimodal with center of symmetry at $(p-1-r)/2$ or $(p-2-r)/2$. The sum of such polynomials is again unimodal. □

5 The Charney-Davis quantity

In [9] Reiner, Stanton and Welker defined the *Charney-Davis quantity* of a graded naturally labeled poset (P, ω) of rank r to be

$$CD(P, \omega) = (-1)^{(p-1-r)/2} W(P, \omega; -1).$$

We define it in the exact same way for sign-graded posets. Since, by Corollary 2.4, the particular labeling does not matter we write $CD(P)$. Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be any permutation. We say that π is *alternating* if $\pi_1 > \pi_2 < \pi_3 > \cdots$ and *reverse alternating* if $\pi_1 < \pi_2 > \pi_3 < \cdots$. Let (P, ω) be a canonically labeled sign-graded poset. If $\pi \in \mathcal{L}(P, \omega)$ then we may write π as $\pi = w_0 w_1 \cdots w_k$ where w_i are maximal words with respect to the property: If a and b are letters of w_i then $\rho(\omega^{-1}(a)) = \rho(\omega^{-1}(b))$. The words w_i are called the *components* of π. The following theorem is well known, see for example [5, 11, 13], and gives the Charney-Davis quantity of an anti-chain.

Proposition 5.1. *Let $n \geq 0$ be an integer. Then $(-1)^{(n-1)/2} A_n(-1)$ is equal to 0 if n is even and equal to the number of (reverse) alternating permutations of the set $\{1, 2, \ldots, n\}$ if n is odd.*

Theorem 5.2. *Let (P, ω) be a canonically labeled sign-graded poset. Then the Charney-Davis quantity, $CD(P)$, is equal to the number of reverse alternating permutations in $\mathcal{L}(P, \omega)$ such that all components have an odd number of letters.*

Proof. It suffices to prove the theorem when (P, ω) is saturated. By Proposition 3.4 we know that

$$(P, \omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,$$

where the A_is are anti-chains. Thus $CD(P) = CD(A_0)CD(A_1)\cdots CD(A_k)$. Let $\pi = w_0 w_1 \cdots w_k \in \mathcal{L}(P, \omega)$ where w_i is a permutation of $\omega(A_i)$. Then π is a reverse alternating permutation such that all components have an odd number of letters if and only if, for all i, w_i is reverse alternating if i is even and alternating if i is odd. Hence, by Proposition 5.1, the number of such permutations is indeed $CD(A_0)CD(A_1)\cdots CD(A_k)$. $\square$

If $h(t)$ is any polynomial with integer coefficients and $h(t) \in S^d$, it follows that $h(t)$ has integer coefficients in the basis $t^i(1+t)^{d-2i}$. Thus we know that if (P, ω) is sign-graded of rank r, then

$$W(P, \omega; t) = \sum_{i=0}^{\lfloor (p-r-1)/2 \rfloor} a_i(P, \omega) t^i (1+t)^{p-r-1-2i},$$

where $a_i(P, \omega)$ are nonnegative integers. By Theorem 5.2 we have a combinatorial interpretation of the $a_{(p-r-1)/2}(P, \omega)$. A similar but more complicated interpretation of $a_i(P, \omega)$, $i = 0, 1, \ldots, \lfloor (p-r-1)/2 \rfloor$ can be deduced from Proposition 3.4 and the work in [5, 11]. We omit this.

6 The right mode

Let $f(x) = a_0 + a_1x + \cdots + a_dx^d$ be a polynomial with real coefficients. The *mode*, mode(f), of f is the average value of the indices i such that $a_i = \max\{a_j\}_{j=0}^d$. One can easily compute the mode of a polynomial with real non-positive zeros only:

Theorem 6.1. *[4] Let f be a polynomial with real non-positive zeros only and with positive leading coefficient. Then*

$$\Big|\frac{f'(1)}{f(1)} - \mathrm{mode}(f)\Big| < 1.$$

It is known, see [2, 14, 17], that

$$W(P, \omega; x) = \sum_{i=1}^{p} e_i(P, \omega)x^{i-1}(1-x)^{p-i},$$

where $e_i(P, \omega)$ is the number of surjective (P, ω)-partitions $\sigma : P \to \{1, 2, \dots, i\}$. A simple calculation gives

$$\frac{W'(P, \omega; 1)}{W(P, \omega; 1)} = p - 1 - \frac{e_{p-1}(P, \omega)}{e_p(P, \omega)}. \tag{6.1}$$

If P is ω-graded of rank r we know by Theorem 4.2 that $\mathrm{mode}(W(P, \omega; x)) = (p-r-1)/2$. The Neggers-Stanley conjecture, Theorem 6.1 and (6.1) suggest that $2e_{p-1}(P, \omega) = (p + r - 1)e_p(P, \omega)$. Stanley [14] proved this for graded posets and it generalizes to sign-graded posets:

Proposition 6.2. *Let P be ω-graded of rank r. Then*

$$2e_{p-1}(P, \omega) = (p + r - 1)e_p(P, \omega).$$

Proof. The identity follows when expanding $\Omega(P, \omega; t)$ in powers of t using Theorem 2.3. See [14, Corollary 19.4] for details. □

7 A characterization of sign-graded posets

Here we give a characterization of sign-graded posets along the lines of the characterization of graded posets given by Stanley in [14]. Let (P, ϵ) be a labeled poset. Define a function $\delta = \delta_\epsilon : P \to \mathbb{Z}$ by

$$\delta(x) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i)\},$$

where $x = x_0 \prec x_1 \prec \cdots \prec x_\ell$ is any saturated chain starting at x and ending at a maximal element x_ℓ. Define a map $\Phi = \Phi_\epsilon : \mathcal{A}(\epsilon) \to \mathbb{Z}^P$ by

$$\Phi\sigma = \sigma + \delta.$$

We have

$$\delta(x) \geq \delta(y) + \epsilon(x, y). \tag{7.1}$$

This means that $\Phi\sigma(x) > \Phi\sigma(y)$ if $\epsilon(x, y) = 1$ and $\Phi\sigma(x) \geq \Phi\sigma(y)$ if $\epsilon(x, y) = -1$. Thus $\Phi\sigma$ is a $(P, -\epsilon)$-partition provided that $\Phi\sigma(x) > 0$ for all $x \in P$. But $\Phi\sigma$ is order reversing so it attains its minimum at maximal elements and for maximal elements, z, we have $\Phi\sigma(z) = \sigma(z)$. This shows that $\Phi : \mathcal{A}(\epsilon) \to \mathcal{A}(-\epsilon)$ is an injection.

The *dual*, (P^*, ϵ^*), of a labeled poset (P, ϵ) is defined by $x <_{P^*} y$ if and only if $y <_{P^*} x$, with labeling defined by $\epsilon^*(y, x) = -\epsilon(x, y)$. We say that P is *dual ϵ-consistent* if P^* is ϵ^*-consistent.

Proposition 7.1. *Let (P, ϵ) be labeled poset. Then $\Phi_\epsilon : \mathcal{A}(\epsilon) \to \mathcal{A}(-\epsilon)$ is a bijection if and only if P is dual ϵ-consistent.*

Proof. If P is dual ϵ-consistent then P is also dual $-\epsilon$-consistent and $\delta_{-\epsilon}(x) = -\delta_\epsilon(x)$ for all $x \in P$. Thus the if part follows since the inverse of Φ_ϵ is $\Phi_{-\epsilon}$.

For the only if direction note that P is dual ϵ-consistent if and only if for all $(x, y) \in E$ we have

$$\delta(x) = \delta(y) + \epsilon(x, y)$$

Hence, if P is not dual ϵ-consistent then by (7.1), there is a covering relation $(x_0, y_0) \in E$ such that either $\epsilon(x_0, y_0) = 1$ and $\delta(x_0) \geq \delta(y_0) + 2$ or $\epsilon(x_0, y_0) = -1$ and $\delta(x_0) \geq \delta(y_0)$.

Suppose that $\epsilon(x_0, y_0) = 1$. It is clear that there is a $\sigma \in \mathcal{A}(-\epsilon)$ such that $\sigma(x_0) = \sigma(y_0) + 1$. But then

$$\sigma(x_0) - \delta(x_0) \leq \sigma(y_0) - \delta(y_0) - 1,$$

so $\sigma - \delta \notin \mathcal{A}(\epsilon)$.

Similarly, if $\epsilon(x_0, y_0) = -1$ then we can find a partition $\sigma \in \mathcal{A}(-\epsilon)$ with $\sigma(x_0) = \sigma(y_0)$, and then

$$\sigma(x_0) - \delta(x_0) \leq \sigma(y_0) - \delta(y_0),$$

so $\sigma - \delta \notin \mathcal{A}(\epsilon)$. ☐

Let (P, ϵ) be a labeled poset. Define $r(\epsilon)$ by

$$r(\epsilon) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i) : x_0 \prec x_1 \prec \cdots \prec x_\ell \text{ is maximal}\}.$$

We then have:

$$\begin{aligned} \max\{\Phi\sigma(x) : x \in P\} &= \max\{\sigma(x) + \delta_\epsilon(x) : x \text{ is minimal}\} \\ &\leq \max\{\sigma(x) : x \in P\} + r(\epsilon). \end{aligned}$$

So if we let $\mathcal{A}_n(\epsilon)$ be the (P, ϵ)-partitions with largest part at most n we have that $\Phi_\epsilon : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ is an injection. A labeling ϵ of P is said to satisfy the *λ-chain condition* if for every $x \in P$ there is a maximal chain $c : x_0 \prec x_1 \prec \cdots \prec x_\ell$ containing x such that $\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i) = r(\epsilon)$.

Lemma 7.2. *Suppose that n is a nonnegative integer such that $\Omega(P,\epsilon;n)\neq 0$. If*

$$\Omega(P,-\epsilon;n+r(\epsilon))=\Omega(P,\epsilon;n)$$

then ϵ satisfies the λ-chain condition.

Proof. Define $\delta^*: P\to\mathbb{Z}$ by

$$\delta^*(x)=\max\{\sum_{i=1}^{\ell}\epsilon(x_{i-1},x_i): x_0\prec x_1\prec\cdots\prec x_\ell = x\},$$

where the maximum is taken over all maximal chains starting at a minimal element and ending at x. Then

$$\delta(x)+\delta^*(x)\leq r(\epsilon) \tag{7.2}$$

for all x, and ϵ satisfies the λ-chain condition if and only if we have equality in (7.2) for all $x\in P$. It is easy to see that the map $\Phi^*:\mathcal{A}_n(\epsilon)\to\mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ defined by

$$\Phi^*\sigma(x)=\sigma(x)+r(\epsilon)-\delta^*(x),$$

is well-defined and is an injection. By (7.2) we have $\Phi\sigma(x)\leq\Phi^*\sigma(x)$ for all σ and all $x\in P$, with equality if and only if x is in a maximal chain of maximal weight. This means that in order for $\Phi:\mathcal{A}_n(\epsilon)\to\mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ to be a bijection it is necessary for ϵ to satisfy the λ-chain condition. □

Theorem 7.3. *Let ϵ be a labeling of P. Then*

$$\Omega(P,\epsilon;t)=(-1)^p\Omega(P,\epsilon;-t-r(\epsilon))$$

if and only if P is ϵ-graded of rank $r(\epsilon)$.

Proof. The "if" part is Theorem 2.3, so suppose that the equality of the theorem holds. By reciprocity we have

$$(-1)^p\Omega(P,\epsilon;-t-r(\epsilon))=\Omega(P,-\epsilon;t+r(\epsilon)),$$

and since $\Phi_\epsilon:\mathcal{A}_n(\epsilon)\to\mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ is an injection it is also a bijection. By Proposition 7.1 we have that P is dual ϵ-consistent and by Lemma 7.2, we have that all minimal elements are members of maximal chains of maximal weight. In other words P is ϵ-graded. □

It should be noted that it is not necessary for P to be ϵ-graded in order for $W(P,\epsilon;t)$ to be symmetric. For example, if (P,ϵ) is any labeled poset then the W-polynomial of the disjoint union of (P,ϵ) and $(P,-\epsilon)$ is easily seen to be symmetric. However, we have the following:

Corollary 7.4. *Suppose that*

$$\Omega(P,\epsilon;t)=\Omega(P,-\epsilon;t+s),$$

for some $s\in\mathbb{Z}$. Then $-r(-\epsilon)\leq s\leq r(\epsilon)$, with equality if and only if P is ϵ-graded.

Proof. We have an injection $\Phi_\epsilon:\mathcal{A}_n(\epsilon)\to\mathcal{A}_{n+r(\epsilon)}(-\epsilon)$. This means that $s\leq r(\epsilon)$. The lower bound follows from the injection $\Phi_{-\epsilon}$, and the statement of equality follows from Theorem 7.3. □

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