# The Scattering Matrix of a Graph


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**Abstract**

Recently, Smilansky expressed the determinant of the bond scattering matrix of a graph by means of the determinant of its Laplacian. We present another proof for this Smilansky's formula by using some weighted zeta function of a graph. Furthermore, we reprove a weighted version of Smilansky's formula by Bass' method used in the determinant expression for the Ihara zeta function of a graph.


## 1 Introduction

Graphs treated here are finite. Let $G = (V(G), E(G))$ be a connected graph (possibly multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges $uv$ joining two vertices $u$ and $v$. For $uv \in E(G)$, an arc $(u, v)$ is the oriented edge from $u$ to $v$. Set $R(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $b = (u, v) \in R(G)$, set $u = o(b)$ and $v = t(b)$. Furthermore, let $\hat{b} = (v, u)$ be the *inverse* of $b = (u, v)$.

A *path $P$ of length $n$* in $G$ is a sequence $P = (b_1, \cdots , b_n)$ of $n$ arcs such that $b_i \in R(G)$, $t(b_i) = o(b_{i+1})(1 \leq i \leq n-1)$, where indices are treated *mod* $n$. Set $\mid P \mid= n$, $o(P) = o(b_1)$ and $t(P) = t(b_n)$. Also, $P$ is called an *$(o(P), t(P))$-path*. We say that a path $P = (b_1, \cdots , b_n)$ has a *backtracking* or *back-scatter* if $\hat{b}_{i+1} = b_i$ for some $i(1 \leq i \leq n-1)$. A $(v, w)$-path is called a *$v$-cycle* (or *$v$-closed path*) if $v = w$. The *inverse cycle* of a cycle $C = (b_1, \cdots , b_n)$ is the cycle $\hat{C} = (\hat{b}_n, \cdots , \hat{b}_1)$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \cdots , e_m)$ and $C_2 = (f_1, \cdots , f_m)$ are called *equivalent* if there exists $k$ such that $f_j = e_{j+k}$ for all $j$. The inverse cycle of $C$ is in general not equivalent to $C$. Let $[C]$ be the equivalence class which contains a cycle $C$. Let $B^r$ be the cycle obtained by going $r$ times around a cycle $B$. Such a cycle is called a *power* of $B$. A cycle $C$ is *reduced* if $C$ has no backtracking.

Furthermore, a cycle $C$ is *primitive* if it is not a power of a strictly smaller cycle. Note that each equivalence class of primitive, reduced cycles of a graph $G$ corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, u)$ of $G$ at a vertex $u$ of $G$. Furthermore, an equivalence class of primitive cycles of a graph $G$ is called a *primitive periodic orbit* of $G$(see [13]).

The *Ihara zeta function* of a graph $G$ is a function of a complex variable $t$ with $\mid t \mid$ sufficiently small, defined by

$$\mathbf{Z}(G, t) = \mathbf{Z}_G(t) = \prod_{[p]} (1 - t^{|p|})^{-1},$$

where $[p]$ runs over all primitive periodic orbits without back-scatter of $G$(see [8]).

Ihara zeta functions of graphs started from Ihara zeta functions of regular graphs by Ihara [8]. Originally, Ihara presented $p$-adic Selberg zeta functions of discrete groups, and showed that its reciprocal is a explicit polynomial. Serre [12] pointed out that the Ihara zeta function is the zeta function of the quotient $T/\Gamma$ (a finite regular graph) of the one-dimensional Bruhat-Tits building $T$ (an infinite regular tree) associated with $GL(2, k_p)$.

A zeta function of a regular graph $G$ associated with a unitary representation of the fundamental group of $G$ was developed by Sunada [15,16]. Hashimoto [7] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial.

**Theorem 1 (Bass)** *Let $G$ be a connected graph. Then the reciprocal of the zeta function of $G$ is given by*

$$\mathbf{Z}(G, t)^{-1} = (1 - t^2)^{r-1} \det(\mathbf{I} - t\mathbf{C}(G) + t^2(\mathbf{D} - \mathbf{I})),$$

*where $r$ and $\mathbf{C}(G)$ are the Betti number and the adjacency matrix of $G$, respectively, and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = v_i = \deg u_i$ where $V(G) = \{u_1, \cdots, u_n\}$.*

Various proofs of Bass' Theorem were given by Stark and Terras [14], Foata and Zeilberger [4], Kotani and Sunada [9].

Let $G$ be a connected graph. We say that a path $P = (b_1, \cdots, b_n)$ has a *bump* at $t(b_i)$ if $b_{i+1} = \hat{b}_i$ $(1 \leq i \leq n)$. The *cyclic bump count* $cbc(\pi)$ of a cycle $\pi = (\pi_1, \cdots, \pi_n)$ is

$$cbc(\pi) = \mid \{ i = 1, \cdots, n \mid \pi_i = \hat{\pi}_{i+1} \} \mid,$$

where $\pi_{n+1} = \pi_1$. Then the *Bartholdi zeta function* of $G$ is a function of two complex variables $u, t$ with $\mid u \mid, \mid t \mid$ sufficiently small, defined by

$$\zeta_G(u, t) = \zeta(G, u, t) = \prod_{[C]} (1 - u^{cbc(C)} t^{|C|})^{-1},$$

where $[C]$ runs over all primitive periodic orbits of $G$(see [1]). If $u = 0$, then the Bartholdi zeta function of $G$ is the Ihara zeta function of $G$.

Bartholdi [1] gave a determinant expression of the Bartholdi zeta function of a graph.

**Theorem 2 (Bartholdi)** *Let $G$ be a connected graph with $n$ vertices and $m$ unoriented edges. Then the reciprocal of the Bartholdi zeta function of $G$ is given by*

$$\zeta(G,u,t)^{-1} = (1-(1-u)^2t^2)^{m-n}\det(\mathbf{I} - t\mathbf{C}(G) + (1-u)(\mathbf{D}-(1-u)\mathbf{I})t^2).$$

In the case of $u=0$, Theorem 2 implies Theorem 1.

Sato [11] defined a new zeta function of a graph by using not an infinite product but a determinant.

Let $G$ be a connected graph and $V(G) = \{u_1, \cdots, u_n\}$. Then we consider an $n \times n$ matrix $\tilde{\mathbf{C}} = (w_{ij})_{1 \leq i,j \leq n}$ with $ij$ entry the complex variable $w_{ij}$ if $(u_i, u_j) \in R(G)$, and $w_{ij} = 0$ otherwise. The matrix $\tilde{\mathbf{C}} = \tilde{\mathbf{C}}(G)$ is called the *weighted matrix* of $G$. For each path $P = (u_{i_1}, \cdots, u_{i_r})$ of $G$, the *norm* $w(P)$ of $P$ is defined as follows: $w(P) = w_{i_1 i_2} w_{i_2 i_3} \cdots w_{i_{r-1} i_r}$. Furthermore, let $w(u_i, u_j) = w_{ij},\ u_i, u_j \in V(G)$ and $w(b) = w_{ij}, b = (u_i, u_j) \in R(G)$.

Let $G$ be a connected graph with $n$ vertices and $m$ unoriented edges, and $\tilde{\mathbf{C}} = \tilde{\mathbf{C}}(G)$ a weighted matrix of $G$. Two $2m \times 2m$ matrices $\mathbf{B} = \mathbf{B}(G) = (\mathbf{B}_{e,f})_{e,f \in R(G)}$ and $\mathbf{J}_0 = \mathbf{J}_0(G) = (\mathbf{J}_{e,f})_{e,f \in R(G)}$ are defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} w(f) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise} \end{cases}, \mathbf{J}_{e,f} = \begin{cases} 1 & \text{if } f = \hat{e}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the *zeta function* of $G$ is defined by

$$\mathbf{Z}_1(G,w,t) = \det(\mathbf{I}_n - t(\mathbf{B} - \mathbf{J}_0))^{-1}.$$

If $w(e) = 1$ for any $e \in R(G)$, then the zeta function of $G$ is the Ihara zeta function of $G$.

**Theorem 3 (Sato)** *Let $G$ be a connected graph, and let $\tilde{\mathbf{C}} = \tilde{\mathbf{C}}(G)$ be a weighted matrix of $G$. Then the reciprocal of the zeta function of $G$ is given by*

$$\mathbf{Z}_1(G,w,t)^{-1} = (1-t^2)^{m-n}\det(\mathbf{I}_n - t\tilde{\mathbf{C}}(G) + t^2(\tilde{\mathbf{D}} - \mathbf{I}_n)),$$

*where $n =| V(G) |$, $m =| E(G) |$ and $\tilde{\mathbf{D}} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \sum_{o(b)=u_i} w(e)$, $V(G) = \{u_1, \cdots, u_n\}$.*

The spectral determinant of the Laplacian on a quantum graph is closely related to the Ihara zeta function of a graph(see [3,5,6,13]) .

Smilansky [13] considered spectral zeta functions and trace formulas for (discrete) Laplacians on ordinary graphs, and expressed some determinant on the bond scattering matrix of a graph $G$ by using the characteristic polynomial of its Laplacian.

Let $G$ be a connected graph with $n$ vertices and $m$ edges, $V(G) = \{u_1, \ldots, u_n\}$ and $R(G) = \{b_1, \ldots, b_m, b_{m+1}, \ldots, b_{2m}\}$ such that $b_{m+j} = \hat{b}_j (1 \leq j \leq m)$.

The *Laplacian (matrix)* $\mathbf{L} = \mathbf{L}(G)$ of $G$ is defined by

$$\mathbf{L} = \mathbf{L}(G) = -\mathbf{C}(G) + \mathbf{D}.$$

Let $\lambda$ be a eigenvalue of $\mathbf{L}$ and $\psi = (\psi_1, \dots, \psi_n)$ the eigenvector corresponding to $\lambda$. For each arc $b = (u_j, u_l)$, one associates a *bond wave function*

$$\psi_b(x) = a_b \mathrm{e}^{i\pi x/4} + a_{\hat{b}} \mathrm{e}^{-i\pi x/4}, \ x = \pm 1$$

under the condition

$$\psi_b(1) = \psi_j, \psi_b(-1) = \psi_l.$$

We consider the following three conditions:

1. *uniqueness*: The value of the eigenvector at the vertex $u_j$, $\psi_j$, computed in the terms of the bond wave functions is the same for all the arcs emanating from $u_j$.

2. *$\psi$ is an eigenvector of* $\mathbf{L}$;

3. *consistency*: The linear relation between the incoming and the outgoing coefficients (1) must be satisfied simultaneously at all vertices.

By the uniqueness, we have

$$a_{b_1} \mathrm{e}^{i\pi/4} + a_{\hat{b}_1} \mathrm{e}^{-i\pi/4} = a_{b_2} \mathrm{e}^{i\pi/4} + a_{\hat{b}_2} \mathrm{e}^{-i\pi/4} = \cdots = a_{b_{v_j}} \mathrm{e}^{i\pi/4} + a_{\hat{b}_{v_j}} \mathrm{e}^{-i\pi/4},$$

where $b_1, b_2, \dots, b_{v_j}$ are arcs emanating from $u_j$, and $v_j = \deg u_j$, $i = \sqrt{-1}$.

By the condition 2, we have

$$-\sum_{k=1}^{v_j} (a_{b_k} \mathrm{e}^{-i\pi/4} + a_{\hat{b}_k} \mathrm{e}^{i\pi/4}) = (\lambda - v_j) \frac{1}{v_j} \sum_{k=1}^{v_j} (a_{b_k} \mathrm{e}^{i\pi/4} + a_{\hat{b}_k} \mathrm{e}^{-i\pi/4}).$$

Thus, for each arc $b$ with $o(b) = u_j$,

$$a_b = \sum_{t(c)=u_j} \sigma_{b,c}^{(u_j)}(\lambda) a_c, \tag{1}$$

where

$$\sigma_{b,c}^{(u_j)}(\lambda) = i(\delta_{\hat{b},c} - \frac{2}{v_j} \frac{1}{1 - i(1 - \lambda/v_j)}),$$

and $\delta_{\hat{b},c}$ is the Kronecker delta. The *bond scattering matrix* $\mathbf{U}(\lambda) = (U_{ef})_{e,f \in R(G)}$ of $G$ is defined by

$$U_{ef} = \begin{cases} \sigma_{e,f}^{(t(f))} & \text{if } t(f) = o(e), \\ 0 & \text{otherwise.} \end{cases}$$

By the consistency, we have

$$\mathbf{U}(\lambda)\mathbf{a} = \mathbf{a},$$

where $\mathbf{a} = {}^t(a_{b_1}, a_{b_2}, \dots, a_{b_{2m}})$. This holds if and only if

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = 0.$$

**Theorem 4 (Smilansky)** *Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then the characteristic polynomial of the bond scattering matrix of $G$ is given by*

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = \frac{2^m i^n \det(\lambda \mathbf{I}_n + \mathbf{C}(G) - \mathbf{D})}{\prod_{j=1}^{n}(v_j - iv_j + \lambda i)} = \prod_{[p]}(1 - a_p(\lambda)),$$

*where $[p]$ runs over all primitive periodic orbits of $G$, and*

$$a_p(\lambda) = \sigma_{b_1,b_n}^{(t(b_n))} \sigma_{b_n,b_{n-1}}^{(t(b_{n-1}))} \cdots \sigma_{b_2,b_1}^{(t(b_1))}, \ p = (b_1, b_2, \ldots, b_n).$$

In this paper, we reprove Smilansky's formula for the characteristic polynomial of the bond scattering matrix of a graph and its weighted version by using some zeta functions of a graph. In Section 2, we consider a new zeta function of a graph $G$, and present another proof of Smilansky's formula for some determinant on the bond scattering matrix of a graph by means of the Laplacian of $G$. Furthermore, we give Smilansky's formula for the case of a regular graph by using Bartholdi zeta function of a graph. In Section 3, we present a decomposition formula for some determinant on the bond scattering matrix of a semiregular bipartite graph. In Section 4, we give another proof for a weighted version of the above Smilansky's formula by Bass' method used in the determinant expression for the Ihara zeta function of a graph. In Section 5, we express a new zeta function of a graph by using the Euler product.

# 2 The scattering matrix of a graph

We present a proof of Theorem 4 by using Theorem 3, which is different from a proof in [13].

**Theorem 5 (Smilansky)** *Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then, for the bond scattering matrix of $G$,*

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = \frac{2^m i^n \det(\lambda \mathbf{I}_n + \mathbf{C}(G) - \mathbf{D})}{\prod_{j=1}^{n}(v_j - iv_j + \lambda i)}.$$

**Proof**. Let $G$ be a connected graph with $n$ vertices and $m$ edges, $V(G) = \{u_1, \cdots, u_n\}$ and $R(G) = \{b_1, \ldots, b_m, \hat{b}_1, \ldots, \hat{b}_m\}$. Set $v_j = \deg u_j$ and

$$x_j = x_{u_j} = \frac{2}{v_j} \frac{1}{1 - i(1 - \lambda/v_j)}$$

for each $j = 1, \ldots, n$. Then we consider a $2m \times 2m$ matrix $\mathbf{B} = (B_{ef})_{e,f \in R(G)}$ given by

$$B_{ef} = \begin{cases} x_{o(f)} & \text{if } t(e) = o(f), \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3, we have

$$\det(\mathbf{I}_{2m} - u(\mathbf{B} - \mathbf{J}_0)) = (1-u^2)^{m-n} \det(\mathbf{I}_n - u\mathbf{W}_x(G) + u^2(\mathbf{D}_x - \mathbf{I}_n)),$$

where $\mathbf{W}_x(G) = (w_{jk})$ and $\mathbf{D}_x = (d_{jk})$ are given as follows:

$$w_{jk} = \begin{cases} x_j & \text{if } (u_j, u_k) \in R(G), \\ 0 & \text{otherwise} \end{cases}, d_{jk} = \begin{cases} v_j x_j & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\det(\mathbf{I}_{2m} - u({}^t\mathbf{B} - {}^t\mathbf{J}_0)) = (1-u^2)^{m-n} \det(\mathbf{I}_n - u\mathbf{W}_x(G) + u^2(\mathbf{D}_x - \mathbf{I}_n)), \tag{2}$$

where ${}^t\mathbf{B}$ is the transpose of $\mathbf{B}$. Note that

$$v_j x_j = \frac{2}{1 - i(1 - \lambda/v_j)} \ (1 \le j \le n).$$

But, since

$$i\mathbf{U}(\lambda) + \mathbf{J}_0 = {}^t\mathbf{B},$$

we have

$${}^t\mathbf{B} - {}^t\mathbf{J}_0 = i\mathbf{U}(\lambda).$$

Substituting $u = -i$ in (2), we obtain

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = 2^{m-n} \det(\mathbf{I}_n + i\mathbf{W}_x(G) - (\mathbf{D}_x - \mathbf{I}_n)). \tag{3}$$

Now, we have

$$\mathbf{W}_x(G) = \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{bmatrix} \mathbf{C}(G)$$

and

$$\mathbf{D}_x = \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{bmatrix} \mathbf{D}.$$

Let

$$\mathbf{X} = \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{bmatrix}.$$

Then it follows that

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = 2^{m-n} \det(2\mathbf{I}_n + i\mathbf{X}\mathbf{C}(G) - \mathbf{X}\mathbf{D})$$

$$= 2^{m-n} i^n \det \mathbf{X} \det(-2i\mathbf{X}^{-1} + \mathbf{C}(G) + i\mathbf{D}) = \frac{2^m i^n \det(-2i\mathbf{X}^{-1} + \mathbf{C}(G) + i\mathbf{D})}{\prod_{j=1}^n (v_j - iv_j + \lambda i)}.$$

Since $2x_j^{-1} = v_j - iv_j + \lambda i$, we have

$$-2i\mathbf{X}^{-1} = -i(1-i)\mathbf{D} + \lambda \mathbf{I}_n$$

and so

$$-2i\mathbf{X}^{-1} + \mathbf{C}(G) + i\mathbf{D} = \lambda \mathbf{I}_n + \mathbf{C}(G) - \mathbf{D}.$$

Hence

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = \frac{2^m i^n \det(\lambda \mathbf{I}_n + \mathbf{C}(G) - \mathbf{D})}{\prod_{j=1}^n (v_j - iv_j + \lambda i)}.$$

Q.E.D.

We present some determinant on the bond scattering matrix of a regular graph $G$ by using the Bartholdi zeta function of $G$.

**Corollary 1 (Smilansky)** *Let $G$ be an $r$-reguar graph with $n$ vertices and $m$ edges. Then, for the bond scattering matrix of $G$,*

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = 2^m i^n (r - ir + \lambda i)^{-n} \det(\lambda \mathbf{I}_n + \mathbf{C}(G) - r\mathbf{I}_n).$$

**Proof**. Let $G$ be an $r$-regular graph with $n$ vertices and $m$ edges, $V(G) = \{u_1, \cdots, u_n\}$ and $R(G) = \{b_1, \ldots, b_m, \hat{b}_1, \ldots, \hat{b}_m\}$. Then we have

$$x = x_j = x_{u_j} = \frac{2}{r} \frac{1}{1 - i(1 - \lambda/r)}$$

for each $j = 1, \ldots, n$. Thus, each $\sigma_{b,c}^{(t(c))}(\lambda)$ in (1) are given by

$$\sigma_{b,c}^{(t(c))} = \begin{cases} -ix & \text{if } t(c) = o(b), \\ i(1-x) & \text{if } c = \hat{b}, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 4, we have

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda))^{-1} = \prod_{[p]} (1 - a_p(\lambda))^{-1},$$

where $[p]$ runs over all primitive periodic orbits of $G$. Since

$$a_p(\lambda) = \sigma_{b_1,b_n}^{(t(b_n))} \sigma_{b_n,b_{n-1}}^{(t(b_{n-1}))} \cdots \sigma_{b_2,b_1}^{(t(b_1))}, \; p = (b_1, b_2, \ldots, b_n),$$

we have

$$\begin{aligned} \det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) &= \prod_{[p]} \left( 1 - \left( i(1-x) \right)^{cbc(p)} (-ix)^{|p| - cbc(p)} \right)^{-1} \\ &= \prod_{[p]} \left( 1 - \left( \frac{i(1-x)}{-ix} \right)^{cbc(p)} (-ix)^{|p|} \right)^{-1}. \end{aligned}$$

Now, let

$$u = \frac{i(1-x)}{-ix},\ t = -ix.$$

By Theorem 2, since $u = 1 + i/t$, we have

$$\begin{aligned}\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) &= (1-(1-u)^2t^2)^{m-n}\det(\mathbf{I}_n - t\mathbf{C}(G) + (1-u)t^2(r\mathbf{I}_n - (1-u)\mathbf{I}_n)) \\ &= 2^{m-n}\det(\mathbf{I}_n - t\mathbf{C}(G) - i(rt+i)\mathbf{I}_n) \\ &= 2^{m-n}\det(2\mathbf{I}_n - t(\mathbf{C}(G) + ir\mathbf{I}_n)) \\ &= 2^{m-n}(-t)^n\det(-2/t\mathbf{I}_n + \mathbf{C}(G) + ir\mathbf{I}_n)\end{aligned}$$

Since

$$-\frac{2}{t} = -i(r - ri + \lambda i),$$

we have

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = 2^{m-n}i^n(r - ri + \lambda)^{-n}\det(\lambda\mathbf{I}_n + \mathbf{C}(G) - r\mathbf{I}_n).$$

Q.E.D.

# 3 The scattering matrix of a semiregular bipartite graph

We present a decomposition formula for some determinant on the scattering matrix of a semiregular bipartite graph.

A graph $G$ is called *bipartite*, denoted by $G = (V_1, V_2)$ if there exists a partition $V(G) = V_1 \cup V_2$ of $V(G)$ such that $uv \in E(G)$ if and only if $u \in V_1$ and $v \in V_2$. A bipartite graph $G = (V_1, V_2)$ is called $(q_1+1, q_2+1)$-*semiregular* if $\deg_G v = q_i + 1$ for each $v \in V_i (i = 1, 2)$. For a $(q_1+1, q_2+1)$-semiregular bipartite graph $G = (V_1, V_2)$, let $G^{[i]}$ be the graph with vertex set $V_i$ and an edge between two vertices in $G^{[i]}$ if there is a path of length two between them in $G$ for $i = 1, 2$. Then $G^{[1]}$ is $(q_1+1)q_2$-regular, and $G^{[2]}$ is $(q_2+1)q_1$-regular.

By Theorem 5, we obtain the following result.

**Theorem 6** *Let $G = (V_1, V_2)$ be a connected $(q_1+1, q_2+1)$-semiregular bipartite graph with $\nu$ vertices and $\epsilon$ edges. Set $\mid V_1 \mid = n$, $\mid V_2 \mid = m (n \leq m)$. Then*

$$\det(\mathbf{I}_{2\epsilon} - \mathbf{U}(\lambda)) = 2^m i^n (\lambda - q_2 - 1)^{m-n}\frac{\prod_{j=1}^{n}(\lambda^2 - (q_1+q_2-2)\lambda + (q_1+1)(q_2+1) - \lambda_j^2)}{((q_1+1)(1-i)+\lambda i)^n((q_2+1)(1-i)+\lambda i)^m}.$$

*where $Spec(G) = \{\pm\lambda_1, \cdots, \pm\lambda_n, 0, \cdots, 0\}$.*

**Proof**. The argument is an analogue of Hashimoto's method [7].

By Theorem 5, we have

$$\det(\mathbf{I}_{2\epsilon}-\mathbf{U}(\lambda))=\frac{2^{\epsilon}i^{\nu}\det(\lambda\mathbf{I}_{\nu}+\mathbf{C}(G)-\mathbf{D})}{((q_1+1)(1-i)+\lambda i)^n((q_2+1)(1-i)+\lambda i)^m}.$$

Let $V_1=\{u_1,\cdots,u_n\}$ and $V_2=\{s_1,\cdots,s_m\}$. Arrange vertices of $G$ as follows: $u_1,\cdots,u_n;v_1,\cdots,v_m$. We consider the matrix $\mathbf{C}(G)$ under this order. Then, with the definition, we can see that

$$\mathbf{C}(G)=\begin{bmatrix}\mathbf{0} & \mathbf{B}\\ {}^t\mathbf{B} & \mathbf{0}\end{bmatrix}.$$

Since $\mathbf{C}(G)$ is symmetric, there exists a orthogonal matrix $\mathbf{U}\in U(m)$ such that

$$\mathbf{B}\mathbf{U}=\begin{bmatrix}\mathbf{C} & \mathbf{0}\end{bmatrix}=\begin{bmatrix}\mu_1 & & 0 & 0 & \cdots & 0\\ & \ddots & & \vdots & & \vdots\\ \star & & \mu_n & 0 & \cdots & 0\end{bmatrix}.$$

Now, let

$$\mathbf{P}=\begin{bmatrix}\mathbf{I}_n & \mathbf{0}\\ \mathbf{0} & \mathbf{U}\end{bmatrix}.$$

Then we have

$${}^t\mathbf{P}\mathbf{C}(G)\mathbf{P}=\begin{bmatrix}\mathbf{0} & \mathbf{F} & \mathbf{0}\\ {}^t\mathbf{F} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},$$

where ${}^t\mathbf{F}$ is the transpose of $\mathbf{F}$. Furthermore, we have

$${}^t\mathbf{P}\mathbf{D}\mathbf{P}=\mathbf{D}.$$

Thus,

$$\begin{aligned}&\det(\mathbf{I}_{2\epsilon}-\mathbf{U}(\lambda))\\ &=\frac{2^m i^n(\lambda-q_2-1)^{m-n}}{((q_1+1)(1-i)+\lambda i)^n((q_2+1)(1-i)+\lambda i)^m}\det\begin{bmatrix}(\lambda-q_1-1)\mathbf{I}_n & -\mathbf{F}\\ -{}^t\mathbf{F} & (\lambda-q_2-1)\mathbf{I}_n\end{bmatrix}\\ &=\frac{2^m i^n(\lambda-q_2-1)^{m-n}}{((q_1+1)(1-i)+\lambda i)^n((q_2+1)(1-i)+\lambda i)^m}\\ &\quad\times\det\begin{bmatrix}(\lambda-q_1-1)\mathbf{I}_n & \mathbf{0}\\ -{}^t\mathbf{F} & (\lambda-q_2-1)\mathbf{I}_n-(\lambda-q_1-1)^{-1t}\mathbf{F}\mathbf{F}\end{bmatrix}\\ &=\frac{2^m i^n(\lambda-q_2-1)^{m-n}}{((q_1+1)(1-i)+\lambda i)^n((q_2+1)(1-i)+\lambda i)^m}\det\left((\lambda-q_1-1)(\lambda-q_2-1)\mathbf{I}_n-{}^t\mathbf{F}\mathbf{F}\right).\end{aligned}$$

Since $\mathbf{C}(G)$ is symmetric, ${}^t\mathbf{F}\mathbf{F}$ is Hermitian and positive definite, i.e., the eigenvalues of ${}^t\mathbf{F}\mathbf{F}$ are of form:

$$\lambda_1^2,\cdots,\lambda_n^2(\lambda_1,\cdots,\lambda_n\geq 0).$$

Therefore it follows that

$$\det(\mathbf{I}_{2\epsilon} - \mathbf{U}(\lambda)) = 2^m i^n (\lambda - q_2 - 1)^{m-n} \frac{\prod_{j=1}^{n}(\lambda^2 - (q_1 + q_2 - 2)\lambda + (q_1 + 1)(q_2 + 1) - \lambda_j^2)}{((q_1 + 1)(1 - i) + \lambda i)^n((q_2 + 1)(1 - i) + \lambda i)^m}.$$

But, we have

$$\det(\lambda \mathbf{I} - \mathbf{C}(G)) = \lambda^{(m-n)} \det(\lambda^2 \mathbf{I} - {}^t\mathbf{F}\mathbf{F}),$$

and so

$$Spec(G) = \{\pm\lambda_1, \cdots, \pm\lambda_n, 0, \cdots, 0\}.$$

Therefore, the result follows. Q.E.D.

# 4 A weighted version of the scattering matrix of a graph

Let $G$ be a connected graph with $n$ vertices and $m$ unoriented edges, and $\tilde{\mathbf{C}} = \tilde{\mathbf{C}}(G)$ a symmertic weighted matrix of $G$ with all nonnegative elements. Then $\tilde{\mathbf{C}}(G)$ is called a *non-negative* symmetric weighted matrix of $G$. Set $V(G) = \{u_1, \cdots, u_n\}$, $R(G) = \{b_1, \ldots, b_m, \hat{b}_1, \ldots, \hat{b}_m\}$. and

$$v_j = \sum_{o(b)=u_j} w(b) \ for \ j = 1, \ldots, n.$$

Smilansky [13] considered a weighted version of the characteristic polynomial of the bond scattering matrix of a regular graph $G$, and expressed it by using the characteristic polynomial of its weighted Laplacian of $G$.

The *weighted bond scattering matrix* $\mathbf{U}(\lambda) = (U_{ef})_{e,f \in R(G)}$ of $G$ is defined by

$$U_{ef} = \begin{cases} i(\delta_{\hat{e},f} - x_{t(f)}\sqrt{w(e)}\sqrt{w(f)}) & \text{if } t(f) = o(e), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$x_j = x_{u_j} = \frac{2}{v_j} \frac{1}{1 - i(1 - \lambda/v_j)}$$

for each $j = 1, \ldots, n$.

Smilansky [13] stated a formula for some determinant on the weighted scattering matrix of a graph $G$ without a proof.

**Theorem 7 (Smilansky)** *Let $G$ be a connected graph with $n$ vertices and $m$ unoriented edges and $\tilde{\mathbf{C}}(G)$ a non-negative symmetric weighted matrix of $G$. Then, for the weighted scattering matrix of $G$,*

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = \frac{2^m i^n \det(\lambda \mathbf{I}_n + \tilde{\mathbf{C}}(G) - \tilde{\mathbf{D}})}{\prod_{j=1}^{n}(v_j - iv_j + \lambda i)}.$$

**Proof**. The argument is an analogue of Bass' method [2].

Let $\rho$ be a unitary representation of $\Gamma$, and $d$ the degree of $\rho$. Furthermore, let $V(G) = \{u_1, \cdots, u_n\}$ and $R(G) = \{b_1, \cdots, b_m, b_{m+1}, \cdots, b_{2m}\}$ such that $b_{m+i} = \hat{b}_i (1 \leq i \leq m)$. Let $\mathbf{K} = (\mathbf{K}_{i,j})_{1 \leq i \leq 2l; 1 \leq j \leq n}$ be the $2l \times n$ matrix defined as follows:

$$\mathbf{K}_{i,j} := \begin{cases} \sqrt{w(b_i)} & \text{if } t(b_i) = u_j, \\ 0 & \text{otherwise.} \end{cases}$$

Next we define two $2m \times n$ matrices $\mathbf{L} = (\mathbf{L}_{i,j})_{1 \leq i \leq 2m; 1 \leq j \leq n}$ and $\mathbf{H} = (\mathbf{H}_{i,j})_{1 \leq i \leq 2m; 1 \leq j \leq n}$ as follows:

$$\mathbf{L}_{i,j} := \begin{cases} \sqrt{w(b_i)} x_{u_j} & \text{if } o(b_i) = u_j, \\ 0 & \text{otherwise.} \end{cases}, \mathbf{H}_{i,j} := \begin{cases} \sqrt{w(b_i)} & \text{if } o(b_i) = u_j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\mathbf{L} = \mathbf{H} \begin{bmatrix} x_{u_1} & & 0 \\ & \ddots & \\ 0 & & x_{u_n} \end{bmatrix} = \mathbf{HX}. \tag{4}$$

Then we have

$$\mathbf{L}^t \mathbf{K} = {}^t\mathbf{B}. \tag{5}$$

and

$${}^t\mathbf{HK} = \tilde{\mathbf{C}}(G), \tag{6}$$

where two matrices $\mathbf{B} = (B_{ef})_{e,f \in R(G)}$ and $\tilde{\mathbf{C}}(G) = (w_{us})_{u,s \in V(G)}$ are given by

$$B_{ef} := \begin{cases} x_{t(e)} \sqrt{w(e) w(f)} & \text{if } t(e) = o(f), \\ 0 & \text{otherwise.} \end{cases}, w_{uv} := \begin{cases} w(u,s) & \text{if } (u,s) \in R(G), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore,

$${}^t\mathbf{HH} = \tilde{\mathbf{D}}. \tag{7}$$

Next, we have

$${}^t\mathbf{KL} = {}^t\mathbf{W}_x(G) \tag{8}$$

and

$${}^t\mathbf{HL} = \mathbf{D}_x, \tag{9}$$

where two matrices $\mathbf{W}_x = ((w_x)_{us})_{u,s \in V(G)}$ and $\mathbf{D}_x = (d_{us})_{u,s \in V(G)}$ are given by

$$(w_x)_{us} := \begin{cases} w(u,s) x_u & \text{if } (u,s) \in R(G), \\ 0 & \text{otherwise.} \end{cases}, d_{us} := \begin{cases} v_u x_u & \text{if } u = s, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & w(b_1) x_{o((\hat{b}_1)} \oplus \cdots \oplus w(b_m) x_{o(\hat{b}_m)} \\ w(b_1) x_{o(b_1)} \oplus \cdots \oplus w(b_m) x_{o(b_m)} & \mathbf{0} \end{bmatrix}$$

and

$$\mathbf{T} = \mathbf{B} - \mathbf{J}.$$

Then we have

$$\mathbf{L}{}^t\mathbf{H} = {}^t\mathbf{T}{}^t\mathbf{J}_0 + (w(b_1)x_{o(b_1)} \oplus \cdots \oplus w(\hat{b}_m)x_{o(\hat{b}_m)}). \tag{10}$$

We introduce two $(2m+n) \times (2m+n)$ matrices as follows:

$$\mathbf{P} = \begin{bmatrix} (1-u^2)\mathbf{I}_n & -{}^t\mathbf{K} + u\ {}^t\mathbf{H} \\ \mathbf{0} & \mathbf{I}_{2m} \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} \mathbf{I}_n & {}^t\mathbf{K} - u\ {}^t\mathbf{H} \\ u\mathbf{L} & (1-u^2)\mathbf{I}_{2m} \end{bmatrix}$$

By (8) and (9), we have

$$\begin{aligned} \mathbf{PQ} &= \begin{bmatrix} (1-u^2)\mathbf{I}_n - u\ {}^t\mathbf{KL} + u^2\ {}^t\mathbf{HL} & \mathbf{0} \\ u\mathbf{L} & (1-u^2)\mathbf{I}_{2m} \end{bmatrix} \\ &= \begin{bmatrix} (1-u^2)\mathbf{I}_n - u\ {}^t\mathbf{W}_x(G) + u^2\mathbf{D}_x & \\ u\mathbf{L} & (1-u^2)\mathbf{I}_{2m} \end{bmatrix}. \end{aligned}$$

By (5) and (10),

$$\mathbf{QP} = \begin{bmatrix} (1-u^2)\mathbf{I}_n & \mathbf{0} \\ u(1-u^2)\mathbf{L} & -u\mathbf{L}{}^t\mathbf{K} + u^2\mathbf{L}{}^t\mathbf{H} + (1-u^2)\mathbf{I}_{2m} \end{bmatrix}.$$

Since

$$w(b_1)x_{o(b_1)} \oplus \cdots \oplus w(\hat{b}_m)x_{o(\hat{b}_m)} = {}^t\mathbf{J}{}^t\mathbf{J}_0$$

and $({}^t\mathbf{J}_0)^2 = \mathbf{I}_{2m}$, we have

$$\begin{aligned} & -u\mathbf{L}{}^t\mathbf{K} + u^2\mathbf{L}{}^t\mathbf{H} + (1-u^2)\mathbf{I}_{2m} \\ = \ & \mathbf{I}_{2m} - u({}^t\mathbf{T} + {}^t\mathbf{J}) + u^2({}^t\mathbf{T}{}^t\mathbf{J}_0 + {}^t\mathbf{J}{}^t\mathbf{J}_0 - {}^t\mathbf{J}_0{}^t\mathbf{J}_0) \\ = \ & (\mathbf{I}_{2m} - u({}^t\mathbf{T} + {}^t\mathbf{J} - {}^t\mathbf{J}_0))(\mathbf{I}_{2m} - u{}^t\mathbf{J}_0). \end{aligned}$$

Thus,

$$\mathbf{QP} = \begin{bmatrix} (1-u^2)\mathbf{I}_n & \mathbf{0} \\ u(1-u^2)\mathbf{L} & (\mathbf{I}_{2m} - u({}^t\mathbf{T} + {}^t\mathbf{J} - {}^t\mathbf{J}_0))(\mathbf{I}_{2m} - u{}^t\mathbf{J}_0) \end{bmatrix}.$$

Since $\det(\mathbf{PQ}) = \det(\mathbf{QP})$, we have

$$\begin{aligned} & (1-u^2)^{2m} \det(\mathbf{I}_n - u\ {}^t\mathbf{W}_x(G) + (\mathbf{D}_x - \mathbf{I}_n)u^2) \\ & = (1-u^2)^n \det(\mathbf{I}_{2m} - u({}^t\mathbf{T} + {}^t\mathbf{J} - {}^t\mathbf{J}_0)) \det(\mathbf{I}_{2m} - u\ {}^t\mathbf{J}_0). \end{aligned}$$

But,

$$\begin{aligned} \det(\mathbf{I}_{2m} - u\ {}^t\mathbf{J}_0) &= \det\left(\begin{bmatrix} \mathbf{I}_m & u\mathbf{I}_m \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}\right) \det\left(\begin{bmatrix} \mathbf{I}_m & -u\mathbf{I}_m \\ -u\mathbf{I}_m & \mathbf{I}_m \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} (1-u^2)\mathbf{I}_m & \mathbf{0} \\ * & \mathbf{I}_m \end{bmatrix}\right) = (1-u^2)^m. \end{aligned}$$

Therefore it follows that

$$(1-u^2)^{2m}\det(\mathbf{I}_n - u\,{}^t\mathbf{W}_x(G) + (\mathbf{D}_x - \mathbf{I}_n)u^2) = (1-u^2)^{(m+n)}\det(\mathbf{I}_{2m} - u({}^t\mathbf{T} + {}^t\mathbf{J} - {}^t\mathbf{J}_0)).$$

Hence

$$\det(\mathbf{I}_{2m} - u({}^t\mathbf{B} - {}^t\mathbf{J}_0)) = (1-u^2)^{(m-n)}\det(\mathbf{I}_n - u\mathbf{W}_x(G) + (\mathbf{D}_x - \mathbf{I}_n))u^2). \tag{11}$$

But, since

$$i\mathbf{U}(\lambda) + \mathbf{J}_0 = {}^t\mathbf{B},$$

we have

$${}^t\mathbf{B} - {}^t\mathbf{J}_0 = i\mathbf{U}(\lambda).$$

Substituting $u = -i$ in (11), we obtain

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = 2^{m-n}\det(\mathbf{I}_n + i\mathbf{W}_x(G) - (\mathbf{D}_x - \mathbf{I}_n)). \tag{12}$$

By (4), (6) and (8), we have

$$\mathbf{W}_x(G) = {}^t\mathbf{L}\mathbf{K} = {}^t\mathbf{X}{}^t\mathbf{H}\mathbf{K} = \mathbf{X}\tilde{\mathbf{C}}(G).$$

Furthermore, by (4), (7) and (9), we have

$$\mathbf{D}_x = {}^t\mathbf{L}\mathbf{H} = {}^t\mathbf{X}{}^t\mathbf{H}\mathbf{H} = \mathbf{X}\tilde{\mathbf{D}}.$$

Thus, we have

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = 2^{m-n}\det(2\mathbf{I}_n + \mathbf{X}\tilde{\mathbf{C}}(G) + i\mathbf{X}\tilde{\mathbf{D}})$$
$$= 2^{m-n}i^n\det\mathbf{X}\det(-2i\mathbf{X}^{-1} + \tilde{\mathbf{C}}(G) + i\tilde{\mathbf{D}}) = \frac{2^m i^n\det(2\mathbf{X}^{-1} + i\tilde{\mathbf{C}}(G) - \tilde{\mathbf{D}})}{\prod_{j=1}^n(v_j - iv_j + \lambda i)}.$$

Since $2x_j^{-1} = v_j - iv_j + \lambda i$, we have

$$-2i\mathbf{X}^{-1} = -i(1-i)\tilde{\mathbf{D}} + \lambda\mathbf{I}_n$$

and so

$$-2i\mathbf{X}^{-1} + \tilde{\mathbf{C}}(G) + i\tilde{\mathbf{D}} = \lambda\mathbf{I}_n + \tilde{\mathbf{C}}(G) - \tilde{\mathbf{D}}.$$

Hence

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = \frac{2^m i^n\det(\lambda\mathbf{I}_n + \tilde{\mathbf{C}}(G) - \tilde{\mathbf{D}})}{\prod_{j=1}^n(d_j - id_j + \lambda i)}.$$

Q.E.D.

Let $G$ be a connected graph and $\tilde{\mathbf{C}} = \tilde{\mathbf{C}}(G)$ a weighted matrix of $G$. Then $G$ is called a *$r$-regular weighted graph* if $\sum_{o(b)=u} w(b) = r$ for each $u \in V(G)$.

By Theorem 7, the following result holds.

**Corollary 2** *Let $G$ be an $r$-reguar weighted graph with $n$ vertices and $m$ edges, and $\tilde{\mathbf{C}}(G)$ a non-negative symmetric weighted matrix of $G$. Then*

$$\det(\mathbf{I}_{2m} - \mathbf{U}(\lambda)) = 2^m i^n (r - ir + \lambda i)^{-n} \det(\lambda \mathbf{I}_n + \tilde{\mathbf{C}}(G) - r\mathbf{I}_n).$$

Let $G = (V_1, V_2)$ be a bipartite graph. Then $G$ is called a $(q_1+1, q_2+1)$-*semiregular* weighted bipartite graph if $\sum_{o(e)=v} w(e) = q_i + 1$ for each $v \in V_i (i = 1, 2)$.

Similarly to the proof of Theorem 6, the following result holds.

**Corollary 3** *Let $G = (V_1, V_2)$ be a connected $(q_1+1, q_2+1)$-semiregular weighted bipartite graph with $\nu$ vertices and $\epsilon$ edges, and $\tilde{\mathbf{C}} = \tilde{\mathbf{C}}(G)$ a real symmetric weighted matrix of $G$. Set $\mid V_1 \mid = n$, $\mid V_2 \mid = m (n \leq m)$. Then*

$$\det(\mathbf{I}_{2\epsilon} - \mathbf{U}(\lambda)) = 2^m i^n (\lambda - q_2 - 1)^{m-n} \frac{\prod_{j=1}^{n} (\lambda^2 - (q_1 + q_2 - 2)\lambda + (q_1+1)(q_2+1) - \lambda_j^2)}{((q_1+1)(1-i) + \lambda i)^n ((q_2+1)(1-i) + \lambda i)^m}.$$

*where $Spec(\tilde{\mathbf{C}}(G)) = \{\pm\lambda_1, \cdots, \pm\lambda_n, 0, \cdots, 0\}$.*

# 5 The Euler product for a new zeta function

We present the Euler product for a new zeta function of a graph.

Foata and Zeilberger [4] gave a new proof of Bass's Theorem by using the algebra of Lyndon words. Let $X$ be a finite nonempty set, $<$ a total order in $X$, and $X^*$ the free monoid generated by $X$. Then the total order $<$ on $X$ derive the lexicographic order $<$ on $X^*$. A *Lyndon word* in $X$ is defined to a nonempty word in $X^*$ which is prime, i.e., not the power $l^r$ of any other word $l$ for any $r \geq 2$, and which is also minimal in the class of its cyclic rearrangements under $<$(see [9]). Let $L$ denote the set of all Lyndon words in $X$.

Let $\mathbf{F}$ be a square matrix whose entries $b(x, x')(x, x' \in X)$ form a set of commuting variables. If $w = x_1 x_2 \cdots x_m$ is a word in $X^*$, define

$$\beta(w) = b(x_1, x_2) b(x_2, x_3) \cdots b(x_{m-1}, x_m) b(x_m, x_1).$$

Furthermore, let

$$\beta(L) = \prod_{l \in L} (1 - \beta(l)).$$

The following theorem played a central role in [4].

**Theorem 8 (Foata and Zeilbereger)** $\beta(L) = \det(\mathbf{I} - \mathbf{F})$.

Let $G$ be a connected graph and $\tilde{\mathbf{C}}(G)$ a weighted matrix of $G$. Then, let $w(e, f)$ be the $(e, f)$-array of the matrix $\mathbf{B} - \mathbf{J}_0$.

**Theorem 9** *Let $G$ be a connected graph, and let $\tilde{\mathbf{C}} = \tilde{\mathbf{C}}(G)$ be a weighted matrix of $G$. Then the reciprocal of the zeta function of $G$ is given by*

$$\mathbf{Z}_1(G, w, t) = \prod_{[p]} (1 - w_p t^{|p|})^{-1},$$

*where $[p]$ runs over all primitive periodic orbits of $G$, and*

$$w_p = w(b_1, b_2) w(b_2, b_3) \cdots w(b_{n-1}, b_n),\ \ p = (b_1, b_2, \ldots, b_n)$$

**Proof**. Let $R(G) = \{b_1, \cdots, b_{2m}\}$ such that $b_{m+j} = \hat{b}_j (1 \leq j \leq m)$, and $b_1 < b_2 < \cdots < b_{2m}$ a total order of $R(G)$. We consider the free monid $R(G)^*$ generated by $R(G)$, and the lexicographic order on $R(G)^*$ derived from $<$. If a cycle $p$ is primitive, then there exists a unique cycle in $[p]$ which is a Lyndon word in $R(G)$.

For $z \in R(G)^*$, let

$$\beta(z) = \begin{cases} w_z t^{|z|} & \text{if } z \text{ is a primitive cycle,} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\beta(L) = \prod_{l \in L} (1 - \beta(l)) = \prod_{[p]} (1 - w_p t^{|p|}),$$

where $[p]$ runs over all primitive periodic orbits of $G$. Furthermore, we define variables $b(x, x')(x, x' \in R(G))$ as follows:

$$b(x, x') = \begin{cases} w(x, x') & \text{if } t(x) = o(x'), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 8 implies that

$$\prod_{[p]} (1 - w_p t^{|p|}) = \det(\mathbf{I} - t\mathbf{F}) = \det(\mathbf{I} - t(\mathbf{B} - \mathbf{J}_0)).$$

Q.E.D.

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