Two-person symmetric whist

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Abstract

We introduce a two-person perfect information model of trick taking games. A set of cards is distributed between two players, and play proceeds in tricks with the obligation to follow suit, as in many real-world card games.

We assume that in each suit, the two players have the same number of cards. Under this assumption, we show how to assign a value from a certain semigroup to each single-suit card distribution in such a way that the outcome of a multi-suit deal under optimal play is determined by the sum of the values of the individual suits.

1 Two-person whist

The game of two-person whist is played with a deck of cards. Each card belongs to a *suit*, and within each suit, the cards are ordered by *rank*. Real-world card packs sometimes have four suits with thirteen cards in each suit. We will not restrict ourselves to the "standard" deck, and in any case, the french suited 52 card pack too is just one of many variations. There does not even have to be the same number of cards in each suit.

The cards are distributed between the two players, so that both players receive the same number of cards. We assume that both players have complete information about the situation. One of the players is said to have the *lead*. The player who has the lead plays, or *leads*, one of his cards. The other player, in response to this, plays one of his cards. If possible, he has to *follow suit*, that is, he has to play a card in the same suit as the card that was led. The player who played the highest card in the suit that was led wins the trick, and obtains the lead. The cards that have been played are removed, and play continues until all cards have been played. Each player tries to win as many tricks as possible.

This game is a pure form of a common type of card game, *trick taking games*. Trick taking games exist in many different forms, and their history goes back to the early fifteenth century. Here we assume that the game is played between two players, and further that it is played with perfect information. The assumption of perfect information is often not realistic in a given situation, but a general understanding of the game probably has to be based on knowledge of the playing technique in its perfect information counterpart.

In this paper, our approach is based on evaluating each suit separately, and then adding the values of the individual suits to obtain a value for the complete card distribution. We focus on a special case where this idea works well.

1.1 The symmetric case

Throughout the paper, we assume that *in each suit* the players have the same number of cards. Such a card distribution is called *symmetric*. If this condition is satisfied initially, then the player not on lead will always be able to follow suit, so the symmetry will never be broken. In a symmetric deal, the number of tricks where the lead is in a given suit is determined in advance, and does not depend on how the cards are played. The advantage of studying symmetric deals is that in a given deal, the effect of a particular suit on the game as a whole can be measured and evaluated by comparing play and outcome with the deal obtained by removing the suit from both hands.

1.2 Conventions

We assume that the game is played between two players called East and West. Our sympathies are usually with West. This convention is customary in combinatorial game theory, where the players are called Left and Right. The author thinks it is more in the spirit of card games to use the labels West and East. When we speak of the *outcome* of a deal, we mean the number of tricks that West will take with optimal play from both sides. When possible, we use the standard ranks from 2 to 10, Jack, Queen, King, and Ace.

2 Aim of the paper

From the point of view of computational complexity, a game such as whist can be regarded as solved when a polynomial time algorithm is found that computes the game-theoretical value of any given deal, as well as an optimal move in any given situation. A polynomial time (in fact almost linear time) algorithm for computing the outcome of single-suit whist was given in [8].

We do not solve symmetric multi-suit whist in this sense (see however the discussion in Section 17). Instead we show how to assign values from a certain semigroup to individual suits in such a way that the sum of the values of the suits in a multi-suit game determines the outcome of the game under optimal play. This includes assigning a rational number to each single-suit deal reflecting the average value of this suit in a multi-suit game. The theory developed for symmetric multi-suit whist includes the technique known to bridge players as *elimination and throw-in*.

Whist is not a combinatorial game in the strict sense, since the move-order is not alternating, and the objective is to win as many tricks as possible, rather than to make the last move. Therefore the theory developed in [1, 2] does not apply directly to whist. However, as readers familiar with combinatorial game theory will necessarily notice, we make use of many of the ideas and methods of this theory. Some of the concepts that we introduce, like mean value, simplicity, numbers, and infinitesimals, have direct counterparts in the theory of combinatorial games as developed in [1, 2].

3 Background and motivating examples

Two-person whist played with a single suit was solved by the author in [8]. We will not make use of this solution, but in principle, the outcome of a single-suit card distribution under optimal play can be regarded as known. A reasonable approach to the symmetric multi-suit game would be to count the number of tricks we can take in each suit, and then add these numbers together. This approach, although too naive in general, obviously works well in many cases. Consider the following deal:

West :	*East* :	
♠ A K	♠ Q J	
♡ A J	♡ K Q	(1)
♢ K 10 9	♢ A Q J	

West can count two tricks in spades and one trick in each of hearts and diamonds. To evaluate the trick-taking potential in spades and hearts we do not even have to take into account how the lead will pass between the players during the game, since they will produce the same number of tricks regardless of how the cards are played. In diamonds, West clearly cannot get more than one trick. On the other hand, as soon as East leads a diamond, whether high or low, West will be certain to win a trick with the king. West can therefore refuse to play diamonds as long as possible. If at the end he is on lead with only the three diamond tricks left to play, he can lead a small diamond, and then score his king in one of the last two tricks. He can therefore consider the diamond king to be worth one trick. On the deal as a whole, West will be able to take $2 + 1 + 1 = 4$ tricks.

In other cases, the outcome of a single-suit game depends on the initial location of the lead. An elementary fact about the single-suit game, proved in [3], is that having the lead is never an advantage, but on the other hand may cost at most one trick. The solution in [8] is based on assigning to each deal a number, which is half of an integer. This number is a measure of the number of tricks that West can take, and if not an integer, it should be rounded to an integer in favor of the player not on lead. Hence this number represents the mean value of the number of tricks that West can take with and without the lead. The simplest case of a non-integral value is the following:

West :	*East* :	
A Q	K J	(2)

The value of this card distribution for West can be described by the number 3/2, which means that West will get 1 trick if he has the lead, but 2 tricks if East has the lead. Interestingly, from a multi-suit perspective, the number 3/2 also happens to represent the mean value of this card distribution in another sense, analogous to the concept of mean value of a combinatorial game. Consider for example:

West :	*East* :
♠ A Q	♠ K J
♡ A Q	♡ K J
♢ A Q	♢ K J
♣ A Q	♣ K J

(3)

Whether or not West will be able to score the queen in a particular suit depends only on who makes the first lead in the suit. If East leads a certain suit, West will immediately be able to cash two tricks in that suit. If West leads the suit, then East will win a trick with the king, immediately or later. With correct play, whenever East is on lead, West will cash two tricks in the suit led. Then West on lead will cash the ace of another suit and continue with the queen. East gets a trick for his king, and the lead is back with East. Hence in this case, West will get 6 of the 8 tricks, regardless of the initial position of the lead, and in general, with any number of suits with this distribution, West will score 3/2 times the number of suits, rounded to an integer in favor of the player not on lead.

It is natural to conjecture that the outcome of a deal in which every suit has a non-integral value in this sense can be determined by adding the values and rounding to the nearest integer. The following theorem is proved later in a more general context.

Theorem 3.1. *Suppose that we assign the number $n + 1/2$ to any single-suit deal in which West will take n tricks with the lead and $n + 1$ tricks with East on lead. Then in a multi-suit deal where every single-suit component is of this type, the outcome under optimal play is obtained by summing the numbers assigned to each suit, and if the sum is not an integer, rounding in favor of the player not on lead.*

For example, in the deal

West :	*East* :
♠ A Q	♠ K J
♡ A K J	♡ Q 10 9
♢ A J 9	♢ K Q 10

(4)

we count $1 + 1/2$ for the spades, $2 + 1/2$ for the hearts, and $1 + 1/2$ for the diamonds. Note that whenever East leads the diamond king, West should play low. This adds to $5 + 1/2$. Consequently, West can take five tricks with the lead, and six tricks if East is on lead.

If we add a club suit to make it

$$\begin{array}{ll} West: & East: \\ \spadesuit\ \mathsf{A\ Q} & \spadesuit\ \mathsf{K\ J} \\ \heartsuit\ \mathsf{A\ K\ J} & \heartsuit\ \mathsf{Q\ 10\ 9} \\ \diamondsuit\ \mathsf{A\ J\ 9} & \diamondsuit\ \mathsf{K\ Q\ 10} \\ \clubsuit\ \mathsf{K\ 10\ 9\ 8} & \clubsuit\ \mathsf{A\ Q\ J\ 7} \end{array} \tag{5}$$

the sum will be $(1 + 1/2) + (2 + 1/2) + (1 + 1/2) + (1 + 1/2) = 7$, indicating that West will get 7 of the tricks regardless of the initial position of the lead.

It becomes clear from a few examples that the situation can be more complicated if the deal contains suits which played separately would yield the same number of tricks regardless of the position of the lead. We can try to evaluate the deal

$$\begin{array}{ll} West: & East: \\ \spadesuit\ \mathsf{A\ Q} & \spadesuit\ \mathsf{K\ J} \\ \heartsuit\ \mathsf{A} & \heartsuit\ \mathsf{K} \end{array} \tag{6}$$

to $(1+1/2)+1 = 2+1/2$, but in fact, West will not get more than two tricks even if East has the lead, since East will simply transfer the lead to West by playing hearts. With

$$\begin{array}{ll} West: & East: \\ \spadesuit\ \mathsf{A\ Q} & \spadesuit\ \mathsf{K\ J} \\ \heartsuit\ \mathsf{A} & \heartsuit\ \mathsf{K} \\ \diamondsuit\ \mathsf{K} & \heartsuit\ \mathsf{A} \end{array} \tag{7}$$

it is an advantage to have the lead. Apparently the number $2 + 1/2$ should be rounded in favor of the player on lead in this case.

As the following example shows, it cannot be consistent to assign the value n to every single-suit deal that, played by itself, produces n tricks for West. Hence there is no analogue of Theorem 3.1 for suits of this type.

$$\begin{array}{ll} West: & East: \\ \spadesuit\ \mathsf{A\ K\ 10} & \spadesuit\ \mathsf{Q\ J\ 9} \\ \heartsuit\ \mathsf{A\ K\ 10} & \heartsuit\ \mathsf{Q\ J\ 9} \end{array} \tag{8}$$

Here each suit would be worth two tricks for West, if played separately, since even if East is on lead, he can secure one trick by leading a high card. On the other hand, in the deal as a whole, West can take five of the six tricks, provided East has the initial lead. In order not to give West a cheap trick immediately, East will lead one of his honors, say the spade queen. West wins the trick and plays ace, king, and ten of hearts. This way East gets the lead (unless he surrenders by playing the queen and jack of hearts under West's ace and king), and is forced to lead spades a second time. This gives West a trick for the spade ten.

In view of Theorem 3.1, we can conjecture that it is consistent to assign the value $n + 1/2$ to any deal, single or multi-suit, that produces n tricks for West on lead, and $n+1$ tricks for West with East on lead. The following two theorems as well as Theorem 3.1 are derived as corollaries of Theorem 12.1.

Theorem 3.2. *It is consistent to assign the value $n + 1/2$ to any deal where West gets n tricks with the lead and $n + 1$ tricks without the lead, in the sense that whenever a deal can be split into components of this type, the outcome of the deal as a whole will be the sum of the values of the components, rounded in favor of the player not on lead.*

Theorem 3.3. *It is consistent to assign the value $n + 1/2$ to any deal where West gets n tricks with the lead and $n+1$ tricks without the lead, in the sense that whenever a deal has this property, we can assign values to its single suit components that add up to the value of the whole deal (and so that the value of a suit still depends only on the distribution of the cards in that suit).*

If this is correct, then the deal (8) should have value $5 + 1/2$, and consequently, the individual suits should have value $2 + 1/4$. Indeed, under reasonable assumptions on correct play, we can see that the mean value of A K 10 versus Q J 9 ought to be $2 + 1/4$. Whenever East is on lead, he will lead a queen or a jack. West takes the trick and continues with ace, king, and ten of a different suit, putting East back on lead. This continues until half the suits are played out completely, and the remaining half are distributed A 10 versus J 9 (or equivalently). This combination is equivalent to A Q versus K J discussed earlier. West will now be able to take a trick with half of his remaining tens. This means that he gets an extra trick for every four suits. Careful analysis shows that the number of tricks that West gets with n suits distributed this way is indeed $(2 + 1/4)n$ rounded to the nearest integer, and if $n \equiv 2 \pmod 4$, rounded in favor of the player not on lead.

4 The numerical value of a card distribution

In the last section, we assigned a numerical value to certain single- and multi-suit deals, namely those that occur as components of deals where having the lead costs a trick. At the same time, it seems even more natural to evaluate a suit distributed for example A versus K, A K versus Q J or A J versus K Q, to the number of tricks that the suit is bound to produce. As we prove later, these card combinations cannot occur in a deal where having the lead is a disadvantage.

The following theorem has served as a working hypothesis that has motivated the approach taken in the paper. It combines the two ways of assigning numbers to individual suits, and thus generalizes Theorems 3.2 and 3.3. This theorem too is a consequence of the main theorem (Theorem 12.1). It defines what we will refer to as the *numerical value* of a card distribution.

Theorem 4.1. *To every symmetric deal D, we can assign a number $N(D)$ called the* numerical value *of D, satisfying the following "axioms":*

1. *The numerical value of a multi-suit card distribution is the sum of the numerical values of its single suit components.*

2. *Regardless of the location of the lead, the outcome of a deal differs by at most $1/2$ from its numerical value.*

We do not prove at this point that Axioms 1 and 2 are consistent. Instead, we assume throughout Sections 4, 5 and 6 the existence of a function N satisfying Theorem 4.1, and derive some of its properties. Throughout Sections 4–7, whenever a statement is labeled *Corollary*, it means that it will follow from Theorem 4.1, that is, from the consistency of Axioms 1 and 2. The following two statements follow from Axiom 2 since the intervals $[m-1/2, m+1/2]$ and $[n-1/2, n+1/2]$ have nonempty intersection if $|m-n| \geq 2$, and intersect only in the point $n+1/2$ if $m-n=1$.

Corollary 4.2. *The difference in outcome of a deal with East and West on lead respectively is at most one trick.*

Corollary 4.3. *If D is a deal in which West gets n tricks with one of the players on lead, and $n+1$ tricks with the other player on lead, then $N(D) = n + 1/2$.*

4.1 The mean value of a deal

Let D be a deal, and let m be a positive integer. We let $m \cdot D$ denote a deal which consists of m copies of D. That is, for each single-suit component of D, the deal $m \cdot D$ has m suits with the same card distribution. By Axiom 1, $N(m \cdot D) = m \cdot N(D)$. If we let a_m be the outcome of $m \cdot D$ with, say, West on lead, then by Axiom 2,

$$|a_m - m \cdot N(D)| \leq 1/2.$$

If we divide by m and let $m \to \infty$, we obtain

$$\frac{a_m}{m} \to N(D).$$

From this, it follows that $N(D)$ is uniquely determined by Axioms 1 and 2. $N(D)$ is the mean value of the number of tricks that West will get per copy of D when a large number of copies of D are played simultaneously. The numerical value of a deal is therefore analogous to the mean value of a combinatorial game.

Theorem 4.4. *There is at most one function N satisfying Axioms 1 and 2.*

Corollary 4.5. *A deal which always gives West n tricks regardless of how the cards are played must have numerical value n.*

This follows since the deal must have mean value n.

Corollary 4.6. *If D is a deal with n cards on each hand, and $\overline{D}$ is the deal obtained by switching the East and West hands, then $N(D) + N(\overline{D}) = n$.*

This follows since this property obviously holds for the mean value.

5 Numerical values of some card distributions

We now show how to compute the numerical values of some card distributions using Axioms 1 and 2. The combination A versus K always gives West one trick. By Corollary 4.5, the numerical value must be 1. Similarly, $N(\mathsf{K},\ \mathsf{A}) = 0$. For 2-card distributions, Corollary 4.5 gives

$$N(\mathsf{A\ K},\ \mathsf{Q\ J}) = 2$$

and

$$N(\mathsf{A\ J},\ \mathsf{K\ Q}) = 1.$$

By axiom 2,

$$N(\mathsf{A\ Q},\ \mathsf{K\ J}) = 1 + 1/2.$$

The numerical values of the remaining 2-card deals follow from Corollary 4.6.

5.1 Three-card deals

The values

$$N(\mathsf{A\ K\ Q},\ \mathsf{J\ 10\ 9}) = 3,$$

$$N(\mathsf{A\ K\ 9},\ \mathsf{Q\ J\ 10}) = 2$$

and

$$N(\mathsf{A\ 10\ 9},\ \mathsf{K\ Q\ J}) = 1$$

follow from Corollary 4.5. The values

$$N(\mathsf{A\ K\ J},\ \mathsf{Q\ 10\ 9}) = 2 + 1/2$$

and

$$N(\mathsf{A\ J\ 9},\ \mathsf{K\ Q\ 10}) = 1 + 1/2$$

follow immediately from Axiom 2. Notice that in the case of A J 9 versus K Q 10, if East has the lead and starts with the king or the queen, West will get two tricks by playing low in the first trick.

In the case of A Q J versus K 10 9, West will get two tricks regardless of the location of the lead. This does not prove that the numerical value of this deal is 2. However, we can prove this by considering the following 2-suit deal:

West :	*East* :	
♠ A Q J	♠ K 10 9	(9)
♡ K J	♡ A Q	

Here both players will try to avoid leading hearts. If West has the lead, and leads spades, he can either cash the ace and continue with another spade, or lead one of the smaller spades immediately. In any case, East will cash his king of spades in one of the two first tricks, and then lead another spade. West will be forced to lead hearts, which restricts him to 2 tricks.

If on the other hand East has the lead, and starts with a spade, then West will cash two spade tricks and lead his third spade. Either East has played his king of spades under West's ace, or he is now forced to lead hearts. In any case West gets 3 tricks.

This shows that the numerical value of the deal as a whole is $2 + 1/2$. Since the value of the heart suit is already known to be $1/2$, it follows that

$$N\,(\mathsf{A\ Q\ J},\ \mathsf{K\ 10\ 9}) = 2.$$

The situation would have been similar if the distribution of the spades had been A Q 10 versus K J 9 or A Q 9 versus K J 10. Hence

$$N\,(\mathsf{A\ Q\ 10},\ \mathsf{K\ J\ 9}) = N\,(\mathsf{A\ Q\ 9},\ \mathsf{K\ J\ 10}) = 2.$$

From the deal (8), we know that the numerical value of

West :	*East* :	
♠ A K 10	♠ Q J 9	(10)
♡ A K 10	♡ Q J 9	

must be $4 + 1/2$, since West will get 4 tricks with the lead and 5 tricks with East on lead. Hence

$$N\,(\mathsf{A\ K\ 10},\ \mathsf{Q\ J\ 9}) = 2 + 1/4.$$

Similarly, with

West :	*East* :	
♠ K Q 9	♠ A J 10	(11)
♡ K Q 9	♡ A J 10	

West will get 2 tricks with the lead, but 3 tricks if East has the lead, as the reader may verify. The strategy is similar to that of (8). When East attacks one of the suits, West will use the other suit to transfer the lead back to East and force him to lead a second time from the same suit. Hence

$$N\,(\mathsf{K\ Q\ 9},\ \mathsf{A\ J\ 10}) = (2 + 1/2)/2 = 1 + 1/4.$$

The remaining three-card deals are obtained from the deals above by switching the East and West hands.

5.2 Examples with more than three cards

The example

$$N\,(\mathsf{A\ K\ 10},\ \mathsf{Q\ J\ 9}) = 2 + 1/4$$

can be generalized in an obvious way. Consider the 4-suit deal

West :	*East* :	
♠ A K Q 8	♠ J 10 9 7	
♡ A K Q 8	♡ J 10 9 7	(12)
♢ A K Q 8	♢ J 10 9 7	
♣ A K Q 8	♣ J 10 9 7	

West has 12 easy tricks. We claim that if East has the lead, West will be able to score a thirteenth trick with one of his eights. If East leads the spade jack say, then West will take this trick, cash the ace, king, queen of hearts and lead his fourth heart. East gets the lead, and he can do no better than lead from a new suit, say the jack of diamonds. West takes the trick, and plays four rounds of clubs, putting East on lead with the last one. The situation is now equivalent to (8) with East on lead.

We will not go through all possible lines of play, but the reader can convince himself that there is no way West can get 13 tricks if he has the lead in (12). It follows that

$$N\,(\mathsf{A\ K\ Q\ 8,\ J\ 10\ 9\ 7}) = (12 + 1/2)/4 = 3 + 1/8.$$

Similarly, we have

$$N\,(\mathsf{A\ K\ Q\ J\ 6,\ 10\ 9\ 8\ 7\ 5}) = 4 + 1/16,$$

$$N\,(\mathsf{A\ K\ Q\ J\ 10\ 4,\ 9\ 8\ 7\ 6\ 5\ 3}) = 5 + 1/32,$$

and so on.

6 Exits and stoppers

Theorem 4.1 specifies the outcome of a deal in terms of its numerical value except when this value is half way between two integers. In this section we look at some examples of deals whose numerical value is half of an odd integer, in order to find the factors that determine whether the outcome is obtained by rounding the numerical value up or down. We already know that the rounding may depend on the location of the lead. In the example

West :	*East* :	
♠ K J	♠ A Q	(13)

the numerical value is 1/2, and this should be rounded in favor of the player not on lead. The numerical value of the deal

West :	*East* :	
♠ K J	♠ A Q	(14)
♡ K	♡ A	

is still 1/2, but here West gets a spade trick whether or not he has the lead. This is because he has what bridge players call an *exit card*. The king of hearts does not win a trick, but it provides West with a possibility to transfer the lead to East. The deal

West :	*East* :	
♠ K J	♠ A Q	
♡ A	♡ K	(15)
♢ K	♢ A	

shows an example of a so called *elimination and throw-in*. West on lead can eliminate East's exit card, the king of hearts, by cashing the ace. Then he exits with the king of diamonds. East is "thrown in" and has to lead spades. If East has the lead, he will do the same thing to West: cash the ace of diamonds before leading hearts.

If both players have exits, it can apparently sometimes be an advantage to have the lead. Some exit suits are better than others though, as the following examples show. The distribution A J versus K Q provides West with an exit, since the numerical value is 1, and in

$$\begin{array}{ll} West: & East: \\ \spadesuit\ \mathsf{K\ J} & \spadesuit\ \mathsf{A\ Q} \\ \heartsuit\ \mathsf{A\ J} & \heartsuit\ \mathsf{K\ Q} \end{array} \tag{16}$$

West gets a spade trick whether or not he has the lead. If we give East too an exit,

$$\begin{array}{ll} West: & East: \\ \spadesuit\ \mathsf{K\ J} & \spadesuit\ \mathsf{A\ Q} \\ \heartsuit\ \mathsf{A\ J} & \heartsuit\ \mathsf{K\ Q} \\ \diamondsuit\ \mathsf{A} & \diamondsuit\ \mathsf{K} \end{array} \tag{17}$$

we would expect a situation where the lead is an advantage. However, we discover that West always gets a spade trick. West on lead can cash the red aces before putting East on lead with a second heart. Suppose now that East has the lead. He can attack West's exit by leading a heart, but West takes with the ace, and now West has time to cash the ace of diamonds, eliminating East's exit, before playing the jack of hearts.

Here the ace of hearts acts by defending West's exit in hearts. It temporarily stops East from cashing his heart trick, giving West time to eliminate East's exit in diamonds before playing his own exit in hearts. Such a card will be called a *stopper*.

Note that the number of exits does not matter:

$$\begin{array}{ll} West: & East: \\ \spadesuit\ \mathsf{K\ J} & \spadesuit\ \mathsf{A\ Q} \\ \heartsuit\ \mathsf{K} & \heartsuit\ \mathsf{A} \\ \diamondsuit\ \mathsf{A} & \diamondsuit\ \mathsf{K} \\ \clubsuit\ \mathsf{A} & \clubsuit\ \mathsf{K} \end{array} \tag{18}$$

The fact that East has two exits while West has only one is irrelevant, since West on lead can eliminate both the diamonds and the clubs before exiting in hearts.

However, the number of stoppers does matter:

$$\begin{array}{ll} West: & East: \\ \spadesuit\ \mathsf{K\ J} & \spadesuit\ \mathsf{A\ Q} \\ \heartsuit\ \mathsf{A\ J} & \heartsuit\ \mathsf{K\ Q} \\ \diamondsuit\ \mathsf{K\ Q} & \diamondsuit\ \mathsf{A\ J} \\ \clubsuit\ \mathsf{K\ Q} & \clubsuit\ \mathsf{A\ J} \end{array} \tag{19}$$

Here East has two stoppers, and West has only one. This gives East an edge in the fight for the second spade trick. If West leads a club or a diamond, say a diamond, then East takes with the ace and returns a heart. His ace of clubs now guarantees that he will have time to cash his heart trick before playing the jack of clubs. This ensures him two spade tricks.

7 The semigroup of states

7.1 Definitions

We introduce an additive notation for card distributions. A single-suit deal is a partition of a finite totally ordered set into two sets E and W of the same cardinality, the East and West hands. This is denoted by $[W, E]$, where W and E are the West and East hands, respectively.

A multi-suit deal is a formal sum of single-suit deals. We denote the set of multi-suit deals by $\mathbf{D}$. Hence $\mathbf{D}$ is the free abelian monoid over the set of single-suit deals.

Addition is commutative, that is, we do not distinguish between the individual suits. The deal

$$[\mathsf{K}, \mathsf{A}] + [\mathsf{K\ J}, \mathsf{A\ Q}],$$

for instance, may represent either of

West :	*East* :
♠ K J	♠ A Q
♡ K	♡ A

and

West :	*East* :
♠ K	♠ A
♡ K J	♡ A Q

In order to represent a state in the game in such a way that the outcome under optimal play from a game-state is a function of the state, we need to include not only the remaining cards on the two hands, but also the number of tricks that West has already taken. Therefore we let a *state* be a sum

$$m + D,$$

where m is an integer, and D is a multi-suit deal. Hence the set S of states is the direct sum

$$\mathbf{Z} \oplus \mathbf{D}$$

of the integers with the set of multi-suit card distributions.

If D is a state, then the *outcome* $\chi(D)$ of D is the pair (m, n), where m is the number of tricks that West takes under optimal play if he has the lead initially, and n is the number of tricks he takes if East has the lead. Hence χ maps S to $\mathbf{Z} \times \mathbf{Z}$. Obviously χ is not a semigroup homomorphism. Our approach is to describe χ by factoring it through a semigroup homomorphism.

When we speak of a *deal*, we mean just a card distribution. Technically, this is a state where West has not already taken any tricks, that is, the integer part of the state is zero. If D is a deal, then we let $|D|$ denote the number of cards on each of the hands, that is, the number of tricks to be played. We let $\overline{D}$ be the deal obtained by switching the East and West hands. Obviously, if $\chi(D) = (m, n)$, then $\chi(\overline{D}) = (|D| - n, |D| - m)$.

7.2 Equivalence and order of states

If D and E are two states, then we say that D is equivalent to E, and write $D \equiv E$, if for every state F, $\chi(D + F) = \chi(E + F)$. In other words, two states are equivalent if they behave in the same way under addition.

If (m, n) and (m', n') are two pairs of integers, we say that $(m, n) \leq (m', n')$ if $m \leq m'$ and $n \leq n'$. If D and E are states, then we say that $D \leq E$ if for every state F, $\chi(D+F) \leq \chi(E+F)$. Clearly $D \equiv E$ if and only if $D \leq E$ and $E \leq D$. Hence this gives a partial ordering on the quotient $S/\equiv$.

Addition is well-defined on the equivalence classes under $\equiv$, and $S/\equiv$ is an abelian semigroup. If D, E and F are states, and $D \leq E$, then $D + F \leq E + F$.

7.3 Examples established by strategy-stealing

Some properties of the ordering of states can be established by simple strategy-stealing arguments.

Example 7.1.

$$[\mathsf{K},\ \mathsf{A}] \geq 0.$$

Proof. We have to show that if D is a state, then

$$\chi(D + [\mathsf{K},\ \mathsf{A}]) \geq \chi(D),$$

in other words, West will get at least as many tricks in $D + [\mathsf{K},\ \mathsf{A}]$ as in D, both when he has the lead and when East has the lead.

West can steal an optimal strategy for D when playing $D + [\mathsf{K},\ \mathsf{A}]$ by pretending that the extra suit is not there. If at any point East leads his ace in the extra suit, West plays his king, and since the lead stays with East, West can continue to play as in D, pretending that the extra suit was never there. If East does not lead the extra suit, then neither does West until possibly in the last trick. This strategy will give West at least as many tricks in $D + [\mathsf{K},\ \mathsf{A}]$ as he can take in D. □

One would perhaps think that $[\mathsf{K},\ \mathsf{A}] \equiv 0$, but this is not true. The deal $[\mathsf{K},\ \mathsf{A}]$, although it does not have any trick-taking potential in itself, may give West the opportunity to put East on lead. This in turn may produce an extra trick in another suit. We have:

$$\chi\left([\mathsf{K},\ \mathsf{A}] + [\mathsf{K}\ \mathsf{J},\ \mathsf{A}\ \mathsf{Q}]\right) = (1, 1),$$

while

$$\chi([\mathsf{K\ J},\ \mathsf{A\ Q}]) = (0,1).$$

This shows that

$$[\mathsf{K},\ \mathsf{A}] > 0.$$

Example 7.2.

$$[\mathsf{K},\ \mathsf{A}] + [\mathsf{K},\ \mathsf{A}] \equiv [\mathsf{K},\ \mathsf{A}]\,.$$

Proof. It follows from Example 7.1 that

$$[\mathsf{K},\ \mathsf{A}] + [\mathsf{K},\ \mathsf{A}] \geq [\mathsf{K},\ \mathsf{A}]\,.$$

We need to establish the opposite inequality. We do this by showing that if D is any deal, then when playing $D + [\mathsf{K},\ \mathsf{A}] + [\mathsf{K},\ \mathsf{A}]$, East can steal an optimal strategy for $D + [\mathsf{K},\ \mathsf{A}]$. Whenever East on lead is required to cash the ace in an optimal strategy for $D + [\mathsf{K},\ \mathsf{A}]$, he will cash both aces in $D + [\mathsf{K},\ \mathsf{A}] + [\mathsf{K},\ \mathsf{A}]$. Whenever West leads one of the two kings in $D + [\mathsf{K},\ \mathsf{A}] + [\mathsf{K},\ \mathsf{A}]$, East will take with the corresponding ace, and immediately cash the other. Then he will steal the strategy that he would use in $D + [\mathsf{K},\ \mathsf{A}]$ if West leads the king. This will hold West to the same number of tricks in both cases. □

These two examples show that the quotient semigroup $S/\equiv$ cannot be embedded into a group.

7.4 Further examples that follow from Theorem 4.1

It seems obvious that

$$[\mathsf{A},\ \mathsf{K}] > [\mathsf{K},\ \mathsf{A}]\,,$$

but it is surprisingly difficult to prove this, or even to prove that

$$[\mathsf{A},\ \mathsf{K}] \geq 0. \tag{20}$$

To prove the inequality (20), we need to consider $D + [\mathsf{A},\ \mathsf{K}]$, where D is an arbitrary state, and show that West always gets at least as many tricks as when playing D. If West has the lead, this is obvious by strategy-stealing: West can cash the ace and then continue with an optimal strategy for D. However, if East has the lead, the problem is that East can transfer the lead to West by playing the king. Although this gives West an extra trick compared to playing D, it is not clear how West on lead can copy a strategy for D with East on lead, even if he can afford to give back one trick.

However, by Corollary 4.2, having the lead may cost at most one trick. Hence the inequality (20) follows from the consistency of Axioms 1 and 2. More generally, we have:

Corollary 7.3. *Let $D = [W, E]$, and $D' = [W\backslash\{x\} \cup \{y\}, E]$, where $x < y$. That is, D' is obtained from D by replacing one card on the West hand by a higher card. Then $D \leq D'$.*

In other words, a higher card is always at least as good as a smaller one.

Proof. We have to consider playing the two sums $D + E$ and $D' + E$, for an arbitrary state E. When playing $D' + E$, we let West steal an optimal strategy for $D + E$. West can pretend that the card y is the card x, until the optimal strategy for $D + E$ requires him to play the card x. Then instead he will play the card y. If East's card in that trick is not between x and y, West can go on pretending that the card he played was the card x. If East's card is between x and y, then West has taken an extra trick compared to playing $D+E$. West has now obtained the lead, so he cannot go on stealing the strategy for $D+E$, but by Corollary 4.2, having the lead will cost him at most one trick compared to not having the lead, so West's total number of tricks will be at least the same as when playing $D + E$. □

We can also establish that

$$[\mathsf{A\ J,\ K\ Q}] > 1$$

by strategy-stealing. When playing $[\mathsf{A\ J,\ K\ Q}] + D$, West can avoid leading from the suit until possibly when D is empty, and whenever East leads the suit, West takes the first trick with the ace and returns the jack. This way the lead stays with East, and West can continue stealing the strategy for D.

By Corollary 7.3, we can establish the ordering of all two-card single-suit deals, and their positions relative to the integers:

Corollary 7.4.

$$0 < [\mathsf{Q\ J,\ A\ K}] < [\mathsf{K\ J,\ A\ Q}] < [\mathsf{K\ Q,\ A\ J}] < 1 < [\mathsf{A\ J,\ K\ Q}] < [\mathsf{A\ Q,\ K\ J}] < [\mathsf{A\ K,\ Q\ J}] < 2. \quad (21)$$

8 Values

We introduce a certain semigroup whose elements will be referred to as *values*. Our aim is to prove that this semigroup is isomorphic to $S/\equiv$. For reasons that are discussed in Section 19, we are constructing the values, and the mapping from states to values "by hand", before proving any of their properties. In this section, we just define the set of values, and its structure of addition, negation, order and simplicity, without proving anything. The discussion should therefore be taken as an attempt to informally motivate these definitions, based on the examples given earlier.

8.1 The semigroup of infinitesimals

8.1.1 Unprotected exits

We denote the value of an unprotected exit by ε. An exit for East is denoted by $-\varepsilon$. As indicated by Example 7.2, we must have $\varepsilon + \varepsilon = \varepsilon$. Hence the sum of any number of exits of the same sign equals a single exit of that sign. However, signs do not cancel. Instead, the sum of a positive and a negative exit, or any number of such, has the "fuzzy" value $\pm\varepsilon$. The values of unprotected exits form a semigroup with the four elements 0, ε, $-\varepsilon$ and $\pm\varepsilon$.

8.1.2 Exits protected by stoppers

Stoppers add like integers. However, the full semigroup of infinitesimals is not isomorphic to a direct sum of the integers with the semigroup of unprotected exits described above. The reason for this is that on the one hand, one cannot have a stopper without having an exit, and on the other hand, there are some equivalences to take into account. If the total number of stoppers in a deal (added with signs) is positive, so that West has more stoppers than East, then it does not matter whether East has an exit or not. For instance,

$$[\mathsf{A\ J,\ K\ Q}] + [\mathsf{A,\ K}] \equiv [\mathsf{A\ J,\ K\ Q}] + 1.$$

We indicate the number of stoppers of an exit with an index. Hence ε_k is the value of an exit for West, protected by k stoppers. An exit for East with k stoppers is denoted ε_{-k}, but can also be regarded as the negative of ε_k, that is, $-\varepsilon_k$. For consistency, the unprotected exits can be written with an index of zero.

8.1.3 The set of infinitesimals

The semigroup of infinitesimals consists of the elements 0, ε_0, $-\varepsilon_0$, $\pm\varepsilon_0$, and ε_k, for nonzero integers k.

8.1.4 Addition of infinitesimals

The infinitesimals are added as follows: 0 is the additive identity. A sum all of whose terms are ε_0 equals ε_0. Similarly, a sum all of whose terms are $-\varepsilon_0$ equals $-\varepsilon_0$. All other sums are evaluated by summing the indices. If the sum of the indices is a nonzero integer k, then the sum equals ε_k. If the indices sum to zero, the sum is $\pm\varepsilon_0$.

8.1.5 The negative of an infinitesimal

As is indicated by the notation, there is a notion of negative of an infinitesimal. We let $-(\varepsilon_0) = -\varepsilon_0$, $-(\pm\varepsilon_0) = \pm\varepsilon_0$, and for nonzero integers k, $-\varepsilon_k = \varepsilon_{-k}$. The negative of an infinitesimal is not in general an additive inverse. Negation only has the weaker property of being an automorphism with respect to addition, that is, $-(\alpha+\beta) = (-\alpha) + (-\beta)$. We still use the shorthand $\alpha - \beta$ for $\alpha + (-\beta)$.

8.1.6 Ordering of infinitesimals

The values of unprotected exits are ordered according to

$$-\varepsilon_0 < 0 < \varepsilon_0$$

and

$$-\varepsilon_0 < \pm\varepsilon_0 < \varepsilon_0,$$

with 0 and $\pm\varepsilon_0$ incompatible. If k is a positive integer, then the values of unprotected exits are greater than $-\varepsilon_k$, but smaller than ε_k. If p and q are two nonzero integers, and $p < q$, then $\varepsilon_p < \varepsilon_q$.

8.2 The semigroup V of values

Rational numbers that can be written with a power of 2 in the denominator will be referred to as *numbers* as in [1, 2]. A *value* is a sum of a number and an infinitesimal. We let V denote the set of values. Hence V is the direct sum of the group of numbers and the semigroup of infinitesimals. Values are added and negated in the obvious way: If q and r are numbers, and x and y are infinitesimals, then $(q + x) + (r + y) = (q + r) + (x + y)$, and $-(q + x) = (-q) + (-x)$. This implies that if α and β are arbitrary values, then $-(\alpha + \beta) = (-\alpha) + (-\beta)$. Negation is therefore a semigroup automorphism.

8.2.1 Ordering of values

We define the ordering of values as follows: If q and r are numbers, and x and y are infinitesimals, then $q + x \leq r + y$ if and only if either $q < r$, or $q = r$ and $x \leq y$. The ordering satisfies $\alpha \leq \beta$ if and only if $-\beta \leq -\alpha$, for all α and β.

8.3 Simplicity

There is a notion of *simplicity* of values, similar to the corresponding concept for combinatorial games discussed in [2] and [1]. We classify values from simple to more complex in the following way:

- The simplest values are the *half-integers*, that is, the numbers of the form $k/2$ for integers k.
- For numbers other than half-integers, a number with smaller denominator is simpler than a number with greater denominator.
- Numbers are simpler than other values.
- Values of the form $q + \varepsilon_k$ and $q - \varepsilon_k$, where q is a number, are simpler when k has smaller absolute value.
- Values of the form $q \pm \varepsilon_0$ are more complex than other values.

We can describe this structure by arranging values in complexity classes, labeled by ordinals, as follows:

- Half-integers have complexity 0.

- Numbers with minimal denominator 2^k, for $k \geq 2$, have complexity $k-1$.
- Values of the form $q+\varepsilon_k$ and $q-\varepsilon_k$ have complexity $\omega+k$.
- Values of the form $q \pm \varepsilon_0$ have complexity $\omega+\omega$.

Since the complexity classes are well-ordered, every nonempty set of values has an element of minimal complexity.

9 Rounding a value to an integer

The value of a deal should determine its outcome. In Section 10, we construct a function $\mathrm{val} : S \to V$ assigning values to states. We now define a function ρ mapping a value to the corresponding outcome, that is, mapping values to ordered pairs of integers. The idea is then to prove that $\chi = \rho \circ \mathrm{val}$.

The function $\rho : V \to \mathbf{Z} \times \mathbf{Z}$ is called the *rounding function* since first of all it rounds the numerical value to the nearest integer. Let α be a value. Then

$$\rho(\alpha) = \begin{cases} (n,n), & \text{if } n-1/2 < \alpha < n+1/2, \\ (n,n+1), & \text{if } \alpha = n+1/2, \\ (n+1,n), & \text{if } \alpha = n+1/2 \pm \varepsilon_0. \end{cases}$$

Note that the ordering of values can be characterized by $\alpha \leq \beta$ if and only if for every value γ, $\rho(\alpha+\gamma) \leq \rho(\beta+\gamma)$. This is what we should expect in view of the definition in Section 8 of the ordering of states.

If A is any nonempty discrete set of numbers, then we can define an analogous function rounding values to pairs of elements of A. Let q be a number and x an infinitesimal. Then $\rho_A(q+x) = (a,a)$ if a is the unique element of A closest to q, while if there is a tie between two elements a and b of A, and $a < q < b$, then

$$\rho_A(q+x) = \begin{cases} (a,a), & \text{if } x \text{ is negative}, \\ (b,b), & \text{if } x \text{ is positive}, \\ (a,b), & \text{if } x = 0, \\ (b,a), & \text{if } x = \pm\varepsilon_0. \end{cases}$$

10 The mapping $\mathrm{val} : S \to V$

The mapping $\mathrm{val} : S \to V$ is a semigroup homomorphism which fixes the integers. Hence to define it, we need only specify its values on single-suit states. This is done inductively.

We let a *labeled value* be a pair (P,x), where P is one of the symbols E or W (for East and West), and x is a value. A labeled value will represent a situation where the player P has the lead in a deal with value x.

In our analysis, an implicit hypothesis is that it is advantageous to have the lead, except if the value of the deal is a number. We therefore introduce the following ordering of labeled values:

- $(P, x) < (Q, y)$ if $x < y$,
- $(E, x) < (W, x)$ if x is not a number,
- $(E, x) > (W, x)$ if x is a number.

10.1 Reductions

By a *single-suit state* we mean a state which is the sum of an integer and a single-suit deal. A single-suit state $D = m + [W, E]$ is said to be an n-card state if the hands W and E have n cards each. If $W = \{W_1, \ldots, W_n\}$ and $E = \{E_1, \ldots, E_n\}$, where $W_1 < \cdots < W_n$ and $E_1 < \cdots < E_n$, then we define the *reduction* $D_{i,j}$ of D to be the state into which D will be transformed if in the first trick West plays the card W_i and East plays the card E_j. That is,

$$D_{i,j} = m + [W \backslash \{W_i\}, E \backslash \{E_j\}] + \begin{cases} 1, \text{ if } W_i > E_j \\ 0, \text{ if } W_i < E_j \end{cases}$$

We now let D be an n-card single-suit deal, and suppose that $\mathrm{val}(D')$ has been defined for every deal D' with fewer than n cards, and in particular, on all the reductions $D_{i,j}$ of D. For $1 \leq i, j \leq n$, we let

$$\alpha_{i,j} = (P, \mathrm{val}(D_{i,j})),$$

where P is the player who gets the lead if West plays his ith card and East plays his jth card. We let

$$\mathrm{maxmin}(D) = \max_i \min_j \alpha_{i,j},$$

and

$$\mathrm{minmax}(D) = \min_j \max_i \alpha_{i,j}.$$

Obviously $\mathrm{maxmin}(D) \leq \mathrm{minmax}(D)$.

10.2 The left and right bounds on $\mathrm{val}(D)$

For a nonempty single-suit deal D, we define two sets $L(D)$ and $R(D)$ of values, which in a certain sense correspond to the left and right options of a combinatorial game.

Let D be as in the previous section, and suppose that we have defined $\mathrm{maxmin}(D)$ and $\mathrm{minmax}(D)$. Then $L(D)$ is defined as follows: Let q be a number. Then $q \in L(D)$ if $\mathrm{maxmin}(D) \geq (E, q)$. Moreover, let x be a nonzero infinitesimal. Then $q + x \in L(D)$ if $\mathrm{maxmin}(D) \geq (E, q)$ and $\mathrm{minmax}(D) \geq (W, q + x)$.

The set $R(D)$ is defined similarly: If q is a number, then $q \in R(D)$ if $\mathrm{minmax}(D) \leq (W, q)$, and if x is a nonzero infinitesimal, then $q - x \in R(D)$ if $\mathrm{minmax}(D) \leq (W, q)$ and $\mathrm{maxmin}(D) \leq (E, q - x)$.

Notice that a value cannot belong to $L(D)$ or $R(D)$, unless its numerical part belongs to $L(D)$ or $R(D)$ respectively. Notice also that if $x \in L(D)$ and $y \le x$, then $y \in L(D)$. Similarly, if $x \in R(D)$ and $y \ge x$, then $y \in R(D)$.

10.3 The interval $I(D)$

Let D be as above. We let $I(D)$ be the set of values that do not belong to either of $L(D)$ and $R(D)$. The set $I(D)$ is an *interval* in the sense that if x, y, and z are values such that $x \le y \le z$, and x and z belong to $I(D)$, then so does y.

We prove that under certain conditions, an interval has a unique simplest element.

Theorem 10.1. *If a nonempty interval contains at most one half-integer, then it has a unique simplest element.*

Proof. Let I be a nonempty interval, and suppose that I contains at most one half-integer. Since the complexity classes are well-ordered, there is an element of I with minimal complexity. To prove uniqueness, it therefore suffices to show that if x and y are two distinct values of the same complexity, then unless they are half-integers, there is a simpler value between them.

Suppose first that x and y are numbers. Then x and y can be written $m/2^k$ and $n/2^k$ respectively, where m and n are distinct odd integers, and $k \ge 2$. Between two distinct odd numbers there is always an even number. Hence between x and y there is a number of the form $2a/2^k = a/2^{k-1}$. This number has smaller denominator, and is therefore simpler than x and y.

Suppose now that x and y are not numbers. Then we can assume that they have the same numerical part q, since otherwise there is a number between them. We must therefore have $x = q - \varepsilon_k$ and $y = q + \varepsilon_k$, or the other way around, for some integer k. It follows that the number q is between x and y. □

Lemma 10.2. *If D is a nonempty single-suit deal, then $I(D)$ is nonempty. Moreover, $I(D)$ always contains a value whose infinitesimal part is distinct from $\pm\varepsilon$.*

Proof. If the numerical part of $\text{maxmin}(D)$ is strictly smaller than the numerical part of $\text{minmax}(D)$, then there is a number strictly between them, and this number must belong to $I(D)$. Suppose therefore that $\text{maxmin}(D)$ and $\text{minmax}(D)$ have the same numerical part q. If q does not belong to $I(D)$, then it must belong either to $L(D)$ or to $R(D)$. By symmetry it suffices to consider the case that $q \in L(D)$, that is, $\text{maxmin}(D) \ge (E, q)$. Then $\text{minmax}(D) \ge (E, q) > (W, q)$. Hence $q \notin R(D)$. It follows that no value with numerical part q belongs to $R(D)$.

For some $P = E$ or W and some nonnegative infinitesimal y, we have $\text{minmax}(D) = (P, q + y)$. Hence if x is an infinitesimal greater than y, then $\text{minmax}(D) < (W, q + x)$. It follows that $q + x \notin L(D)$, and hence that $q + x \in I(D)$. □

10.4 Definition of val(D)

We let val(D) be an element of $I(D)$ of minimal complexity. We prove later on that there is no ambiguity in this definition, that is, there is always a unique simplest element of $I(D)$. For the moment we know that $I(D)$ contains at least one element of minimal complexity, so we can think of the function val as being defined up to a number of choices. This completes the definition of val(D) for every state D. We also notice that for single-suit deals, the value always has infinitesimal part distinct from $\pm\varepsilon$.

11 Reduced and refined whist

For technical reasons, we introduce two auxiliary games which have the same form as whist, but with slightly different objectives. We first briefly discuss a game which we call *reduced whist*. This game is similar to the game of whist, except that the last trick does not count. Hence in an n-card deal, the objective is to take as many as possible of the first $n-1$ tricks.

Lemma 11.1. *In single-suit reduced whist, there is an optimal strategy that always saves the smallest card on the hand for the last trick. Hence when playing single-suit reduced whist, the players can start by removing the smallest card from their hands, and then play as in ordinary whist with the remaining cards.*

Proof. We prove this by induction on the number of cards on the hand. Consider an optimal strategy for playing a certain single-suit deal of reduced whist. We modify this strategy so that it never uses the smallest card before the last trick.

If in the first trick, the strategy requires us to play a card higher than the smallest one, then after the first trick we can, by induction, use a strategy that saves the smallest card for the last trick. Suppose therefore that the strategy requires us to play the smallest card in the first trick. Then instead, we play the next to smallest card. If our opponent plays a card which is not between our smallest and next to smallest card, then this will not make any difference. The same player will win the trick, and by induction, we can after the first trick use a strategy that makes no use of our smallest card. Hence it does not matter whether our smallest remaining card is the originally smallest card or another one.

Suppose now that in the first trick, our opponent plays a card which is between our smallest and next to smallest card. Then we have won a trick that we would not have won with the given strategy, and we have obtained the lead. By induction, we can assume that from the second trick on, both players will use a strategy that saves the smallest card for the last trick. Hence we can assume that after the first trick, both players remove their smallest remaining cards from their hands, and continue as in ordinary single-suit whist. Hence compared to the given strategy, it will do us no harm to have wasted a higher card in the first trick. The only difference in the situation is that we have obtained the lead, whereas with the given strategy we would not have had the lead. On the other hand, we have won the first trick, so in order to prove the lemma, we only have to prove that in ordinary single-suit whist, having the lead cannot cost more than one trick compared to not having the lead. This is what the following theorem tells us. □

To complete the proof, we cite a theorem from [3]:

Theorem 11.2. *In single-suit whist, having the lead is never an advantage, but may cost at most one trick compared to not having the lead.*

The following theorem holds also for multi-suit deals:

Theorem 11.3. *There is a strategy which is at the same time optimal for whist and reduced whist.*

We prove this theorem by introducing yet another form of whist, called *refined* whist. This game has the property that its optimal strategies are precisely the common optimal strategies of whist and reduced whist. Since there is an optimal strategy for refined whist, this proves Theorem 11.3.

It turns out that the outcome of refined whist is better approximated by the value of the deal, than is the outcome of whist. The game is defined by the following minor adjustment of the scoring: For every trick except the last one, West scores one point for winning, and no points for losing. In the last trick, West gets $3/4$ of a point for winning the trick, and $1/4$ for losing it. Alternatively, we can regard this as a bonus of $1/4$ for not having the lead when the game is over, and a punishment of $-1/4$ for having the lead. To make things consistent, we should therefore regard the zero state as having refined outcome $(-1/4, 1/4)$.

Theorem 11.4. *An optimal strategy for refined whist is optimal for both whist and reduced whist.*

Proof. Taking at least n tricks in whist is equivalent to scoring at least $n - 1/4$ in refined whist. Taking at least n tricks in reduced whist is equivalent to scoring at least $n + 1/4$ in refined whist. □

This motivates the name "refined". Note that scoring at least $n + 1/4$ in refined whist cannot be expressed in terms of the outcome of whist. For example, if we have the lead with A Q versus K J, we can score $1 + 1/4$ in refined whist by starting with the ace, but in whist, it is still optimal to lead the queen, which scores only $3/4$ in refined whist.

If we know how to play refined whist, we also know how to play whist. It is therefore sufficient to study the game of refined whist. Every result about this game will yield as a corollary the corresponding result for whist.

Next we show that, at least for single suit hands, the converse also holds: If we know how to play single-suit whist (and by [8] we do), then we also know how to play refined single-suit whist.

We cite another theorem about single-suit whist that was proved in [3]. We refer to this theorem as the *monotonicity principle*.

Theorem 11.5. *In single-suit whist, a higher card is always at least as good as a smaller one. In other words, if D and D' are single-suit deals, and D' is obtained from D by replacing a card on the West hand by a higher one, then $\chi(D) \leq \chi(D')$.*

We prove an analogous (but slightly refined) result for refined whist.

Lemma 11.6. *In single-suit refined whist, having the lead is never an advantage, but may cost at most half a point.*

For the proof of this lemma, we introduce the following notation: If D is a nonempty single-suit deal, we let D_0 denote the deal obtained by deleting the smallest card on each of the two hands. Notice that the first statement of the lemma is obvious by strategy-stealing. If our opponent has the lead, we can still play any card we want to.

Proof. Let D be a single-suit deal with n cards on each hand. Suppose that West can score at least $k + 1/4$ when East has the lead. Then he must be able to take at least k of the first $n - 1$ tricks. By Lemma 11.1, West's smallest card cannot help him in doing this. Hence West must be able to take at least k tricks in D_0 if East has the lead. Now suppose that West has the lead in D. He can then lead his smallest card in the first trick. If East wins this trick, then by monotonicity, the situation is at least as good for West as when playing D_0 with East on lead. Hence West can take at least k tricks. If on the other hand West wins the first trick, then by Theorem 11.2, he can take at least $k - 1$ more tricks. In any case, West will get a total of at least k tricks, and thereby a score of at least $k - 1/4$.

By switching the East and West hands, we see that if West can score at least $k + 3/4$ when East has the lead, then he must be able to score at least $k + 1/4$ when he has the lead himself. □

Next we show that the outcome of single-suit refined whist is determined by the value of the deal. If D is a deal, we let $\chi'(D)$ be the outcome of D in refined whist. We let ρ' be the function that rounds a value to the nearest rational number with a minimal denominator of exactly 4, with the same tie-break rules as ρ.

Theorem 11.7. *If D is a single-suit deal, then*

$$\chi'(D) = \rho'(\mathrm{val}(D)).$$

We prove this theorem by induction on the number of cards in D, simultaneously with the following theorem:

Theorem 11.8. *If D is a single-suit deal, then $I(D)$ contains at most one number of the form $k/2$ for $k \in \mathbf{Z}$.*

Here we introduce a notational convention that will also be convenient in the following sections. If D is a state, and we consider the possible lines of play from D, we let D' denote the state into which D reduces after the first trick. This means that D' is obtained from D by deleting the two cards played in the first trick, and adding 1 if West won the trick. Hence D' depends not only on D, but also on the choices made by East and West in the first trick.

Proof of Theorems 11.7 and 11.8. Let D be a single-suit deal. Suppose that the two statements hold for every single-suit deal with fewer cards than D, and in particular for every reduction of D.

We first show that there cannot be two half-integers in $I(D)$. Suppose for a contradiction that k is an integer such that both $k/2$ and $(k+1)/2$ belong to $I(D)$. Then $\mathrm{maxmin}(D) \leq (W, k/2)$. Hence if West has the lead, East can always make sure that after the first trick, either $\mathrm{val}(D') < k/2$, or $\mathrm{val}(D') = k$ with West still on lead. Hence West can score at most $k/2 - 1/4$ with the lead. On the other hand, if East has the lead, then since $\mathrm{minmax}(D) \geq (E, (k+1)/2)$, West can make sure that on any lead from East, either $\mathrm{val}(D') > (k+1)/2$, or $\mathrm{val}(D') = (k+1)/2$ with East still on lead. In any case, West will score at least $(k+1)/2 + 1/4 = k/2 + 3/4$ with East on lead. This contradicts Lemma 11.6. Hence there is at most one half-integer in $I(D)$.

We now turn to the statement of Theorem 11.7. The statement clearly holds when $D = 0$, since $\chi'(0) = (-1/4, 1/4)$. Again let k be an integer. Suppose that $\mathrm{val}(D) > k/2$. We have to show that with the lead, West can score at least $k/2 + 1/4$. We must have $k/2 \in L(D)$. Hence $\mathrm{maxmin}(D) \geq (E, k/2)$. This means that there is a card that West can lead, so that no matter what East does, either $\mathrm{val}(D') > k/2$, or $\mathrm{val}(D') = k/2$ with East on lead. By induction, West can score at least $k/2 + 1/4$.

Suppose now that East is on lead and that $\mathrm{val}(D) \geq k/2$. Then $k/2$ cannot belong to $R(D)$. Hence $\mathrm{minmax}(D) > (W, k/2)$. This means that on any lead from East, West has a reply such that either $\mathrm{val}(D') > k/2$, or $\mathrm{val}(D') = k/2$ with East still on lead. By induction, West can score at least $k/2 + 1/4$. □

Hence there is no ambiguity in the definition of $\mathrm{val}(D)$. We remark that the non-refined version of Theorem 11.7 is an immediate consequence:

Theorem 11.9. *If D is a single-suit deal, then*

$$\chi(D) = \rho(\mathrm{val}(D)).$$

In this context, we make another observation which can be useful for computing the value of a deal.

Theorem 11.10. *If D is a single-suit deal, and k is an integer, then* $\mathrm{val}(D) = k$ *if and only if* $\mathrm{val}(D_0) = k - 1/2$.

Proof. Let D be an n-card single-suit deal. Suppose that $\mathrm{val}(D_0) = k - 1/2$. Then $\chi(D_0) = (k-1, k)$. If West has the lead in D, then he can lead his smallest card. By the same argument as in the proof of Lemma 11.6, West will be able to take at least k tricks. Hence he can score at least $k - 1/4$ with the lead in refined whist in D. If East has the lead in D, then West can use the same strategy as in D_0 to get at least k of the $n-1$ first tricks. Hence he can score at least $k + 1/4$ in refined whist in D. This shows that $\chi'(D) \geq (k - 1/4, k + 1/4)$, which implies that $\mathrm{val}(D) \geq k$. By changing the roles of East and West, we obtain the reverse inequality. Hence we have shown that $\mathrm{val}(D_0) = k - 1/2$ implies $\mathrm{val}(D) = k$.

Suppose now that $\mathrm{val}(D) = k$. Then $\chi'(D) = (k - 1/4, k + 1/4)$. If West has the lead, he must therefore be able to take at least k tricks. This implies that he must be able to take at least $k - 1$ tricks in D_0. If East is on lead, then West can take k of the first $n - 1$ tricks. Hence he can take at least k tricks in D_0. This shows that $chi(D_0) \geq (k - 1, k)$, which implies that $\mathrm{val}(D_0) \geq k - 1/2$. Again we obtain the reverse inequality by changing the roles of East and West. □

12 The main theorem

Our main theorem states that the map $\mathrm{val} : S \to V$ determines the outcome of symmetric multi-suit whist under optimal play. We prove this theorem, and at the same time, the refined version.

Theorem 12.1. *If D is a state, then*

$$\chi(D) = \rho(\mathrm{val}(D)).$$

Moreover,

$$\chi'(D) = \rho'(\mathrm{val}(D)).$$

We prove this by induction on the total number of cards. Notice that it suffices to prove the second statement. The proof is divided into a number of lemmas. Again we use the notation D' for the state into which D is transformed after the first trick.

12.1 The single-suit lemmas

The first four of these lemmas deal with the situation when West has the lead.

Lemma 12.2. *Suppose that D is a single-suit deal, and that*

$$\mathrm{val}(D) = q - x,$$

where q is a number and x is a positive infinitesimal. Then there is a card that West can lead such that regardless of East's reply, either $\mathrm{val}(D') > q - x$, or $\mathrm{val}(D') = q - x$ with West still on lead.

Proof. We have $q \in R(D)$. Hence $\mathrm{minmax}(D) \leq (W, q)$. Since $q - x \notin R(D)$, we have

$$\mathrm{maxmin}(D) > (E, q - x).$$

The statement follows. □

Lemma 12.3. *Suppose that D is a single-suit deal, and that $\mathrm{val}(D) = q + x$, where q is a number and x is a positive infinitesimal. Then West has a lead such that on every reply from East, either $\mathrm{val}(D') > q$, or $\mathrm{val}(D') = q$ and East gets the lead.*

Proof. We have $q \in L(D)$. Hence

$$\text{maxmin}(D) \geq (E, q).$$

The statement follows. $\square$

Lemma 12.4. *Suppose that D is a single-suit deal such that*

$$\text{val}(D) = \frac{a}{2^k},$$

where a is an odd integer and k is an integer greater than 1. Then West has a lead such that regardless of East's reply, either $\text{val}(D') > (a-1)/2^k$*, or* $\text{val}(D') = (a-1)/2^k$ *and East gets the lead.*

Proof. Since a is odd, the number $(a-1)/2^k$ can be written with a denominator of at most 2^{k-1}. This number is therefore simpler, and smaller, than $\text{val}(D)$. It follows that $(a-1)/2^k \in L(D)$. Hence $\text{maxmin}(D) \geq (E, (a-1)/2^k)$. The lemma follows. $\square$

Lemma 12.5. *Suppose that D is a nonempty single-suit deal, and that* $\text{val}(D) = k/2$*, where k is an integer. Then West has a lead such that for every reply from East, either* $\text{val}(D') > (k-1)/2$*, or* $\text{val}(D') = (k-1)/2$ *with East on lead.*

Proof. Since there can be at most one half-integer in $I(D)$, $(k-1)/2$ must belong to $L(D)$. Hence $\text{maxmin}(D) \geq (E, (k-1)/2)$. The statement follows. $\square$

The following four lemmas concern the situation when East has the lead.

Lemma 12.6. *Suppose that D is a single-suit deal, and that* $\text{val}(D) = q + \varepsilon_0$*. Then on any lead from East, West can make sure that either* $\text{val}(D') > q$*, or* $\text{val}(D') = q$ *with East still on lead.*

Proof. Since $q \notin R(D)$, $\text{minmax}(D) > (W, q)$. $\square$

Lemma 12.7. *Suppose that D is a single-suit deal with value $q + \varepsilon_k$, where q is a number and k is a positive integer. Then on any lead from East, West can make sure that either* $\text{val}(D') \geq q + \varepsilon_k$*, or* $\text{val}(D') = q + \varepsilon_{k-1}$ *with West on lead.*

Proof. Since $q + \varepsilon_{k-1}$ is smaller and simpler than $\text{val}(D)$, it belongs to $L(D)$. In particular, $\text{minmax}(D) \geq (W, q + \varepsilon_{k-1})$. The statement follows. $\square$

Lemma 12.8. *Suppose that D is a single-suit deal with value $q - x$, where q is a number and x is a positive infinitesimal. Then on any lead from East, West can make sure that either* $\text{val}(D') > q - x$*, or* $\text{val}(D') = q - x$ *with West on lead.*

Proof. Since the number q is greater and simpler than $q - x$, it belongs to $R(D)$. On the other hand $q - x \notin R(D)$. Hence $\text{maxmin}(D) > (E, q - x)$. The statement follows. $\square$

Lemma 12.9. *Suppose that D is a single-suit deal whose value is a number q. Then on any lead from East, West can make sure that either* $\text{val}(D') > q$*, or* $\text{val}(D') = q$ *with East still on lead.*

Proof. We have $q \notin R(D)$. Hence $\text{minmax}(D) > (W, q)$. The statement follows. $\square$

12.2 Proof of the main theorem

We now put together the results of the eight single-suit lemmas to obtain a proof of Theorem 12.1. It suffices to prove the second (refined) assertion. The proof uses induction on the total number of cards. With the convention $\chi'(n) = (n - 1/4, n + 1/4)$ for every integer n, the statement of Theorem 12.1 is true for the integers, that is, for states with no cards. Suppose that E is a nonempty multi-suit deal, and suppose that the statement of Theorem 12.1 has been established for every deal with fewer cards. Suppose first that West has the lead. Then we have to prove that if m is an integer, and $\mathrm{val}(E) \geq m/2 \pm \varepsilon_0$, then West can score at least $m/2 + 1/4$ in refined whist.

Suppose first that there is a single-suit component D of E whose value has negative infinitesimal part. Then by Lemma 12.2, West can lead a card of this suit such that either $\mathrm{val}(E') > \mathrm{val}(E)$, or $\mathrm{val}(E') = \mathrm{val}(E)$ with West still on lead. By induction, West will get at least $m/2 + 1/4$ points.

Next suppose that there is no suit whose value has negative infinitesimal part, but that there is a suit D whose value has positive infinitesimal part, say $q + x$, where q is a number and x is a positive infinitesimal. Then the value of E, being the sum of the values of the single-suit components, must also have positive infinitesimal part. Hence $\mathrm{val}(E) \geq m/2 + \varepsilon_0$. By Lemma 12.3, West can lead a card from the suit D such that either $\mathrm{val}(D') > q$, or $\mathrm{val}(D') = q$ with East on lead after the first trick. For the compound deal E this means that either $\mathrm{val}(E') > m/2$, or $\mathrm{val}(E') = m/2$ with East on lead. By induction, West can score $m/2 + 1/4$.

It remains to consider the case that every single-suit component of E has a value which is a number. In this case the value of E is also a number. Suppose first that there is a single-suit component whose value has a denominator of at least 4. Let D be a suit whose value has maximal denominator among all the single-suit components of E, say 2^k. Then the value of E can also be written as a fraction with a denominator of 2^k. In order for the hypothesis to be satisfied, the value of E must therefore be at least $m/2 + 1/2^k$. By Lemma 12.4, West can lead a card from the suit D such that the value of D decreases by at most $1/2^k$, with East getting the lead if the decrease is as much as $1/2^k$. Hence either $\mathrm{val}(E') > m/2$, or $\mathrm{val}(E') = m/2$ with East on lead after the first trick. By induction, West scores at least $m/2 + 1/4$.

Now consider the case that every suit in E has a value which is half an integer. Then for the hypothesis to hold, the value of E must be at least $m/2 + 1/2$. It therefore suffices to show that West can lead a card such that the value of the suit led decreases by at most $1/2$, with East getting the lead if the decrease is as large as $1/2$. This is exactly what Lemma 12.5 tells us.

We have completed the case that West is on lead, and we turn to the case that East is on lead. We have to show that if the value of E is at least $m/2$, then West can score at least $m/2 + 1/4$.

Suppose first that East leads a card from a suit D with value $q + \varepsilon_0$. Then the value of E cannot be $m/2$, so it must be at least $m/2 + \varepsilon_0$. By Lemma 12.6, West has a reply such that either $\mathrm{val}(D') \geq q + \varepsilon_0$, or $\mathrm{val}(D') = q$ with East still on lead. This implies that either $\mathrm{val}(E') \geq m/2 + \varepsilon_0$, or $\mathrm{val}(E') = m/2$ with East on lead after the first trick. By induction, West will score at least $m/2 + 1/4$.

Suppose now that East leads a card from a suit D with value $q + \varepsilon_k$, for some positive integer k. Since there is a suit with such a value, the value of E cannot be $m/2$ or $m/2 + \varepsilon_0$, and must therefore be at least $m/2 + \varepsilon_1$. By Lemma 12.7, West has a reply such that either $\mathrm{val}(D') \geq q + \varepsilon_k$, or $\mathrm{val}(D') = q + \varepsilon_{k-1}$ with West on lead. In the first case, $\mathrm{val}(E') \geq \mathrm{val}(E)$, and in the second case, the value of E' will be at least $m/2 \pm \varepsilon_0$. In both cases it follows by induction that West will score at least $m/2 + 1/4$.

Suppose now that East leads a card from a suit D with negative infinitesimal part. Again the value of E cannot be a number. By Lemma 12.8, West can make sure that either the value of D increases, or is stays the same with West getting the lead. By induction, West will get at least $m/2 + 1/4$ points.

Finally suppose East leads a card from a suit whose value is a number. By Lemma 12.9, West can then make sure that either the value of this suit increases, or it stays the same with East still on lead. This means that either $\mathrm{val}(E') > m/2$, or $\mathrm{val}(E') = m/2$ with East on lead after the first trick. By induction, West will score at least $m/2 + 1/4$.

This completes the proof of Theorem 12.1. We can now prove Theorem 4.1, and thereby establish all of its corollaries, including Theorems 3.1, 3.2 and 3.3.

Proof of Theorem 4.1. For every state D, we define $N(D)$ to be the numerical part of $\mathrm{val}(D)$. Since val is a semigroup homomorphism, so is N. In particular, Axiom 1 is satisfied. By Theorem 12.1, Axiom 2 is satisfied. □

13 Values of some single-suit deals

We now rigorously establish the values of some special single-suit deals.

Lemma 13.1.

$$\mathrm{val}([\mathsf{K},\ \mathsf{A}]) = \varepsilon_0.$$

Proof. The only reduction of this deal is to zero. Hence

$$\mathrm{minmax}([\mathsf{K},\ \mathsf{A}]) = \mathrm{maxmin}([\mathsf{K},\ \mathsf{A}]) = (E, 0).$$

It follows that $L([\mathsf{K},\ \mathsf{A}]) = \{\alpha \in V : \alpha \leq 0\}$, while $R([\mathsf{K},\ \mathsf{A}])$ is the set of all values with positive numerical part. Hence

$$I([\mathsf{K},\ \mathsf{A}]) = \{\pm\varepsilon_0, \varepsilon_0, \varepsilon_1, \varepsilon_2, \dots\}$$

is the set of all nonnegative nonzero infinitesimals, of which ε_0 is the simplest. □

This can be generalized to:

Theorem 13.2. *If D is a nonempty single-suit deal, where every card on East's hand is higher than every card on West's hand, then* $\mathrm{val}(D) = \varepsilon_0$.

Proof. There is essentially only one reduction of D, and supposing that $|D| \geq 2$, this reduction is to a deal of the same type. By induction, we can assume that the value of the reduction is ε_0. Hence $\text{minmax}(D) = \text{maxmin}(D) = (E, \varepsilon_0)$. It follows that the sets $L(D)$ and $R(D)$ are equal to $L([\mathsf{K},\ \mathsf{A}])$ and $R([\mathsf{K},\ \mathsf{A}])$ respectively, and therefore that

$$\text{val}(D) = \text{val}([\mathsf{K},\ \mathsf{A}]) = \varepsilon_0.$$

□

Theorem 13.3. *Let D be a single-suit deal in which West holds the k highest cards, and where every other card on West's hand is smaller than every card on East's hand. Provided that West holds the smallest card, we have*

$$\text{val}(D) = k + \varepsilon_k.$$

Proof. We prove this by induction on k. The case $k = 0$ is equivalent to Theorem 13.2. If $k \geq 1$, there are essentially two different reductions of D. If West plays a high card, the deal reduces to 1 plus a deal of the same type with k replaced by $k - 1$. By induction we can assume that the value of such a deal is $1 + (k - 1) + \varepsilon_{k-1} = k + \varepsilon_{k-1}$. If West plays a small card, then either the hand reduces to a hand of the same type with one fewer small card on West's hand, or to a deal where all West's cards are high. In the former case, we can assume, by making a simultaneous induction on the total number of cards, that the value of the reduction has already been evaluated to $k + \varepsilon_k$. In the latter case, we can use Theorem 13.2 with the East and West hands switched. Since only k cards remain, and the deal has value ε_0 from East's perspective, the value must be $k - \varepsilon$. We can conclude that $\text{maxmin}(D) = \text{minmax}(D) = (W, k + \varepsilon_{k-1})$ or $(E, k + \varepsilon_k)$ depending on whether West has one or more small cards. In any case,

$$L(D) = \{\alpha \in V : \alpha < k + \varepsilon_k\},$$

while $R(D)$ is the set of all values with numerical part greater than k. Hence

$$I(D) = \{k + \varepsilon_k, k + \varepsilon_{k+1}, k + \varepsilon_{k+2}, \dots\}.$$

The simplest value in $I(D)$ is $k + \varepsilon_k$. □

Theorem 13.4. *Let*

$$D_1 = [\mathsf{A\ Q},\ \mathsf{K\ J}]\,,$$

$$D_2 = [\mathsf{A\ K\ 10},\ \mathsf{Q\ J\ 9}]\,,$$

$$D_3 = [\mathsf{A\ K\ Q\ 8},\ \mathsf{J\ 10\ 9\ 7}]\,,$$

etc. With the standard numbering, D_k is the $(k+1)$-card deal in which West holds the cards $2, k+3, k+4, \dots, 2k+2$, and East holds $1, 3, 4, 5, \dots, k+2$. Then

$$\text{val}(D_k) = k + 1/2^k.$$

Proof. We have $\chi(D_1) = (1,2)$. By Theorem 12.1, it follows that $\mathrm{val}(D_1) = 1 + 1/2$. Consider D_k, for $k \geq 2$. There are essentially four different reductions. If both players play their highest cards (or any equivalent cards), then the deal reduces to 1 plus a deal equivalent to D_{k-1}. By induction on k, we can assume that this reduction has value $1 + (k-1) + 1/2^{k-1} = k + 1/2^{k-1}$. If West plays a high card and East plays his smallest card, then by Theorem 13.3, the value of the reduction is $1 + (k-1) + \varepsilon_{k-1} = k + \varepsilon_{k-1}$. If West plays his smallest card, then the deal reduces to a deal where all West's cards are high. The value of this deal is $k - \varepsilon_0$ or $k + 1 - \varepsilon_0$ depending on whether East played one of his higher cards or not. It follows that

$$\mathrm{maxmin}(D_k) = \max\left((E, k - \varepsilon_0), (W, k + \varepsilon_{k-1})\right) = (W, k + \varepsilon_{k-1}),$$

and that

$$\mathrm{minmax}(D_k) = \min\left((W, k + 1 - \varepsilon_0), (W, k + 1/2^{k-1})\right) = (W, k + 1/2^{k-1}). \qquad (22)$$

Hence $I(D)$ is the set of values whose numerical part is strictly between k and $k + 1/2^{k-1}$. The simplest of these values is the number $k + 1/2^k$. □

In a similar way, one can prove that the deals

$$[\mathsf{K\ Q\ 9,\ A\ J\ 10}]\,,$$
$$[\mathsf{K\ Q\ J\ 7,\ A\ 10\ 9\ 8}]\,,$$
$$[\mathsf{K\ Q\ J\ 10\ 5,\ A\ 9\ 8\ 7\ 6}]\,,$$

have values $1 + 1/4$, $2 + 1/8$, $3 + 1/16$, an so on.

14 $S/\equiv$ is isomorphic to V

The introduction of values in Section 8 was motivated only informally by examples of playing technique. We now show that the semigroup of values is isomorphic to the semigroup of equivalence classes of states. This justifies our construction of values, since it shows that V is the simplest structure for which a statement like Theorem 12.1 can hold. Our construction is just as complicated as it needs to be to characterize the game.

Theorem 14.1. *The function* $\mathrm{val} : S \to V$ *is surjective.*

Proof. Theorem 13.4 shows that the values $1/2^k$, for positive integers k, all occur as values of states. It follows that all numbers are values of states. By Theorems 13.2 and 13.3, we have

$$\begin{aligned}
\mathrm{val}([\mathsf{K,\ A}]) &= \varepsilon, \\
\mathrm{val}([\mathsf{A,\ K}]) &= 1 - \varepsilon, \\
\mathrm{val}([\mathsf{A\ J,\ K\ Q}]) &= 1 + \varepsilon_1, \\
\mathrm{val}([\mathsf{K\ Q,\ A\ J}]) &= 1 - \varepsilon_1.
\end{aligned}$$

By combining integers and suits with these distributions, all infinitesimals can be obtained. It follows that all values occur as values of states. □

Here we have allowed states with negative integral part. The question of which values occur as values of states with integer part zero seems much more complicated. Obviously no negative values can then occur. Moreover, there are many positive values that can easily be seen not to occur. For instance, it is clear that no value between ε_0 and $1/2$ occurs: Either there is a card on West's hand which is higher than some card on East's hand, and then West will be able to take at least one trick if East has the lead, making the value of the deal at least $1/2$, or all East's cards are high, in which case the value of the deal is ε_0.

Theorem 14.2. *The function* val : $S \to V$ *factors through the equivalence relation* $\equiv$. *In other words, equivalent states have the same value.*

In view of Theorem 14.1, it suffices to prove the following statement:

Lemma 14.3. *If x and y are distinct values, then there is a value z such that $\rho(x+z) \neq \rho(y+z)$.*

We prove this for general rounding functions, assuming only that there are at least two distinct numbers to round to. This will make the results of this section apply to refined whist as well.

Lemma 14.4. *Let A be a discrete set of numbers containing at least two elements. If x and y are distinct values, then there is a value z such that $\rho_A(x+z) \neq \rho_A(y+z)$.*

Proof. Let a and b be distinct elements of A with $a < b$, and let $m = (a+b)/2$. If x and y have distinct numerical parts, then there is a number q between them. If we take $z = m - q$, then $x + z$ and $y + z$ will be rounded in different directions. Suppose therefore that x and y have the same numerical part, say r. If $x - (m - r)$ and $y - (m - r)$ are rounded to the same number, then the infinitesimal parts of x and y must either both be positive, or both negative. Suppose they are both positive. Then there are distinct nonnegative integers k and l such that $x = r + \varepsilon_k$ and $y = r + \varepsilon_l$. If we let $z = m - r - \varepsilon_k$, then $\rho_A(x+z) = (b, a)$, while $\rho_A(y+z) = (a, a)$ or (b, b) depending on whether k is greater or smaller than l. □

Theorem 14.5. *The mapping* val : $S/\equiv \to V$ *is injective. In other words, if two states have the same value, then they are equivalent.*

Proof. Let D, E and F be states, and suppose D and E have the same value. Then $D+F$ and $E + F$ have the same value. By Theorem 12.1, $D + F$ and $E + F$ have the same outcome. Since this holds for every F, we have $D \equiv E$. □

Theorem 14.6. *The function* val *is well-defined on equivalence classes of states, and provides an isomorphism between $S/\equiv$ and V.*

Proof. This follows from Theorems 14.1, 14.2 and 14.5. □

The fact that the function val : $S \to V$ provides an isomorphism between $S/\equiv$ and V whether we take whist or refined whist as the basis for equivalence between states shows that these two games are equivalent in a very precise and natural sense: Two states are equivalent, or one is smaller than the other, with respect to one of the games precisely when the same relation holds with respect to the other game.

15 The structure of an optimal strategy

From the single-suit lemmas, we can make a few observations. The optimal strategy which is implicit in the proof of Theorem 12.1 uses the following method for choosing a lead: First, choose the suit to lead from. The suit is chosen as follows:

1. If there is a suit whose value has negative infinitesimal part, choose any such suit.
2. If there is no suit whose value has negative infinitesimal part, but at least one suit whose value has positive infinitesimal part, choose any suit with positive infinitesimal part.
3. If the value of every suit is a number, then choose any suit whose value is at least as complex (that is, has at least the same denominator) as the value of the whole deal.

Second, a card to lead is chosen from that suit. When choosing a card to lead, only the local information of the card distribution in that suit is taken into account. Hence it is possible to make a table of leads, so that from the same card distribution, we always make the same lead. For instance, we can decide that when leading from [A Q, K J], we always lead the ace, while from [A J 10 7, K Q 9 8], we always lead the jack.

Bridge players will recognize a certain similarity to the way a defender chooses the opening lead in bridge. The choice of suit is often difficult, but once a suit has been chosen, the particular card is determined more or less on the basis only of the cards in that suit. This idea is implicit in systems like the 11- and 10–12 rules, and many books on defence contain tables of opening leads from various card combinations. Actually, such a table would recommend either the jack or the ten from A-J-10-7.

From the way we choose the suit to lead, we see that the so called Number Avoidance Theorem of combinatorial game theory [1] is valid also for whist. In [1], this theorem is stated as:

Never move in a Number, unless there is Nothing else to do.

This should be interpreted in the weak sense that it is possible to play optimally according to this rule. In the case of whist, we have the following theorem:

Theorem 15.1. *If D and E are states such that* $\mathrm{val}(D)$ *is a number, but* $\mathrm{val}(E)$ *is not, then there is a card in E which is an optimal lead from $D + E$.*

Hence the rule:

Never lead from a suit whose value is a Number, unless there is Nothing else to do.

The following examples show that the best reply to a lead may depend not only on the distribution in the suit that was led, but also on the distribution of the cards in the other suits. If East leads the spade king from

West :	*East* :	
♠ A J 9	♠ K Q 10	(23)

then West should hold off in order to take the next two tricks. Suppose now that East leads the spade king from

West :	*East* :	
♠ A J 9	♠ K Q 10	
♡ A	♡ K	(24)
♢ K	♢ A	

If West plays low, East can cash the diamond ace and exit in hearts. This holds West down to two tricks. On the other hand if West takes the first trick with the spade ace, he can cash his heart trick and exit in diamonds, for a total of three tricks.

The flaw of this example is that East's spade lead is incorrect. East should immediately eliminate diamonds and exit in hearts. This will always hold West down to two tricks. If West decides to duck whenever East leads spades, he will still get as many tricks as he was entitled to in the first place. The author does not know whether there is an example where, under optimal play from East, West may have to look at the other suits before deciding on his reply.

16 Examples

We give some examples which are a little more complicated than those given elsewhere in the paper, and where our theory can be used to find an optimal strategy, and to prove its optimality. We do not prove our assertions about the values of the suits involved, but in most cases, these can easily be verified using Theorem 11.10 and the theorems of Section 13.

We first consider a deal whose value is a number. This deal shows that it can sometimes be necessary to play low to a lead, even if it does not matter for the number of tricks we get in the suit that is being led:

West :	*East* :	
♠ A J 9 8	♠ $\underline{\text{K}}$ Q 10 7	
♡ A K 10	♡ Q J 9	(25)
♢ K Q 9	♢ A J 10	

East leads the spade king. We can evaluate the spade suit to 2, the heart suit to $2 + 1/4$, and the diamonds to $1 + 1/4$. This sums to $5 + 1/2$, and since East has the lead, we expect to be able to take six tricks. If we take the first trick with the ace of spades, then the spade suit reduces to $1 +$ [J 9 8, Q 10 7], which has the value 2. The problem is that West gets the lead, so on the deal as a whole, West will now only be able to take five tricks. On the other hand, if West plays low in the first trick, the spade suit reduces to [A J 9, Q 10 7]. This still has the value 2, but now East is on lead, which means that West will be able to take six tricks. Whenever East plays another spade, West can cash two spade tricks and exit with the last spade. This way, East will have to make the first move in the red suits. If East leads the queen of hearts, West takes the trick and returns the diamond king. If instead East leads a small diamond, West takes the trick and clears the hearts with ace, king, and ten. No matter what East does, West will either get three heart tricks or two diamond tricks.

Here the spade suit produces two tricks almost regardless of how the cards are played. Still it is essential that West plays low in the first trick in order to get four tricks in the other suits.

Our next deal is an example of "exiting" in a suit with positive infinitesimal part, when there is no suit with negative infinitesimal part, in order to force the opponent to be the first to lead a suit with numerical value.

West :	*East* :	
♠ A K Q 8	♠ J 10 9 7	
♡ A Q 8 7	♡ K J 10 9	(26)
♢ A J 10 7	♢ K Q 9 8	
♣ K Q 9	♣ A J 10	

We evaluate the spades to $3 + 1/8$, the hearts to $2 + \varepsilon_2$, the diamonds to $2 + 1/8$, and the clubs to $1 + 1/4$. This sums to $8 + 1/2 + \varepsilon_2$, which rounds to 9. Hence West should be able to take nine tricks regardless of who has the initial lead.

If West has the lead, there is only one way to accomplish this. A lead in either of the "numerical" suits, spades, diamonds or clubs, will weaken that suit. Instead, West has to exit by playing a small heart. When East takes the trick with the nine of hearts, the value of the heart suit will go down to 2, but East gets the lead.

We can then distinguish two main lines: If East leads the spade jack, West will take the trick and continue with a middle diamond (the only possibility!), while if East leads the diamond king, West will take the trick and play four rounds of spades, putting East on lead with the last one.

Finally, we give an example of a "hot" deal where both sides have exits, and the first phase of the game is a race to knock out the opponent's stoppers. West to play and take eleven tricks:

West :	*East* :	
♠ K Q 10 9 8	♠ A J 7 6 5	
♡ A K 9	♡ Q J 10	(27)
♢ A J 9 7 5 3	♢ K Q 10 8 6 4	
♣ K Q 10 8 6	♣ A J 9 7 5	

The spades are actually an exit for East, with two stoppers. Hence they are worth $3 - \varepsilon_2$. The hearts provide an exit for West, with value $2 + \varepsilon_2$. It can be shown, for example with the methods of [8], that the outcome of the club suit played separately is

(2, 3). It follows that the value of the clubs is $2 + 1/2$. Then it follows from Theorem 11.10 that the diamonds have value 3. This gives a total of $10 + 1/2 \pm \varepsilon_0$. West should therefore be able to score eleven tricks, provided he has the lead.

When both sides have exits, the correct strategy is to attack the opponent's exit suits, that is, the suits whose values have negative infinitesimal part. In this case the only way for West to take eleven tricks is to lead a spade. On the other hand, it does not matter which one. West should not be afraid of giving East a cheap trick for the spade jack, since there is no way to prevent this anyway. East's natural defense is to play hearts consistently, but West is one step ahead, and will be able to clear the spades before exiting with the nine of hearts.

Now the character of the game changes from a hot race to knock out stoppers to a cold game of numbers. East has the lead, and may set up a trap by leading the diamond king. West is certain to get three diamond tricks whether he takes the first one or not, but if he falls for the temptation and wins the first diamond trick with the ace, East will secure three club tricks by ducking when West leads the king. Instead, West has to let East win the first diamond trick. After that he will easily get eleven tricks.

Notice that it will not do to start with three rounds of hearts. When East gets the lead in the third trick, he can exit with a small spade. If West then returns the spade king, East will of course play low.

17 Computing the value of a card distribution

By inspecting the definition of the value of a deal given in Section 10, we find that there is an algorithm for computing it. Clearly, with a reasonable representation of the elements of V, addition in V can be computed efficiently. Hence the problem of computing the value of a deal reduces to that of computing the values of its single-suit components. Let D be a single-suit deal. If we assume that the values of all reductions $D_{i,j}$ of D have been computed, we can determine $\text{minmax}(D)$ and $\text{maxmin}(D)$. Then we can find the set $I(D)$. Clearly $\text{val}(D)$ can now be computed. The running time of this algorithm is exponential in the size of D, since the computation of the value of an n-card deal is reduced to the computation of values of n^2 $(n-1)$-card deals. The branching factor n^2 can be improved to $O(n)$ by using the fact that when responding to a lead, it is optimal either to take the trick as cheaply as possible, or to play low, but the running time is still exponential.

A combinatorial game can be considered solved when a polynomial time algorithm has been found that for any position in the game computes the outcome of the game under optimal play, as well as an optimal move for the player whose turn it is to play. Often these two computational problems can be reduced to each other in polynomial time (see for example [3]), and are therefore considered equivalent.

In this sense, two-person symmetric whist is not yet solved. However, we strongly believe that an efficient algorithm for computing values exists. In support of this claim, we prove the following:

Theorem 17.1. *There is an algorithm running in time $O(n \log n)$ that detects if a single-suit deal has an integral or half-integral value.*

Proof. The proof of this uses an algorithm given in [8] that computes the outcome of single-suit whist in time $O(n \log n)$. Clearly a single-suit deal D has value $n + 1/2$ if and only if its outcome is $(n, n+1)$. By Theorem 11.10, a single-suit deal D has value n if and only if the deal D_0 obtained by removing the smallest card from each hand has outcome $(n-1, n)$. □

It was proved in [7] that if the cards in a single suit of size $2n$ are distributed randomly, then with the notation of [7, 8], the probability that $H(D) = H(D_0)$ tends to 1 as $n \to \infty$. If $H(D) = H(D_0)$, then either D or D_0 has an outcome that depends on the position of the lead, and hence a value which is half an odd integer. It follows that the value of D is half of an integer.

Hence we have an algorithm that computes the value of a random single-suit deal quickly with very high probability, and which, if it fails, still gives bounds of the form $n/2 < \mathrm{val}(D) < (n+1)/2$.

18 Values of single-suit deals

We make a few remarks on the question of which values occur as values of single-suit deals. This question is probably related to the problem of effectively computing the value of a single-suit deal. As mentioned in the previous section, we do not at present have a satisfactory answer to this problem. The sequence of card distributions given in Theorem 13.4 provides, for every k, a single-suit deal whose value has fractional part $1/2^k$.

The following example, discovered through a computer search, shows that fractional parts not on the form $1/2^k$ or $1 - 1/2^k$ are also possible:

$$[13\ 12\ 9\ 8\ 7\ 2\ 1,\ 14\ 11\ 10\ 6\ 5\ 4\ 3]$$

To verify that this deal has value $3 + 3/8$, we can play the sum

$$[13\ 12\ 9\ 8\ 7\ 2\ 1,\ 14\ 11\ 10\ 6\ 5\ 4\ 3] + [\mathsf{A\ K\ Q}\ 8,\ \mathsf{J}\ 10\ 9\ 7]\,.$$

By considering the various lines of play, we can convince ourselves that West will get 6 tricks with the lead, and 7 tricks with East on lead. Hence the sum has value $6 + 1/2$. Since [A K Q 8, J 10 9 7] is known to have value $3 + 1/8$, it follows that the other term has value $3 + 3/8$.

We offer two conjectures, both supported by computer analysis of all single-suit deals with at most 10 cards on each hand:

Conjecture 18.1. *If the value of a single-suit deal is of the form $q + x$, where q is a number and x is a positive infinitesimal, then q must be a nonnegative integer, and $x = \varepsilon_q$.*

We could easily prove this by induction if we could prove that we cannot have

$$\text{maxmin}(D) = \text{minmax}(D) = (E, q)$$

unless $q = 0$. In other words, this conjecture boils down to proving that no "new" exits are created from suits with numerical value after [K, A], so that all suits with positive infinitesimal part are descendants of the ε_0 created on day one.

If this conjecture is true, then suits with non-numerical values can indeed be described using the concept of stoppers, as is implicitly suggested in the discussion in Section 6.

Conjecture 18.2. *If the value of a single-suit deal is a number which is not an integer, then the fractional part of this number must be of the form* $1/2^k$, $1 - 1/2^k$, $3/8$ *or* $5/8$.

We also state another conjecture, whose solution seems to require a deeper understanding of the values of single-suit deals. The single-suit case of this conjecture was proved in [8].

Conjecture 18.3. *If the opponent leads a card that we can beat with a card smaller than the highest remaining card in the suit, then it is always optimal to do so. Hence with the standard numbering of the cards, the only situation where it can be necessary to refuse to take a trick is when we have the ace, and the opponent leads from a sequence containing the king and queen.*

19 The non-absoluteness of values in whist

This section deals with an aspect of the theory of whist that makes it fundamentally different from the classical theory of combinatorial games. The classical theory deals with games played under the *normal playing convention.* With this convention, the move-order is alternating, and the winner of a game is the player who makes the last move.

With the normal playing convention, every game G has an inverse $-G$ with the property that $G+(-G)$ is a second player win. The relation $H \leq G$ is then defined as meaning that $G + (-H)$ is a win for Left if Right makes the first move. This means that it is a property intrinsic to the games G and H. If we are given two games, we can determine whether one of them is greater than the other simply by computing the outcome of their difference. Similarly, the games G and H are considered equivalent (in [2] they are even said to be *equal*), if and only if $G + (-H)$ is a second player win.

It is easy to prove from these definitions that G and H are equal in this respect if and only if for every game K, $G + K$ has the same outcome under optimal play as $H + K$.

Since in general a whist deal has no additive inverse, we have taken the latter property as our definition of equivalence. Hence we have defined two card distributions D and E to be equivalent if for every card distribution F, $D + F$ has the same outcome as $E + F$. This means that in order to prove that two deals are equivalent, we potentially have to investigate an infinite set of other deals. Such a problem can therefore be difficult even if the two deals in question are simple. Indeed, we had to do a certain amount of work to prove for instance that

$$[\mathsf{A\ Q\ 9,\ K\ J\ 10}] \equiv [\mathsf{A\ Q\ J,\ K\ 10\ 9}]\,,$$

and that

$$[\mathsf{A,\ K}] > 0.$$

For this reason, we have no general methods for rigorously establishing algebraic properties and order relationships in the quotient semigroup $S/\equiv$ until we have constructed the semigroup V and the mapping val. Somewhat surprisingly, it was much easier to prove that

$$[\mathsf{K,\ A}] > 0.$$

Apparently some statements about the ordering of deals can be proved easily by strategy-stealing, while other statements are more difficult and seem to require the development of a general theory of the game. The former are statements that hold universally, regardless of the class of games under consideration, while the latter depend on restrictions like for example the assumption that the relative rank of two cards is determined by a total ordering of the cards within a suit.

Suppose for example that in a particular suit P, there are four cards which beat each other cyclically, in a scissors-paper-stone-like way. West holds the cards a and c, while East holds b and d, and a beats b which beats c which beats d which beats a. If this suit is played on its own, the player not on lead will take both tricks. This already shows that such a suit cannot be assigned a value and incorporated in the theory of symmetric whist. Moreover, if such a situation is admissible, then it is no longer true that a higher card is at least as good as a smaller one. Indeed, we can disprove the statement that

$$[\mathsf{A,\ K}] \geq 0,$$

by adding P to both sides. The outcome of $[\mathsf{A,\ K}] + P$ is $(1,1)$ while the outcome of $0 + P$ is $(0,2)$.

In the same way, we can prove that

$$[\mathsf{A\ Q\ 9,\ K\ J\ 10}] \not\equiv [\mathsf{A\ Q\ J,\ K\ 10\ 9}]\,.$$

If East is on lead in $[\mathsf{A\ Q\ 9,\ K\ J\ 10}] + P$, he will get only one trick. On the other hand, in $[\mathsf{A\ Q\ J,\ K\ 10\ 9}] + P$, he can get two tricks by sacrificing the king under West's ace.

However, the assumption that cards within a suit are totally ordered is not necessary for the strategy-stealing arguments that prove for instance that $[\mathsf{K,\ A}] \geq 0$ and that

$$[\mathsf{A\ Q\ J,\ K\ 10\ 9}] \leq 2 \leq [\mathsf{A\ Q\ 9,\ K\ J\ 10}]\,.$$

These relations therefore seem to be intrinsic to the games in question.

Hence we can make a precise distinction between on the one hand "absolute" statements provable by strategy-stealing arguments requiring analysis of only the card distributions involved in the actual statement, and on the other hand "non-absolute" statements that require a general theory of the game.

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20 Appendix: Table of values of single-suit deals

The following is a table of values of single-suit deals with up to four cards, where East has the ace. The values of the single-suit deals where West has the ace can be obtained through the identity $\mathrm{val}(D) = |D| - \mathrm{val}(\overline{D})$.

West	East	value
K	A	ε_0
Q J	A K	ε_0
K J	A Q	$1/2$
K Q	A J	$1 - \varepsilon_1$
J 10 9	A K Q	ε_0
Q 10 9	A K J	$1/2$
Q J 9	A K 10	$3/4$
Q J 10	A K 9	$1 - \varepsilon_2$
K 10 9	A Q J	1
K J 9	A Q 10	1
K J 10	A Q 9	1
K Q 9	A J 10	$1 + 1/4$
K Q 10	A J 9	$1 + 1/2$
K Q J	A 10 9	$2 - \varepsilon_1$

West	East	value
10 9 8 7	A K Q J	ε_0
J 9 8 7	A K Q 10	$1/2$
J 10 8 7	A K Q 9	$3/4$
J 10 9 7	A K Q 8	$7/8$
J 10 9 8	A K Q 7	$1-\varepsilon_3$
Q 9 8 7	A K J 10	1
Q 10 8 7	A K J 9	1
Q 10 9 7	A K J 8	1
Q 10 9 8	A K J 7	1
Q J 8 7	A K 10 9	$1+1/4$
Q J 9 7	A K 10 8	$1+1/2$
Q J 9 8	A K 10 7	$1+1/2$
Q J 10 7	A K 9 8	$1+1/2$
Q J 10 8	A K 9 7	$1+3/4$
Q J 10 9	A K 8 7	$2-\varepsilon_2$
K 9 8 7	A Q J 10	$1+\varepsilon_1$
K 10 8 7	A Q J 9	$1+1/4$
K 10 9 7	A Q J 8	$1+1/2$
K 10 9 8	A Q J 7	$1+1/2$
K J 8 7	A Q 10 9	$1+1/2$
K J 9 7	A Q 10 8	$1+1/2$
K J 9 8	A Q 10 7	$1+1/2$
K J 10 7	A Q 9 8	$1+1/2$
K J 10 8	A Q 9 7	$1+3/4$
K J 10 9	A Q 8 7	$2-\varepsilon_2$
K Q 8 7	A J 10 9	$1+1/2$
K Q 9 7	A J 10 8	$1+3/4$
K Q 9 8	A J 10 7	$1+7/8$
K Q 10 7	A J 9 8	2
K Q 10 8	A J 9 7	2
K Q 10 9	A J 8 7	2
K Q J 7	A 10 9 8	$2+1/8$
K Q J 8	A 10 9 7	$2+1/4$
K Q J 9	A 10 8 7	$2+1/2$
K Q J 10	A 9 8 7	$3-\varepsilon_1$