An Identity Generator: Basic Commutators

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Abstract

We introduce a group theoretical tool on which one can derive a family of identities from sequences that are defined by a recursive relation. As an illustration it is shown that

$$\sum_{i=1}^{n-1} F_{n-i}F_i^2 = \frac{1}{2}\sum_{i=1}^{n}(-1)^{n-i}(F_{2i}-F_i) = \binom{F_{n+1}}{2} - \binom{F_n}{2},$$

where $\{F_n\}$ denotes the sequence of Fibonacci numbers.

1 Preliminaries and Introduction

We start our work with recalling some basic facts about the structural properties of words in a free group; cf. [1]. Let F be the free group generated by the set $X = \{x_1, \ldots, x_n\}$. Marshall Hall [1] introduced a family of words in F, which are known as basic commutators and play an essential role. Every basic commutator u has a weight, denoted by $\omega(u)$, which is a natural number. Also, the basic commutators can be ordered generally with respect to their weight.

Definition. (Basic Commutators)

1) $x_1, \ldots, x_n$ are basic commutators of weight 1 and are ordered with respect to each other (here $x_1 < \cdots < x_n$),

2) if the basic commutators of weights less than n are defined, then the basic commutators of weight n are $w = [u, v] = u^{-1}v^{-1}uv$, where

i) u, v are basic commutators and $\omega(u) + \omega(v) = n$,

ii) $u > v$ and if $u = [s, t]$ then $t \leq v$.

If $\omega(u) < n$ then $u < w$. The basic commutators of weight n are ordered arbitrarily with respect to each other.

The following theorem of Marshall Hall plays a basic role in the study of basic commutators. Recall that the commutator subgroups $\gamma_k(G)$ in a group G are defined recursively by $\gamma_1(G) = G$ and

$$\gamma_{i+1}(G) = [\gamma_i(G), G] = \langle [x, g]; x \in \gamma_i(G), g \in G \rangle,$$

for all $i \geq 1$. We refer the reader to [1] for some basic properties of $\gamma_k(G)$.

Theorem 1.1. *(Marshall Hall [1, Theorem 11.2.4]) If F is the free group with free generators $x_1, \ldots, x_n$ and if $c_1, \ldots, c_m$ is the sequence of basic commutators of weights $1, \ldots, k$, then an arbitrary element w of F has a unique representation*

$$w = c_1^{a_1} \cdots c_m^{a_m} \pmod{\gamma_{k+1}(F)},$$

where $a_1, \ldots, a_m$ are integers. Moreover, the basic commutators of weight k form a basis for the free abelian group $\gamma_k(F)/\gamma_{k+1}(F)$.

In this paper, we introduce a general strategy on the discovery of almost number theoretical identities using a word-based combinatorics. As an illustration it is shown that

$$\sum_{i=1}^{n-1} F_{n-i}F_i^2 = \frac{1}{2}\sum_{i=1}^{n}(-1)^{n-i}(F_{2i} - F_i) = \binom{F_{n+1}}{2} - \binom{F_n}{2},$$

where $\{F_n\}$ denotes the sequence of Fibonacci numbers.

2 Main Results

To explain our method, let F be the free group of finite rank generated by X and $\{w_n\}$ be a recursively defined sequence of words in F. Also, let $k \geq 1$ and $c_1, \ldots, c_m$ be the sequence of basic commutators of weights $1, \ldots, k$. Then, by Theorem 1.1, w_n has a unique representation

$$w_n = c_1^{a_{1,n}} \cdots c_m^{a_{m,n}} \pmod{\gamma_{k+1}(F)}, \tag{1}$$

where $a_{1,n}, \ldots, a_{m,n}$ are integers. Since $\{w_n\}$ is recursively defined, we may assume that $w_n = W_n(w_1, \ldots, w_{n-1}, X)$, where W_n is a word on $w_1, \ldots, w_{n-1}$ and elements of X. Suppose that $i \geq 1$ and $a_{j,k}$'s are known for all j such that $\omega(c_j) < \omega(c_i)$ and all $k \geq 1$. Feeding the representation (1) of $w_1, \ldots, w_{n-1}$ in w_n one observes that $a_{i,n}$ can be obtained recursively by $a_{i,1}, \ldots, a_{i,n-1}$, i.e., $\{a_{i,n}\}_{n=1}^{\infty}$ is also a recursive sequence. Now, by solving the recursive sequences $\{w_n\}$ and $\{a_{i,n}\}_{n=1}^{\infty}$, we obtain $a_{i,n}$ in two different forms from which we obtain an identity. An identity which is obtained in this way is called the c_i-identity of $\{w_n\}$. It is evident that different methods in solving the sequences $\{w_n\}$ and $\{a_{i,n}\}_{n=1}^{\infty}$ would give different identities. To be more tangible what it means, in Theorem 2.2 we obtain a $[y, x]$-identity in details.

Throughout this paper, F denotes the free group of rank 2 generated by x and y. In this case, $x < y < [y, x]$ would denotes the basic commutators of weights $1, 1, 2$, respectively. In what follows we use frequently the well-known identities $yx = xy[y,x]$, $[xy,z] = [x,z]^y[x,z]$ and $[x,yz] = [x,z][x,y]^z$, where x, y and z are elements of an arbitrary group G. As a direct consequence of these identities we can prove

Lemma 2.1. *For any group G and elements $x, y \in G$*

i) $y^n x^m = x^m y^n [y,x]^{mn} \pmod{\gamma_3(G)}$;

ii) $(xy)^n = x^n y^n [y,x]^{\binom{n}{2}} \pmod{\gamma_3(G)}$.

Now, we explain the first example in details. Let $w_1 = x^a y^c$, $w_2 = x^b y^d$ and $w_{n+2} = w_n^u w_{n+1}^v$, where a, b, c, d, u, v are integers and $n \geq 0$. Also, let $\bar{F} = F/\gamma_3(F)$ and $\bar{w} = w\gamma_3(F)$, for each $w \in F$. Then, by Theorem 1.1, there are unique integers a_n, b_n and c_n such that

$$\bar{w}_n = \bar{x}^{a_n} \bar{y}^{b_n} [\bar{y}, \bar{x}]^{c_n},$$

for all $n \geq 1$.

To obtain the $[y,x]$-identity of $\{w_n\}$ we need some more notations. To do this, let $\{L_n\}, \{L'_n\}$ be the sequences recursively defined by the rules $L_{n+2} = uL_n + vL_{n+1}$ and $L'_{n+2} = uL'_n + vL'_{n+1}$, where $L_0 = 0$, $L_1 = u$, $L'_0 = 1$, $L'_1 = v$ and $n \geq 0$. Moreover, Let $\{G_n\}, \{G'_n\}$ be sequences recursively defined by $G_{n+2} = uG_n + vG_{n+1}$ and $G'_{n+2} = uG'_n + vG'_{n+1}$, where $G_1 = a$, $G_2 = b$, $G'_1 = c$, $G'_2 = d$ and $n \geq 1$.

Utilising the notations above, we have

Theorem 2.2.

$$\sum_{i=1}^{n} L'_{n-i} \left[\binom{u}{2} G_i G'_i + \binom{v}{2} G_{i+1} G'_{i+1} + uv G_{i+1} G'_i \right] \tag{2}$$

$$= u \sum_{i=1}^{n} (-u)^{n-i} \left[L_{i-1} L'_{i-1} + v \binom{L'_{i-1}}{2} \right] \begin{vmatrix} a & b \\ c & d \end{vmatrix} + ac \binom{L_n}{2} + bd \binom{L'_n}{2} + bc L_n L'_n,$$

for all $n \geq 1$

To prove Theorem 2.2, we need the following lemmas.

Lemma 2.3. *If $n \geq 0$, then $L_{n+1} = uL'_n$ and $L'_{n+1} = L_n + vL'_n$.*

Proof. By definition $L_1 = u = uL'_0$, $L_2 = uv = uL'_1$, $L'_1 = v = L_0 + vL'_0$ and $L'_2 = u + v^2 = L_1 + vL'_1$. Now, if $n > 1$ and the result hold for $n-2$ and $n-1$, then

$$L_{n+2} = uL_n + vL_{n+1} = u(uL'_{n-1} + vL'_n) = uL'_{n+1},$$
$$L'_{n+2} = uL'_n + vL'_{n+1} = L_{n+1} + vL'_{n+1},$$

as required. □

Lemma 2.4. *Let k and n be nonnegative integers. Then*

i) $\bar{w}_n^k = \bar{x}^{ka_n} \bar{y}^{kb_n} [\bar{y}, \bar{x}]^{kc_n + \binom{k}{2} a_n b_n}$;

ii) $[\bar{w}_{n+1}, \bar{w}_n] = [\bar{y}, \bar{x}]^{(-u)^{n-1}(ad-bc)}$.

Proof. i) It is obvious by Lemma 2.1(ii).

ii) If $n = 1$, then $[\bar{w}_{n+1}, \bar{w}_n] = [\bar{w}_2, \bar{w}_1] = [\bar{x}^b\bar{y}^d, \bar{x}^a\bar{y}^c] = [\bar{y}, \bar{x}]^{ad-bc}$. Now, if $n > 1$, then

$$[\bar{w}_{n+1}, \bar{w}_n] = [\bar{w}_{n-1}^u \bar{w}_n^v, \bar{w}_n] = [\bar{w}_n, \bar{w}_{n-1}]^{-u}$$

and the result follows inductively. □

Proof of Theorem 2.2. To prove identity (2), we calculate c_{n+2} in two different ways.

1) First, we count c_{n+2} directly by solving $\{c_n\}$. If $n \geq 1$, then by Lemmas 2.1(i) and 2.4(i)

$$\begin{aligned}
\bar{w}_{n+2} &= \bar{w}_n^u \bar{w}_{n+1}^v \\
&= \bar{x}^{ua_n}\bar{y}^{ub_n}[\bar{y},\bar{x}]^{uc_n+\binom{u}{2}a_nb_n}\bar{x}^{va_{n+1}}\bar{y}^{vb_{n+1}}[\bar{y},\bar{x}]^{vc_{n+1}+\binom{v}{2}a_{n+1}b_{n+1}} \\
&= \bar{x}^{ua_n}\bar{y}^{ub_n}\bar{x}^{va_{n+1}}\bar{y}^{vb_{n+1}}[\bar{y},\bar{x}]^{uc_n+vc_{n+1}+\binom{u}{2}a_nb_n+\binom{v}{2}a_{n+1}b_{n+1}} \\
&= \bar{x}^{ua_n}\bar{x}^{va_{n+1}}\bar{y}^{ub_n}[\bar{y},\bar{x}]^{uva_{n+1}b_n}\bar{y}^{vb_{n+1}}[\bar{y},\bar{x}]^{uc_n+vc_{n+1}+\binom{u}{2}a_nb_n+\binom{v}{2}a_{n+1}b_{n+1}} \\
&= \bar{x}^{ua_n+va_{n+1}}\bar{y}^{ub_n+vb_{n+1}}[\bar{y},\bar{x}]^{uc_n+vc_{n+1}+\binom{u}{2}a_nb_n+\binom{v}{2}a_{n+1}b_{n+1}+uva_{n+1}b_n}.
\end{aligned}$$

Hence

$$\begin{aligned}
a_{n+2} &= ua_n + va_{n+1}, \\
b_{n+2} &= ub_n + vb_{n+1}, \\
c_{n+2} &= uc_n + vc_{n+1} + \binom{u}{2}a_nb_n + \binom{v}{2}a_{n+1}b_{n+1} + uva_{n+1}b_n.
\end{aligned}$$

It follows from the definitions of $\{a_k\}, \{b_k\}$ and $\{G_k\}, \{G'_k\}$ that $a_k = G_k$ and $b_k = G'_k$, for all $k \geq 1$. Let $d_{k+2} = \binom{u}{2}a_kb_k + \binom{v}{2}a_{k+1}b_{k+1} + uva_{k+1}b_k$, for all $k \geq 1$. Then $c_{n+2} = uc_n + vc_{n+1} + d_{n+2} = L_1c_n + L'_1c_{n+1} + L'_0d_{n+2}$. Now, suppose that $1 \leq k < n$ and

$$c_{n+2} = L_kc_{n-k+1} + L'_kc_{n-k+2} + L'_{k-1}d_{n-k+3} + \cdots + L'_0d_{n+2}.$$

Then

$$\begin{aligned}
c_{n+2} &= L_kc_{n-k+1} + L'_kc_{n-k+2} + L'_{k-1}d_{n-k+3} + \cdots + L'_0d_{n+2} \\
&= L_kc_{n-k+1} + L'_k(uc_{n-k} + vc_{n-k+1} + d_{n-k+2}) + L'_{k-1}d_{n-k+3} + \cdots + L'_0d_{n+2} \\
&= L_{k+1}c_{n-k} + L'_{k+1}c_{n-k+1} + L'_kd_{n-k+2} + \cdots + L'_0d_{n+2}
\end{aligned}$$

and so by induction we obtain

$$\begin{aligned}
c_{n+2} &= L_nc_1 + L'_nc_2 + L'_{n-1}d_3 + \cdots + L'_0d_{n+2} \\
&= L'_{n-1}d_3 + \cdots + L'_0d_{n+2} = \sum_{i=1}^{n} L'_{n-i}d_{i+2},
\end{aligned}$$

as $c_1 = c_2 = 0$. Therefore

$$c_{n+2} = \sum_{i=1}^{n} L'_{n-i}\left[\binom{u}{2}G_iG'_i + \binom{v}{2}G_{i+1}G'_{i+1} + uvG_{i+1}G'_i\right]. \qquad (3)$$

2) Now, we count c_{n+2} in a different way by solving $\{w_n\}$. Put

$$\alpha_i = (-u)^{n-i}\left[uL_{i-1}L'_{i-1} + uv\binom{L'_{i-1}}{2}\right]\begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

for $i = 1, \ldots, n$. Clearly $\alpha_1 = 0$ and so $\bar{w}_{n+2} = \bar{w}_n^u \bar{w}_{n+1}^v = \bar{w}_n^{L_1}\bar{w}_{n+1}^{L'_1}[\bar{y}, \bar{x}]^{\alpha_1}$. We will show that for $i = 1, \ldots, n$,

$$\bar{w}_{n+2} = \bar{w}_{n-i+1}^{L_i}\bar{w}_{n-i+2}^{L'_i}[\bar{y}, \bar{x}]^{\alpha_1+\cdots+\alpha_i}. \tag{4}$$

If (4) holds for i, then using Lemmas 2.1(i,ii) and 2.4(ii)

$$\begin{aligned}
\bar{w}_{n+2} &= \bar{w}_{n-i+1}^{L_i}\bar{w}_{n-i+2}^{L'_i}[\bar{y}, \bar{x}]^{\alpha_1+\cdots+\alpha_i} \\
&= \bar{w}_{n-i+1}^{L_i}(\bar{w}_{n-i}^u\bar{w}_{n-i+1}^v)^{L'_i}[\bar{y}, \bar{x}]^{\alpha_1+\cdots+\alpha_i} \\
&= \bar{w}_{n-i+1}^{L_i}\bar{w}_{n-i}^{uL'_i}\bar{w}_{n-i+1}^{vL'_i}[\bar{w}_{n-i+1}^v, \bar{w}_{n-i}^u]^{\binom{L'_i}{2}}[\bar{y}, \bar{x}]^{\alpha_1+\cdots+\alpha_i} \\
&= \bar{w}_{n-i+1}^{L_i}\bar{w}_{n-i}^{uL'_i}\bar{w}_{n-i+1}^{vL'_i}[\bar{y}, \bar{x}]^{\alpha_1+\cdots+\alpha_i+(-u)^{n-i-1}uv\binom{L'_i}{2}(ad-bc)} \\
&= \bar{w}_{n-i}^{uL'_i}\bar{w}_{n-i+1}^{L_i}[\bar{w}_{n-i+1}^{L_i}, \bar{w}_{n-i}^{uL'_i}]\bar{w}_{n-i+1}^{vL'_i}[\bar{y}, \bar{x}]^{\alpha_1+\cdots+\alpha_i+(-u)^{n-(i+1)}uv\binom{L'_i}{2}(ad-bc)} \\
&= \bar{w}_{n-i}^{uL'_i}\bar{w}_{n-i+1}^{L_i+vL'_i}[\bar{y}, \bar{x}]^{\alpha_1+\cdots+\alpha_i+(-u)^{n-(i+1)}\left(uL_iL'_i+uv\binom{L'_i}{2}\right)(ad-bc)} \\
&= \bar{w}_{n-i}^{L_{i+1}}\bar{w}_{n-i+1}^{L'_{i+1}}[\bar{y}, \bar{x}]^{\alpha_1+\cdots+\alpha_{i+1}}.
\end{aligned}$$

By replacing i by n in (4) and using Lemma 2.1(i,ii), we get

$$\begin{aligned}
\bar{w}_{n+2} &= \bar{w}_1^{L_n}\bar{w}_2^{L'_n}[\bar{y}, \bar{x}]^{\alpha_1+\cdots+\alpha_n} \\
&= (x^a y^c)^{L_n}(x^b y^d)^{L'_n}[\bar{y}, \bar{x}]^{\alpha_1+\cdots+\alpha_n} \\
&= x^{aL_n}y^{cL_n}x^{bL'_n}y^{dL'_n}[\bar{y}, \bar{x}]^{\alpha_1+\cdots+\alpha_n+ac\binom{L_n}{2}+bd\binom{L'_n}{2}} \\
&= x^{aL_n+bL'_n}y^{cL_n+dL'_n}[\bar{y}, \bar{x}]^{\alpha_1+\cdots+\alpha_n+ac\binom{L_n}{2}+bd\binom{L'_n}{2}+bcL_nL'_n}.
\end{aligned}$$

Therefore

$$c_{n+2} = \alpha_1 + \cdots + \alpha_n + ac\binom{L_n}{2} + bd\binom{L'_n}{2} + bcL_nL'_n. \tag{5}$$

Now, the equations (3) and (5) imply the identity (2), which is the $[y, x]$-identity of $\{w_n\}$. □

Corollary 2.5. *For any $n > 0$*

$$\sum_{i=1}^{n-1} F_{n-i}F_i^2 = \frac{1}{2}\sum_{i=1}^{n}(-1)^{n-i}(F_{2i} - F_i). \tag{6}$$

Proof. By putting $u = v = a = d = 1$ and $b = c = 0$ in identity (2), we get $L_n = F_n$, $L'_n = F_{n+1}$, $G_n = F_{n-2}$, $G'_n = F_{n-1}$ and so

$$\sum_{i=1}^{n} F_{n+1-i}F_{i-1}^2 = \sum_{i=1}^{n}(-1)^{n-i}\left(F_{i-1}F_i + \binom{F_i}{2}\right).$$

Now, $\sum_{i=1}^{n} F_{n+1-i}F_{i-1}^2 = \sum_{i=1}^{n-1} F_{n-i}F_i^2$ and $F_{i-1}F_i + \binom{F_i}{2} = \frac{1}{2}(F_{2i} - F_i)$, which completes the proof. $\square$

Corollary 2.6. *For any* $n > 0$

$$\sum_{i=1}^{n-1} F_{n-i}F_iF_{i+1} = \binom{F_{n+1}}{2}. \tag{7}$$

Proof. Put $u = v = a = b = d = 1$ and $c = 0$ in identity (2). $\square$

Corollary 2.7. *For any* $n > 0$

$$\sum_{i=1}^{n-1} F_{n-i}F_i^2 = \binom{F_{n+1}}{2} - \binom{F_n}{2}. \tag{8}$$

Proof. By Corollary 2.6, we have

$$\begin{aligned}
\sum_{i=1}^{n-1} F_{n-i}F_i^2 &= \sum_{i=1}^{n-1} F_{n-i}F_i(F_{i+1} - F_{i-1}) \\
&= \sum_{i=1}^{n-1} F_{n-i}F_iF_{i+1} - \sum_{i=1}^{n-1} F_{n-i}F_iF_{i-1} \\
&= \sum_{i=1}^{n-1} F_{n-i}F_iF_{i+1} - \sum_{i=1}^{n-2} F_{n-1-i}F_iF_{i+1} \\
&= \binom{F_{n+1}}{2} - \binom{F_n}{2}.
\end{aligned}$$

$\square$

Similar to Corollary 2.7, one we can prove the following result.

Corollary 2.8. *For any* $n > 0$

$$\sum_{i=1}^{n-1} F_{n-i}F_{2i} = \binom{F_n}{2} + \binom{F_{n+1}}{2}. \tag{9}$$

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References

[1] M. Hall, *The Theory of Groups*, Macmillan, New York, 1955.