# Lower Bound for the Size of Maximal Nontraceable Graphs *

Marietjie Frick, Joy Singleton
University of South Africa,
P.O. Box 392, Unisa, 0003,
South Africa.
**e-mail:** frickm@unisa.ac.za singlje@unisa.ac.za

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**Abstract**

Let $g(n)$ denote the minimum number of edges of a maximal nontraceable graph of order $n$. Dudek, Katona and Wojda (2003) showed that $g(n) \geq \lceil \frac{3n-2}{2} \rceil - 2$ for $n \geq 20$ and $g(n) \leq \lceil \frac{3n-2}{2} \rceil$ for $n \geq 54$ as well as for $n \in I = \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$. We show that $g(n) = \lceil \frac{3n-2}{2} \rceil$ for $n \geq 54$ as well as for $n \in I \cup \{12, 13\}$ and we determine $g(n)$ for $n \leq 9$.
**Keywords:** maximal nontraceable, hamiltonian path, traceable, nontraceable, nonhamiltonian

## 1 Introduction

We consider only simple, finite graphs $G$ and denote the vertex set, the edge set, the order and the size of $G$ by $V(G)$, $E(G)$, $v(G)$ and $e(G)$, respectively. The *open neighbourhood* of a vertex $v$ in $G$ is the set $N_G(v) = \{x \in V(G) : vx \in E(G)\}$. If $U$ is a nonempty subset of $V(G)$ then $\langle U \rangle$ denotes the subgraph of $G$ induced by $U$.

A graph $G$ is *hamiltonian* if it has a *hamiltonian cycle* (a cycle containing all the vertices of $G$), and *traceable* if it has a *hamiltonian path* (a path containing all the vertices of $G$). A graph $G$ is *maximal nonhamiltonian* (MNH) if $G$ is not hamiltonian, but $G+e$ is hamiltonian for each $e \in E(\overline{G})$, where $\overline{G}$ denotes the complement of $G$. A graph $G$ is *maximal nontraceable* (MNT) if $G$ is not traceable, but $G+e$ is traceable for each $e \in E(\overline{G})$.

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In 1978 Bollobás [1] posed the problem of finding the least number of edges, $f(n)$, in a MNH graph of order $n$. Bondy [2] had already shown that a MNH graph with order $n \geq 7$ that contained $m$ vertices of degree 2 had at least $(3n+m)/2$ edges, and hence $f(n) \geq \lceil 3n/2 \rceil$ for $n \geq 7$. Combined results of Clark, Entringer and Shapiro [3], [4] and Lin, Jiang, Zhang and Yang [7] show that $f(n) = \lceil 3n/2 \rceil$ for $n \geq 19$ and for $n = 6, 10, 11, 12, 13, 17$. The values of $f(n)$ for the remaining values of $n$ are also given in [7].

Let $g(n)$ denote the minimum number of edges in a MNT graph of order $n$. Dudek, Katona and Wojda [5] proved that

$$g(n) \geq \lceil \tfrac{3n-2}{2} \rceil - 2 \text{ for } n \geq 20$$

and showed, by construction, that

$$g(n) \leq \lceil \tfrac{3n-2}{2} \rceil \text{ for } n \geq 54$$

as well as for $n \in I = \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$.

We prove, using a method different from that in [5], that

$$g(n) \geq \lceil \tfrac{3n-2}{2} \rceil \text{ for } n \geq 10.$$

We also construct graphs of order $n = 12, 13$ with $\lceil \frac{3n-2}{2} \rceil$ edges and thus show that

$$g(n) = \lceil \tfrac{3n-2}{2} \rceil \text{ for } n \geq 54 \text{ as well as for } n \in I \cup \{12, 13\}.$$

We also determine $g(n)$ for $n \leq 9$.

## 2 Auxiliary Results

In this section we present some results concerning MNT graphs, which we shall use, in the next section, to prove that a MNT graph of order $n \geq 10$ has at least $\frac{3n-2}{2}$ edges. The first one concerns the lower bound for the number of edges of MNH graphs. It is the combination of results proved in [2] and [7].

**Theorem 1** *(Bondy and Lin, Jiang, Zhang and Yang) If $G$ is a MNH graph of order $n$, then $e(G) \geq \frac{3n}{2}$ for $n \geq 6$.*

The following lemma, which we proved in [6], will be used frequently.

**Lemma 2** *Let $Q$ be a path in a MNT graph $G$. If $\langle V(Q) \rangle$ is not complete, then some internal vertex of $Q$ has a neighbour in $G - V(Q)$.*

**Proof.** Let $u$ and $v$ be two nonadjacent vertices of $Q$. Then $G + uv$ has a hamiltonian path $P$. Let $x$ and $y$ be the two endvertices of $Q$ and suppose no internal vertex of $Q$ has a neighbour in $G - V(Q)$. Then $P$ has a subpath $R$ in $\langle V(Q)\rangle + uv$ and $R$ has either one or both endvertices in $\{x, y\}$. If $R$ has only one endvertex in $\{x, y\}$, then $P$ has an endvertex in $Q$. In either case the path obtained from $P$ by replacing $R$ with $Q$ is a hamiltonian path of $G$. ∎

The following lemma is easy to prove.

**Lemma 3** *Suppose $T$ is a cutset of a connected graph $G$ and $A_1, ..., A_k$ are components of $G - T$.*

*(a) If $k \geq |T| + 2$, then $G$ is nontraceable.*

*(b) If $G$ is MNT then $k \leq |T| + 2$.*

*(c) If $G$ is MNT and $k = |T| + 2$, then $\langle T \cup A_i\rangle$ is complete for $i = 1, 2, ..., k$.*

**Proof.** (a) and (b) are obvious. If (c) is not true, then there is an $i$ such that $\langle T \cup A_i\rangle$ has two nonadjacent vertices $x$ and $y$. But then $T$ is a cutset of the graph $G + xy$ and $(G + xy) - T$ has $|T| + 2$ components and hence $G + xy$ is nontraceable, by (a). ∎

The proof of the following lemma is similar to the previous one.

**Lemma 4** *Suppose $B$ is a block of a connected graph $G$.*

*(a) If $B$ has more than two cut-vertices, then $G$ is nontraceable.*

*(b) If $G$ is MNT, then $B$ has at most three cut-vertices.*

*(c) If $G$ is MNT and $B$ has exactly three cut-vertices, then $G$ consists of exactly four blocks, each of which is complete.*

In [6] we proved some results concerning the degrees of the neighbours of the vertices of degree 2 in a 2-connected MNT graph, which enabled us to show that the average degree of the vertices in a 2-connected MNT graph is at least 3. We now restate those results in a form that is applicable also to MNT graphs which are not 2-connected. (Note that in a 2-connected graph no two vertices of degree 2 are adjacent to one another.)

**Lemma 5** *If $G$ is a connected MNT graph and $v \in V(G)$ with $d(v) = 2$, then the neighbours of $v$ are adjacent. Also, one of the neighbours has degree at least 4 and the other neighbour has degree 2 or at least 4.*

**Proof.** Let $N_G(v) = \{x_1, x_2\}$ and let $Q$ be the path $x_1 v x_2$. Since $N_G(v) \subseteq Q$, it follows from Lemma 2 that $\langle V(Q)\rangle$ is a complete graph; hence $x_1$ and $x_2$ are adjacent.

Since $G$ is connected and nontraceable, at least one of $x_1$ and $x_2$ has degree bigger that 2. Suppose $d(x_1) > 2$ and let $z \in N(x_1) - \{v, x_2\}$. If $Q$ is the path $z x_1 v x_2$ then, since $d(v) = 2$, the graph $\langle V(Q)\rangle$ is not complete and hence it follows from Lemma 2 that $d(x_1) \geq 4$. Similarily if $d(x_2) > 2$, then $d(x_2) \geq 4$ . ∎

**Lemma 6** *Suppose $G$ is a connected MNT graph with distinct nonadjacent vertices $v_1$ and $v_2$ such that $d(v_1) = d(v_2) = 2$.*

*(a) If $v_1$ and $v_2$ have exactly one common neighbour $x$, then $d(x) \geq 5$.*

*(b) If $v_1$ and $v_2$ have the same two neighbours $x_1$ and $x_2$, then $N_G(x_1) - \{x_2\} = N_G(x_2) - \{x_1\}$ and $d(x_1) = d(x_2) \geq 5$.*

**Proof.** (a) Let $N(v_i) = \{x, y_i\}$; $i = 1, 2$. It follows from Lemma 5 that $x$ is adjacent to $y_i$; $i = 1, 2$. Let $Q$ be the path $y_1v_1xv_2y_2$. Since $\langle V(Q)\rangle$ is not complete, it follows from Lemma 2 that $x$ has a neighbour in $G - V(Q)$. Hence $d(x) \geq 5$.

(b) From Lemma 5 it follows that $x_1$ and $x_2$ are adjacent. Let $Q$ be the path $x_2v_1x_1v_2$. $\langle V(Q)\rangle$ is not complete since $v_1$ and $v_2$ are nonadjacent. Thus it follows from Lemma 2 that $x_1$ has a neighbour in $G - V(Q)$. Now suppose $p \in N_{G-V(Q)}(x_1)$ and $p \notin N_G(x_2)$. Then a hamiltonian path $P$ in $G + px_2$ contains a subpath of either of the forms given in the first column of Table 1. Note that $i, j \in \{1, 2\}$; $i \neq j$ and that $L$ represents a subpath of $P$ in $G - \{x_1, x_2, v_1, v_2, p\}$. If each of the subpaths is replaced by the corresponding subpath in the second column of the table we obtain a hamiltonian path $P'$ in $G$, which leads to a contradiction.

| Subpath of $P$ | Replace with |
|---|---|
| $v_ix_1v_jx_2p$ | $v_ix_2v_jx_1p$ |
| $v_ix_1Lpx_2v_j$ | $v_ix_2v_jx_1Lp$ |

Table 1

Hence $p \in N_G(x_2)$. Thus $N_G(x_1) - \{x_2\} \subseteq N_G(x_2) - \{x_1\}$. Similarly $N_G(x_2) - \{x_1\} \subseteq N_G(x_1) - \{x_2\}$. Thus $N_G(x_1) - \{x_2\} = N_G(x_2) - \{x_1\}$ and hence $d(x_1) = d(x_2)$. Now let $Q$ be the path $px_1v_1x_2v_2$. Since $\langle V(Q)\rangle$ is not complete, it follows from Lemma 2 that $x_1$ or $x_2$ has a neighbour in $G - V(Q)$. Hence $d(x_1) = d(x_2) \geq 5$. ■

**Lemma 7** *Suppose $G$ is a connected MNT graph of order $n \geq 6$ and that $v_1, v_2$ and $v_3$ are vertices of degree 2 in $G$ having the same neighbours, $x_1$ and $x_2$. Then $G - \{v_1, v_2, v_3\}$ is complete and hence $e(G) = \frac{1}{2}(n^2 - 7n + 24)$.*

**Proof.** The set $\{x_1, x_2\}$ is a cutset of $G$. Thus according to Lemma 3 $G - \{v_1, v_2, v_3\} = K_{n-3}$. Hence $e(G) = \frac{1}{2}(n-3)(n-4) + 6$. ■

By combining the previous three results we obtain

**Theorem 8** *Suppose $G$ is a connected MNT graph without vertices of degree 1 or adjacent vertices of degree 2. If $G$ has order $n \geq 7$ and $m$ vertices of degree 2, then $e(G) \geq \frac{1}{2}(3n + m)$.*

**Proof.** If $G$ has three vertices of degree 2 having the same two neighbours then, by Lemma 7, $m = 3$ and

$$e(G) = \tfrac{1}{2}(n^2 - 7n + 24) \geq \tfrac{1}{2}(3n + m) \text{ when } n \geq 7.$$

We now assume that $G$ does not have three vertices of degree 2 that have the same two neighbours. Let $v_1, ..., v_m$ be the vertices of degree 2 in $G$ and let $H = G - \{v_1, ..., v_m\}$. Then by Lemmas 5 and 6 the minimum degree, $\delta(H)$ of $H$ is at least 3. Hence

$$e(G) = e(H) + 2m \geq \tfrac{3}{2}(n - m) + 2m = \tfrac{1}{2}(3n + m). \qquad \blacksquare$$

# 3 The minimum size of a MNT graph

Our aim is to determine the exact value of $g(n)$. By consulting the Atlas of Graphs [8], one can see, by inspection, that $g(2) = 0$, $g(3) = 1$, $g(4) = 2$, $g(5) = 4$, $g(6) = 6$ and $g(7) = 8$ (see Fig. 3).

We now give a lower bound for $g(n)$ for $n \geq 8$.

**Theorem 9** *If $G$ is a MNT graph of order $n$, then*

$$e(G) \geq \begin{cases} 10 & \text{if } n = 8 \\ 12 & \text{if } n = 9 \\ \frac{3n-2}{2} & \text{if } n \geq 10. \end{cases}$$

**Proof.** If $G$ is not connected, then $G = K_k \cup K_{n-k}$, for some positive integer $k < n$ and then, clearly, $e(G) > \frac{3n-2}{2}$ for $n \geq 8$. Thus we assume that $G$ is connected.

We need to prove that the sum of the degrees of the vertices of $G$ is at least $3n - 2$. In view of Theorem 8, we let

$$M = \{v \in V(G) \mid d(v) = 2 \text{ and no neighbour of } v \text{ has degree } 2\}.$$

The remaining vertices of degree 2 can be dealt with simultaneously with the vertices of degree 1. We let

$$S = \{v \in V(G) - M \mid d(v) = 2 \text{ or } d(v) = 1\}.$$

If $S = \emptyset$, then it follows from Theorem 8 that $e(G) \geq \frac{1}{2}(3n + m)$. Thus we assume that $S \neq \emptyset$.

We observe that, if $H$ is a component of the graph of $\langle S \rangle$, then either $H \cong K_1$ or $H \cong K_2$ and $N_G(H) - V(H)$ consists of a single vertex, which is a cut-vertex of $G$.

An example of such a graph $G$ is depicted in the figure below.

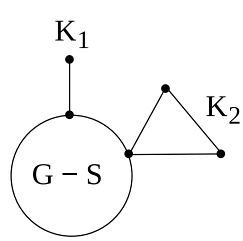


Fig. 1

Let $s = |S|$. By Lemma 4 the graph $\langle S \rangle$ has at most three components. We thus have three cases:

**CASE 1.** $\langle S \rangle$ has exactly three components, say $H_1, H_2, H_3$:

In this case the neighbourhoods of $H_1, H_2, H_3$ are pairwise disjoint; hence $G$ has three cut-vertices. Hence it follows from Lemma 4 that $G - S$ is a complete graph of order at least 3. Futhermore, for every possible value of $s$, the number of edges in $G$ incident with the vertices in $S$ is $2s - 3$. Thus

$$e(G) = \binom{n-s}{2} + 2s - 3 \text{ for } s = 3, 4, 5 \text{ or } 6;\ \ s \le n - 3.$$

An easy calculation shows that, for each possible value of $s$,

$$e(G) \ \ \ge \ \ \begin{cases} 10 & \text{if } n = 8 \\ 12 & \text{if } n = 9 \\ \frac{3n-2}{2} & \text{if } n \ge 10. \end{cases}$$

This case is a Zelinka Type II construction, cf. [9]. The graphs of smallest size of order 8 and 9 given by this construction are depicted in Fig. 3.

**CASE 2.** $\langle S \rangle$ has exactly two components, say $H_1, H_2$:

In this case the number of edges in $G$ incident with the vertices in $S$ is $2s - 2$.

**Subcase 2.1.** $N_G(H_1) = N_G(H_2)$:

Then it follows from Lemma 3 that $G - S$ is a complete graph. Hence

$$e(G) = \binom{n-s}{2} + 2s - 2 \text{ for } s = 2, 3 \text{ or } 4.$$

Thus

$$e(G) \ \ \ge \ \ \begin{cases} 12 & \text{if } n = 8 \\ 16 & \text{if } n = 9 \\ \frac{3n-2}{2} & \text{if } n \ge 10 \end{cases}$$

This case is a Zelinka Type I construction, cf. [9].

**Subcase 2.2.** $N_G(H_1) \neq N_G(H_2)$:

Let $N_G(H_i) = y_i$, $i = 1, 2$ and $y_1 \neq y_2$.

If $y_1y_2 \notin E(G)$ then $G + y_1y_2$ has a hamiltonian path $P$. But then $P$ has one endvertex in $H_1$ and the other in $H_2$ and contains the edge $y_1y_2$; hence $V(G - S) = \{y_1, y_2\}$. But then $G$ is disconnected. This contradiction shows that $y_1y_2 \in E(G)$.

Now $G - S$ is not complete, otherwise $G$ would be traceable. Since $G + vw$, where $v$ and $w$ are nonadjacent vertices in $V(G - S)$, contains a hamiltonian path with one endvertex in $H_1$ and the other in $H_2$ and $y_1y_2 \in E(G)$, it follows that $(G - S) + vw$ has a hamiltonian cycle. Hence $G-S$ is either hamiltonian or MNH. We consider these two cases separately:

**Subcase 2.2.1.** $G-S$ is hamiltonian:
Then no hamiltonian cycle in $G-S$ contains $y_1y_2$, otherwise $G$ would be traceable. Thus $d_{G-S}(y_i)\geq 3$ for $i=1,2$.

It also follows from Lemma 3 that no vertex $v\in M$ can be adjacent to both $y_1$ and $y_2$ since the graph $\langle V(H_i)\cup T\rangle$, where $T=\{y_1,y_2\}$ is not complete, for $i=1,2$. If $v\in M$ is adjacent to to one of the $y_i$'s for $i=1,2$, say $y_1$, then, since the neighbours of $v$ are adjacent, it follows that $d_{G-M-S}(y_1)\geq 3$.

It follows from our definition of $M$ and $S$ that $N_G(M)\cap S=\emptyset$. Since $G-M$ is not a complete graph, it follows from Lemma 7 that $M$ does not have three vertices that have the same neighbourhood in $G$. Hence, by Lemmas 5 and 6, the minimum degree of the graph $G-M-S$ is at least 3.

Now, for $n\geq 8$

$$\begin{aligned}
e(G) &= e(G-M-S)+2m+2s-2\\
&\geq \frac{1}{2}\left(3\left(n-m-s\right)\right)+2m+2s-2\\
&= \frac{1}{2}\left(3n+m+s-4\right)\\
&\geq \frac{3n-2}{2},\ \text{since } s\geq 2.
\end{aligned}$$

**Subcase 2.2.2.** $G-S$ is nonhamiltonian:
Then $G-S$ is MNH (as shown above); hence it follows from Theorem 1, that $e(G-S)\geq\frac{3}{2}(n-s)$ for $n-s\geq 6$.

Thus, for $n-s\geq 6$ and $n\geq 8$

$$\begin{aligned}
e(G) &= e(G-S)+2s-2\\
&\geq \frac{1}{2}(3(n-s))+2s-2\\
&= \frac{1}{2}(3n+s-4)\\
&\geq \frac{3n-2}{2},\ \text{since } s\geq 2.
\end{aligned}$$

From [7] we have

$$e(G-S)\ \geq\ \begin{cases} 6 & \text{for } n-s=5\\ 4 & \text{for } n-s=4.\end{cases}$$

Thus

$$e(G)\ \geq\ \begin{cases} 12 & \text{for } n=9 \text{ and } n-s=5\\ 10 & \text{for } n=8 \text{ and } n-s=5 \text{ or } n-s=4.\end{cases}$$

The smallest MNH graphs $F_4$ and $F_5$ of order 4 and 5 respectively, are depicted in Fig. 2; cf. [7]. The graphs $G_8$ and $G_9$ (see Fig. 3) are obtained, respectively, by using $F_4$ with $s = 4$ or $F_5$ with $s = 3$, and $F_5$ with $s = 4$.

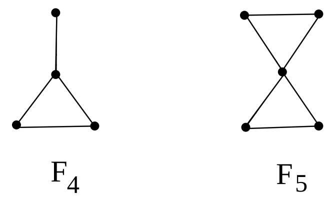


Fig. 2

**CASE 3.** $\langle S \rangle$ has exactly one component, say $H$:
Since

$$\sum_{v \in S} d_G(v) = 3s - 2, \text{ for } s = 1, 2$$

it follows that

$$\begin{aligned}
e(G) &= e(G - M) + 2m \\
&= \frac{1}{2}\left( \sum_{v \in V(G-M)-S} d_{G-M}(v) + \sum_{v \in S} d_{G-M}(v) \right) + 2m \\
&\geq \frac{1}{2}\left(3\left(n - m - s\right) + 3s - 2\right) + 2m \\
&= \frac{1}{2}\left(3n + m - 2\right) \\
&\geq \frac{3n-2}{2}.
\end{aligned}$$

■

From the previous theorem we have $g(8) = 10$, $g(9) = 12$ and $g(n) \geq \lceil \frac{3n-2}{2} \rceil$ for $n \geq 10$. The MNT graphs $G_n$ of order $n$ with $g(n)$ edges, for $n \leq 9$ are given in Fig. 3.

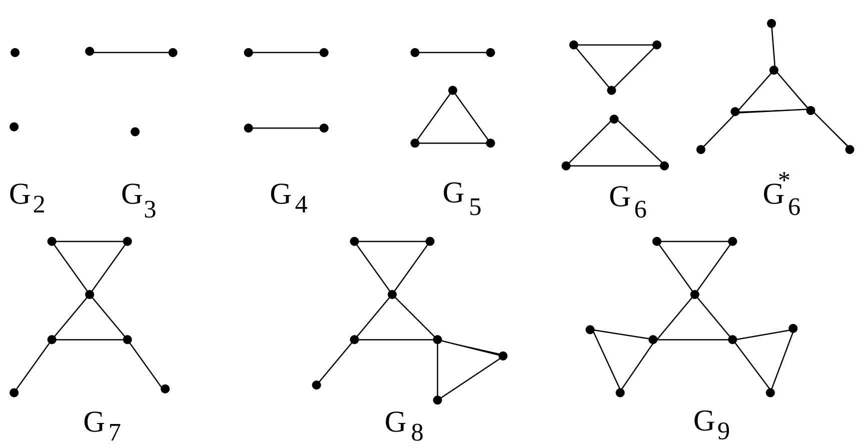


Fig. 3

In [5] Dudek, Katona and Wojda constructed, for every $n \geq 54$ as well as for every $n \in I = \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$, a MNT graph of size $\lceil \frac{3n-2}{2} \rceil$ in the following way: Consider a cubic MNH graph $G$ with the property that
(1) there is an edge $y_1y_2$ of $G$, such that $N(y_1) \cap N(y_2) = \emptyset$, and
(2) $G + e$ has a hamiltonian cycle containing $y_1y_2$ for every $e \in E(\overline{G})$.

Now take two graphs $H_1$ and $H_2$, with $H_1 \cong K_1$ and $H_2 \cong K_1$ or $H_2 \cong K_2$ and join each vertex of $H_i$ to $y_i$; $i = 1, 2$. The new graph is a MNT graph of order $v(G) + 2$ and size $e(G) + 2$ or of order $v(G) + 3$ and size $e(G) + 4$.

It follows from results in [3] and [4] that for every even $n \geq 52$ as well as for $n \in \{20, 28, 36, 38, 40, 44, 46, 48\}$ there exists a cubic MNH graph of order $n$ that satisfies (1) and (2). Thus this construction provides MNT graphs of order $n$ and size $\lceil \frac{3n-2}{2} \rceil$ for every $n \geq 54$ as well as for every $n \in I$.

We determined, by using the Graph Manipulation Package developed by Siqinfu and Sheng Bau*, that the Petersen graph also satisfies the above property. Hence, according to the above construction, there are also MNT graphs of order $n$ and size $\lceil \frac{3n-2}{2} \rceil$ for $n = 12, 13$.

Thus $g(n) = \lceil \frac{3n-2}{2} \rceil$ for $n \geq 54$ as well as for every $n \in I \cup \{12, 13\}$.

It remains an open problem to find $g(n)$ for $n = 10, 11$ and those values of $n$ between 13 and 54 which are not in $I$.

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