New infinite families of almost-planar crossing-critical graphs

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Abstract

We show that, for all choices of integers $k > 2$ and m, there are simple 3-connected k-crossing-critical graphs containing more than m vertices of each even degree $\leq 2k-2$. This construction answers one half of a question raised by Bokal, while the other half asking analogously about vertices of odd degrees at least 7 in crossing-critical graphs remains open. Furthermore, our newly constructed graphs have several other interesting properties; for instance, they are almost planar and their average degree can attain any rational value in the interval $\left[3+\frac{1}{5}, 6-\frac{8}{k+1}\right)$.

Keywords: crossing number, graph drawing, crossing-critical graph.

1 Introduction

We assume that the reader is familiar with basic terms of graph theory. In a *drawing* of a graph G the vertices of G are points and the edges are simple curves joining their endvertices. Moreover, it is required that no edge passes through a vertex (except at its ends), and no three edges cross in a common point. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of crossing points of edges in a drawing of G in the plane.

For $k \geq 1$, we say that a graph G is *k-crossing-critical* if $\mathrm{cr}(G) \geq k$ but $\mathrm{cr}(G-e) < k$ for each edge $e \in E(G)$. It is important to study crossing-critical graphs in order to understand structural properties of the crossing number problem. The only 1-crossing-critical graphs are, by the Kuratowski theorem, subdivisions of K_5 and $K_{3,3}$. The first construction of an infinite family of 2-crossing-critical simple 3-connected graphs was by

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Kochol [8] (Figure 8), improving previous construction by Širáň [12]. Many more crossing-critical constructions have appeared since.

It has been noted by D. Bokal (personal communication and preprint of [2]) that typical constructions of infinite families of simple 3-connected k-crossing-critical graphs create bounded numbers (wrt. k) of vertices of degrees other than $3, 4, 5$, or 6. Actually, the existence of such 2-crossing-critical families with many degree-5 vertices has been established by Bokal only recently. Bokal's natural question thus was, what about occurrence of other vertex degree values in infinite families of k-crossing-critical graphs? We positively answer one half of his question in Theorem 3.1 and Proposition 2.1;

- namely we construct, for all $k > 2$, infinite families of simple 3-connected almost-planar k-crossing-critical graphs which contain arbitrary numbers of vertices of each *even degree* $4, 6, 8, \ldots, 2k - 2$.

The analogous question about occurrence of vertices of odd degrees ≥ 7 in k-crossing-critical graphs remains open, and it appears to be significantly harder than the even case. One should also note that a (still open) question about the existence of an infinite family of simple 5-regular crossing-critical graphs was raised long before by Richter and Thomassen [9].

Usual constructions of crossing-critical graphs use an approach that can be described as a "Möbius twist"—they create graphs embeddable on a Möbius band which thus have to be twisted for drawing in the plane. We offer a quite different approach in Section 2, which extends our older construction [4], resulting in graphs that are *almost-planar* (sometimes called "near planar"), i.e. they can be made planar by deleting just one edge. As an easy corollary of this new and very flexible construction;

- we also produce almost-planar crossing-critical families with any prescribed average degree from $\left[3 + \frac{1}{5},\, 6 - \frac{8}{k+1}\right)$,

see in Theorem 4.1 and Corollaries 4.2, 4.3.

2 "Belt" constructions

An illustrating example of crossing-critical graphs constructed in our older work [4] is shown in Figure 1. The construction in [4] used vertices of degrees 4 or 3, and now we generalize it to allow more flexible structure and, particularly, vertices of arbitrary even degrees.

For easier notation, we (in the coming definitions) consider embeddings in the plane $\mathcal{P}$ with removed open disc $\mathcal{X}$. We say that a closed curve (loop) γ is of *type-*$\mathcal{X}$ if the homotopy type of γ in $\mathcal{P} \setminus \mathcal{X}$ is to "wind once around $\mathcal{X}$". Having two loops γ, δ of type-$\mathcal{X}$, we write $\gamma \preceq \delta$ if γ separates $\mathcal{X}$ from $\delta \setminus \gamma$ (meaning γ is "*nested*" inside δ).

Crossed belt graphs. A plane graph F_0 is a *plane k-belt* graph if it can be constructed as a connected edge-disjoint union of k embedded "*belt*" cycles $C_1 \cup C_2 \cup \cdots \cup C_k = F_0$, where all $C_1, \ldots, C_k$ are of type-$\mathcal{X}$ nested as $C_1 \preceq C_2 \preceq \cdots \preceq C_k$.

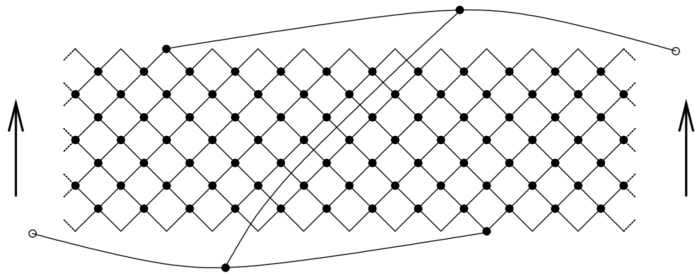

Figure 1: A simple 3-connected almost-planar 8-crossing-critical graph [4]. (The "grid-belt" is wraps around a cylinder without twist.)

A path $R \subseteq F_0$ connecting a vertex p of C_1 to a vertex q of C_k is *radial* if, for each $1 < i \leq k$, R intersects $C_i \cup \cdots \cup C_k$ in a subpath (with one end q). Informally, a radial path of F_0 has to "proceed straight across F_0" from C_1 to C_k. A vertex of F_0 is *accumulation* if its degree is at least 6 in F_0, i.e. if it is contained in at least three of the cycles $C_1, \ldots, C_k$.

Furthermore, a planar k-belt graph is *proper* if there are four distinct vertices $s_1, t_1 \in V(C_1) \setminus V(C_2)$ and $s_2, t_2 \in V(C_k) \setminus V(C_{k-1})$, and the following is true:

(B1) No radial path of F_0 starting in s_1 or t_1 contains an accumulation vertex. In particular, no accumulation vertex exists on the cycle C_k.

(B2) Let $P_2, P_2' \subseteq C_k$ be the two paths with the ends s_2, t_2 on C_k. Then every radial path of F_0 strating in s_1 (in t_1) hits C_k first in an internal vertex of P_2 (of P_2', respectively).

(B3) Let $P_1, P_1' \subseteq C_1$ be analogously the two paths with the ends s_1, t_1 on C_1. There exist collections of k pairwise disjoint radial paths in F_0, all disjoint from s_1, t_1 and all starting on P_1 (on P_1', respectively).

A graph F is a *crossed k-belt* if it is $F = F_0 \cup S_0 \cup S_1 \cup S_2$, where

– F_0 is a proper planar k-belt graph as above;
– S_1 is a path with the ends s_1, t_1 internally disjoint from F_0 and S_2 is a path with the ends s_2, t_2 internally disjoint from $F_0 \cup S_1$; and
– S_0 is a path disjoint from F_0, connecting a vertex of S_1 to one of S_2.

This lengthy definition is illustrated in Figure 2. Notice that a crossed 1-belt graph is always a subdivision of $K_{3,3}$, and that removing an edge of S_0 from a crossed k-belt graph leaves it planar. Particularly, the graph in Figure 1 is a crossed 8-belt graph without accumulation vertices, and we call this special case a "*square-grid*" 8-belt graph. We aim to show that crossed k-belt graphs are k-crossing-critical with the exception of $k = 2$. (This exception is remarkable in view of successful research progress into the structure of 2-crossing-critical graphs.)

For better understanding we first discuss the conditions (B1), (B2) and (B3) imposed on our graphs. (B1) is generally unavoidable, as a nontrivial (counter)example violating

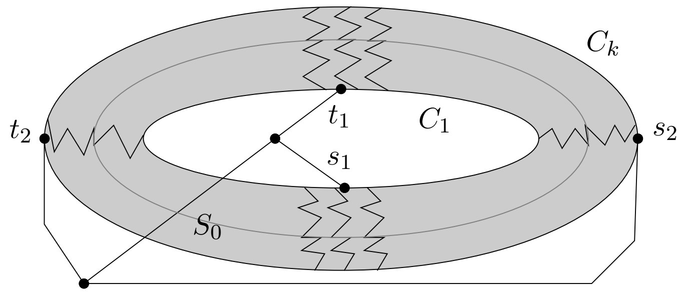

Figure 2: An illustration of the definition of a *crossed k-belt* graph. (The "zig-zag" lines are examples of radial paths as discussed in the definition.)

(B1) in Figure 3 shows. The other two conditions are, on the other hand, necessary mainly due to our inductive proof in the next section. (B2) establishes the base cases $k = 1, 3$ of the induction—violating (B2), one could easily construct planar graphs for $k = 1$ or graphs of crossing number 2 for $k = 3$. Perhaps, (B2) might not be necessary for higher values of k, but without Lemma 3.3 we could hardly start our induction. Finally, (B3) gives a sort of "sufficient interconnection" between the cycles $C_1, \ldots, C_k$ (we obviously cannot allow those to be disjoint), and then (B3) is the key ingredience in the inductive step in Theorem 3.1.

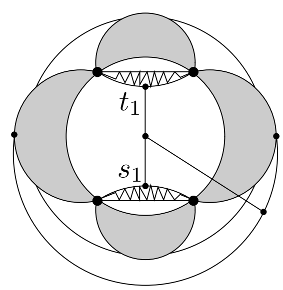

Figure 3: A sketch of a graph similar to crossed k-belt (with four "bad" accumulation vertices) which has crossing number 13 for large values of k.

The cruical property which motivated our construction, and which (in half) answers the aforementioned question of Bokal, is stated now:

Proposition 2.1 *Let $k > 3$ be an integer. For every integer m there is a crossed k-belt graph which is simple 3-connected and which contains more than m vertices of each of degrees $\ell = 4, 6, 8, \ldots, 2k - 2$.*

Proof. In this case a picture is worth more than thousand words. Figure 4 shows local modifications of the square-grid 8-belt graph which produce accumulation vertices of degrees 14 and 12 while preserving its simplicity and connectivity. It is straightforward to generalize this picture to any $k > 3$ and all degrees $\ell = 6, 8, \ldots, 2k - 2$. Starting from a sufficiently large square-grid k-belt graph F, we can produce in this way F' with

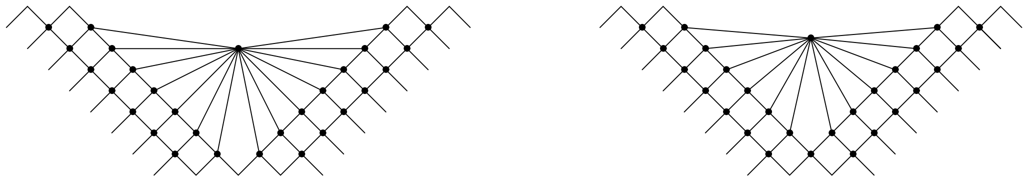

Figure 4: Examples of accumulation vertices.

arbitrarily many accumulation vertices of each degree $\ell = 6, 8, \ldots, 2k-2$, all of which are "sufficiently far" from the vertices s_1, t_1 as in the condition (B1). $\square$

3 Crossing-criticality

We continue to use the notation from the definition of k-belt graphs also in this section. Now we come to the main result of our paper.

Theorem 3.1 *For $k \geq 3$, every crossed k-belt graph is k-crossing-critical.*

Proof. Let F be our k-belt graph, considered with notation as in the definition above. In one direction, by a straightforward induction we argue that any crossed k-belt graph, $k \geq 1$, can be drawn such that the only crossings occur between the path S_0 and each of the belt cycles $C_1, \ldots, C_k$ once. This is trivial for $k = 1$. For $k > 1$, we draw a $(k-1)$-belt subgraph $F' \subset F$ from Lemma 3.2 with $k-1$ crossings between S_0 and each of the belt cycles $C_2, \ldots, C_k$, in a way that one end of S_0 is inside the set $\mathcal{X}$ (see the definition of type-$\mathcal{X}$ in Section 2) and the other end of S_0 is in the face of C_k not with $\mathcal{X}$. By definition the remaining cycle C_1 is nested inside each cycle C_i, $i > 1$, and so to obtain an analogous drawing of (whole) F it is enough to add one more crossing of S_0 with C_1 since C_1 is also of type-$\mathcal{X}$. Furthermore, using analogous arguments, it is easy to verify that deleting any edge e of F allows us to draw $F - e$ with fewer than k crossings.

Conversely, we assume an arbitrary drawing $\mathcal{F}$ of F, and we want to prove that $\mathcal{F}$ has at least k edge crossings. There are two possibilities—either C_1 is drawn uncrossed in $\mathcal{F}$, or some edge of C_1 is crossed in $\mathcal{F}$. In the first case, assuming $k \geq 4$, we will argue that $\mathrm{cr}(\mathcal{F}) \geq k$ straight away.

Let $Q_1, \ldots, Q_k$ and $R_1, \ldots, R_k$ be the collections of disjoint radial paths established in (B3), ordered such that Q_1 and R_1 are the closest ones to s_1. Also using (B3), there exist Q_0 a radial path starting in s_1 and R_0 a radial path starting in t_1, none of Q_0, R_0 intersecting more than one of $Q_1, \ldots, Q_k$ and $R_1, \ldots, R_k$. Then there exist $k-2$ pairwise edge-disjoint paths $T_i \subseteq (Q_i \cup C_{i+2} \cup R_i) - V(R_0)$ for $i = 1, 2, \ldots, k-2$ in F, such that each T_i intersects C_1 in two single vertices (T_i-ends) which separate s_1 from t_1 on C_1. Notice that these T_i need not actually use sections of Q_i or R_i if closer accumulation vertices between C_1 and C_{i+2} exist (still respecting (B1)), but in this particular setting such paths T_i always exist. Their key properties are that $T_1, \ldots, T_{k-2}$ are internally disjoint from C_1, and that all of them intersect $Q_0 - V(C_1)$.

Analogously, we obtain two more such edge-disjoint paths $T_{k-1} \subseteq (Q_{k-1} \cup C_k \cup R_{k-1}) - V(Q_0)$ and $T_k \subseteq (Q_k \cup C_{k-1} \cup R_k) - V(Q_0)$, both intersected by $R_0 - V(C_1)$. Thus all $T_1, \dots, T_k$ belong to the same connected component of $F - V(C_1)$ as $C_k \cup Q_0 \cup R_0$ does, where C_k is disjoint from C_1 by (B1). Furthermore, $S_1 - s_1 - t_1$ also belongs to the component with C_k. So, if C_1 is drawn uncrossed in $\mathcal{F}$, then all S_1 and $T_1, \dots, T_k$ are drawn in the same face of C_1, and hence S_1 has to cross each of the edge-disjoint paths $T_1, \dots, T_k$ by Jordan's curve theorem, witnessing $\mathrm{cr}(\mathcal{F}) \geq k$.

Otherwise, there is an edge f of C_1 which is crossed in $\mathcal{F}$. We apply Lemma 3.2 to F and f, so obtaining a crossed $(k-1)$-belt subgraph F' of $F - f$, and conclude by induction that $\mathrm{cr}(\mathcal{F}) \geq 1 + \mathrm{cr}(F') = 1 + (k-1) = k$ if the claim holds true in the base case $k = 3$. Hence we can finish the proof of the theorem with further Lemma 3.3 which takes care of $k = 3$. □

Lemma 3.2 *Let F be a crossed k-belt graph as above, and choose any $f \in E(C_1)$. Then $F - f$ contains a crossed $(k-1)$-belt subgraph F' having $C_2, \dots, C_k$ as its collection of belt cycles.*

Proof. We refer to the notation in the definition of belt graphs. Let s_1', t_1' denote vertices of $C_1 \cap C_2$ connected across $C_1 - V(C_2) - f$ to s_1, t_1, respectively. Then $s_1', t_1' \notin C_3$ thanks to (B1). Notice that for at least one of s_1', t_1' we have a choice of two possibilities at each "side" of s_1 or t_1, and so we can ensure that not both s_1', t_1' intersect the same one collection of radial paths from (B3).

Let F_0' denote the subgraph of F induced on $V(C_2) \cup \dots \cup V(C_k)$, and let path S_1' be the prolongation of S_1 on $C_1 - f$ with the ends s_1', t_1'. We claim that $F' = F_0' \cup S_1' \cup S_2 \cup S_0$ is a crossed $(k-1)$-belt graph: The properties (B1) and (B2) are easily inherited by F' since radial paths starting in s_1' or t_1' form a subset of those starting in s_1 or t_1. (B3) is then satisfied thanks to our choice of s_1' or t_1' above. □

Lemma 3.3 *Any crossed 3-belt graph is 3-crossing-critical.*

Proof. We adapt some of the ideas of Theorem 3.1 to this special case of $k = 3$. Let $\mathcal{F}$ be again a drawing of F. Say, if both cycle C_1 and C_3 are crossed in $\mathcal{F}$, then this case accounts for two distinct crossings—even if C_1 crossed C_3, these two disjoint cycles would have to cross twice. So let $f \in E(C_1)$ and $f' \in E(C_3)$ be edges of distinct crossings in $\mathcal{F}$. We can now successively apply Lemma 3.2 to F and f, then f'. The result is a 1-belt graph $F'' \supset C_2$ (avoiding the crossings on f, f') which is a subdivision of nonplanar $K_{3,3}$ thanks to (B2), and hence we conclude $\mathrm{cr}(\mathcal{F}) \geq 2 + 1 = 3$ in this case.

The other possible case is that C_1 or C_3 is uncrossed in $\mathcal{F}$. Considering uncrossed C_1, we turn the definition of a 3-belt graph F into a symmetric one by establishing the following properties:

(B1+) There is clearly no accumulation vertex at all in F.

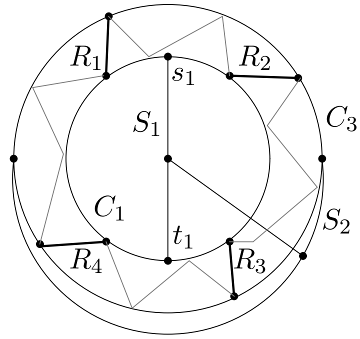

Figure 5: The "core" scheme of a crossed 3-belt graph, cf. (B2+).

(B2+) Let $P_1, P_1' \subseteq C_1$ and $P_2, P_2' \subseteq C_3$ be the paths as in (B2) and (B3) above. There are pairwise disjoint paths $R_1, R_2, R_3, R_4 \subseteq C_2$ connecting internal vertices, in order, of P_1 to P_2, of P_1' to P_2, of P_1' to P_2', and of P_1 to P_2'. This fact follows rather easily from previous (B2) and (B3) when $k = 3$. See in Figure 5.

Analogously to Theorem 3.1, there are paths $T_1 \subseteq R_1 \cup C_3 \cup R_2$ and $T_2 \subseteq R_3 \cup C_3 \cup R_4$ such that the ends of each one T_1 or T_2 separate s_1 from t_1 on C_1. Again, the paths T_1, T_2 must be drawn in the same face of the uncrossed cycle C_1 in $\mathcal{F}$ as the path S_1 is, and hence they account for two crossings on S_1. If, moreover, the cycle C_3 is uncrossed in $\mathcal{F}$, then we get by symmetry another two crossings on S_2, and conclude $\mathrm{cr}(\mathcal{F}) \geq 2 + 2 = 4$. Hence C_3 has got some crossings, and if such a crossing is not with S_1, we are done again as $\mathrm{cr}(\mathcal{F}) \geq 2 + 1 = 3$. So it remains to consider that the only two crossings on C_3 are those with S_1, and then another crossing with S_2 or C_2 must exist on S_1 as well. Thus $\mathrm{cr}(\mathcal{F}) \geq 3$. □

4 Average degrees

Although the main motivation for our k-belt construction of crossing-critical graphs was to answer a part of Bokal's [2, Section 6, preprint] question, the critical graph families we obtain are so rich and flexible that they deserve further consideration and applications.

We look here at one particular question studied in a series of papers [11, 10, 2]: Salazar constructed infinite families of k-crossing-critical graphs with average degree equal to any rational in the interval $[4, 6)$. Then Pinontoan and Richter [10] extended this to the interval $(3.5, 4)$, and finally Bokal [2] has found k-crossing-critical families for any rational average degree in the interval $(3, 6)$. (Average degrees ≤ 3 or > 6 cannot occur for infinite families, and the average degree 6 remains an open case.)

Using our construction and Theorem 3.1, we duplicate Salazar's result in Theorem 4.1 within the restricted subclass of almost-planar crossing-critical graphs, and further extend this in the subsequent corollaries.

Theorem 4.1 *For every odd $k > 3$ there are infinitely many simple 3-connected crossed k-belt graphs with the average degree equal to any given rational value in the interval $\left[4,\, 6 - \frac{8}{k+1}\right)$.*

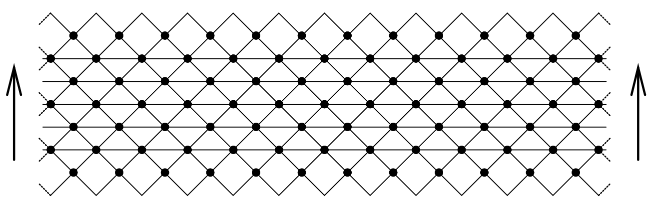

Figure 6: An approach to a plane 13-belt graph with accumulation vertices of degree 6.

Proof. Figure 6 illustrates a construction of a plane graph F_1 that fulfills all conditions of the definition of a plane 13-belt graph except (B1). *Splitting* of a vertex is a simple-graph inverse (not necessarily unique) of the edge-contraction operation. Figure 7 shows details of two "splitting" operations which can be applied to any accumulation vertex of F_1. These both preserve simplicity and 3-connectivity of F_1, and can be used to eventually construct a proper 13-belt graph from F_1.

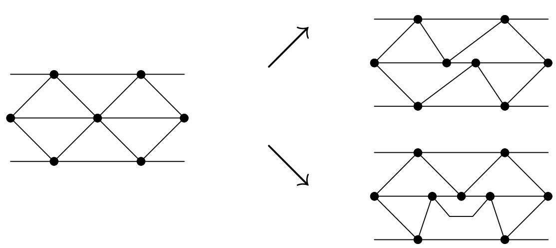

Figure 7: Details of *single-split* (top) and *double-split* (bottom) operations in the graph from Figure 6.

The construction of F_1 from Figure 6 can easily be generalized for any odd $k > 3$. Let ℓ be the length of the C_1-cycle in F_1, and let the number of accumulation vertices from F_1 that are single-split during the construction of F_0 be m and the number of double-split accumulation vertices be m'. Admissible values of m and m' in our construction are at most the total number of accumulation vertices $m+m' \leq \ell(k-3)/2$, and at least $m \geq 4k^2$ since it is enough to single-split $2k^2$ accumulation vertices from F_1 near each of s_1, t_1 to satisfy (B1) of a proper k-belt graph.

An easy calculation shows that F_0 has $\ell(k+1)/2 + m + 2m'$ vertices, and so F has $\ell(k+1)/2 + m + 2m' + 6$ vertices. The average degree of F is

$$d_{avg}(F) = \frac{6k\ell - 2\ell + 4m + 12m' + 36}{k\ell + \ell + 2m + 4m' + 12} = 6 - \frac{8\ell + 8m + 12m' + 36}{k\ell + \ell + 2m + 4m' + 12}\,. \qquad (1)$$

Now choose any rational $d_{avg} \in \left[4,\, 6 - \frac{8}{k+1}\right)$. Then setting $d_{avg} = 6 - \frac{p}{q} = 6 - \frac{cp}{cq}$ in (1) gives a system of two linear equations in two unknowns ℓ, m and parameters k, c, m', which is nonsingular for each $k \neq 1$. Its solution is

$$\ell = \frac{c}{4k-4}(4q-p) - \frac{m'+3}{k-1}\,, \qquad m = \frac{cp}{8} - \frac{12(m'+3)}{8} - \ell\,.$$

The expressions show that choosing our parameters as $m'+3 = 2(k-1)$ and $c = c'\cdot 8(k-1)$ leads always to integer values of ℓ and m as

$$\ell = c'(8q-2p) - 2\,, \qquad m = c'\big((k+1)p - 8q\big) - 3k + 5\,. \tag{2}$$

By the choice $6 - \frac{p}{q} \in \big[4,\, 6 - \frac{8}{k+1}\big)$ it is easy to show in (2) that always $m + m' \leq \ell(k-3)/2 - 3$, and since $(k+1)p - 8q > 0$ it follows that for sufficiently large choices of c' we get also $m \geq 4k^2$. Thus we get from (2) an infinite sequence of admissible pairs ℓ, m (note fixed k and $m' = 2k-5$), defining each one a crossed k-belt graph F with average degree exactly $6 - \frac{p}{q}$ as needed. This holds for any fixed odd $k > 3$. □

Our restriction to odd values of k was just for our comfort. We can easily overcome it using a powerful "zip-product" construction of Bokal [1, 2]. In our *restricted case*; having two simple graphs G_1, G_2 with cubic vertices $u_i \in V(G_i)$ and their neighbors denoted by r_i, s_i, t_i, the *zip product* G of G_1 and G_2, according to the chosen vertices u_1, u_2 and their neighbors, is the disjoint union of $G_1 - u_1$ and $G_2 - u_2$ with added three edges r_1r_2, s_1s_2, t_1t_2. A cubic vertex u_1 in G_1 with the neighbors r_1, s_1, t_1 has *two coherent bundles* if there are two vertices $v, w \in V(G_1 - u_1)$ such that there exist six pairwise edge-disjoint paths, three of them from v and the other three from w to each of r_1, s_1, t_1. We shall use Bokal's [2, Theorem 21];

- if the above graphs G_i, $i = 1, 2$ are k_i-crossing-critical where $\mathrm{cr}(G_i) = k_i$, and u_i have two coherent bundles in G_i, then their zip product G is $(k_1 + k_2)$-crossing-critical.

Corollary 4.2 *For every $k \geq 5$ there are infinitely many simple 3-connected almost-planar $2k$-crossing-critical graphs with the average degree equal to any given rational value in the interval $\big[4,\, 6 - \frac{8}{k+1}\big)$.*

Proof. We take two disjoint copies G_1, G_2 of a graph resulting from Theorem 4.1. It is easy to check that the (unique) cubic vertex v_1 of G_1, which is a neighbor of s_1, t_1 as in Figure 2, has two coherent bundles. (This fact is implicitly contained already in [2, Section 6].) Let f_1 denote the edge of v_1 not incident with s_1, t_1, and let v_2, f_2 be the corresponding elements in G_2. Recall that $G_i - f_i$ is planar. Then the zip product G of G_1 and G_2 at v_1, v_2, matching edges f_1, f_2 into f of G, is $2k$-crossing-critical by [2], and $G - f$ is planar. To achieve the same average degree of the product as that of G_1, we finally double-split one more accumulation vertex in G_1. □

Furthermore, we can lower the average degree of almost-planar crossing-critical graphs down to 3.2. For that we recall an old construction of Kochol [8]: His 3-connected 2-crossing-critical graphs consist of $2m+1$ copies of a pentagon joined together as in Figure 8. Notice that also these graphs are almost-planar—just delete the marked edge f, and their average degree equals $3 + \frac{1}{5}$. They can be nicely combined with our construction in Theorem 4.1 using zip product, too.

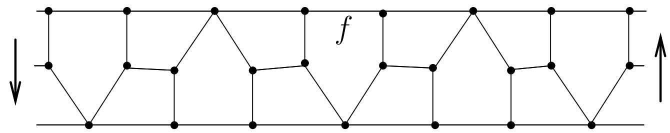

Figure 8: The 2-crossing-critical family of Kochol [8]; with "twisted" winding around a Möbius band.

Corollary 4.3 *For every $k \geq 12$ (odd $k \geq 7$) there are infinitely many simple 3-connected almost-planar k-crossing-critical graphs with the average degree equal to any given rational value in the interval $\left(3+\frac{1}{5},\, 4\right)$.*

Proof. We consider first odd $k \geq 7$, and denote by F_1 the graph sketched in Figure 6, made as a union of $k-2$ cycles with the first cycle of length ℓ. Then we construct a simple 3-connected crossed $(k-2)$-belt graph F from F_1 after double-splitting $\ell+1$ accumulation vertices of F_1 and single-splitting remaining accumulation vertices. Hence F has $n=(k-1)\ell+(\ell+1)+6=k\ell+7$ vertices and degree sum $4n-6$ (note that all vertices of F are of degree 4 except six of degree 3). We again denote by v_1 the cubic vertex of F, which is a neighbor of s_1, t_1 as in Figure 2.

We also denote by G Kochol's graph (Figure 8) on $10m-5$ vertices, and by w one end the edge f in G. It is again easy to check that w has two coherent bundles in G, and so we may apply zip product here: Let H be the result of the zip product of F and G at v_1, w, such that H is almost-planar and $(k-2+2)$-crossing-critical by [2]. A direct calculation shows that H has $k\ell+7+10m-5-2=k\ell+10m$ vertices and its degree sum is $4(k\ell+7)-6+32m-16-6=4k\ell+32m$. Hence expressing its average degree as

$$\frac{4k\ell+32m}{k\ell+10m}=4-\frac{p}{q}$$

leads to an equation

$$m\cdot(8q-10p)=\ell\cdot kp\,,$$

which clearly has infinitely many admissible integral solutions ℓ, m for all choices of $4-\frac{p}{q}\in\left(3+\frac{1}{5},\, 4\right)$.

On the other hand, for even $k \geq 12$ we may apply an analogous construction starting from the graphs of Corollary 4.2. ☐

5 Additional remarks

First, we remind readers that our Theorem 3.1 gives an answer only to a half of the question originally asked by Bokal, and so we repeat the other part which remains open:

Question 5.1 (Bokal) *For which odd values of $d \geq 7$ are there infinite families of simple 3-connected k-crossing-critical graphs having arbitrarily many vertices of degree d?*

Second, although our subsequent results in Section 4 are not quite new, they bring some interesting advantages over previous [2, 10, 11]. Prominently, we are constructing such crossing-critical graphs as almost-planar which was not the case of previous constructions. Our construction works with all (not too small) values of k, and not only with sporadic large k's as, say [11], and we approach the upper–boundary value of 6 with much smaller values of k than [2]. Though, in connection with Corollary 4.3 it is interesting to ask the next.

Question 5.2 *Do there exist infinite families of almost-planar k-crossing-critical graphs with average degree below $3 + \frac{1}{5}$?*

Third, we have shown [5] that all k-crossing-critical graphs have path-width bounded in k. This result has been followed by a conjecture of Richter and Salazar; that k-crossing-critical graphs have bandwidth bounded in k. The close relation of this conjecture to our topic appears clear when one notices a positive answer would imply that maximal degree of k-crossing-critical graphs is bounded in k. We, however, are not strong supporters of it (particularly since an analogous claim for the projective plane is false [6]), and so we ask:

Question 5.3 *Do k-crossing-critical graphs have maximal degree bounded by a function of k?*

One may, as well, ask whether can all k-crossing-critical graphs be "nicely characterized"? Recent signals suggest that such a characterization is not far in the case of $k = 2$, but values of $k > 3$ appear hopeless. At least one could hope an asymptotic characterization of almost-planar crossing-critical is feasible. In this relation the following question occurs naturally. (We note that for non-critical graphs, the questioned claim is false [3, 7].)

Question 5.4 *Is it true that for every almost-planar k-crossing-critical graph G there is an optimal drawing of G with all the crossings concentrated on one edge of G?*

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