Binary words containing infinitely many overlaps

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Abstract

We characterize the squares occurring in infinite overlap-free binary words and construct various α power-free binary words containing infinitely many overlaps.

1 Introduction

If α is a rational number, a word w is an α *power* if there exists words x and x', with x' a prefix of x, such that $w = x^n x'$ and $\alpha = n + |x'|/|x|$. We refer to $|x|$ as a *period* of w. An α^+ *power* is a word that is a β power for some $\beta > \alpha$. A word is α *power-free* (resp. α^+ *power-free*) if none of its subwords is an α power (resp. α^+ power). A 2 power is called a *square*; a 2^+ power is called an *overlap*.

Thue [18] constructed an infinite overlap-free binary word; however, Dekking [8] showed that any such infinite word must contain arbitrarily large squares. Shelton and Soni [17] characterized the overlap-free squares, but it is not hard to show that there are some overlap-free squares, such as 00110011, that cannot occur in an infinite overlap-free binary word. In this paper, we characterize those overlap-free squares that do occur in infinite overlap-free binary words.

Shur [16] considered the bi-infinite overlap-free and 7/3 power-free binary words and showed that these classes of words were identical. There have been several subsequent papers [1, 10, 11, 14] that have shown various similarities between the classes of overlap-free binary words and 7/3 power-free binary words. Here we contrast the two classes of words by showing that there exist one-sided infinite $7/3$ power-free binary words containing infinitely many overlaps. More generally, we show that for any real number $\alpha > 2$ there exists a real number β arbitrarily close to α such that there exists an infinite β^+ power-free binary word containing infinitely many β powers.

All binary words considered in the sequel will be over the alphabet $\{0, 1\}$. We therefore use the notation $\overline{w}$ to denote the *binary complement* of w; that is, the word obtained from w by replacing 0 with 1 and 1 with 0.

2 Properties of the Thue-Morse morphism

In this section we present some useful properties of the *Thue-Morse morphism*; *i.e.*, the morphism μ defined by $\mu(0) = 01$ and $\mu(1) = 10$. It is well-known [12, 18] that the *Thue-Morse word*

$$\mathbf{t} = \mu^\omega(0) = 0110100110010110 \cdots$$

is overlap-free.

The following property of μ is easy to verify.

Lemma 1. *Let x and y be binary words. Then x is a prefix (resp. suffix) of y if and only if $\mu(x)$ is a prefix (resp. suffix) of $\mu(y)$.*

Brandenburg [6] proved the following useful theorem, which was independently rediscovered by Shur [16].

Theorem 2 (Brandenburg; Shur). *Let w be a binary word and let $\alpha > 2$ be a real number. Then w is α power-free if and only if $\mu(w)$ is α power-free.*

The following sharper version of one direction of this theorem (implicit in [10]) is also useful.

Theorem 3. *Suppose $\mu(w)$ contains a subword u of period p, with $|u|/p > 2$. Then w contains a subword v of length $\lceil |u|/2 \rceil$ and period $p/2$.*

Karhumäki and Shallit [10] gave the following generalization of the factorization theorem of Restivo and Salemi [15]. The extension to infinite words is clear.

Theorem 4 (Karhumäki and Shallit). *Let $x \in \{0, 1\}^*$ be α power-free, $2 < \alpha \leq 7/3$. Then there exist $u, v \in \{\epsilon, 0, 1, 00, 11\}$ and an α power-free $y \in \{0, 1\}^*$ such that $x = u\mu(y)v$.*

3 Overlap-free squares

Let

$$A = \{00, 11, 010010, 101101\}$$

and let

$$\mathcal{A} = \bigcup_{k \geq 0} \mu^k(A).$$

Pansiot [13] and Brlek [7] gave the following characterization of the squares in $\mathbf{t}$.

Theorem 5 (Pansiot; Brlek). *The set of squares in* $\mathbf{t}$ *is exactly the set* $\mathcal{A}$.

We can use this result to prove the following.

Proposition 6. *For any position* i*, there is at most one square in* $\mathbf{t}$ *beginning at position* i.

Proof. Suppose to the contrary that there exist distinct squares x and y that begin at position i. Without loss of generality, suppose that x and y begin with 0. Then by Theorem 5, $x = \mu^p(u)$ and $y = \mu^q(v)$, for some p, q and $u, v \in \{00, 010010\}$. Suppose $p \leq q$ and let $w = \mu^{q-p}(v)$. By Lemma 1, either u is a proper prefix of w or w is a proper prefix of u, neither of which is possible for any choice of $u, v \in \{00, 010010\}$. □

The set $\mathcal{A}$ does not contain all possible overlap-free squares. Shelton and Soni [17] characterized the overlap-free squares (the result is also attributed to Thue in [4]).

Theorem 7 (Shelton and Soni). *The overlap-free binary squares are the conjugates of the words in* $\mathcal{A}$.

Some overlap-free squares cannot occur in any infinite overlap-free binary word, as the following lemma shows.

Lemma 8. *Let* $x = \mu^k(z)$ *for some* $k \geq 0$ *and* $z \in \{011011, 100100\}$*. Then* xa *contains an overlap for all* $a \in \{0, 1\}$.

Proof. It is easy to see that $x = uvvuvv$ for some $u, v \in \{0,1\}^*$, where u and v begin with different letters. Thus one of $uvvuvva$ or vva is an overlap. □

We can characterize the squares that can occur in an infinite overlap-free binary word. Let

$$B = \{001001, 110110\}$$

and let

$$\mathcal{B} = \bigcup_{k \geq 0} \mu^k(B).$$

Theorem 9. *The set of squares that can occur in an infinite overlap-free binary word is* $\mathcal{A} \cup \mathcal{B}$*. Furthermore, if* $\mathbf{w}$ *is an infinite overlap-free binary word containing a subword* $x \in \mathcal{B}$*, then* $\mathbf{w}$ *begins with* x *and there are no other occurrences of* x *in* $\mathbf{w}$.

Proof. Let $\mathbf{w}$ be an infinite overlap-free binary word beginning with a square $yy \notin \mathcal{A} \cup \mathcal{B}$. Suppose further that yy is a smallest such square that can be extended to an infinite overlap-free word. If $|y| \leq 3$, then $yy \notin \mathcal{A} \cup \mathcal{B}$ is one of 011011 or 100100, neither of which can be extended to an infinite overlap-free word by Lemma 8.

We assume then that $|y| > 3$. Since, by Theorem 7, yy is a conjugate of a word in $\mathcal{A}$, we have two cases.

Case 1: $yy = \mu(zz)$ for some $z \in \{0, 1\}^*$. By Theorem 4, $\mathbf{w} = \mu(zz\mathbf{w}')$ for some infinite $\mathbf{w}'$, where $zz\mathbf{w}'$ is overlap-free. Thus zz is a smaller square not in $\mathcal{A} \cup \mathcal{B}$ that can be extended to an infinite overlap-free word, contrary to our assumption.

Case 2: $yy = a\mu(zz')\overline{a}$ for some $a \in \{0, 1\}$ and $z, z' \in \{0, 1\}^*$. By Theorem 4, yy is followed by a in $\mathbf{w}$, and so yya is an overlap, contrary to our assumption.

Since both cases lead to a contradiction, our assumption that $yy \notin \mathcal{A} \cup \mathcal{B}$ must be false.

To see that each word in $\mathcal{A} \cup \mathcal{B}$ does occur in some infinite overlap-free binary word, note that Allouche, Currie, and Shallit [2] have shown that the word $\mathbf{s} = 001001\overline{\mathbf{t}}$ is overlap-free. Now consider the words $\mu^k(\mathbf{s})$ and $\mu^k(\overline{\mathbf{s}})$, which are overlap-free for all $k \geq 0$.

Finally, to see that any occurrence of $x \in \mathcal{B}$ in $\mathbf{w}$ must occur at the beginning of $\mathbf{w}$, we note that by an argument similar to that used in Lemma 8, ax contains an overlap for all $a \in \{0, 1\}$, and so x occurs at the beginning of $\mathbf{w}$. ◻

4 Words containing infinitely many overlaps

In this section we construct various infinite α power-free binary words containing infinitely many overlaps. We begin by considering the infinite 7/3 power-free binary words.

Proposition 10. *For all $p \geq 1$, an infinite 7/3 power-free word contains only finitely many occurrences of overlaps with period p.*

Proof. Let $\mathbf{x}$ be an infinite 7/3 power-free word containing infinitely many overlaps with period p. Let $k \geq 0$ be the smallest integer satisfying $p \leq 3 \cdot 2^k$. Suppose $\mathbf{x}$ contains an overlap w with period p starting in a position $\geq 2^{k+1}$. Then by Theorem 4, we can write

$$\mathbf{x} = u_1\mu(u_2) \cdots \mu^{k-1}(u_k)\mu^k(\mathbf{y}),$$

where each $u_i \in \{\epsilon, 0, 1, 00, 11\}$. The overlap w occurs as a subword of $\mu^k(\mathbf{y})$. By Lemma 3, $\mathbf{y}$ contains an overlap with period $p/2^k \leq 3$. But any overlap with period ≤ 3 contains a 7/3 power. Thus, $\mathbf{x}$ contains a 7/3 power, a contradiction. ◻

The following theorem provides a striking contrast to Shur's result [16] that the bi-infinite 7/3 power-free words are overlap-free.

Theorem 11. *There exists a 7/3 power-free binary word containing infinitely many overlaps.*

Proof. We define the following sequence of words: $A_0 = 00$ and $A_{n+1} = 0\mu^2(A_n)$, $n \geq 0$. The first few terms in this sequence are

$$\begin{aligned} A_0 &= 00 \\ A_1 &= 001100110 \\ A_2 &= 0011001101001100101100110100110010110 \\ &\vdots \end{aligned}$$

We first show that in the limit as $n \to \infty$, this sequence converges to an infinite word $\mathbf{a}$. It suffices to show that for all n, A_n is a prefix of A_{n+1}. We proceed by induction on n. Certainly, $A_0 = 00$ is a prefix of $A_1 = 0\mu^2(00) = 001100110$. Now $A_n = 0\mu^2(A_{n-1})$, $A_{n+1} = 0\mu^2(A_n)$, and by induction, A_{n-1} is a prefix of A_n. Applying Lemma 1, we see that A_n is a prefix of A_{n+1}, as required.

Note that for all n, A_{n+1} contains $\mu^{2n}(A_1)$ as a subword. Since A_1 is an overlap with period 4, $\mu^{2n}(A_1)$ contains 2^{2n} overlaps with period 2^{2n+2}. Thus, $\mathbf{a}$ contains infinitely many overlaps.

We must show that $\mathbf{a}$ does not contain a 7/3 power. It suffices to show that A_n does not contain a 7/3 power for all $n \geq 0$. Again, we proceed by induction on n. Clearly, $A_0 = 00$ does not contain a 7/3 power. Consider $A_{n+1} = 0\mu^2(A_n)$. By induction, A_n is 7/3 power-free, and by Theorem 2, so is $\mu^2(A_n)$. Thus, if A_{n+1} contains a 7/3 power, such a 7/3 power must occur as a prefix of A_{n+1}. Note that A_{n+1} begins with 00110011. The word 00110011 cannot occur anywhere else in A_{n+1}, as that would imply that A_{n+1} contained a cube 000 or 111, or the 5/2 power 1001100110. If A_{n+1} were to begin with a 7/3 power with period ≥ 8, it would contain two occurrences of 00110011, contradicting our earlier observation. We conclude that the period of any such 7/3 power is less than 8. Checking that no such 7/3 power exists is now a finite check and is left to the reader. ☐

In fact, we can prove the following stronger statement.

Theorem 12. *There exist uncountably many 7/3 power-free binary words containing infinitely many overlaps.*

Proof. For a finite binary sequence b, we define an operator g_b on binary words recursively by

$$\begin{aligned} g_\epsilon(w) &= w \\ g_{0b}(w) &= \mu^2(g_b(w)) \\ g_{1b}(w) &= 0\mu^2(g_b(w)). \end{aligned}$$

Note that $g_b(0)$ always starts with a 0, so that for any finite binary words p and b, $g_p(0)$ is always a prefix of $g_{pb}(0)$. Since $g_0(0)$ is not a prefix of $g_1(0)$, $g_{p0}(0)$ is not a prefix of $g_{p1}(0)$ for any p, so that distinct b give distinct words. Given an infinite binary sequence $\mathbf{b} = b_1b_2b_3\cdots$ where the $b_i \in \{0,1\}$, define an infinite binary sequence $w_\mathbf{b}$ to be the limit of

$$g_\epsilon(00), g_{b_1}(00), g_{b_1b_2}(00), g_{b_1b_2b_3}(00), \ldots$$

By an earlier argument, each $w_{\mathbf{b}}$ is 7/3 power-free. Since $g_1(00) = 001100110$ is an overlap, $g_{b1}(00) = g_b(001100110)$ ends with an overlap for any finite word b. Thus, each 1 in $\mathbf{b}$ introduces an overlap in $w_{\mathbf{b}}$. Since uncountably many binary sequences contain infinitely many 1's, uncountably many of the $w_{\mathbf{b}}$ are 7/3 power-free words containing infinitely many overlaps. □

Next, we show that the sequence $\mathbf{a}$ constructed in the proof of Theorem 11 is an automatic sequence (in the sense of [3]).

Proposition 13. *The sequence* $\mathbf{a}$ *is* 4*-automatic.*

Proof. We show that $\mathbf{a} = g(h^\omega(0))$, where h and g are the morphisms defined by

$$\begin{array}{rcl} h(0) &=& 0134 \\ h(1) &=& 2134 \\ h(2) &=& 3234 \\ h(3) &=& 2321 \\ h(4) &=& 3421 \end{array} \quad \text{and} \quad \begin{array}{rcl} g(0) &=& 0 \\ g(1) &=& 0 \\ g(2) &=& 0 \\ g(3) &=& 1 \\ g(4) &=& 1. \end{array}$$

We make some observations concerning 2-letter subwords: The sequence $h^\omega(0)$ clearly does not contain any of the words 11, 14, 22, 24, 31, 33, 41 or 44. In fact, neither 12 nor 43 appears as a subword either: Words 12 and 43 do not appear internally in $h(i)$, $0 \le i \le 4$; therefore, if 43 appears in $h^n(0)$, it must 'cross the boundary' in one of $h(12)$, $h(14)$, $h(22)$ or $h(24)$. Since 14, 22 and 24 do not appear in $h^\omega(0)$, word 43 can only appear in $h^n(0)$ as a descendant of a subword 12 in $h^{n-1}(0)$. However, the situation is symmetrical; word 12 can only appear in $h^n(0)$ as a descendant of a subword 43 in $h^{n-1}(0)$. By induction, neither 43 nor 12 ever appears.

The point of the previous paragraph is that

$h(0)$ always occurs in the context $h(0)2$
$h(1)$ always occurs in the context $h(1)2$
$h(2)$ always occurs in the context $h(2)2$
$h(3)$ always occurs in the context $h(3)3$
$h(4)$ always occurs in the context $h(4)3$

The word $h^\omega(0)$ can thus be parsed in terms of a new morphism f:

$$\begin{array}{rcl} f(0) &=& 1342 \\ f(1) &=& 1342 \\ f(2) &=& 2342 \\ f(3) &=& 3213 \\ f(4) &=& 4213. \end{array}$$

The parsing in terms of f works as follows: If we write $h^\omega(0) = 0w$, then $w = f(0w)$. It is useful to rewrite this relation in terms of the finite words $h^n(0)$. For non-negative integer n let x_n be the unique letter such that $h^n(0)x_n$ is a prefix of $h^\omega(0)$. Thus $x_0 = 1$, $x_1 = 2$, etc. We then have

$$h^n(0)x_n = 0f(h^{n-1}(0)), \quad n \geq 1. \tag{1}$$

Since for all $a \in \{0, 1, 2, 3, 4\}$, $g(f(a)) = \mu^2(g(a))$, we have $g(f(u)) = \mu^2(g(u))$ for all words u. Therefore, applying g to (1)

$$\begin{aligned} g(h^n(0)x_n) &= g(0f(h^{n-1}(0))) \\ &= g(0)g(f(h^{n-1}(0))) \\ &= 0\mu^2(g(h^{n-1}(0))), \quad n \geq 1. \end{aligned}$$

From this relation we show by induction that A_n is the prefix of $g(h^{n+1}(0))$ of length $(4^{n+1} + 3 \cdot 4^n - 1)/3$. Certainly, $A_0 = 00$ is the prefix of length 2 of $g(h(0)) = 0011$. Consider $A_n = 0\mu^2(A_{n-1})$. We can assume inductively that A_{n-1} is the prefix of $g(h^n(0))$ of length $(4^n + 3 \cdot 4^{n-1} - 1)/3$. Writing $g(h^n(0)) = A_{n-1}z$ for some z, we have

$$\begin{aligned} g(h^{n+1}(0)x_{n+1}) &= 0\mu^2(g(h^n(0))) \\ &= 0\mu^2(A_{n-1}z) \\ &= A_n\mu^2(z), \end{aligned}$$

for some x_{n+1}, whence A_n is a prefix of $g(h^{n+1}(0))$. Since $|A_n| = 4|A_{n-1}| + 1$, we have $|A_n| = (4^{n+1} + 3 \cdot 4^n - 1)/3$, as required. □

The result of Theorem 11 can be strengthened even further.

Theorem 14. *For every real number $\alpha > 2$ there exists a real number β arbitrarily close to α, such that there is an infinite β^+ power-free binary word containing infinitely many β powers.*

Proof. Let $s \geq 3$ be a positive integer, and let $r = \lfloor \alpha + 1 \rfloor$. Let t be the largest positive integer such that $r - t/2^s > \alpha$, and such that the word obtained by removing a prefix of length t from $\mu^s(0)$ begins with 00. Let $\beta = r - t/2^s$. Since $\alpha \geq r - 1$, we have $t < 2^s$. Also, $\mu^3(0) = 01101001$ and $\mu^3(1) = 10010110$ are of length 8, and both contain 00 as a subword; it follows that $|\alpha - \beta| \leq 8/2^s$, so that by choosing large enough s, β can be made arbitrarily close to α.

We construct sequences of words A_n, B_n and C_n. Define $C_0 = 00$. For each $n \geq 0$:

1. Let $A_n = 0^{r-2}C_n$.

2. Let $B_n = \mu^s(A_n)$.

3. Remove the first t letters from B_n to obtain a new word C_{n+1} beginning with 00.

Since each A_n begins with the r power 0^r, each $B_n = \mu^s(A_n)$ begins with an r power of period 2^s. Removing the first t letters ensures that C_{n+1} commences with an $(r2^s - t)/2^s$ power, viz., a β power. The limit of the C_n gives the desired infinite word. Let us check that this limit exists:

Let w be the word consisting of the first t letters of $\mu^s(0)$. Since all the A_n commence with 0 by construction, all the B_n commence with $\mu^s(0)$, and hence with w. This means that $B_n = wC_{n+1}$ for each n.

We show that A_n is always a prefix of A_{n+1} by induction. Certainly A_0 is a prefix of A_1. Assume that A_{n-1} is a prefix of A_n. Since $A_n = 0^{r-2}C_n$ and $A_{n+1} = 0^{r-2}C_{n+1}$, A_n is a prefix of A_{n+1} if C_n is a prefix of C_{n+1}. Since $B_{n-1} = wC_n$ and $B_n = wC_{n+1}$, C_n is a prefix of C_{n+1} if B_{n-1} is a prefix of B_n. By Lemma 1, B_{n-1} is a prefix of B_n if A_{n-1} is a prefix of A_n, which is our inductive assumption. We conclude that A_n is a prefix of A_{n+1}.

It follows that C_n is a prefix of C_{n+1} for $n \geq 0$, so that the limit of the C_n exists. It will thus suffice to prove the following claim:

Claim: The A_n, B_n and C_n satisfy the following:

1. The word C_n contains no β^+ powers.
2. The only β^+ power in A_n is 0^r.
3. Any β^+ powers in B_n appear only in the prefix $\mu^s(0^r)$.

Certainly C_0 contains no β^+ powers, and since $\beta > r - 1$, the only β^+ power in A_0 is 0^r. Suppose then that the claim holds for A_n and C_n.

Now suppose that $B_n = \mu^s(0^{r-2})\mu^s(C_n)$ contains a β^+ power u with period p. Since C_n contains no β^+ powers, Theorem 2 ensures that $\mu^s(C_n)$ contains no β^+ powers. We can therefore write $B_n = xuy$ where $|x| < |\mu^s(0^{r-2})|$. In other words, u overlaps $\mu^s(0^{r-2})$ from the right. By Theorem 3, the preimage of B_n under μ, i.e., $\mu^{s-1}(A_n)$, contains a β^+ power of length at least $|u|/2$ and period $p/2$. In fact, iterating this argument, A_n contains a β^+ power of period $p/2^s$ of length at least $|u|/2^s$. Since the only β^+ power in A_n is 0^r, with period 1, we see that $p/2^s = 1$, whence $p = 2^s$ and $|u| \leq r2^s$.

Recall that B_n has a prefix $\mu^s(0^r)$ which also has period 2^s, and that this prefix is overlapped by u. It follows that all of xu is a β^+ power with period $p = 2^s$. However, as just argued, this means that $|xu| \leq r2^s = |\mu^s(0^r)|$, so that u is contained in $\mu^s(0^r)$ and part 3 of our claim holds for B_n. We now show that parts 1 and 2 hold for C_{n+1} and A_{n+1} respectively, and the truth of our claim will follow by induction.

Part 1 follows immediately from part 3.

Now suppose that A_{n+1} contains a β^+ power u. Recall that $A_{n+1} = 0^{r-2}C_{n+1}$, and C_{n+1} begins with 00, but contains no β^+ powers. It follows that u is not a subword of C_{n+1}. Therefore, 000 must be a prefix of u. If $u = 0^q$ for some integer q, then $q \leq r$ by the construction of A_{n+1}, and

$$r \geq q > \beta > \alpha > r - 1.$$

This implies that $q = r$, and $u = 0^r$, as claimed. If we cannot write $u = 0^q$, then $|u|_1 \geq 1$. Because u is a 2^+ power, 000 must appear twice in u with a 1 lying somewhere between the two appearances. This implies that 000 is a subword of C_{n+1}, and hence of $B_n = \mu^s(A_n)$. However, no word of the form $\mu(w)$ contains 000. This is a contradiction. □

We conclude by presenting the following open problem.

> Does there exist a characterization (in the sense of [5, 9]) of the infinite 7/3 power-free binary words?

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