How different can two intersecting families be?

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Abstract

To measure the difference between two intersecting families $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ we introduce the quantity $D(\mathcal{F},\mathcal{G}) = |\{(F,G) : F \in \mathcal{F}, G \in \mathcal{G}, F \cap G = \emptyset\}|$. We prove that if $\mathcal{F}$ is k-uniform and $\mathcal{G}$ is l-uniform, then for large enough n and for any $i \neq j$ $\mathcal{F}_i = \{F \subseteq [n] : i \in F, |F| = k\}$ and $\mathcal{F}_j = \{F \subseteq [n] : j \in F, |F| = l\}$ form an optimal pair of families (that is $D(\mathcal{F},\mathcal{G}) \leq D(\mathcal{F}_i,\mathcal{F}_j)$ for all uniform and intersecting $\mathcal{F}$ and $\mathcal{G}$), while in the non-uniform case any pair of the form $\mathcal{F}_i = \{F \subseteq [n] : i \in F\}$ and $\mathcal{F}_j = \{F \subseteq [n] : j \in F\}$ is optimal for any n.

1 Introduction

To answer the question of the title, we first have to find out how to measure the difference between two intersecting set systems $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$. If for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ we have $F \cap G \neq \emptyset$, then $\mathcal{F} \cup \mathcal{G}$ is still an intersecting system of sets, so $\mathcal{F}$ and $\mathcal{G}$ are not really different as they are subfamilies of the same maximal intersecting family. So the evidence of being different (at least in this sense) is a pair $(F,G) : F \in \mathcal{F}, G \in \mathcal{G}, F \cap G = \emptyset$. Therefore it seems natural to introduce

$$D(\mathcal{F},\mathcal{G}) = |\{(F,G) : F \in \mathcal{F}, G \in \mathcal{G}, F \cap G = \emptyset\}|$$

to measure the difference between $\mathcal{F}$ and $\mathcal{G}$. Our purpose is to get optimal set systems $\mathcal{F}^*, \mathcal{G}^*$ (i.e. ones with the following property: $D(\mathcal{F},\mathcal{G}) \leq D(\mathcal{F}^*,\mathcal{G}^*)$ for any pair of intersecting families).

In fact, we will handle two different cases: first, when there is a restriction on the cardinality of the sets belonging to $\mathcal{F}$ and $\mathcal{G}$ ($\mathcal{F}$ and $\mathcal{G}$ will be uniform set systems), and secondly, when there is not. In both cases one can consider these hypergraphs:

$$\mathcal{F}_i = \{F \subseteq [n] : i \in F\}$$

Observe that, when considering the pair $\mathcal{F}_i, \mathcal{F}_j$, all the sets containing both i and j are not contained in any disjoint pairs, so the D-number won't decrease if we throw these sets away getting

$$\mathcal{F}_{i,j} = \{F \subseteq [n] : i \in F, j \notin F\}, \mathcal{F}_{j,i} = \{F \subseteq [n] : j \in G, i \notin G\}$$

These pairs of hypergraphs will be referred as the conjectured hypergraphs/set systems. In section 2, we will show that in the uniform case the conjectured set systems fail to be maximum with respect to $D(\mathcal{F}, \mathcal{G})$ for small n, but they are optimal if n is large enough (depending on the cardinality of the elements of $\mathcal{F}$ and $\mathcal{G}$). Note that in the uniform case $D(\mathcal{F}_i, \mathcal{F}_j) = D(\mathcal{F}_{i,j}, \mathcal{F}_{j,i}) = \binom{n-2}{k-1}\binom{n-k-1}{l-1}$. In section 3, we will prove that in the non-uniform case the conjectured set systems are optimal for all n. This time $D(\mathcal{F}_i, \mathcal{F}_j) = D(\mathcal{F}_{i,j}, \mathcal{F}_{j,i}) = 3^{n-2}$, because when considering a disjoint pair of sets $F_i \in \mathcal{F}_{i,j}, F_j \in \mathcal{F}_{j,i}$, then any $k \in [n] \setminus \{i, j\}$ can be either in F_i or in F_j or in none of them.

2 The Uniform Case

Throughout this section we will assume that $\mathcal{F}$ is k-uniform and $\mathcal{G}$ is l-uniform. Now if $k + l > n$, then there are no disjoint k and l element subsets.

If $k + l \leq n$, but, say, $l > \frac{n}{2}$, then any two l-element subsets meet each other. For any fixed k-element subset there are $\binom{n-k}{l}$ l-element subsets disjoint from this fixed set. So the best one can do is to let $\mathcal{F}$ be the largest intersecting k-uniform set system, and let $\mathcal{G}$ consist of all l-element subsets disjoint from at least one set in $\mathcal{F}$. The Erdős-Ko-Rado theorem [1] says that $\mathcal{F}$ should be all k-element sets containing a fixed element, so then $\mathcal{G}$ should be all l-element sets not containing this fixed element. Thus in this case the conjectured set systems are not optimal.

If $2k = n$ and $k = l$ then any set has only one disjoint pair (considering now only the k-element sets), its complement. So one can put from each pair one set into $\mathcal{F}$ and one into $\mathcal{G}$, and since in this way subsets containg 1 *and* n together (or containing none of them) will be put into $\mathcal{F}$ or $\mathcal{G}$, these families will have more disjoint pairs, than the conjectured systems (and clearly will be maximal ones).

Despite these failures of the conjectured systems, one can state the following

Theorem 2.1. *For any k and l, there exists an $n(k, l)$ such that if $n \geq n(k, l)$ and $\mathcal{F}$, $\mathcal{G}$ are k and l-uniform hypergraphs , then $D(\mathcal{F}, \mathcal{G}) \leq D(\mathcal{F}_0, \mathcal{G}_0)$ where $\mathcal{F}_0, \mathcal{G}_0$ are the conjectured hypergraphs.*

Proof: Case A $\bigcap \mathcal{F} \neq \emptyset$ and $\bigcap \mathcal{G} \neq \emptyset$.

In this case $\bigcap \mathcal{F}$ and $\bigcap \mathcal{G}$ must be disjoint, since otherwise there would be no disjoint sets in $\mathcal{F}$ and $\mathcal{G}$. Let's pick an $i \in \bigcap \mathcal{F}$ and a $j \in \bigcap \mathcal{G}$, and add $\{F \subseteq [n] : i \in F\}$ to $\mathcal{F}$ and $\{G \subseteq [n] : j \in G\}$ to $\mathcal{G}$. In this way we get the conjectured hypergraphs, and clearly $D(\mathcal{F}, \mathcal{G})$ can't decrease.

Case B $\bigcap \mathcal{F} = \emptyset$. (or similarly $\bigcap \mathcal{G} = \emptyset$)

Observe the following two things:

1, if $n \geq k + 2l$ then again by [1] one gets that for a fixed $F \in \mathcal{F}$ the number of sets in $\mathcal{G}$ from which F is disjoint is at most $\binom{n-k-1}{l-1}$, which is the case in the conjectured hypergraphs for all sets in $\mathcal{F}_{i,j}$. So if $|\mathcal{F}| \leq |\mathcal{F}_{i,j}| = \binom{n-2}{k-1}$ then we are done.

2, Since $\bigcap \mathcal{F} = \emptyset$ then as a special case of Theorem3 of [2] we get that

$$|\mathcal{F}| \leq 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$$

and as for large enough n

$$\binom{n-2}{k-1} > 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$$

holds, by the remark made after the first observation we are done. $\square$

3 The Non-Uniform Case

Considering the non-uniform case one can assume that the pair $(\mathcal{F}, \mathcal{G})$ is maximal with respect to the property that all $F \in \mathcal{F}$ have at least one $G \in \mathcal{G}$ disjoint from it (and the same holds for any $G \in \mathcal{G}$). First we state our main theorem.

Theorem 3.1. *For any $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ and for any $n \geq 2$, $D(\mathcal{F}, \mathcal{G}) \leq D(\mathcal{F}_{i,j}, \mathcal{F}_{j,i})$ holds, where $\mathcal{F}_{i,j}, \mathcal{F}_{j,i}$ are one of the conjectured pairs.*

Proof: We begin with

Claim 3.2. *$F \in \mathcal{F} \Leftrightarrow F^c \in \mathcal{G}$, where F^c denotes the complement of F.*

Proof: If $F \in \mathcal{F}$ then there is some $G \in \mathcal{G}$ such that $F \cap G = \emptyset$. This means $G \subseteq F^c$, and as G meets all sets in $\mathcal{G}$, F^c meets them, too. So by maximality $F^c \in \mathcal{G}$. The other direction follows, since we can change the role of $\mathcal{F}$ and $\mathcal{G}$. $\square^{3.2.}$

By virtue of the above claim, we can "forget about" $\mathcal{G}$. But what should we count, and are there any additional conditions on $\mathcal{F}$? Concerning the first question: as for a fixed F we counted the Gs disjoint from it, and since $F \cap G = \emptyset \Leftrightarrow F \subseteq G^c$, by the claim we get, that now for a fixed F we should count the number of $F' \in \mathcal{F} : F \subseteq F'$. (Note that $F \subseteq F$ also counts, because this is for the pair (F, F^c)!) Let's denote this number by $\rho_{\mathcal{F}}(F)$ (and we will omit $\mathcal{F}$ from the index, if it is clear from the context), and put $\rho(\mathcal{F}) = \sum_{F \in \mathcal{F}} \rho_{\mathcal{F}}(F)$.

Now to the other question: since by the above claim we know that $\mathcal{G} = \mathcal{F}^c = \{F^c : F \in \mathcal{F}\}$ and the original conditions were that both $\mathcal{F}$ and $\mathcal{G}$ should be intersecting, we get that $\mathcal{F}$ should be intersecting *and* co-intersecting. So we conclude the following

Claim 3.3. *$max\{D(\mathcal{F}, \mathcal{G}) : \mathcal{F}, \mathcal{G}$ are intersecting$\} = \max\{\rho(\mathcal{F}) : \mathcal{F}$ is intersecting and co-intersecting$\}$.* $\square$

By this claim we are left to show that $\rho(\mathcal{F}) \leq \rho(\mathcal{F}_{i,j})$ whenever $\mathcal{F}$ is an intersecting and co-intersecting family.

Now note that, when counting $\rho(\mathcal{F})$ one counts the pairs (F, F') where $F, F' \in \mathcal{F}$ and $F \subseteq F'$. But this can be done from the point of view of F', that is, if we put $\delta_{\mathcal{F}}(F') = |\{F \in \mathcal{F} : F' \supseteq F\}|$ and $\delta(\mathcal{F}) = \sum_{F \in \mathcal{F}} \delta_{\mathcal{F}}(F)$, then $\rho(\mathcal{F}) = \delta(\mathcal{F})$. With this remark we are able to prove

Lemma 3.4. *If $\mathcal{F}$ is intersecting and co-intersecting, furthermore $\bigcap \mathcal{F} \neq \emptyset$, then $\delta(\mathcal{F}) \leq \delta(\mathcal{F}_{i,j})$.*

Proof: W.l.o.g. one can assume that $1 \in F$ for all $F \in \mathcal{F}$. Consider the hypergraph $\mathcal{F}^* = \{F \setminus \{1\} : F \in \mathcal{F}\}$. Since we removed 1, this need no longer be intersecting, but it is clearly co-intersecting on $[2, ..., n]$, furthermore $\delta_{\mathcal{F}}(F) = \delta_{\mathcal{F}^*}(F \setminus \{1\})$.

It is well-known, that if a hypergraph is maximal co-intersecting, then it contains one set from any pair of complements, and if $F \subseteq F' \in \mathcal{F}^*$, then $F \in \mathcal{F}^*$. So $\delta_{\mathcal{F}^*}(F \setminus \{1\}) = 2^{|F \setminus \{1\}|}$, hence to obtain the largest $\delta(\mathcal{F}^*)$ one should put the most possible large sets into $\mathcal{F}^*$. Again, by [1], we know that for fixed $k \geq \frac{n-1}{2}$ we can put at most $\binom{n-2}{k}$ k-element sets into $\mathcal{F}^*$, but in the case of any conjectured hypergraph (any $\mathcal{F}_{i,j}$, or as we assume now that $1 \in \bigcap \mathcal{F}$ any $\mathcal{F}_{1,j}$) exactly that many sets (now with $k+1$-elements, as we put back 1 to all the sets) are there. So for all k we put the most possible number of large sets into our family when considering the k and $n-1-k$-element complementing pairs. $\square^{3.4.}$

So we will be done, if we can prove

Lemma 3.5. *For any intersecting and co-intersecting family $\mathcal{F}$, there exists another $\mathcal{F}'$ with $\bigcap \mathcal{F}' \neq \emptyset$ and $\rho(\mathcal{F}) \leq \rho(\mathcal{F}')$.*

Before starting the proof of Lemma 3.5., we introduce some notation: the shift operation $\tau_{i,j}$ is defined by

$$\tau_{i,j}(F) = \begin{cases} F \setminus \{j\} \cup \{i\} & \text{if } j \in F, i \notin F \text{and} F \setminus \{j\} \cup \{i\} \notin \mathcal{F} \\ F & \text{otherwise} \end{cases} \tag{1}$$

Put $\tau_{i,j}(\mathcal{F}) = \{\tau_{i,j}(F) : F \in \mathcal{F}\}$.

It is well-known that the shift operation preserves the intersecting and co-intersecting property. It is also known, that starting from any family of sets, performing finitely many shift operation, one can obtain a so-called *left-shifted* family, that is a family for which $\tau_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $i < j$. So in what follows, we can assume that $\mathcal{F}$ is left-shifted, if we can prove the following

Claim 3.6. $\rho(\mathcal{F}) \leq \rho(\tau_{i,j}(\mathcal{F}))$.

Proof: We will consider how $\rho(F)$ changes when performing the operation $\tau_{i,j}$.

Case A If $i, j \in F$ or $i, j \notin F$, then $\tau_{i,j}(F) = F$ and for all $F' \in \mathcal{F}$ with $F \subseteq F'$ we have $F \subseteq \tau_{i,j}(F')$. So $\rho_{\mathcal{F}}(F) \leq \rho_{\tau_{i,j}(\mathcal{F})}(F) = \rho_{\tau_{i,j}(\mathcal{F})}(\tau_{i,j}(F))$.

Case B Let $A \subseteq [n]$ with $i, j \notin A$. Put $F = A \cup \{i\}$ and $F' = A \cup \{j\}$.

Subcase B1 $F \in \mathcal{F}, F' \notin \mathcal{F}$

Now for all $G \supseteq F$ $i \in G$, therefore $G = \tau_{i,j}(G) \supseteq \tau_{i,j}(F) = F$, thus $\rho_{\mathcal{F}}(F) \leq \rho_{\tau_{i,j}(\mathcal{F})}(F) = \rho_{\tau_{i,j}(\mathcal{F})}(\tau_{i,j}(F))$.

Subcase B2 $F \notin \mathcal{F}, F' \in \mathcal{F}$

Now $\tau_{i,j}(F') = F$, and if $F' \subset G \in \mathcal{F}$ with $i \notin G$, then $(G \setminus \{j\} \cup \{i\}) = G' \in \tau_{i,j}(\mathcal{F})$ and clearly $F \subseteq G'$. If $F' \subseteq G$ with $i, j \in G$, then $G = \tau_{i,j}(G) \supseteq F$, thus we conclude, that $\rho_{\mathcal{F}}(F') \leq \rho_{\tau_{i,j}(\mathcal{F})}(F) = \rho_{\tau_{i,j}(\mathcal{F})}(\tau_{i,j}(F'))$.

Subcase B3 $F, F' \in \mathcal{F}$ (thus $\tau_{i,j}(F) = F, \tau_{i,j}(F') = F'$)

Now let $G \in \mathcal{F}$ contain at least one of F, F'. If $i \in G$, then $\tau_{i,j}(G) = G$ contains as many of F, F' as before performing the τ-operation. Otherwise $i \notin G, j \in G$ and G contains only F'. So, putting $G' = G \setminus \{j\} \cup \{i\}$, if $G' \notin \mathcal{F}$, then $\tau_{i,j}(G) = G'$ and $G' \supseteq F$, while if $G' \in \mathcal{F}$, then $\tau_{i,j}(G) = G$ and still $F' \subseteq G$. So we get $\rho_{\tau_{i,j}(\mathcal{F})}(F) + \rho_{\tau_{i,j}(\mathcal{F})}(F') \geq \rho_{\mathcal{F}}(F) + \rho_{\mathcal{F}}(F')$.

So for sets of type of the first case $\rho(F)$ doesn't decrease, and we can partition the sets of type of the second case into "pairs" (of which one may be missing) for which the sum of $\rho(F)$s doesn't decrease. $\square^{3.6.}$

Further notations:

$$\mathcal{F} + \mathcal{G} = \{F \cup G : F \in \mathcal{F}, G \in \mathcal{G}\},\ \mathcal{F} - \mathcal{G} = \{F \setminus G : F \in \mathcal{F}, G \in \mathcal{G}\}$$

$$\Delta\mathcal{F} = \mathcal{F} - \mathcal{F};\ \mathrm{Sub}\mathcal{F} = \{S : S \subseteq F \in \mathcal{F}\}$$

And we will write $\mathbf{1} + \mathcal{F}$ if $\mathcal{G}$ consists of one single set containing only 1.

Now we can return to the proof of Lemma 3.5. In the proof we will use the basic ideas of [3].

Proof of Lemma 3.5. For arbitrary $\mathcal{F}$ intersecting and co-intersecting family we have to define another one of which each set has an element in common. Now let $\mathcal{F} = \mathcal{F}^0 \cup^* \mathcal{F}^1$, where $\mathcal{F}^1 = \{F \in \mathcal{F} : 1 \in F\}$ and $\mathcal{F}^0 = \{F \in \mathcal{F} : 1 \notin F\}$. Put $\mathcal{F}' = \mathcal{F}^1 \cup (\mathbf{1} + \mathrm{Sub}\mathcal{F}^0)$.

We have to prove that i, $\bigcap \mathcal{F}' \neq \emptyset$ (and therefore it is intersecting), ii, $\mathcal{F}'$ is co-intersecting and iii, $\rho(\mathcal{F}') \geq \rho(\mathcal{F})$.

i, is clear, as by definition $1 \in F$ for all $F \in \mathcal{F}'$.

To prove ii, we will use that $\mathcal{F}$ is left-shifted (and maximal).

Claim 3.7. $\mathbf{1} + \mathcal{F}^0 \subset \mathcal{F}^1$

Proof: Since for any $F \in \mathcal{F}^0$ $F' = \{1\} \cup F \supset F$, F' meets all sets in $\mathcal{F}$. We have to show, that there is no $G \in \mathcal{F}$ such that $F' \cup G = [n]$. Suppose to the contrary that such a G exists. Note that $1 \notin G$, because otherwise $G \cup F = [n]$ would hold, contradicting the co-intersecting property of $\mathcal{F}$. Now as $\mathcal{F}$ is intersecting, there is $j \in F \cap G$. But since $\mathcal{F}$ is left-shifted, $G \setminus \{j\} \cup \{1\} = G' \in \mathcal{F}$. But then $G' \cup F = [n]$ would hold - a contradiction. $\square^{3.7.}$

By Claim 3.7. we know that all new sets in $\mathcal{F}'$ are subsets of one of the old sets (that is a set from $\mathcal{F}$), therefore as $\mathcal{F}$ was co-intersecting, so is $\mathcal{F}'$.

It remains to prove iii,. For this purpose we will define an injective mapping $f : \mathcal{F}^0 \to \mathbf{1} + \Delta\mathcal{F}^0$ (observe that $\Delta\mathcal{F}^0 \subseteq \mathrm{Sub}\mathcal{F}^0$!) such that for all $F \in \mathcal{F}^0$ $\rho_{\mathcal{F}'}(f(F)) \geq \rho_{\mathcal{F}}(F)$. This is clearly enough, because $\mathcal{F}^1 \subseteq \mathcal{F}'$, so $\rho(F)$ can't decrease for any $F \in \mathcal{F}^1$ (and if $F_1, F_2 \in \mathcal{F}^0$, then $\{1\} \cup F_1 \setminus F_2$ is disjoint from F_2, so, by the intersecting property of $\mathcal{F}$, it is not an element of $\mathcal{F}^1$, so we won't count twice any $\rho(F)$).

To define f (using the notation of [3]) let $k = \min\{|I| : I = F_1 \cap F_2; F_1, F_2 \in \mathcal{F}^0\}$ (note, that I is not empty, as $\mathcal{F}$ is intersecting!) and fix F_1, F_2 with $I = F_1 \cap F_2 : |I| = k$.

Now consider the following partition of $\mathcal{F}^0$:

$$\mathcal{C} = \{F \in \mathcal{F}^0 : I \not\subseteq F\}; \mathcal{A} = \{F \in \mathcal{F}^0 : I \subseteq F, \text{there is } F' \in \mathcal{F}^0 \text{ with } F \cap F' = I\};$$

$$\mathcal{B} = \mathcal{F}^0 \setminus (\mathcal{A} \cup \mathcal{C})$$

For a better understanding, $F \in \mathcal{B}$ if $I \subset F$ and whenever there is a set $F' \in \mathcal{F}^0$ with $I \subseteq F'$, then F should meet F' outside I, as well. Note that $\mathcal{A}$ is not empty, since $F_1, F_2 \in \mathcal{A}$. Now for any $A \in \mathcal{A}$ let $f(A) = (A \setminus I) \cup \{1\}$. (Observe that for any $A \in \mathcal{A}$ there is $A' \in \mathcal{A} \subseteq \mathcal{F}^0$ with $A \cap A' = I$, $A \setminus I = A \setminus A'$, so $f(A) \in \mathbf{1} + \Delta\mathcal{F}^0$ as required!) As all $A \in \mathcal{A}$ contain I, f is injective restricted to $\mathcal{A}$.

To show that $\rho_{\mathcal{F}'}(f(A)) \geq \rho_{\mathcal{F}}(A)$, observe that $f(A) \subset A \cup \{1\}$. Therefore if $A \subset F \in \mathcal{F}$ and $1 \in F$ (that is $F \in \mathcal{F}^1$, therefore $F \in \mathcal{F}'$, too) , then $f(A) \subset F$, as well, so the part of $\rho(A)$ which comes from the Fs in $\mathcal{F}^1$ can't decrease.

We have to handle the sets $A \subset F \in \mathcal{F}^0$. To do this let $(F \setminus I) \cup \{1\} = F'$. Then $F' \in \mathcal{F}'$ and $f(A) \subseteq F'$ by definition. If $F \neq G$ then $F' \neq G'$, because we took the same set I away from both (and $I \subseteq F, G$), and 1 was neither in F nor in G. We still have to point out that F' is not equal to any $G \in F^1$, G containing A for any $F \in \mathcal{F}^0$ (because in that case we would take into account that containing relation twice when counting $\rho(f(A))$). But this is clear, because a G of this form contains I (as $I \subseteq A$), and $F' \cap I = \emptyset$ by definition (and as we pointed out I is not empty).

To finish the proof we need to continue this procedure now considering the remaining sets, that is $\mathcal{B} \cup \mathcal{C}$. So we define a new I' and a new k' now only considering sets in $\mathcal{B} \cup \mathcal{C}$, then get a new partition $\mathcal{A}', \mathcal{B}', \mathcal{C}'$ with respect to this new I' and new k', and define f on $\mathcal{A}'$ with the help of I', and then start again with $\mathcal{B}' \cup \mathcal{C}'$... This procedure ends after finitely many steps, as the $\mathcal{A}$s are never empty, so there is strictly less and less remainder. In each step f is injective, the only difficulty is to assure for sets A, B on which f is defined at different steps $f(A) = f(B)$ can't happen. This is clearly done by

Claim 3.8. $(\mathcal{A} - \{I\}) \cap \Delta(\mathcal{B} \cup \mathcal{C}) = \emptyset$

$\mathcal{A} - \{I\}$ is the set of the f-images defined at a step (if we don't count 1, which is an element of all images). For a set B on which f is defined later, the image is of the form $B \setminus I' = B \setminus B'$ (again without 1), so it is in $\Delta(\mathcal{B} \cup \mathcal{C})$. Therefore by the claim we will be really done.

Proof: This is in fact the lemma in [3], but to be self-contained we repeat the proof.

Case1: $A \in \mathcal{A}, B \in \mathcal{B}, F \in \mathcal{B} \cup \mathcal{C}$.

By the definition of $\mathcal{B}$, B must meet A outside of I, too. Therefore $F \setminus B$ doesn't contain this (these) element(s), while $A \setminus I$ does.

Case2: $A \in \mathcal{A}, C \in \mathcal{C}, F \in \mathcal{B} \cup \mathcal{C}$

By the definition of $\mathcal{C}$, C doesn't contain I, therefore by the minimality of $|I|$, C must meet A outside of I, too. The rest is as in Case1. $\square^{3.8.}$ $\square^{3.5.}$ $\square^{3.1.}$

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