H-free graphs of large minimum degree

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Abstract

We prove the following extension of an old result of Andrásfai, Erdős and Sós. For every fixed graph H with chromatic number $r+1 \geq 3$, and for every fixed $\epsilon > 0$, there are $n_0 = n_0(H, \epsilon)$ and $\rho = \rho(H) > 0$, such that the following holds. Let G be an H-free graph on $n > n_0$ vertices with minimum degree at least $\left(1 - \frac{1}{r-1/3} + \epsilon\right)n$. Then one can delete at most $n^{2-\rho}$ edges to make G r-colorable.

1 Introduction

Turán's classical Theorem [11] determines the maximum number of edges in a K_{r+1}-free graph on n vertices. It easily implies that for $r \geq 2$, if a K_{r+1}-free graph on n vertices has minimum degree at least $(1 - \frac{1}{r})n$, then it is r-colorable (in fact, it is a complete r-partite graph with equal color classes). The following stronger result has been proved by Andrásfai, Erdős and Sós [2].

Theorem 1.1 ([2]) *If G is a K_{r+1}-free graph of order n with minimum degree $\delta(G) > \left(1 - \frac{1}{r-1/3}\right)n$ then G is r-colorable.*

The following construction shows that this is tight. Let G be a graph whose vertex set is the disjoint union of $r+3$ sets $U_1, U_2, \ldots, U_5$ and $V_1, V_2 \ldots, V_{r-2}$, in which $|U_i| = \frac{1}{3r-1}n$ for all i and $|V_j| = \frac{3}{3r-1}n$ for all j. Each vertex of V_j is adjacent to all vertices but the other members of V_j and each vertex of U_i is adjacent to all vertices of $U_{(i+1) \bmod 5}$, $U_{(i-1) \bmod 5}$

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and $\cup_j V_j$. All vertices in this graph have degree $\frac{3r-4}{3r-1}n = \left(1 - \frac{1}{r-1/3}\right)n$ and it is easy to see that G contains no K_{r+1}, and is not r-colorable.

Turán's result has been extended by Erdős-Stone [6] and by Erdős-Simonovits [4] showing that for $r \geq 2$, for any fixed graph H of chromatic number $\chi(H) = r + 1$ and for any fixed $\epsilon > 0$, any H-free graph on n vertices cannot have more than $(1 - \frac{1}{r} + \epsilon)\binom{n}{2}$ edges provided n is sufficiently large as a function of H and ϵ. Moreover, it is known that if an H-free graph on a large number n of vertices has at least $(1 - \frac{1}{r})\binom{n}{2}$ edges, then one can delete $o(n^2)$ of its edges to make it r-colorable.

It therefore seems natural to try to extend Theorem 1.1 from complete graphs K_{r+1} to general graphs H. Such an extension for critical graphs, i.e., H which have an edge whose removal decreases its chromatic number, has been proved in [5]. In the present short paper we handle the general case. Our main results are the following. Let $K_{r+1}(t)$ be the complete $(r+1)$-partite graph with t vertices in each vertex class.

Theorem 1.2 *Let $r \geq 2, t \geq 1$ be integers and let $\epsilon > 0$. Then there exist $n_0 = n_0(r, t, \epsilon)$ such that if G is a $K_{r+1}(t)$-free graph of order $n \geq n_0$ with minimum degree $\delta(G) \geq \left(1 - \frac{1}{r-1/3} + \epsilon\right)n$, then one can delete at most $O\left(n^{2-1/(4r^{2/3}t)}\right)$ edges to make G r-colorable.*

Corollary 1.3 *Let H be a fixed graph on h vertices with chromatic number $r + 1 \geq 3$, suppose $\epsilon > 0$ and let G be an H-free graph of sufficiently large order $n > n_0(h, \epsilon)$ with minimum degree $\delta(G) \geq \left(1 - \frac{1}{r-1/3} + \epsilon\right)n$. Then one can delete at most $O\left(n^{2-1/(4r^{2/3}h)}\right)$ edges to make G r-colorable.*

As shown by the example above, the fraction $1 - \frac{1}{r-1/3} = \frac{3r-4}{3r-1}$ is tight in general. It is also not difficult to see that indeed in general some $O(n^{2-\rho})$ edges have to be deleted to make the graph G r-colorable, though the best possible value of $\rho = \rho(K_{r+1}(t))$ may well be slightly better than the one we obtain. The problem of determining the behavior of the best possible value of ρ, as well as that of deciding if the ϵn-term can be replaced by $O(1)$, remain open.

A weaker version of Corollary 1.3 is proved in [1], where it is applied to prove the NP-hardness of various edge-deletion problems. This version asserts that there are some $\gamma = \gamma(H) > 0$ and $\mu = \mu(H) > 0$ so that the following holds. For any H-free graph G on n vertices with minimum degree at least $(1 - \gamma)n$, one can delete $O(n^{2-\mu})$ edges from G to make it r-colorable. Theorem 1.2 supplies the asymptotically best possible value of $\gamma(K_{r+1}(t))$ for all admissible r and t.

2 Proofs

In this section we prove our main theorem. Let G be a $K_{r+1}(t)$-free graph of order n with minimum degree $\delta(G) \geq \left(1 - \frac{1}{r-1/3} + \epsilon\right)n$. We assume throughout the proof that n is sufficiently large. We first establish the following weaker bound.

Lemma 2.1 *G can be made r-partite by deleting $o(n^2)$ edges.*

The proof of this statement is a standard application of Szemerédi's Regularity Lemma and we refer the interested reader to the comprehensive survey of Komlós and Simonovits [8], which discusses various results proved by this powerful tool.

We start with a few definitions, most of which follow [8]. Let $G = (V, E)$ be a graph, and let A and B be two disjoint subsets of $V(G)$. If A and B are non-empty, define the *density of edges* between A and B by $d(A, B) = \frac{e(A,B)}{|A||B|}$. For $\gamma > 0$ the pair (A, B) is called *γ-regular* if for every $X \subset A$ and $Y \subset B$ satisfying $|X| > \gamma|A|$ and $|Y| > \gamma|B|$ we have $|d(X, Y) - d(A, B)| < \gamma$. An *equitable partition* of a set V is a partition of V into pairwise disjoint classes $V_1, \cdots, V_k$ of almost equal size, i.e., $\big||V_i| - |V_j|\big| \le 1$ for all i, j. An equitable partition of the set of vertices V of G into the classes $V_1, \cdots, V_k$ is called *γ-regular* if $|V_i| \le \gamma|V|$ for every i and all but at most γk^2 of the pairs (V_i, V_j) are γ-regular. The above partition is called *totally γ-regular* if all the pairs (V_i, V_j) are γ-regular. The following celebrated lemma was proved by Szemerédi in [10].

Lemma 2.2 *For every $\gamma > 0$ there is an integer $M(\gamma)$ such that every graph of order $n > M(\gamma)$ has a γ-regular partition into k classes, where $k \le M(\gamma)$.*

In order to apply the Regularity Lemma we need to show the existence of a complete multipartite subgraph in graphs with a totally γ-regular partition. This is established in the following well-known lemma, see, e.g., [8].

Lemma 2.3 *For every $\eta > 0$ and integers r, t there exist $0 < \gamma = \gamma(\eta, r, t)$ and $n_0 = n_0(\eta, r, t)$ with the following property. If G is a graph of order $n > n_0$ and $(V_1, \cdots, V_{r+1})$ is a totally γ-regular partition of vertices of G such that $d(V_i, V_j) \ge \eta$ for all $i < j$, then G contains a complete $(r + 1)$-partite subgraph $K_{r+1}(t)$ with parts of size t.*

Proof of Lemma 2.1. We use the Regularity Lemma given in Lemma 2.2. For every constant $0 < \eta < \epsilon/4$ let $\gamma = \gamma(\eta, r, t) < \eta^2$ be sufficiently small to guarantee that the assertion of Lemma 2.3 holds. Consider a γ-regular partition $(U_1, U_2, \ldots U_k)$ of G. Let G' be a new graph on the vertices $1 \le i \le k$ in which (i, j) is an edge iff (U_i, U_j) is a γ-regular pair with density at least η. Since G is a $K_{r+1}(t)$-free graph, by Lemma 2.3, G' contains no clique of size $r + 1$. Call a vertex of G' *good* if there are at most ηk other vertices j such that the pair (U_i, U_j) is not γ-regular, otherwise call it *bad*. Since the number of non-regular pairs is at most $\gamma\binom{k}{2} \le \eta^2 k^2/2$ we have that all but at most ηk vertices are good. By the definition of "good" and by the assumption on the minimum degree of G, the degree of each good vertex in G' is at least $\left(1 - \frac{1}{r-1/3} + \epsilon\right) k - 2\eta k - 1$, since deletion of the edges from non-regular pairs and sparse pairs can decrease the degree by at most ηk each and the deletion of edges inside the sets U_i can decrease it by 1. By deleting all bad vertices we obtain a K_{r+1}-free graph on at most k vertices with minimum degree at least

$$\left(1 - \frac{1}{r - 1/3} + \epsilon\right) k - 3\eta k - 1 \ge \left(1 - \frac{1}{r - 1/3} + \epsilon\right) k - 4\eta k > \left(1 - \frac{1}{r - 1/3}\right) k.$$

Therefore, by the result of Andrásfai, Erdős and Sós [2] mentioned as Theorem 1.1 in the introduction, this graph is r-partite. This implies that to make G r-partite it suffices to delete at most $\gamma n^2 + \eta n^2 + (\eta n) \cdot n + k \cdot (n/k)^2 \leq 3\eta n^2 + n^2/k = o(n^2)$ edges. $\square$

Consider a partition $(V_1, \ldots, V_r)$ of the vertices of G into r parts which maximizes the number of crossing edges between the parts. Then for every $x \in V_i$ and $j \neq i$ the number of neighbors of x in V_i is at most the number of its neighbors in V_j, as otherwise by shifting x to V_j we increase the number of crossing edges. By the above discussion, we have that this partition satisfies that $\sum_i e(V_i) = o(n^2)$. Call a vertex x of G *typical* if $x \in V_i$ has at most $\epsilon n/2$ neighbors in V_i. Note that there are at most $o(n)$ non-typical vertices in G and, in particular, every part V_i contains a typical vertex. By definition, the degree of this vertex outside V_i is at least $\left(\frac{3r-4}{3r-1} + \epsilon\right) n - \epsilon n/2 = \left(\frac{3r-4}{3r-1} + \epsilon/2\right) n$ and at most $n - |V_i|$. Therefore, for all $1 \leq i \leq r$

$$\begin{aligned} |V_i| &\leq n - \left(\frac{3r-4}{3r-1} + \epsilon/2\right) n = \left(\frac{3}{3r-1} - \epsilon/2\right) n \qquad (1) \\ |V_i| &\geq n - \sum_{j \neq i} |V_j| \geq n - (r-1)\left(\frac{3}{3r-1} - \epsilon/2\right) n \geq \left(\frac{2}{3r-1} + \epsilon/2\right) n. \end{aligned}$$

Our next lemma reduces further the possible number of non-typical vertices in G.

Lemma 2.4 *Each V_i contains at most $O(1)$ non-typical vertices.*

To prove this statement we need the following two claims.

Claim 2.5 *Let $y_1, \ldots, y_k$ be an arbitrary set of $k \leq r-1$ typical vertices outside V_j, such that each y_i belongs to a different part of the partition. Then V_j contains at least $\frac{2}{3r-1}n$ vertices adjacent to all vertices y_i.*

Proof. It is enough to prove this statement for $k = r-1$, since the addition of $r-1-k$ typical vertices y_i from the remaining parts can only decrease the size of the common neighborhood. Thus, without loss of generality, we assume that $V_j = V_r$ and $y_i \in V_i, 1 \leq i \leq r-1$. Since every y_i is a typical vertex it has at most $\epsilon n/2$ neighbors in V_i and hence at most $\epsilon n/2 + (n - |V_i| - |V_r|)$ neighbors outside V_r. This implies that the number of neighbors of y_i in V_r is at least

$$\begin{aligned} d_{V_r}(y_i) &\geq d(y_i) - \Big((1+\epsilon/2)n - |V_i| - |V_r|\Big) \\ &\geq \left(\frac{3r-4}{3r-1} + \epsilon\right) n - \Big((1+\epsilon/2)n - |V_i| - |V_r|\Big) \\ &> |V_r| + |V_i| - \frac{3}{3r-1} n \end{aligned}$$

By definition, there are at most $|V_r| - d_{V_r}(y_i) < \frac{3}{3r-1}n - |V_i|$ non-neighbors of y_i in V_r. Delete from V_r any vertex, which is not a neighbor of either $y_1, y_2, \ldots, y_{r-1}$. The remaining set is adjacent to every vertex y_i and has size at least

$$\begin{aligned} |V_r| - \sum_i \big(|V_r| - d_{V_r}(y_i)\big) &> |V_r| - \sum_{i \le r-1} \left(\frac{3}{3r-1}n - |V_i|\right) \\ &= \sum_{i=1}^{r} |V_i| - (r-1)\frac{3}{3r-1}n \\ &= n - \frac{3r-3}{3r-1}n = \frac{2}{3r-1}n. \end{aligned}$$

□

Claim 2.6 *For every non-typical vertex $x \in V_i$ there are at least $\big(\epsilon n/3\big)^r$ cliques $y_1, \ldots, y_r$ of size r such that $y_j \in V_j$ for all $1 \le j \le r$ and all vertices y_j are adjacent to x.*

Proof. Without loss of generality let $i = 1$ and let $x \in V_1$ be a non-typical vertex. Since for every $j \neq 1$ the number of neighbors of x in V_j is at least as large as the number of its neighbors in V_1 we have that

$$\begin{aligned} d_{V_j}(x) &\ge \frac{d_{V_j}(x) + d_{V_1}(x)}{2} \ge \frac{1}{2}\left(\Big(\frac{3r-4}{3r-1} + \epsilon\Big)n - (r-2)\max_i |V_i|\right) \\ &> \frac{1}{2}\left(\Big(\frac{3r-4}{3r-1} + \epsilon\Big)n - (r-2)\frac{3}{3r-1}n\right) \\ &= \left(\frac{1}{3r-1} + \epsilon/2\right) n. \end{aligned}$$

To construct the r-cliques satisfying the assertion of the claim, first observe, that since x is non-typical it has at least $\epsilon n/2$ neighbors in V_1 and at least $\epsilon n/2 - o(n) > \epsilon n/3$ of these neighbors are typical. Choose y_1 to be an arbitrary typical neighbor of x in V_1 and continue. Suppose at step $1 \le k \le r-1$ we already have a k-clique $y_1, \ldots, y_k$ such that $y_i \in V_i$ for all i and all vertices y_i are adjacent to x. Let U_{k+1} be the set of common neighbors of $y_1, \ldots, y_k$ in V_{k+1}. Then, by the previous claim we have that $|U_{k+1}| \ge \frac{2}{3r-1}n$. Therefore, there are at least

$$d_{V_{k+1}}(x) + |U_{k+1}| - |V_{k+1}| \ge \left(\frac{1}{3r-1} + \epsilon/2\right) n + \frac{2}{3r-1}n - \frac{3}{3r-1}n = \epsilon n/2$$

common neighbors of the vertices y_i and x in V_{k+1}. Moreover, at least $\epsilon n/2 - o(n) > \epsilon n/3$ of them are typical and we can choose y_{k+1} to be any of them. Therefore at the end of the process we indeed obtained at least $\big(\epsilon n/3\big)^r$ r-cliques with the desired property. □

Proof of Lemma 2.4. Suppose that the number of non-typical vertices in V_i is at least $t\big(3/\epsilon\big)^r$. Consider an auxiliary bipartite graph F with parts W_1, W_2, where W_1 is the set of some $s = t\big(3/\epsilon\big)^r$ non-typical vertices in V_i, W_2 is the family of all n^r r-element subsets of $V(G)$ such that $x \in W_1$ is adjacent to the subset Y from W_2 iff Y is an r-clique in

G with exactly one vertex in every V_j and all vertices of Y are adjacent to x. By the previous claim, F has at least $e(F) \geq s\big(\epsilon n/3\big)^r = tn^r$ edges and therefore the average degree of a vertex in W_2 is at least $d_{av} = e(F)/|W_2| = e(F)/n^r \geq t$. By the convexity of the function $f(z) = \binom{z}{t}$, we can find t vertices $x_1, \ldots, x_t$ in W_1 such that the number of their common neighbors in W_2 is at least

$$m \geq \frac{\sum_{Y \in W_2} \binom{d(Y)}{t}}{\binom{s}{t}} \geq n^r \frac{\binom{d_{av}}{t}}{s^t} = \Omega\big(n^r\big).$$

Thus we proved that G contains t vertices $X = \{x_1, \ldots, x_t\}$ and a family of r-cliques $\mathcal{C}$ of size $m = \Omega\big(n^r\big)$ such that every clique in $\mathcal{C}$ is adjacent to all vertices in X. Next we need the following well-known lemma which appears first implicitly in Erdős [3] (see also, e.g., [7]). It states that if an r-uniform hypergraph on n vertices has $m = \Omega\big(n^r\big)$ edges, then it contains a complete r-partite r-uniform hypergraph with parts of size t. By applying this statement to $\mathcal{C}$, we conclude that there are r disjoint set of vertices $A_1, \ldots, A_r$ each of size t such that every r-tuple $a_1, \ldots, a_r$ with $a_i \in A_i$ forms a clique which is adjacent to all vertices in X. The restriction of G to $X, A_1, \ldots, A_r$ forms a complete $(r+1)$-partite graph with parts of size t each. This contradiction shows that there are less than $t\big(3/\epsilon\big)^r = O(1)$ non-typical vertices in V_i and completes the proof of the lemma. □

Lemma 2.7 *Let s be a fixed integer and let $U_1, \ldots, U_k$ be subsets of typical vertices of sizes $|U_1| = 2s$ and $|U_2| = \ldots = |U_k| = s$, which belong to k different parts of the partition of G. Without loss of generality, suppose that $U_i \subset V_i$ and let $U = \cup_{i=1}^k U_i$ and $W = \cup_{j>k} V_j$. Then G contains a complete bipartite graph with parts $U' \subset U$ and $W' \subset W$ such that $|U'| \geq \left(k + \frac{3(r-k)-2}{3(r-k)}\right) s$ and $|W'| = \Omega(n)$.*

Proof. Since every typical vertex $x \in V_i$ has $d_{V_i}(x) \leq \epsilon n/2$, we obtain that the number of its neighbors in W is at least

$$\begin{aligned} d_W(x) &\geq d(v) - d_{V_i}(x) - \sum_{j \leq k, j \neq i} |V_j| \\ &\geq d(v) - \epsilon n/2 + |V_i| - \sum_{j \leq k} |V_j| \\ &\geq \left(\frac{3r-4}{3r-1} + \epsilon\right) n - \epsilon n/2 + |V_i| - \big(n - |W|\big) \\ &\geq |W| + |V_i| - \frac{3}{3r-1} n. \end{aligned}$$

Note that $|W| + \sum_{i=1}^k |V_i| = n$ and also by (1) we have $|W| = \sum_{j>k} |V_j| \leq (r-k)\frac{3}{3r-1}n$ and $|V_1| \geq \left(\frac{2}{3r-1} + \epsilon/2\right) n$. All these facts together give the following estimate on the number of edges between U and W

$$\begin{aligned}
e(U,W) &= \sum_{x\in U} d_W(x) = \sum_{i=1}^{k}\sum_{x\in U_i} d_W(x) \geq \sum_{i=1}^{k}\left(|W|+|V_i|-\frac{3}{3r-1}n\right)|U_i| \\
&= \left((k+1)|W|+|V_1|+\sum_{i=1}^{k}|V_i|-(k+1)\frac{3}{3r-1}n\right)s \\
&\geq \left(k|W|+\left(\frac{2}{3r-1}+\epsilon/2\right)n+\left(|W|+\sum_{i=1}^{k}|V_i|\right)-\frac{3k+3}{3r-1}n\right)s \\
&= \left(k|W|+\epsilon n/2+\frac{3(r-k)-2}{3r-1}n\right)s \\
&\geq \left(k+\frac{3(r-k)-2}{3(r-k)}\right)|W|s+\Omega(n).
\end{aligned}$$

Since U has constant size and $d_U(y)\leq |U|$ for all $y\in W$, we conclude that there are at least

$$\frac{e(U,W)-\left(k+\frac{3(r-k)-2}{3(r-k)}\right)s\cdot|W|}{|U|}=\Omega(n)$$

vertices in W whose degree in U is larger than $\left(k+\frac{3(r-k)-2}{3(r-k)}\right)s$. To complete the proof, note that the number of subsets of U is also bounded by a constant and therefore at least $\Omega(n)$ such vertices will have the same set of neighbors U' in U. $\Box$

Finally we need the following simple estimate.

Lemma 2.8 *For all integers $r\geq 2$ we have the following inequality*

$$\frac{1}{3}\cdot\frac{4}{6}\cdots\frac{3r-5}{3r-3}>\frac{1}{4r^{2/3}}\,.$$

Proof. Let $x=\prod_{j=2}^{r-1}\frac{3j-2}{3j}$, $y=\prod_{j=2}^{r-1}\frac{3j-3}{3j-1}$ and let $z=\prod_{j=2}^{r-1}\frac{3j-4}{3j-2}$. Since $\frac{3j-2}{3j}>\frac{3j-3}{3j-1}>\frac{3j-4}{3j-2}$ and all three products have the same number of terms we have that $x>y>z$. Therefore

$$x^3>zyx=\frac{2}{4}\cdot\frac{3}{5}\cdot\frac{4}{6}\cdots\frac{3r-7}{3r-5}\cdot\frac{3r-6}{3r-4}\cdot\frac{3r-5}{3r-3}=\frac{2\cdot 3}{(3r-4)(3r-3)}>\frac{2}{3r^2}.$$

This implies the assertion of the lemma, since $\frac{1}{3}\cdot\frac{4}{6}\cdots\frac{3r-5}{3r-3}=x/3>\frac{1}{3}\left(\frac{2}{3r^2}\right)^{1/3}>\frac{1}{4r^{2/3}}$. $\Box$

Having finished all the necessary preparations, we are now ready to complete the proof of Theorem 1.2. Without loss of generality, suppose that V_1 spans at least $2n^{2-1/(4r^{2/3}t)}$ edges. By Lemma 2.4, only at most $O(n)$ of these edges are incident to non-typical vertices. Therefore the set of typical vertices in V_1 spans at least $n^{2-1/(4r^{2/3}t)}$ edges. By the well known result of Kövari, Sós and Turán [9] about the Turán numbers of bipartite graphs, V_1 contains a complete bipartite graph H_1 with parts (A,B) of size $|A|=|B|=s_1=4r^{2/3}t$

all of whose vertices are typical. If there are at least $s_2 = \frac{3r-5}{3r-3}s_1$ typical vertices in one of the remaining parts $V_2, \ldots, V_r$ which are adjacent to two subsets $A' \subset A, B' \subset B$ of size s_2 then we add them to (A', B') to form a complete 3-partite graph H_2 with parts of sizes s_2 and continue.

Suppose that at step $1 \le k \le r-1$ we have a complete $k+1$-partite graph H_k with parts $(A, B, U_2, \ldots, U_k)$ of size s_k each, all of whose vertices are typical and $A, B \subset V_1$. Without loss of generality we can assume that $U_i \subset V_i$ for all $2 \le i \le k$. Put $U_1 = A \cup B$ and let $U = \cup_{i=1}^{k} U_k$ and $W = \cup_{j>k} V_j$. Then, by Lemma 2.7, G contains a complete bipartite subgraph with parts (U', W') such that $U' \subset U, |U'| \ge \left(k + \frac{3(r-k)-2}{3(r-k)}\right) s_k$ and $W' \subset W, |W'| \ge \Omega(n)$. Note that, since all parts of H_k have size s_k, we have that all intersections $U' \cap A, U' \cap B$ or $U' \cap U_i, 2 \le i \le k$ have size at least $|U'| - ks_k \ge \frac{3(r-k)-2}{3(r-k)} s_k = s_{k+1}$. Also, since $|W'| \ge \Omega(n)$ and there are at most $O(1)$ non-typical vertices, there exists an index $j > k$ such that $W' \cap V_j$ contains at least s_{k+1} typical vertices. Let U'_{k+1} be some set of s_{k+1} typical vertices from $W' \cap V_j$. Choose subsets $A' \subset U' \cap A, B' \subset U' \cap B$ and $U'_i \subset U' \cap U_i, i \le k$ all of size s_{k+1}. Then $(A, B, U_2, \ldots, U_{k+1})$ form a complete $k+1$-partite graph H_{k+1} with parts of size s_{k+1} all of whose vertices are typical.

Continuing the above process $r-1$ steps we obtain a complete $(r+1)$-partite graph with parts of sizes

$$s_r = \frac{1}{3} s_{r-1} = \frac{1}{3} \cdot \frac{4}{6} s_{r-2} = \ldots = \frac{1}{3} \cdot \frac{4}{6} \cdots \frac{3r-5}{3r-3} s_1 > \frac{s_1}{4r^{2/3}} = t.$$

This contradicts our assumption that G is $K_{r+1}(t)$-free and shows that every V_i spans at most $O\big(n^{2-1/(4r^{2/3}t)}\big)$ edges. Therefore the number of edges we need to delete to make G r-partite is bounded by $\sum_i e(V_i) \le O\big(n^{2-1/(4r^{2/3}t)}\big)$. This completes the proof of Theorem 1.2. □

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