Circular Digraph Walks, k-Balanced Strings, Lattice Paths and Chebychev Polynomials

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Abstract

We count the number of walks of length n on a k-node circular digraph that cover all k nodes in two ways. The first way illustrates the transfer-matrix method. The second involves counting various classes of height-restricted lattice paths. We observe that the results also count so-called k-balanced strings of length n, generalizing a 1996 Putnam problem.

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1 Introduction: Walks and k-Balanced Binary Strings

Let C_k be a circular digraph that consists of k nodes, namely, $v_0, \ldots, v_{k-1}$. A walk on C_k of length n is simply a sequence of $n+1$ nodes $(w_0, \ldots, w_n)$ such that w_i is adjacent to w_{i+1} in C_k for $0 \le i \le n-1$. Notice that we may assign the (clockwise) arcs, between nodes v_i and $v_{(i+1) \pmod k}$ for each $i = 0, \ldots, k-1$, with transition label 1 whereas assign the (counterclockwise) arcs, between $v_{(i+1) \pmod k}$ and v_i for each $i = 0, \ldots, k-1$, with transition label 0. Then each walk on C_k of length n generates a unique binary word of length n. For ease of visualization, we provide Figure 1 as an instance when $k = 4$.

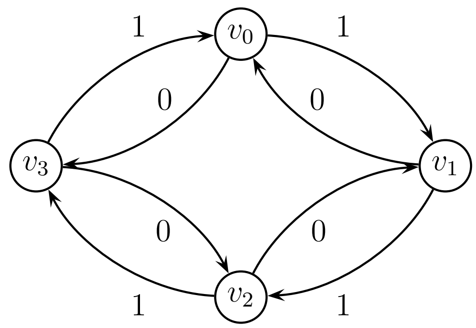


Figure 1: When $k = 4$, an instance of a good walk of length 5 starting from v_0 is $(v_0, v_1, v_2, v_1, v_2, v_3)$. This walk generates the unique binary string 11011.

We now define a "good walk" on C_k as a walk starting from v_0 and visiting all k nodes of C_k. We settle the question of how many good walks exist, by restricting our attention to "bad walks" (i.e. walks that do not cover all nodes).

The binary strings generated by "bad walks" can be placed into a 1-1 correspondence with the so-called $(k-2)$-balanced binary strings. A k-balanced binary string, in turn, is defined as a finite binary string S in which every substring T (of consecutive bits) of S has $-k \le \Delta(S) \le k$, where $\Delta(S)$ denotes the number of 1's minus the number of 0's. For example, 11011 represents an unbalanced string (for 2-balanced binary strings).

A 1996 Putnam problem [1] by Michael Larsen asked for the number of 2-balanced binary strings, and a generalization to k-balanced strings was the motivation for this paper. An explicit-sum solution to the Putnam problem is given in [1] but generalizing it seems unwieldy. Here we focus on generating functions.

The outline of the paper is as follows. In Section 2 we use the transfer-matrix method to obtain the desired generating function as a difference $S_{k+1}(x) - S_k(x)$ where $S_k(x)$ is the sum of the entries in a certain $k \times k$ matrix, and to make a first stab at simplifying $S_k(x)$. In Section 3 we survey the use of Chebychev polynomials to count various classes of height-restricted lattice paths and deduce an alternative expression for the desired generating function as a product $R_k(x)R_{k-1}(x)$. In Section 4 we reconcile the two formulas $S_{k+1}(x) - S_k(x)$ and $R_k(x)R_{k-1}(x)$.

2 The Transfer-Matrix Approach

2.1 The basic result

Theorem 1. *Let A_k denote the tridiagonal $k \times k$ matrix with 1s just above and just below the main diagonal and 0s elsewhere,*

$$A_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \tag{1}$$

and let $S_k(x)$ denote the sum of all the entries in $(I_k - xA_k)^{-1}$ where I_k is the $k \times k$ identity matrix. Then the generating function for "bad walks" of length n on C equals

$$S_{k-1}(x) - S_{k-2}(x).$$

In other words, the generating function for k-balanced strings of length n is

$$f_k(x) = S_{k+1}(x) - S_k(x).$$

Proof Given a "bad walk" $w = (w_0 = v_0, w_1, \ldots, w_n)$ of length n, let

$$\max(w) = \max\{i \colon \{v_0, v_1, \ldots, v_i\} \subseteq \{w_0, \ldots, w_n\}\}.$$

We see that "bad walks" with $\max(w) = r$ are just the walks $w = (w_0, \ldots, w_n)$ on $C \setminus \{v_{r+1}\}$ such that $w_0 = v_0$ and $v_r \in \{w_0, \ldots, w_n\}$. Notice that an arbitrary walk $w = (w_0, \ldots, w_n)$ on $C \setminus \{v_{r+1}\}$ either satisfies $v_r \in \{w_0, \ldots, w_n\}$ or is a walk on $C \setminus \{v_r, v_{r+1}\}$. Thus the walks with $\max(w) = r$ are those that miss $\{v_{r+1}\}$ but don't miss $\{v_r, v_{r+1}\}$. Now $C \setminus \{v_{r+1}\}$ is the path graph on vertex list $v_{r+2}, v_{r+3}, \ldots, v_{k-1}, v_0, v_1, \ldots, v_r$. Note that v_0 is the $(k-1-r)$th entry in the vertex list and the adjacency matrix is A_{k-1}. The transfer-matrix method [2, Theorem 4.7.2] says that the generating function for walks from v_0 to the jth vertex is the $(k-1-r, j)$ entry of $(I_{k-1} - xA_{k-1})^{-1}$. Similarly, the generating function for walks from v_0 to the jth vertex in the path graph $C \setminus \{v_r, v_{r+1}\}$ is the $(k-1-r, j)$ entry of $(I_{k-2} - xA_{k-2})^{-1}$. Taking the difference and summing over r and j yields the result. $\square$

2.2 A determinant formula

Now we obtain an expression for $S_k(x) :=$ sum of entries in $(I_k - xA_k)^{-1}$. Let $\mathsf{U}_k(x)$ denote the "combinatorial" Chebyshev polynomial introduced in the next section.

Since $\det(I_k - xA_k)$ and $\mathsf{U}_k(x)$ both satisfy the recurrence $\mathsf{P}_k(x) = \mathsf{P}_{k-1}(x) - x^2\mathsf{P}_{k-2}(x)$ with initial conditions $\mathsf{P}_0(x) = \mathsf{P}_1(x) = 1$, we conclude that $\det(I_k - xA_k) = \mathsf{U}_k(x)$.

Applying linear algebra, we have $S_k(x) = x_1 + \cdots + x_k$, where the x_i (functions of x) denote the solutions to the equation system

$$(I_k - xA_k)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

By summing up the equations, we find

$$(1-x)x_1 + (1-2x)(x_2 + \cdots + x_{k-1}) + (1-x)x_k = k, \tag{2}$$

and Cramer's rule implies

$$x_1 = x_k = \det\begin{pmatrix} 1 & -x & 0 & 0 & \cdots & 0 \\ 1 & 1 & -x & 0 & \cdots & 0 \\ 1 & -x & 1 & -x & \cdots & \vdots \\ 1 & 0 & -x & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & -x & 1 \end{pmatrix} \Bigg/ \mathsf{U}_k(x). \tag{3}$$

Denote the determinant in the numerator by W_k. Thus from (2) and (3), we have

$$S_k(x) = \frac{k - 2xx_1}{1-2x} = \frac{k\mathsf{U}_k(x) - 2xW_k}{(1-2x)\mathsf{U}_k(x)}.$$

3 The Lattice Path Approach

3.1 Combinatorial Chebychev polynomials

The familiar Chebychev polynomials $T_k(x)$ and $U_k(x)$ (first and second kinds) are defined by $\cos k\theta = T_k(\cos\theta)$ and $\sin(k+1)\theta/\sin\theta = U_k(\cos\theta)$. They occur in diverse areas, as suggested by the subtitle of Theodore Rivlin's book [3]. Their application in combinatorics to lattice path counting is less well known. For this purpose, it is convenient to define modified Chebychev polynomials by

$$\mathsf{T}_k(x) = 2x^k T_k\big(\tfrac{1}{2x}\big), \qquad \mathsf{U}_k(x) = x^k U_k\big(\tfrac{1}{2x}\big).$$

This removes an extraneous power of 2 and reverses the coefficients to produce integer-coefficient polynomials with constant term 1 (except that $\mathsf{T}_0 = 2$) which might be called the *combinatorial Chebychev polynomials*. Both satisfy the defining recurrence $\mathsf{P}_k(x) = \mathsf{P}_{k-1}(x) - x^2\mathsf{P}_{k-2}(x)$, differing only in the initial conditions, and both have simple explicit expressions:

$$\mathsf{T}_k(x) = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \left(\binom{k-j}{j} + \binom{k-j-1}{j-1} \right) x^{2j}, \quad \mathsf{U}_k(x) = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \binom{k-j}{j} x^{2j}.$$

The first few are listed in the following Table.

k	$\mathsf{T}_k(x)$	$\mathsf{U}_k(x)$
0	2	1
1	1	1
2	$1-2x^2$	$1-x^2$
3	$1-3x^2$	$1-2x^2$
4	$1-4x^2+2x^4$	$1-3x^2+x^4$
5	$1-5x^2+5x^4$	$1-4x^2+3x^4$
6	$1-6x^2+9x^4-2x^6$	$1-5x^2+6x^4-x^6$
7	$1-7x^2+14x^4-7x^6$	$1-6x^2+10x^4-4x^6$

Table of combinatorial Chebychev polynomials

3.2 Application to height-restricted lattice paths

Consider lattice paths of upsteps $u = (1, 1)$ and downsteps $d = (1, -1)$. The horizontal line through a path's initial vertex is *ground level* and heights are measured relative to ground level. Thus if the path starts at the x-y origin, ground level is the x-axis. The *height* of a path is the maximum of the heights of its vertices. A *nonnegative* path is one that never dips below ground level. A *balanced* path (not to be confused with k-balanced strings) is one that ends at ground level. A *Dyck* path is a nonnegative balanced path, including the empty path.

The generating function for a given class of paths is $\sum_{n\geq 0} a(n)x^n$ where $a(n)$ is the number of paths of size n: size is taken as "number of steps" except for paths specified to terminate at height k, where size is "# steps $-\,k$" since such a path necessarily contains k upsteps.

The application of combinatorial Chebychev polynomials to count height-restricted lattice paths is given in Table 1. *Here, and in the sequel,* U_k *is short for* $\mathsf{U}_k(x)$ *and so on.*

paths bounded by $y = 0$ and $y = k$

path ends at height	generating function
0	$F_k = \dfrac{\mathsf{U}_k}{\mathsf{U}_{k+1}}$
k	$G_k = \dfrac{1}{\mathsf{U}_{k+1}}$

paths bounded by $y = \pm k$

path ends at height	generating function
0	$\overline{F}_k = \dfrac{\mathsf{U}_k}{\mathsf{T}_{k+1}}$
k	$\overline{G}_k = \dfrac{1}{\mathsf{T}_{k+1}}$

Table 1
Generating functions for some height-restricted
lattice paths with specified terminal height

Thus the first item, $F_k(x)$, is the generating function for Dyck paths of height $\leq k$ with x marking length. The expressions for F_k and G_k are folklore; two early references are [4, 5] and a recent one is [6]. For completeness we briefly outline below proofs for all the items in Table 1.

It is also possible to find corresponding generating functions $H_k(x)$ and $\overline{H}_k(x)$ for paths with no restriction on the height of the terminal vertex. paths bounded by $y = \pm k$. Here it is necessary to distinguish the cases $k = 2m$ is even and $k = 2m + 1$ is odd:

paths bounded by $y = 0$ and $y = k$	paths bounded by $y = \pm k$
$H_{2m} = \dfrac{\mathsf{U}_m + x\mathsf{U}_{m-1}}{\mathsf{T}_{m+1}}$	$\overline{H}_{2m} = \dfrac{(\mathsf{U}_m + x\mathsf{U}_{m-1})^2}{\mathsf{T}_{2m+1}}$
$H_{2m+1} = \dfrac{\mathsf{U}_m}{\mathsf{U}_{m+1} - x\mathsf{U}_m}$	$\overline{H}_{2m+1} = (1 + 2x)\,\dfrac{(\mathsf{U}_m)^2}{\mathsf{T}_{2m+2}}$

Table 2
Generating functions for some height-restricted lattice paths

Sri Gopal Mohanty [7] uses the reflection principle to count paths bounded by $y = s$ and $y = -t$ for arbitrary nonnegative s and t, obtaining explicit sums rather than generating functions.

Proofs for Tables 1 and 2

$\boldsymbol{F}$: A nonempty Dyck path P can be uniquely expressed as uP_1dP_2 where P_1 and P_2 are Dyck paths. The path P has height $\le k$ if and only if P_1 has height $\le k-1$ and P_2 has height $\le k$. This observation translates to a recurrence for the generating function:

$$F_k = 1 + x^2 F_{k-1} F_k,$$

with solution $F_k = \mathsf{U}_k/\mathsf{U}_{k+1}$ because the substitution $\mathsf{U}_k/\mathsf{U}_{k+1}$ for F_k reduces to $\mathsf{U}_{k+1} = \mathsf{U}_k - x^2\mathsf{U}_{k-1}$, equivalent to a well known recurrence for Chebychev polynomials.

$\boldsymbol{G}$: A path bounded by $y=0$ and $y=k$ that terminates at height k has a last upstep to height i for $i=1,2,\ldots,k-1$ and the last upstep to height k is necessarily the last step of the path. Remove these k upsteps to obtain a list of k Dyck paths (some may be empty). The ith path in this list from right to left has height $\le i$ and hence generating function F_i. Thus $G_k = \prod_{i=1}^{k} F_i = 1/\mathsf{U}_{k+1}$.

$\overline{\boldsymbol{F}}$: A balanced path P bounded by $y=\pm k$ is either (i) empty or (ii) starts up or (iii) starts down. In case (ii) P decomposes as uP_1dP_2 where P_1 is a Dyck path of height $\le k-1$ and P_2 is another balanced path bounded by $y=\pm k$. Thus case (ii) contributes $x^2F_{k-1}\overline{F}_k$ and, by symmetry, so does case (iii). Hence

$$\overline{F}_k = 1 + 2x^2 F_{k-1}\overline{F}_k$$

leading to $\overline{F}_k = \mathsf{U}_k/(\mathsf{U}_k - 2x^2\mathsf{U}_{k+1})$ and so, using another well known Chebychev polynomial identity, $\overline{F}_k = \mathsf{U}_k/\mathsf{T}_{k+1}$.

$\overline{\boldsymbol{G}}$: A path bounded by $y=\pm k$ terminating at height k has a last upstep to height i, $1 \le i \le k-1$. Delete these upsteps to obtain a list consisting of a balanced path bounded by $y=\pm k$, followed by $k-1$ Dyck paths of heights $\le k-1,\ \le k-2,\ \ldots,\ \le 1$ respectively. Thus

$$\overline{G}_k = \frac{\mathsf{U}_k}{\mathsf{T}_{k+1}}\frac{\mathsf{U}_{k-1}}{\mathsf{U}_k}\cdots\frac{\mathsf{U}_1}{\mathsf{U}_2} = \frac{1}{\mathsf{T}_{k+1}}.$$

$\boldsymbol{H}$: A path bounded by $y=0$ and $y=k$ with no restriction on the terminal height is either (i) empty or (ii) starts with an upstep and never returns to ground level or (iii) has the form $uPdQ$ where P is a Dyck path of height $\le k-1$ and Q is a path bounded by $y=0$ and $y=k$. The contributions to the generating function H_k are respectively (i) 1, (ii) xH_{k-1}, (iii) $x^2F_{k-1}H_k$. Thus

$$H_k = 1 + xH_{k-1} + x^2F_{k-1}H_k.$$

It is routine, if tedious, to verify that the expressions for H_{2m} and H_{2m+1} in Table 2 satisfy this equation. The $2m$ case, for example, can be verified as follows. Replace T_{m+1} by $\mathsf{U}_{m+1} - x^2\mathsf{U}_{m-1}$ (another Chebychev identity) and revert to the standard Chebychev polynomials $U_k(x)$. With the substitution $y = 1/(2x)$ this reduces matters to verifying that

$$\big(U_m^2(y) - U_{m-1}^2(y)\big)\big(2yU_{2m}(y) - U_{2m-1}(y)\big) = U_m(y)U_{2m}(y)\big(U_{m+1}(y) - U_{m-1}(y)\big),$$

an identity that ultimately depends on the elementary addition formulae for trigonometric functions.

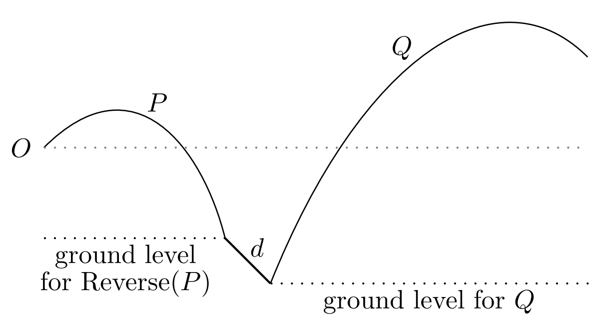
$\overline{\boldsymbol{H}}$: A path bounded by $y = \pm k$ with no restriction on the terminal height is either (i) empty or (ii) starts with an upstep (resp. downstep) and never returns to ground level or (iii) starts with an upstep (resp. downstep) and returns to ground level. Case (ii) "start up" makes a contribution of xH_{k-1} and by symmetry, case (ii) "start down" makes the same contribution. In case (iii) "start up", the path has the form $uPdQ$ where P is a Dyck path of height $\le k-1$ and Q is another path of the kind being counted. Thus case (iii) makes contribution $2x^2F_{k-1}\overline{H}_k$. Hence

$$\overline{H}_k = 1 + 2xH_{k-1} + 2x^2F_{k-1}\overline{H}_k$$

and another trite calculation shows that the expression for $\overline{H}_k$ in Table 2 satisfies this recurrence.

3.3 Application to k-balanced strings

A binary string of, say, Xs and Os can be coded as a lattice path: $X \to u$, $O \to d$. The k-balanced strings of length n translate to lattice paths of n steps with *vertical extent* $\le k$ where vertical extent means "maximum vertex height − minimum vertex height". A recurrence for the generating function $g_k(x)$ for these paths (with x marking number of steps) can be obtained from the following decomposition. Such a path is either nonnegative or else dips below ground level and hence has a first downstep d carrying it to its lowest level (below ground level). The first case gives contribution H_k. In the second case, the path has the form PdQ where the reverse of P is a nonnegative path of height $\le k-1$ and Q is a nonnegative path of height $\le k$ as illustrated.



Hence

$$g_k = H_k + xH_{k-1}H_k.$$

This is the desired generating function but it has an interesting alternative expression. Define a sequence of rational functions $(R_k(x))_{k\geq 1}$ by

$$R_{2m} = \frac{\mathsf{U}_m}{\mathsf{T}_{m+1}}, \quad R_{2m+1} = \frac{\mathsf{U}_{m+1} + x\mathsf{U}_m}{\mathsf{U}_{m+1} - x\mathsf{U}_m}.$$

Then it is easy to check that $H_k\big(1 + xH_{k-1}\big) = R_k R_{k-1},\ k \geq 1$. Thus $g_k = R_k R_{k-1}$ and we have established

Theorem 2. *The generating function for k-balanced binary strings, equivalently for u-d paths of vertical extent $\leq k$, is given by*

$$\begin{cases} \frac{\mathsf{U}_m}{\mathsf{T}_{m+1}} \cdot \frac{\mathsf{U}_m + x\mathsf{U}_{m-1}}{\mathsf{U}_m - x\mathsf{U}_{m-1}} & \text{if } k = 2m \text{ is even;} \\ \frac{\mathsf{U}_{m+1} + x\mathsf{U}_m}{\mathsf{U}_{m+1} - x\mathsf{U}_m} \cdot \frac{\mathsf{U}_m}{\mathsf{T}_{m+1}} & \text{if } k = 2m+1 \text{ is odd.} \end{cases}$$

Remark The expression in Theorem 2 for $g_k = R_k R_{k-1}$ is in lowest terms because $\mathsf{T}_{m+1} = \mathsf{U}_m - 2x^2\mathsf{U}_{m-1}$ and the recurrence $\mathsf{U}_m = \mathsf{U}_{m-1} - x^2\mathsf{U}_{m-2}$ yields by induction that U_m and U_{m-1} are relatively prime.

Remark R_k can be compactly expressed in terms of entries in Tables 1 and 2:

$$R_{2m} = \overline{F}_m, \quad R_{2m+1} = 1 + 2xH_{2m+1}.$$

Thus g_k involves convolutions of paths bounded by $y = \pm\lfloor k/2 \rfloor$ terminating at ground level ($\overline{F}_m$) and nonnegative paths of height $\leq k$ terminating anywhere (H_{2m+1}). A combinatorial explanation would be interesting but does not seem to be obvious.

4 Reconciling the Two Formulas

We have obtained expressions $f_k(x)$ and $g_k(x)$ for the generating function for k-balanced strings in Sections 2 and 3 respectively. We now show that $f_k = g_k$. The proof ultimately depends on the standard identities

$$\begin{aligned} \mathsf{U}_{2k} &= \mathsf{U}_k^2 - x^2\mathsf{U}_{k-1}^2, \\ \mathsf{U}_{2k+1} &= \mathsf{U}_k^2 - 2x^2\mathsf{U}_k\mathsf{U}_{k-1}. \end{aligned} \tag{4}$$

Set

$$P_k = \frac{k\mathsf{U}_k - 2xW_k}{(1-2x)}$$

so that, from Section 2.2, $S_k = P_k/\mathsf{U}_k$.

First, we find an expression for the determinant W_k. By cofactor expansion along the first row, W_k satisfies the defining recurrence

$$W_0 = 0,\ W_1 = 1, \quad W_k = \mathsf{U}_{k-1} + xW_{k-1},$$

with solution, verified using (4),

$$\begin{aligned} W_{2m} &= (\mathsf{U}_m + x\mathsf{U}_{m-1})\mathsf{U}_{m-1}, \\ W_{2m+1} &= (\mathsf{U}_m + x\mathsf{U}_{m-1})\mathsf{U}_m. \end{aligned}$$

Now define two sequences of polynomials $(A_k)_{k\geq 0},\ (B_k)_{k\geq 0}$ by

$$\begin{aligned} A_{2m} &= \mathsf{U}_m - x\mathsf{U}_{m-1}, \qquad & A_{2m+1} &= \mathsf{T}_{m+1} = \mathsf{U}_m - 2x^2\mathsf{U}_{m-1}; \\ B_{2m} &= \mathsf{U}_m + x\mathsf{U}_{m-1}, \qquad & B_{2m+1} &= \mathsf{U}_m. \end{aligned}$$

Thus, in particular, $W_k = B_k B_{k-1}$ whether k is even or odd. Also define a sequence $(C_k)_{k\geq 0}$ of rational functions (actually polynomials) by

$$C_k = \frac{kA_k - 2xB_{k-1}}{1-2x}.$$

It is now easy to verify that

$$\begin{aligned} P_k &= B_k C_k, \\ \mathsf{U}_k &= A_k B_k, \\ R_k &= \frac{B_{k+1}}{A_{k+1}}, \end{aligned}$$

where R_k is as defined in the preceding section, and that

$$C_k A_{k+1} - C_{k-1} A_k \ = \ W_k.$$

Hence

$$f_k = S_{k+1} - S_k = \frac{P_{k+1}}{\mathsf{U}_{k+1}} - \frac{P_k}{\mathsf{U}_k} = \frac{C_{k+1}}{A_{k+1}} - \frac{C_k}{A_k} = \frac{W_{k+1}}{A_{k+1}A_k} = \frac{B_{k+1}B_k}{A_{k+1}A_k} = R_k R_{k-1} = g_k,$$

as required.

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