Erdős-Ko-Rado-Type Theorems for Colored Sets

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Abstract

An Erdős-Ko-Rado-type theorem was established by Bollobás and Leader for q-signed sets and by Ku and Leader for partial permutations. In this paper, we establish an LYM-type inequality for partial permutations, and prove Ku and Leader's conjecture on maximal k-uniform intersecting families of partial permutations. Similar results on general colored sets are presented.

1 Introduction

Erdős, Ko and Rado proved in 1961 [10] that a family of pairwise intersecting k-subsets of an n-set cannot have more members than the family of k-subsets all of which contain a given element a, say, provided $k \leq \lfloor \frac{n}{2} \rfloor$. Bollobás in 1973 [3] established a stronger result—an LYM-type inequality, which says that if $\mathcal{A}$ is an intersecting antichain of subsets of an n-set, then $\sum_{k\geq 1} \frac{f_k}{\binom{n-1}{k-1}} \leq 1$, where f_k denotes the number of sets in $\mathcal{A}$ of size k with $k \leq n/2$. This inequality implies the Erdős-Ko-Rado Theorem. The original LYM inequality says that if $\mathcal{A}$ is an antichain of subsets of an n-set, then $\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq 1$, which yields a simple proof of Sperner's Theorem that $|\mathcal{A}| = \sum_{k=0}^{n} f_k \leq \binom{n}{[\frac{n}{2}]}$. This proof is due independently to Lubell, Yamamoto and Meschalkin, and therefore the inequality is known as the LYM-inequality (see [9] for detail).

In 1972 Katona presented a rather simple proof of the Erdős-Ko-Rado Theorem. By his technique one can usually establish an LYM-type inequality. By employing Katona's technique, in 1997, Bollobás and Leader [4] presented an Erdős-Ko-Rado theorem for q-*signed sets* where $q \geq 2$. A q-signed k-set is a pair (A, f), where $A \subseteq [n]$ is a k-set and f

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is a function from A to $[q]$. A family $\mathcal{F}$ of q-signed k-sets is *intersecting* if for any (A, f), $(B, g) \in \mathcal{F}$ there exists $x \in A \cap B$ such that $f(x) = g(x)$.

Theorem 1.1 *(Bollobás and Leader) Fix a positive integer $k \leq n$, and let $\mathcal{F}$ be an intersecting family of q-signed k-sets on $[n]$, where $q \geq 2$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} q^{k-1}$. Unless $q = 2$ and $k = n$, equality holds if and only if $\mathcal{F}$ consists of all q-signed k-sets (A, f) such that $x_0 \in A$ and $f(x_0) = \varepsilon_0$ for some fixed $x_0 \in [n]$, $\varepsilon_0 \in [q]$.*

Note that a q-signed set can be reformulated as an element of a *generalized Boolean algebra*. Let $M_1, M_2, \ldots, M_n$ be n pairwise disjoint sets of the same cardinality q, say $M_i = \{x_{i,1}, \ldots, x_{i,q}\}$, $i = 1, \ldots, n$. The associated generalized Boolean algebra is defined to be the family

$$\mathcal{B}(n, q) = \{C \subseteq M_1 \cup M_2 \cup \cdots \cup M_n : |C \cap M_i| \leq 1,\ i = 1, \ldots, n\} \tag{1}$$

ordered by containment. Given a k-set $C \in \mathcal{B}(n, q)$, say $C = \{x_{i_1,j_1}, \ldots, x_{i_k,j_k}\}$, we define a unique q-signed k-set (A, f), where $A = \{i_1, \ldots, i_k\}$ and $f(i_t) = j_t$ for $t = 1, \ldots, k$. It is evident that two sets in $\mathcal{B}(n, q)$ are intersecting if and only if the q-signed sets corresponding to them are intersecting. Deza and Frankl in 1983 [6] proved that if $\mathcal{F}$ is a k-uniform intersecting family in $\mathcal{B}(n, q)$, then $|\mathcal{F}| \leq \binom{n-1}{k-1} q^{k-1}$ for $q \geq 2$ and $k = 1, 2, \ldots, n$, which is equivalent to the first part of Theorem 1.1. Engel [8] strengthened the result of Deza and Frankl to an LYM-type inequality as follows.

Theorem 1.2 *(Engel) Assume $q \geq 2$ and let $\mathcal{F} \subseteq \mathcal{B}(n, q)$ be an intersecting antichain with profile $(a_1, \ldots, a_n)$, where $a_k = |\{A \in \mathcal{F} : |A| = k\}|$. Then*

$$\sum_{k=1}^{n} \frac{a_k}{\binom{n-1}{k-1} q^{k-1}} \leq 1.$$

Note that when $\mathcal{F}$ is k-uniform, the inequality above implies $|\mathcal{F}| = a_k \leq \binom{n-1}{k-1} q^{k-1}$. Note also that Erdős, Faigle and Kern in 1992 [11] gave a group-theoretic proof of Theorem 1.2.

Recently, Ku and Leader [15] established an Erdős-Ko-Rado-type theorem for partial permutations. A *k-partial permutation* of $[n]$ is a pair (A, f) where $A \subseteq [n]$ with $|A| = k$ and $f : A \to [n]$ is an injective map. Note that an n-partial permutation of $[n]$ is just a permutation on $[n]$. By S_n we denote the set of all permutations on $[n]$. The intersecting property for partial permutations is defined in the same way as for signed sets, that is, a family $\mathcal{F}$ of partial permutations is *intersecting* if for any (A, f), $(B, g) \in \mathcal{F}$ there exists $x \in A \cap B$ such that $f(x) = g(x)$.

Theorem 1.3 *(Ku and Leader) Fix k, n with $k \leq n - 1$ and let $\mathcal{F}$ be an intersecting family of k-partial permutations. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} \frac{(n-1)!}{(n-k)!}.$$

They also showed that for $8 \le k \le n-3$, equality holds if and only if $\mathcal{F}$ consists of all k-partial permutations (A, f) such that $x_0 \in A$ and $f(x_0) = \varepsilon_0$ for some fixed $x_0,\ \varepsilon_0 \in [n]$. And, they conjectured the following.

Conjecture 1.4 *(Ku and Leader) Equality in Theorem 1.3 holds if and only if $\mathcal{F}$ consists of all k-partial permutations (A, f) such that $x_0 \in A$ and $f(x_0) = \varepsilon_0$ for some fixed $x_0,\ \varepsilon_0 \in [n]$.*

In fact, Theorem 1.3 and Conjecture 1.4 hold for $k = n$.

Theorem 1.5 *Let $\mathcal{F}$ be an intersecting family in S_n. Then*

(i) *(Deza and Frankl [7]) $|\mathcal{F}| \le (n-1)!$.*

(ii) *(Cameron and Ku [5]) Equality in (i) holds if and only if $\mathcal{F}$ is a coset of the stabilizer of a point.*

The result in (ii) was also deduced from a more general result on certain vertex transitive graphs in Larose and Malvenuto's paper [16].

Combining the signed sets and the partial permutations, we introduce the following concepts.

Let N be a fixed finite set, and let $\mathfrak{p}_n$ be a subset of $N^{[n]}$, the set of all maps from $[n]$ to N. Then $\mathfrak{p}_n$ can be regarded as a set of colorings of $[n]$. Define

$$\mathcal{B}(\mathfrak{p}_n) = \{(A, f|_A) : A \subseteq [n], f \in \mathfrak{p}_n\},$$

where $f|_A$ is the restriction of f on A. We simply write the pair $(A, f|_A)$ as (A, f) for short, which will not cause confusions. Define an ordering on $\mathcal{B}(\mathfrak{p}_n)$ as follows:

$$(A, f) \le (B, g) \Leftrightarrow A \subseteq B \text{ and } g|_A = f|_A.$$

With this ordering $\mathcal{B}(\mathfrak{p}_n)$ forms a ranked poset with the rank function $\rho(A, f) = |A|$. By $\mathcal{B}_k(\mathfrak{p}_n)$ we denote the set of all elements of rank k. An element of rank 1 is usually called an *atom*. An *antichain* of $\mathcal{B}(\mathfrak{p}_n)$ is a subset of which no two elements are comparable in $\mathcal{B}(\mathfrak{p}_n)$. For example, $\mathcal{B}_k(\mathfrak{p}_n)$ is an antichain.

From the definition, we see that $\mathcal{B}(\mathfrak{p}_n)$ is determined by the set of colorings $\mathfrak{p}_n$. If $\mathfrak{p}_n$ is the empty set, then $\mathcal{B}(\mathfrak{p}_n)$ is the boolean algebra B_n. Let $\mathfrak{q}_n = [q]^{[n]}$ for a positive integer $q \ge 2$, and let $\mathfrak{s}_n = S_n$. Then $\mathcal{B}(\mathfrak{q}_n)$ is the set of all q-signed sets, and $\mathcal{B}(\mathfrak{s}_n)$ is the set of all partial permutations.

Given an $A \subseteq [n]$, let $[\mathfrak{p}_n]_A$ denote the set of all pairs $(A, f) \in \mathcal{B}(\mathfrak{p}_n)$. We say $\mathfrak{p}_n$ is *regular* if the cardinality of $[\mathfrak{p}_n]_A$ depends only on $|A|$.

In the sequel of this paper, all sets of colorings concerned are assumed regular, and by $[\mathfrak{p}_n]_k$ we denote the cardinality of $[\mathfrak{p}_n]_A$ with $|A| = k$. Thus

$$|\mathcal{B}_k(\mathfrak{p}_n)| = \binom{n}{k}[\mathfrak{p}_n]_k.$$

It is easy to verify that the sets of colorings $\mathfrak{q}_n$ and $\mathfrak{s}_n$ are regular with $[\mathfrak{q}_n]_k = q^k$ and $[\mathfrak{s}_n]_k = \frac{n!}{(n-k)!}$.

A subset $\mathcal{F}$ of $\mathcal{B}(\mathfrak{p}_n)$ is called an *intersecting* family if for any (A, f), $(B, g) \in \mathcal{F}$, there exists $x \in A \cap B$ such that $f(x) = g(x)$, in other words, both (A, f) and (B, g) are greater than the atom $(\{x\}, f_0)$ where f_0 is defined by $f_0(x) = f(x) = g(x)$. The profile $(a_1, a_2, \ldots)$ of $\mathcal{F}$ is given by $a_k = |\{(A, f) \in \mathcal{F} : |A| = k\}|$ for $k = 1, 2, \ldots, n$. We say $\mathcal{F}$ is *k-uniform* if $\mathcal{F} \subseteq \mathcal{B}_k(\mathfrak{p}_n)$. Let α be an atom of $\mathcal{B}(\mathfrak{p}_n)$, and set $\mathcal{S}_k(\alpha) = \{(A, f) \in \mathcal{B}_k(\mathfrak{p}_n) : (A, f) \geq \alpha\}$. Then $\mathcal{S}_k(\alpha)$ is a k-uniform intersecting family, called a *k-star*. The regularity of $\mathfrak{p}_n$ implies that $|\mathcal{S}_k(\alpha)| = \binom{n-1}{k-1}[\mathfrak{p}_{n-1}]_{k-1}$ for each atom α.

For $1 \leq k \leq n$, we say $\mathcal{B}(\mathfrak{p}_n)$ has the *EKR property for rank k* if every k-uniform intersecting family $\mathcal{F}$ satisfies $|\mathcal{F}| \leq \binom{n-1}{k-1}[\mathfrak{p}_{n-1}]_{k-1}$. And, we say $\mathcal{B}(\mathfrak{p}_n)$ has the *uniqueness property for rank k* if equality holds if and only if $\mathcal{F}$ is a k-star. We say $\mathcal{B}(\mathfrak{p}_n)$ satisfies an *LYM-type inequality for rank k* if for each intersecting antichain $\mathcal{F}$ with profile $(a_1, a_2, \ldots, a_k)$, we have

$$\sum_{i=1}^{k} \frac{a_i}{\binom{n-1}{i-1}[\mathfrak{p}_{n-1}]_{i-1}} \leq 1.$$

(Note that the previous notions can be generalized to a ranked poset in a similar way.) Furthermore, we say $\mathcal{B}(\mathfrak{p}_n)$ has the *local EKR property for rank k* if for every $A \subseteq [n]$ with $|A| = k$, $[\mathfrak{p}_n]_A$ has the EKR property, that is, there is an $x_0 \in A$ and a $y_0 \in N$ such that $\{(A, f) : f(x_0) = y_0\}$ is a maximum intersecting family in $[\mathfrak{p}_n]_A$.

Example 1.6 From Theorem 1.1 we see that $\mathcal{B}(\mathfrak{q}_n)$ has the EKR property for rank n. Recall that $\mathfrak{q}_n = [q]^{[n]}$, where q is independent to n. We therefore obtain that $\mathcal{B}(\mathfrak{q}_n)$ has the local EKR property for all ranks $k = 1, 2, \ldots, n$. We believe that $\mathcal{B}(\mathfrak{s}_n)$ also has the local EKR property for every rank $k = 1, 2, \ldots, n$, but it can not follow from the EKR property for rank n, because the domain and the image of $\mathfrak{s}_n$ are dependent.

Remark 1.7 Generally, the local EKR property does not imply the EKR property. For example, when $\mathfrak{p}_n$ is empty, $\mathcal{B}(\mathfrak{p}_n)$ is the boolean algebra B_n. For every $A \subseteq [n]$ with $|A| > n/2$, $[\mathfrak{p}_n]_A$ trivially has the EKR property, but B_n has no the EKR property for ranks greater than $n/2$.

In the next section, we first establish an LYM-type inequality for $\mathcal{B}(\mathfrak{s}_n)$ which deduces Theorem 1.3 immediately, then we prove Conjecture 1.4. Note that our proof of the conjecture does not depend on the LYM-type inequality, but only on the inequality in Theorem 1.3. In Section 3 we discuss the direct product of colorings (as sets), and present a theorem on its EKR property, an LYM-type inequality, and the uniqueness property. As a consequence, we give corresponding results on the direct product of $\mathfrak{q}_n$ and $\mathfrak{s}_n$.

2 On partial permutations

Recall that a partial permutation, as defined in [15], is a pair (A, f), where $A \subseteq [n]$ and f is an injection from A into $[n]$. By our notation, $f \in \mathfrak{s}_n$, and $\mathcal{B}(\mathfrak{s}_n)$ denotes the set of all partial permutations. We first establish an LYM-type inequality for $\mathcal{B}(\mathfrak{s}_n)$. The techniques we use here are based on the ideas from [4, 13, 15], which originally came from Katona [14].

As defined in [15], a *cyclic ordering* of $[n] \times [n]$ is a bijection $\sigma : [n] \times [n] \to [n^2]$. Given such cyclic ordering σ, we may arrange the elements of $[n] \times [n]$ on a cycle of length n^2 in the natural way. Let k, n be positive integers where $k \leq n-1$. A *k-interval* in the cyclic ordering is a sequence of k elements $(x_1, \varepsilon_1), \ldots, (x_k, \varepsilon_k)$ in $[n] \times [n]$ such that $\sigma(x_{i+1}, \varepsilon_{i+1}) = \sigma(x_i, \varepsilon_i) + 1 \pmod{n^2}$ for $1 \leq i \leq k-1$, and denote this k-interval by $[(x_1, \varepsilon_1), \ldots, (x_k, \varepsilon_k)]$. A k-partial permutation (A, f) is *compatible* with a cyclic ordering σ, written as $(A, f) \prec \sigma$, if there is a k-interval $[(x_1, \varepsilon_1), \ldots, (x_k, \varepsilon_k)]$ in the ordering such that $x_i \in A$ and $f(x_i) = \varepsilon_i$ for $i = 1, 2, \ldots, k$.

The following $n!^2$ good cyclic orderings constructed by Ku and Leader in [15] play an essential role for our argument: the *standard* good cyclic ordering τ defined by $\tau(x, \varepsilon) = x + dn$ where $d = \varepsilon - x \pmod{n}$, and other good cyclic orderings $\tau_{\pi\pi'}$ defined by $\tau_{\pi\pi'}(x, \varepsilon) = \tau(\pi(x), \pi'(\varepsilon))$, where $\pi,\ \pi' \in S_n$. Write the set of these good cyclic orderings as $\mathcal{C}_n$.

Lemma 2.1 *Let $k \leq n-1$ be a positive integer. Then every k-partial permutation is exactly compatible with $n^2 k!(n-k)!^2$ good cyclic orderings in $\mathcal{C}_n$.*

Proof. Let (A, f) be any selected k-partial permutation with $A = \{a_1, \ldots, a_k\}$ and $f(A) = \{b_1, \ldots, b_k\}$ where $b_i = f(a_i)$, $i = 1, \ldots, k$. Then, for a $\sigma \in \mathcal{C}_n$, (A, f) is compatible with σ if and only if there is a k-interval of σ, say $[(x_1, \varepsilon_1), \ldots, (x_k, \varepsilon_k)]$, such that $\{(x_1, \varepsilon_1), \ldots, (x_k, \varepsilon_k)\} = \{(a_1, b_1), \ldots, (a_k, b_k)\}$, which says that if $\sigma = \tau_{\pi\pi'}$, then there is a k-interval $[(y_1, \theta_1), \ldots, (y_k, \theta_k)]$ in τ such that

$$\{(y_1, \theta_1), \ldots, (y_k, \theta_k)\} = \{(\pi(a_1), \pi'(b_1)), \ldots, (\pi(a_k), \pi'(b_k))\} \tag{2}$$

as two sets. Clearly, τ has n^2 many k-intervals, and for each one, there are $k!(n-k)!^2$ pairs (π, π')'s satisfying (2), completing the proof. $\square$

Theorem 2.2 *Let $\mathcal{F}$ be an intersecting antichain of partial permutations with profile $(a_1, \ldots, a_{n-1})$. Then*

$$\sum_{k=1}^{n-1} \frac{a_k}{\binom{n-1}{k-1} \frac{(n-1)!}{(n-k)!}} \leq 1.$$

Proof. The argument below is standard, see e.g. [1, p.73]. For each $\sigma \in \mathcal{C}_n$ and each partial permutation (A_i, f_i) in $\mathcal{F}$, define

$$F(\sigma, (A_i, f_i)) = \begin{cases} \frac{1}{|A_i|}, & \text{if } (A_i, f_i) \prec \sigma; \\ 0, & \text{otherwise.} \end{cases}$$

We count $\sum_{i,\sigma} F(\sigma, (A_i, f_i))$ in two different ways. First we have

$$\sum_{i,\sigma} F(\sigma, (A_i, f_i)) = \sum_{\sigma} \sum_{(A_i, f_i) \prec \sigma} \frac{1}{|A_i|}.$$

Consider the inner sum where σ is fixed. Choose (A_j, f_j) from (A_i, f_i)'s compatible with σ such that $\rho(A_j, f_j)$ is the smallest of the $\rho(A_i, f_i)$. Clearly, there are at most $|A_j|$ of the intervals of σ may intersect pairwise, i.e. at most $|A_j|$ terms in the inner sum, each $\leq \frac{1}{|A_j|}$. Therefore the inner sum is at most $|A_j| \cdot \frac{1}{|A_j|} = 1$, and we have

$$\sum_{i,\sigma} F(\sigma, (A_i, f_i)) \leq \sum_{\sigma} 1 = n!^2. \tag{3}$$

On the other hand, we have

$$\sum_{i,\sigma} F(\sigma, (A_i, f_i)) = \sum_{i} \frac{1}{|A_i|} n^2 |A_i|!(n - |A_i|)!^2 = \sum_{k=1}^{n-1} a_k n^2 (k-1)!(n-k)!^2. \tag{4}$$

Comparing (3) and (4), we obtain the desired inequality. □

From Theorem 2.2 it follows immediately that $|\mathcal{F}| \leq \binom{n-1}{k-1}\frac{(n-1)!}{(n-k)!}$ if $\mathcal{F}$ is a k-uniform intersecting family. The theorem below confirms Conjecture 1.4.

Theorem 2.3 *Fix k, n with $k \leq n-1$. Suppose that $\mathcal{F}$ is a k-uniform intersecting family in $\mathcal{B}(\mathfrak{s}_n)$ with $|\mathcal{F}| = \binom{n-1}{k-1}\frac{(n-1)!}{(n-k)!}$. Then $\mathcal{F} = S_k(\alpha)$ for some atom $\alpha \in \mathcal{B}_1(\mathfrak{s}_n)$.*

Proof. From a key observation in the well-known argument of Katona [14] we know that given a $\sigma \in \mathcal{C}_n$, there are at most k of the k-intervals of it may intersect pairwise, since $2k < n^2$. Suppose $|\mathcal{F}| = \binom{n-1}{k-1}\frac{(n-1)!}{(n-k)!}$. Then each $\sigma \in \mathcal{C}_n$ must contain exactly k members of $\mathcal{F}$, and since the corresponding k-intervals must intersect pairwise, all these intervals must contain a fixed element of $[n] \times [n]$. We shall denote this fixed element (depending on $\mathcal{F}$) by $(x^{(\sigma)}, \varepsilon^{(\sigma)})$, and call each k-interval containing $(x^{(\sigma)}, \varepsilon^{(\sigma)})$ in σ an $\mathcal{F}$-interval, which corresponds to an element of $\mathcal{F}$.

Consider the standard ordering τ, and assume without loss of generality that $(x^{(\tau)}, \varepsilon^{(\tau)}) = (n, n)$. Then, in τ, the $(2k-1)$-interval $[(n-k+1, n-k+1), \ldots, (n, n), (1, 2), (2, 3), \ldots, (k-1, k)]$ contains k $\mathcal{F}$-intervals.

Let $\mathcal{C}'_n$ denote the set of good cyclic orderings $\tau_{\pi\pi'}$'s with $\pi(n) = n$ and $\pi'(n) = n$. We claim that $(x^{(\tau_{\pi\pi'})}, \varepsilon^{(\tau_{\pi\pi'})}) = (n, n)$ for any $\tau_{\pi\pi'} \in \mathcal{C}'_n$.

We first prove $(x^{(\tau_{\pi\pi})}, \varepsilon^{(\tau_{\pi\pi})}) = (n, n)$. Set $I = \{(i, i) : 1 \leq i \leq n-1\}$ and $\bar{I} = [n] \times [n] \setminus (I \cup \{(n, n)\})$. Then $(\pi \times \pi)(I) = \{(\pi(i), \pi(i)) : 1 \leq i \leq n-1\} = I$ and $(\pi \times \pi)(\bar{I}) = \bar{I}$. Suppose $(x^{(\tau_{\pi\pi})}, \varepsilon^{(\tau_{\pi\pi})}) \neq (n, n)$. Then $(x^{(\tau_{\pi\pi})}, \varepsilon^{(\tau_{\pi\pi})}) \in I$ or $(x^{(\tau_{\pi\pi})}, \varepsilon^{(\tau_{\pi\pi})}) \in \bar{I}$. If the former, then $\tau_{\pi\pi}$ has an $\mathcal{F}$-interval contained in I, which is clearly disjoint with the $\mathcal{F}$-interval $[(n, n), (1, 2), \ldots, (k-1, k)]$; if the latter, then $\tau_{\pi\pi}$ has an $\mathcal{F}$-interval contained in $\bar{I}$, which is clearly disjoint with the $\mathcal{F}$-interval $[(n-k+1, n-k+1), \ldots, (n, n)]$. It yields contradictions in both cases.

Suppose now $(x^{(\tau_{\pi\pi'})}, \varepsilon^{(\tau_{\pi\pi'})}) \neq (n, n)$ for some $\tau_{\pi\pi'} \in \mathcal{C}'_n$ with $\pi \neq \pi'$. Then $\tau_{\pi\pi'}$ has an $\mathcal{F}$-interval, written as J, which contains no (n, n). From the above discussion we see that $J \not\subset I$ and $J \not\subset \bar{I}$. Set $I \cap J = \{(a_1, a_1), \ldots, (a_r, a_r)\}$ where $1 \leq r < k$. Define a permutation π by $\pi^{-1}(i) = a_i$ for $i \in [n]$ with $a_n = n$. Then $\tau_{\pi\pi} \in \mathcal{C}'_n$, and $\tau_{\pi\pi}$ has an $\mathcal{F}$-interval which is contained in the $(n-1)$-interval $[(a_{r+1}, a_{r+1}), \ldots, (n, n), (a_1, a_2), \ldots, (a_{r-1}, a_r)]$. It is clear that J is disjoint with this $(n-1)$-interval. It yields a contradiction again.

Therefore, we have $(x^{(\tau_{\pi\pi'})}, \varepsilon^{(\tau_{\pi\pi'})}) = (x^{(\tau)}, \varepsilon^{(\tau)}) = (n, n)$ for any $\tau_{\pi\pi'} \in \mathcal{C}'_n$. However, from Lemma 2.1 we know that if (A, f) is any selected k-partial permutation with $n \in A$ and $f(n) = n$, then there are $k!(n-k)!^2$ pairs (π, π')'s such that $\tau_{\pi\pi'} \in \mathcal{C}'_n$ and $(A, f) \prec \tau_{\pi\pi'}$. It follows that $\mathcal{F}$ consists of all k-partial permutations (A, f) with $n \in A$ and $f(n) = n$, as required. □

3 Direct product of colorings

Let $\mathfrak{p}_n$ and $\mathfrak{p}'_n$ be two sets of colorings. As two sets we consider their direct product $\mathfrak{p}_n \times \mathfrak{p}'_n$, whose element (f, g) is regarded as a function on $[n]$. We thus get a new set of colorings from the old ones, and write $\mathcal{B}(\mathfrak{p}_n \times \mathfrak{p}'_n) = \{(A, f, g) : A \subseteq [n], f \in \mathfrak{p}_n, g \in \mathfrak{p}'_n\}$. From definition it is easy to see that $\mathcal{B}(\mathfrak{p}_n \times \mathfrak{p}'_n)$ and $\mathcal{B}(\mathfrak{p}'_n \times \mathfrak{p}_n)$ are isomorphic; $\mathfrak{p}_n \times \mathfrak{p}'_n$ is regular if both $\mathfrak{p}_n$ and $\mathfrak{p}'_n$ are regular, and $[\mathfrak{p}_n \times \mathfrak{p}'_n]_k = [\mathfrak{p}_n]_k [\mathfrak{p}'_n]_k$ for $1 \le k \le n$. More generally, we may consider the product $\mathfrak{p}_n^{(1)} \times \cdots \times \mathfrak{p}_n^{(m)}$ and write an element of $\mathcal{B}(\mathfrak{p}_n^{(1)} \times \cdots \times \mathfrak{p}_n^{(m)})$ as $(A, f_1, \ldots, f_m)$ where $A \subseteq [n]$ and $f_i \in \mathfrak{p}_n^{(i)}$ for $i = 1, \ldots, m$.

We may reformulate $(A, f_1, \ldots, f_m)$ as a matrix $[\alpha_1, \ldots, \alpha_n]$, where $\alpha_i = (a_{1i}, \ldots, a_{mi})^T$ is a column vector defined by

$$a_{ji} = \begin{cases} f_j(i) & \text{if } i \in A, \\ 0 & \text{if } i \notin A, \end{cases} \quad \text{for } j = 1, 2, \ldots, m.$$

The rank of $[\alpha_1, \ldots, \alpha_n]$ is given by the number of nonzero α_i's. Let $M(\mathfrak{p}_n^{(1)} \times \cdots \times \mathfrak{p}_n^{(m)})$ denote the set of all such matrices. An order relation on $M(\mathfrak{p}_n^{(1)} \times \cdots \times \mathfrak{p}_n^{(m)})$ is defined by

$$[\alpha_1, \ldots, \alpha_n] \le [\beta_1, \ldots, \beta_n] \text{ iff } \alpha_i = 0 \text{ (vector) or } \alpha_i = \beta_i \text{ for } i = 1, 2, \ldots, n.$$

Then, as posets, $M(\mathfrak{p}_n^{(1)} \times \cdots \times \mathfrak{p}_n^{(m)})$ is isomorphic to $\mathcal{B}(\mathfrak{p}_n^{(1)} \times \cdots \times \mathfrak{p}_n^{(m)})$, so they both can be regarded as generalizations of the function lattice (see [2] and [12]).

Theorem 3.1 *Let $\mathfrak{p}_n$ and $\mathfrak{p}'_n$ be two sets of regular colorings, and let k be a positive integer with $1 \le k \le n$.*

(i) *If both $\mathcal{B}(\mathfrak{p}_n)$ and $\mathcal{B}(\mathfrak{p}'_n)$ have the EKR property for rank k and one of them has the local EKR property for rank k, then $\mathcal{B}(\mathfrak{p}_n \times \mathfrak{p}'_n)$ also has the EKR property for rank k;*

(ii) *If both $\mathcal{B}(\mathfrak{p}_n)$ and $\mathcal{B}(\mathfrak{p}'_n)$ have the uniqueness property for rank k, then $\mathcal{B}(\mathfrak{p}_n \times \mathfrak{p}'_n)$ also has the uniqueness property for rank k;*

(iii) *If $\mathcal{B}(\mathfrak{p}_n)$ satisfies an LYM-type inequality for rank k, and $\mathcal{B}(\mathfrak{p}'_n)$ has the local EKR properties for ranks from 1 to k, then $\mathcal{B}(\mathfrak{p}_n \times \mathfrak{p}'_n)$ satisfies an LYM-type inequality for rank k.*

Proof. (i) Let $\mathcal{F}$ be a k-uniform intersecting family in $\mathcal{B}(\mathfrak{p}_n \times \mathfrak{p}'_n)$. Put

$$\mathcal{F}_1 = \{(A, f) : \text{there is a } g \in \mathfrak{p}'_n \text{ such that } (A, f, g) \in \mathcal{F}\} \tag{5}$$

and

$$\mathcal{F}_2 = \{(A, g) : \text{there is a } f \in \mathfrak{p}_n \text{ such that } (A, f, g) \in \mathcal{F}\}. \tag{6}$$

Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are k-uniform intersecting families in $\mathcal{B}(\mathfrak{p}_n)$ and $\mathcal{B}(\mathfrak{p}'_n)$, respectively, yielding $|\mathcal{F}_1| \leq \binom{n-1}{k-1}[\mathfrak{p}_{n-1}]_{k-1}$ and $|\mathcal{F}_2| \leq \binom{n-1}{k-1}[\mathfrak{p}'_{n-1}]_{k-1}$. Now, suppose that $\mathcal{B}(\mathfrak{p}'_n)$ has the local EKR property for rank k. Then, for each $(A, f) \in \mathcal{F}_1$, there are at most $[\mathfrak{p}'_{n-1}]_{k-1}$ many $g \in \mathfrak{p}'_n$ such that $(A, f, g) \in \mathcal{F}$, which implies

$$|\mathcal{F}| \leq \binom{n-1}{k-1}[\mathfrak{p}_{n-1}]_{k-1}[\mathfrak{p}'_{n-1}]_{k-1} = \binom{n-1}{k-1}[\mathfrak{p}_{n-1} \times \mathfrak{p}'_{n-1}]_{k-1}, \tag{7}$$

as desired.

(ii) Suppose that $\mathcal{F}$ is a maximum k-uniform intersecting family in $\mathcal{B}(\mathfrak{p}_n \times \mathfrak{p}'_n)$, that is, equality in (7) holds. This implies that $|\mathcal{F}_1| = \binom{n-1}{k-1}[\mathfrak{p}_{n-1}]_{k-1}$ and $|\mathcal{F}_2| = \binom{n-1}{k-1}[\mathfrak{p}'_{n-1}]_{k-1}$, so $\mathcal{F}_i$ is a star, written as $S_k(\alpha_i)$, where $i = 1, 2$. Put $\alpha_1 = (\{x_0\}, f_0) \in \mathcal{B}_1(\mathfrak{p}_n)$ and $\alpha_2 = (\{y_0\}, g_0) \in \mathcal{B}_1(\mathfrak{p}'_n)$. A careful analysis of the situation shows that $x_0 = y_0$ and $\mathcal{F} = S_k(\alpha)$ where $\alpha = (\{x_0\}, f_0, g_0)$, as desired.

(iii) Let $\mathcal{F}$ be an intersecting antichain in $\mathcal{B}(\mathfrak{p}_n \times \mathfrak{p}'_n)$ with profile $(a_1, a_2, \ldots, a_k)$, let $\mathcal{F}_1$ be as defined in (5) with profile $(b_1, b_2, \ldots, b_k)$, and let $\mathcal{F}_2$ be as defined in (6). Since $\mathcal{B}(\mathfrak{p}'_n)$ has the local EKR property from rank 1 to rank k, we have that $a_i \leq b_i[\mathfrak{p}'_{n-1}]_{i-1}$ for $i = 1, 2, \ldots, k$, so

$$\begin{aligned}\sum_{i=1}^{k} \frac{a_i}{\binom{n-1}{i-1}[\mathfrak{p}_{n-1} \times \mathfrak{p}'_{n-1}]_{i-1}} &\leq \sum_{i=1}^{k} \frac{b_i[\mathfrak{p}'_{n-1}]_{i-1}}{\binom{n-1}{i-1}[\mathfrak{p}_{n-1}]_{i-1}[\mathfrak{p}'_{n-1}]_{i-1}} \\ &= \sum_{i=1}^{k} \frac{b_i}{\binom{n-1}{i-1}[\mathfrak{p}_{n-1}]_{i-1}} \leq 1,\end{aligned}$$

as desired. □

As an application we consider $\mathcal{B}(\mathfrak{q}_n \times \mathfrak{s}_n)$. We have known that for each $k \leq n-1$, both $\mathcal{B}(\mathfrak{q}_n)$ and $\mathcal{B}(\mathfrak{s}_n)$ have the EKR property for rank k, the uniqueness property for rank k, and satisfies an LYM-type inequality for rank k, $\mathcal{B}(\mathfrak{q}_n)$ also has the local EKR property for rank k. From Theorem 3.1 we immediately obtain the following

Corollary 3.2 *Let $\mathcal{F}$ be an intersecting antichain in $\mathcal{B}(\mathfrak{q}_n \times \mathfrak{s}_n)$ with profile $(a_1, \ldots, a_{n-1})$. Then*

$$\sum_{k=1}^{n-1} \frac{a_k}{\binom{n-1}{k-1}\frac{(n-1)!}{(n-k)!}q^{k-1}} \leq 1.$$

Equality holds if and only if there is a k with $1 \leq k \leq n-1$ such that $\mathcal{F}$ is k-uniform and $\mathcal{F}$ is a k-star.

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