Generalized Schur Numbers for $x_1 + x_2 + c = 3x_3$

André E. Kézdy
Department of Mathematics
University of Louisville
Louisville, KY 40292 USA
kezdy@louisville.edu

Hunter S. Snevily
Department of Mathematics
University of Idaho
Moscow, ID 83844 USA
snevily@uidaho.edu

Susan C. White
Department of Mathematics
University of Louisville
Louisville, KY 40292 USA
susan.white@louisville.edu

Submitted: Jul 10, 2008; Accepted: Jul 30, 2009; Published: Aug 14, 2009
Mathematics Subject Classification: 05D10

Abstract

Let $r(c)$ be the least positive integer n such that every two coloring of the integers $1, \ldots, n$ contains a monochromatic solution to $x_1 + x_2 + c = 3x_3$. Verifying a conjecture of Martinelli and Schaal, we prove that

$$r(c) = \left\lceil \frac{2\lceil \frac{2+c}{3} \rceil + c}{3} \right\rceil,$$

for all $c \geqslant 13$, and

$$r(c) = \left\lceil \frac{3\lceil \frac{3-c}{2} \rceil - c}{2} \right\rceil,$$

for all $c \leqslant -4$.

Section 1. Introduction

Let $\mathbb{N}$ denote the set of positive integers, and $[a, b] = \{n \in \mathbb{N} : a \leqslant n \leqslant b\}$. A map $\chi : [a, b] \to [1, t]$ is a t-coloring of $[a, b]$. Let L be a system of equations in the variables $x_1, \ldots, x_m$. A positive integral solution $n_1, \ldots, n_m$ to L is *monochromatic* if $\chi(n_i) = \chi(n_j)$, for all $1 \leqslant i, j \leqslant m$. The *$t$-color generalized Schur number of L*, denoted $S_t(L)$, is the least positive integer n, if it exists, such that any t-coloring of $[1, n]$ results in a monochromatic solution to L. If no such n exists, then $S_t(L)$ is ∞.

A classical result of Schur [5] states that $S_t(L) < \infty$ for $L = \{x_1 + x_2 = x_3\}$ and all $t \geqslant 2$. An exercise is to show that $S_4(L) = \infty$ for $L = \{x + y = 3z\}$. Very few generalized Schur numbers are known, but several recent papers have revived interest in determining some of them (for example [1, 2, 3, 4]).

In this paper we answer a conjecture posed by Martinelli and Schaal [3] concerning the 2-color generalized Schur number of the equation $x_1 + x_2 + c = 3x_3$. This number is denoted $r(c)$. Verifying the conjecture, we prove in section that

$$r(c) = \left\lceil \frac{2\lceil\frac{2+c}{3}\rceil + c}{3} \right\rceil,$$

for all $c \geqslant 13$, and we prove in section that

$$r(c) = \left\lceil \frac{3\lceil\frac{3-c}{2}\rceil - c}{2} \right\rceil,$$

for all $c \leqslant -4$. Martinelli and Schaal were motivated to consider a more general equation

$$x_1 + x_2 + c = kx_3,$$

where c is an arbitrary integer and k is a positive integer. They denote the 2-color generalized Schur number of this equation by $r(c, k)$. They prove that $r(c, k) = \infty$ for any odd c and even k, and give a general lower bound. In section we briefly examine this general lower bound.

Section 2. Positive c

In this section we prove that

$$r(c) = \left\lceil \frac{2\lceil\frac{2+c}{3}\rceil + c}{3} \right\rceil, \text{ for all } c \geqslant 13. \tag{1}$$

In their paper, Martinelli and Schaal show that this is a lower bound for $r(c)$ (see Lemma 2 of [3]) so it suffices to prove that this is an upper bound. They also note that for positive values of c less than 13, the bound given by (1) is too small.

It is convenient for us to assume $c > 48$ since this guarantees that M_2 (defined later in Lemma 2) is at least six. The reader can verify the conjecture for values $13 \leqslant c \leqslant 48$. As an example, we will show that the conjecture is true for $c = 24$; a similar argument may be used to verify the conjecture for other values of c. Let $c = 24$. The claim is that $r(24) = 14$. We must show that any 2-coloring of $[1, 14]$ contains a monochromatic solution to $x_1 + x_2 + 24 = 3x_3$. Assume that the two colors used in the coloring of $[1, 14]$ are red and blue. Consider two cases according to whether the values 2 and 9 have the same color or opposite color. If 2 and 9 are the same color, say red, then

$$\begin{aligned}
9 + 9 + 24 &= 3(14) && \text{so we may assume that 14 is blue.}\\
1 + 2 + 24 &= 3\ \ (9) && \text{so we may assume that 1 is blue.}\\
2 + 13 + 24 &= 3(13) && \text{so we may assume that 13 is blue.}\\
1 + 14 + 24 &= 3(13) && \text{is now all blue.}
\end{aligned}$$

If 2 is red and 9 is blue, then

$$\begin{array}{rcll} 9+9+24 &=& 3(14) & \text{so we may assume that 14 is red.} \\ 2+13+24 &=& 3(13) & \text{so we may assume that 13 is blue.} \\ 9+6+24 &=& 3(13) & \text{so we may assume that 6 is red.} \\ 14+4+24 &=& 3(14) & \text{so we may assume that 4 is blue.} \\ 4+11+24 &=& 3(13) & \text{so we may assume that 11 is red.} \\ 6+12+24 &=& 3(14) & \text{so we may assume that 12 is blue.} \\ 9+3+24 &=& 3(12) & \text{so we may assume that 3 is red.} \\ 3+6+24 &=& 3(11) & \text{is now all red.} \end{array}$$

We shall omit further details for values of $c \leqslant 48$.

For the remainder of this section we shall assume that $c > 48$,

$$N = \left\lceil \frac{2\lceil \frac{2+c}{3} \rceil + c}{3} \right\rceil,$$

and $\chi : [1, N] \to \{\text{red}, \text{blue}\}$ is a 2-coloring of the integers in the interval $[1, N]$ such that there is no monochromatic solution to $x_1 + x_2 + c = 3x_3$.

Lemma 1 (Cascade Lemma) *If $x \in [1, N]$, $x \equiv c \pmod 2$, and $x > \frac{c}{2}$, then*

$$\chi(x) = \chi(x-1) = \chi(x-2).$$

Proof. First we prove that $\chi(x) = \chi(x-2)$ by contradiction. Assume $\chi(x) \neq \chi(x-2)$. Without loss of generality, $\chi(x) = \text{red}$ and $\chi(x-2) = \text{blue}$. Because $x \equiv c \pmod 2$, the value $\frac{3x-c}{2}$ is an integer. To avoid a monochromatic solution to $x_1 + x_2 + c = 3x_3$,

$$\left(\frac{3x-c}{2}\right) + \left(\frac{3x-c}{2}\right) + c = 3x \quad \Rightarrow \quad \left(\frac{3x-c}{2}\right) \text{ is blue.}$$

Similarly,

$$\begin{array}{rcl} (2x-c) + x + c = 3x & \Rightarrow & 2x - c \text{ is blue.} \\ \left(\frac{3x-c}{2}\right) + \left(\frac{3x-c}{2} - 6\right) + c = 3(x-2) & \Rightarrow & \left(\frac{3x-c}{2} - 6\right) \text{ is red.} \\ \left(\frac{3x-c}{2} - 6\right) + \left(\frac{3x-c}{2} - 6\right) + c = 3(x-4) & \Rightarrow & (x-4) \text{ is blue.} \\ \left(\frac{3x-c}{2} - 12\right) + \left(\frac{3x-c}{2}\right) + c = 3(x-4) & \Rightarrow & \left(\frac{3x-c}{2} - 12\right) \text{ is red.} \\ \left(\frac{3x-c}{2} - 6\right) + \left(\frac{3x-c}{2} - 12\right) + c = 3(x-6) & \Rightarrow & (x-6) \text{ is blue.} \end{array}$$

Notice that the hypothesis $x > \frac{c}{2}$ and $c > 48$ guarantees that all of the intermediate numbers in these calculations are in the range $1, \ldots, N$. Now there is the following monochromatic solution to $x_1 + x_2 + c = 3x_3$:

$$(2x - c) + (x - 6) + c = 3(x - 2),$$

a contradiction.

Now we prove, also by contradiction, that $\chi(x) = \chi(x-1)$. Without loss of generality, assume $\chi(x) = $ red and $\chi(x-1) = $ blue. Note that the argument above shows that $\chi(x-2) = \chi(x) = $ red. Therefore,

$$\begin{aligned}
x + (2x - c) + c = 3x \quad &\Rightarrow \quad 2x - c \text{ is blue.} \\
\left(\frac{3x-c}{2}\right) + \left(\frac{3x-c}{2}\right) + c = 3x \quad &\Rightarrow \quad \left(\frac{3x-c}{2}\right) \text{ is blue.} \\
(2x - c) + (x - 3) + c = 3(x-1) \quad &\Rightarrow \quad (x-3) \text{ is red.} \\
\left(\frac{3x-c}{2}\right) + \left(\frac{3x-c}{2} - 3\right) + c = 3(x-1) \quad &\Rightarrow \quad \left(\frac{3x-c}{2} - 3\right) \text{ is red.}
\end{aligned}$$

Now there is the following monochromatic solution to $x_1 + x_2 + c = 3x_3$:

$$\left(\frac{3x-c}{2} - 3\right) + \left(\frac{3x-c}{2} - 3\right) + c = 3(x-2),$$

a contradiction. ◇

For positive values of c of the form $c = 9s + t$ $(0 \leqslant t \leqslant 8)$, we have

$$N = \left\lceil \frac{2\lceil\frac{2+c}{3}\rceil + c}{3} \right\rceil = 5s + \begin{cases} 1 & \text{if } t = 0 \text{ or } 1 \\ 2 & \text{if } t = 2 \\ 3 & \text{if } t = 3 \text{ or } 4 \\ 4 & \text{if } t = 5 \text{ or } 6 \\ 5 & \text{if } t = 7 \\ 6 & \text{if } t = 8. \end{cases}$$

Because $c = 9s + t$ is even if and only if $s \equiv t \pmod 2$, the description of N above shows $N \equiv c \pmod 2$ if and only if $c \not\equiv 0, 4,$ or $5 \pmod 9$. A consequence of this and the last part of Lemma 1 is that we can now easily describe a large subinterval of $[1, N]$ that must be monochromatic.

Corollary 1 *The interval $W_1 = [m_1, M_1]$ is monochromatic, where $m_1 := \lceil\frac{c-1}{2}\rceil$ and*

$$M_1 := \begin{cases} N - 1 & \text{if } c \equiv 0, 4, \text{ or } 5 \pmod 9 \\ N & \text{otherwise} \end{cases}$$

Proof. This follows from the prior lemma. ◇

The large monochromatic interval W_1 implies the existence of another large monochromatic interval, as shown in the next lemma.

Lemma 2 (Domino Lemma) *The interval $W_2 = [m_2, M_2]$ is monochromatic with color different than the color on the interval W_1, where $m_2 = 1$ and $M_2 = 3M_1 - m_1 - c$.*

Proof. Corollary 1 implies that the interval $W_1 = [m_1, M_1]$ is monochromatic. Consider the set

$$S = \{t : 1 \leqslant t \leqslant N \text{ and } \alpha + t + c = 3\beta, \text{ for some } \alpha, \beta \in W_1\}.$$

Because all values in W_1 have the same color, all values in S have the same color – the color opposite the one given the values in W_1. It suffices to prove that $[1, M_2] \subseteq S$.

If $\alpha = m_1$ and $\beta = M_1$, then $t = M_2$ so $M_2 \in S$. Suppose now that $1 < t \in S$ via

$$\alpha + t + c = 3\beta, \text{ for some } \alpha, \beta \in W_1.$$

We shall prove that $t - 1 \in S$.

If $\alpha + 1 \notin W_1$ and $\alpha - 2 \notin W_1$, then $M_1 - m_1 \leqslant 1$ which implies $N - 1 - (c-1)/2 \leqslant 1$, and thus $c < 27$, a contradiction. In the case that $\alpha + 1 \notin W_1$ and $\beta - 1 \notin W_1$, it follows that $\alpha = M_1$ and $\beta = m_1$ so

$$\begin{aligned} 1 < t &= 3\beta - \alpha - c \\ &= 3m_1 - M_1 - c \\ &\leqslant 3\left(\frac{c}{2}\right) - (N-1) - c \\ &\leqslant 0, \end{aligned}$$

a contradiction.

So, either $\alpha + 1 \in W_1$ or $\alpha - 2, \beta - 1 \in W_1$. In the former case, the equation $(\alpha + 1) + (t - 1) + c = 3\beta$ implies that $t - 1 \in S$. In the latter case, the equation $(\alpha - 2) + (t - 1) + c = 3(\beta - 1)$ implies $t - 1 \in S$. Either way, $t - 1 \in S$, so $[1, M_2] \subseteq S$ as desired. ◇

Now we are ready to prove the Martinelli-Schaal conjecture for large positive c.

Theorem 1 *Assume $c > 48$ and $N = \left\lceil \frac{2\lceil \frac{2+c}{3} \rceil + c}{3} \right\rceil$. Any 2-coloring of the integers in the interval $[1, N]$ produces a monochromatic solution to $x_1 + x_2 + c = 3x_3$. It follows that $r(c) = N$.*

Proof. Corollary 1 guarantees the interval $W_1 = [m_1, M_1]$ is monochromatic, say red. Lemma 2 ensures the interval $W_2 = [1, M_2]$ is monochromatic of the opposite color, blue.

We now consider the following two cases.

Case 1: $c \not\equiv 0, 4, \text{ or } 5 \pmod 9$.

In this case, as noted earlier, $N \equiv c \pmod 2$ which implies $M_1 = N$. In particular, N is red because it is a member of W_1.

Consider the elements $1, N, \frac{3N-c}{2}, \frac{9N-5c-2}{2}$. Observe that for c of the form $c = 9s + t$ $(0 \leqslant t \leqslant 8)$

$$\frac{9N-5c-2}{2} = \begin{cases} 1 & \text{if } t = 1 \\ 3 & \text{if } t = 2 \\ 5 & \text{if } t = 3 \\ 2 & \text{if } t = 6 \\ 4 & \text{if } t = 7 \\ 6 & \text{if } t = 8. \end{cases}$$

Because $c > 48$, the value $M_2 \geqslant 6$ so $\left(\frac{9N-5c-2}{2}\right)$ is blue. Therefore,

$$1 + \left(\frac{9N-5c-2}{2}\right) + c = 3\left(\frac{3N-c}{2}\right) \qquad \text{implies} \qquad \left(\frac{3N-c}{2}\right) \text{ is red.}$$

Now there is the following monochromatic solution to $x_1 + x_2 + c = 3x_3$:

$$\left(\frac{3N-c}{2}\right) + \left(\frac{3N-c}{2}\right) + c = 3(N).$$

Case 2: $c \equiv 0, 4,$ or $5 \pmod 9$.

In this case, $N \not\equiv c \pmod 2$, which implies $M_1 = N - 1$. In particular, N is not a member of W_1.

Consider the elements $1, N, 2N-c, \frac{3N-c-1}{2}, \frac{3N-c+1}{2}, \frac{9N-5c-5}{2}, \frac{9N-5c+1}{2}$. Observe that for c of the form $c = 9s + t$ $(0 \leqslant t \leqslant 8)$

$$\frac{9N-5c-5}{2} = \begin{cases} 2 & \text{if } t = 0 \\ 1 & \text{if } t = 4 \\ 3 & \text{if } t = 5 \end{cases}$$

Therefore, because $M_2 \geqslant 6$, both $\frac{9N-5c-5}{2}$ and $\frac{9N-5c+1}{2}$ are blue. Consequently,

$$1 + \left(\frac{9N-5c-5}{2}\right) + c = 3\left(\frac{3N-c-1}{2}\right) \qquad \text{implies} \qquad \left(\frac{3N-c-1}{2}\right) \text{ is red,}$$

and

$$1 + \left(\frac{9N-5c+1}{2}\right) + c = 3\left(\frac{3N-c+1}{2}\right) \qquad \text{implies} \qquad \left(\frac{3N-c+1}{2}\right) \text{ is red.}$$

Now $2N - c < M_2 = 3M_1 - m_1 - c = 3(N-1) - m_1 - c$, because $m_1 < N - 3$. So $2N - c$ is also blue. Hence

$$N + (2N - c) + c = 3N \qquad \text{implies} \qquad N \text{ is red.}$$

Now there is the following monochromatic solution to $x_1 + x_2 + c = 3x_3$:

$$\left(\frac{3N-c-1}{2}\right) + \left(\frac{3N-c+1}{2}\right) + c = 3(N).$$

◇

Section 3. Negative c

In this section we prove that

$$r(c) = \left\lceil \frac{3\lceil \frac{3-c}{2} \rceil - c}{2} \right\rceil, \text{ for all } c \leqslant -4. \tag{2}$$

In their paper, Martinelli and Schaal show that this is a lower bound for $r(c)$ (see Lemma 3 of [3]) so it suffices to prove that this is an upper bound. They also note that for negative values of c greater than -4, the bound given by (2) is too small. It is convenient for us to assume $c < -35$; the reason for this assumption is this value conveniently is enough to guarantee $\frac{5-c}{8} > 5$ via Lemma 4. The reader can verify the conjecture for values $-35 \leqslant c \leqslant -4$ as illustrated in the previous section for positive c.

For the remainder of this section we shall assume that $c < -35$,

$$N = \left\lceil \frac{3\lceil \frac{3-c}{2} \rceil - c}{2} \right\rceil,$$

and $\chi : [1, N] \to \{\text{red}, \text{blue}\}$ is a 2-coloring such that there is no monochromatic solution to $x_1 + x_2 + c = 3x_3$.

Lemma 3 *If $x \geqslant 5$, $2x-2-c \leqslant N$, and $x \equiv c$ (mod 2), then $\chi(x) = \chi(x-1) = \chi(x-2)$.*

Proof. We shall argue by contradiction. First assume, to the contrary, that $\chi(x) \neq \chi(x-1)$. Without loss of generality, $\chi(x) = \text{red}$ and $\chi(x-1) = \text{blue}$. By assumption $2x - 2 - c \leqslant N$ and $x \equiv c$ (mod 2), so the following equations involve integers in the interval $[1, N]$:

$$\begin{aligned}
(2x-2-c) + (x-1) + c = 3(x-1) \quad &\Rightarrow \quad 2x-2-c \text{ is red.} \\
\left(\frac{3x-c}{2}\right) + \left(\frac{3x-c}{2}\right) + c = 3x \quad &\Rightarrow \quad \left(\frac{3x-c}{2}\right) \text{ is blue.} \\
\left(\frac{3x-c}{2}\right) + \left(\frac{3x-c}{2} - 3\right) + c = 3(x-1) \quad &\Rightarrow \quad \left(\frac{3x-c}{2} - 3\right) \text{ is red.} \\
(2x-2-c) + (x+2) + c = 3x \quad &\Rightarrow \quad x+2 \text{ is blue.} \\
\left(\frac{3x-c}{2} - 3\right) + \left(\frac{3x-c}{2} + 3\right) + c = 3x \quad &\Rightarrow \quad \left(\frac{3x-c}{2} + 3\right) \text{ is blue.}
\end{aligned}$$

Now the following equation is all blue

$$\left(\frac{3x-c}{2} + 3\right) + \left(\frac{3x-c}{2} + 3\right) + c = 3(x+2),$$

a contradiction. Therefore, $\chi(x) = \chi(x-1)$.

Now let's assume that $\chi(x) \neq \chi(x-2)$. Without loss of generality, $\chi(x) = \text{red} = \chi(x-1)$ and $\chi(x-2) = \text{blue}$. By assumption $x \geqslant 5$, $2x-2-c \leqslant N$ and $x \equiv c \pmod 2$, so the following equations involve integers in the interval $[1, N]$:

$$\begin{aligned}
(2x-2-c) + (x-1) + c = 3(x-1) \quad &\Rightarrow \quad 2x-2-c \text{ is blue.}\\
\left(\frac{3x-c}{2}\right) + \left(\frac{3x-c}{2}\right) + c = 3x \quad &\Rightarrow \quad \left(\frac{3x-c}{2}\right) \text{ is blue.}\\
(2x-2-c) + (x-4) + c = 3(x-2) \quad &\Rightarrow \quad x-4 \text{ is red.}\\
\left(\frac{3x-c}{2}\right) + \left(\frac{3x-c}{2} - 6\right) + c = 3(x-2) \quad &\Rightarrow \quad \left(\frac{3x-c}{2} - 6\right) \text{ is red.}
\end{aligned}$$

Now the following equation is all red

$$\left(\frac{3x-c}{2} - 6\right) + \left(\frac{3x-c}{2} - 6\right) + c = 3(x-4),$$

a contradiction. Therefore, $\chi(x) = \chi(x-1) = \chi(x-2)$, as desired. $\diamond$

In light of Lemma 3, it is natural now to define m this way

$$m := \max\{x : 5 \leqslant x \leqslant N \text{ and } 2x-2-c \leqslant N \text{ and } x \equiv c \pmod 2\}.$$

It is useful to give a lower bound for m. Observe that if m exists, $m \geqslant 5$ by definition.

Lemma 4 *For all $c < -35$, m exists and*

$$m \geqslant \frac{5-c}{8}.$$

Proof. Because of its definition, m is at least 5 and is the maximum integer satisfying $2m-2-c \leqslant N$ and $m \equiv c \pmod 2$. Because we assume $c < -35$, we shall see that m exists. Assuming that the right-hand side of (3) is at least 5, the definition of m shows that

$$m = \begin{cases} \left\lfloor \frac{N+c+2}{2} \right\rfloor & \text{if } \left\lfloor \frac{N+c+2}{2} \right\rfloor \equiv c \pmod 2 \\ \left\lfloor \frac{N+c+2}{2} \right\rfloor - 1 & \text{otherwise.} \end{cases} \tag{3}$$

For values of $c \leqslant -4$, we have the following:

$$4N = 4\left\lceil \frac{3\left\lceil \frac{3-c}{2} \right\rceil - c}{2} \right\rceil = \begin{cases} 12 - 5c & \text{if } c \equiv 0 \pmod 4 \\ 9 - 5c & \text{if } c \equiv 1 \pmod 4 \\ 14 - 5c & \text{if } c \equiv 2 \pmod 4 \\ 11 - 5c & \text{if } c \equiv 3 \pmod 4 \end{cases}$$

From this one can show that

$$8\left(\frac{N+c+2}{2}\right) = \begin{cases} 20 - c & \text{if } c \equiv 0 \pmod 4 \\ 17 - c & \text{if } c \equiv 1 \pmod 4 \\ 22 - c & \text{if } c \equiv 2 \pmod 4 \\ 9 - c & \text{if } c \equiv 3 \pmod 4 \end{cases}$$

Accordingly, to determine whether $\lfloor \frac{N+c+2}{2} \rfloor \equiv c \pmod 2$ there are sixteen cases to consider depending on the residue of c modulo 16. We show the extremal case, $c \equiv 13 \pmod{16}$, and leave the remaining similar cases to the reader.

Assume that $c \equiv 13 \pmod{16}$, say $c = -16p - 3$, for some positive integer p. An easy computation reveals that $N = 20p + 6$. Therefore,

$$\left\lfloor \frac{N+c+2}{2} \right\rfloor = \left\lfloor \frac{4p+5}{2} \right\rfloor = 2p + 2.$$

Note that the floor function caused the fraction to be reduced by a half. Now, to determine m, another reduction is required because $2p+2$ is even, whereas c is odd. Hence $m = 2p+1$; that is $m = \frac{5-c}{8}$. This residue for c modulo 16 causes the greatest reductions and so determines the lower bound for m. Choosing $c < -35$ guarantees that the right-hand side of (3) is indeed at least 5 as needed. ◇

We assume that $c < -35$, since this value conveniently is enough to guarantee $\frac{5-c}{8} > 5$ via Lemma 4; that is, $m \geqslant 6$ since m is an integer.

Corollary 2 *Assume $c < -35$. The interval $[1, m]$ is monochromatic.*

Proof. Apply induction on j to prove that $m-2j-1$ and $m-2j-2$ have the same color as m. The basis case, $j = 0$, states that $m-1$ and $m-2$ have the same color as m, which is a consequence of Lemma 3. Assume now that $j > 0$ and that $m, m-1, \dots, m-2j$ are all monochromatic. Because $m \equiv c \pmod 2$, it follows that $m - 2j \equiv c \pmod 2$. Therefore, if $m - 2j \geqslant 5$, then Lemma 3 applies and shows that $m-2j, m-2j-1, m-2j-2$ all have the same color. Thus, $m, m-1, \dots, 4$ all have the same color, say red. It suffices to show that $1, 2, 3$ are also red. Because $m \geqslant 6$, we have for $i = 3, 2, 1$ in this order,

$$\begin{aligned} (3+i) + (2i-c) + c = 3(i+1) &\quad\Rightarrow\quad 2i - c \text{ is blue.} \\ (i) + (2i-c) + c = 3(i) &\quad\Rightarrow\quad i \text{ is red.} \end{aligned}$$

◇

The monochromatic interval $[1, m]$ forces another large monochromatic interval as the next lemma shows.

Lemma 5 *Define $M = 3 - c - m$. The interval $[M, N]$ is monochromatic with color opposite the color given to elements of the interval $[1, m]$.*

Proof. Set $W = [1, m]$. Consider the set

$$S := \{t : x + t + c = 3y \text{ for some } x, y \in W\}.$$

Observe that because Corollary 2 guarantees that the interval W is monochromatic, the elements of S must all have color opposite the color given to elements in W. So it suffices to show that S contains the interval $[M, N]$.

Notice that $M \in S$ because $1, m \in W$ and, by definition, $m + M + c = 3(1)$. Suppose now that $t \in S$ via $x + t + c = 3y$ for some $1 \leqslant x, y \leqslant m$. We shall prove that $t + 1 \in S$, provided that $t < N$.

If $x - 1 \in W$, then $(x-1) + (t+1) + c = 3y$ shows that $t + 1 \in S$. Otherwise $x \in W$ and $x - 1 \notin W$ implies that $x = 1$. We may assume now that $x = 1$, so in particular, by assumption $x + 2 \in W$ since $m \geqslant 5$. If $y + 1 \in W$, then $(x+2) + (t+1) + c = 3(y+1)$ shows that $t + 1 \in S$. Otherwise $y \in W$ and $y + 1 \notin W$ implies that $y = m$. Therefore, $1 + t + c = 3m$; that is, $t = 3m - c - 1$. Lemma 4 shows $m \geqslant \frac{5-c}{8}$, so

$$\begin{aligned} t &= 3m - c - 1 \\ &\geqslant 3\left(\frac{5-c}{8}\right) - c - 1 \\ &= \frac{7-11c}{8} \\ &> N. \end{aligned}$$

◇

Now we are ready to prove the Martinelli-Schaal conjecture for $c < -35$.

Theorem 2 *Assume $c < -35$ and $N = \left\lceil \frac{3\lceil \frac{3-c}{2} \rceil - c}{2} \right\rceil$. Any 2-coloring of the integers in the interval $[1, N]$ produces a monochromatic solution to $x_1 + x_2 + c = 3x_3$. It follows that $r(c) = N$.*

Proof. For values of $c \leqslant -4$, recall that

$$4N = 4\left\lceil \frac{3\lceil \frac{3-c}{2} \rceil - c}{2} \right\rceil = \begin{cases} 12 - 5c & \text{if } c \equiv 0 \pmod 4 \\ 9 - 5c & \text{if } c \equiv 1 \pmod 4 \\ 14 - 5c & \text{if } c \equiv 2 \pmod 4 \\ 11 - 5c & \text{if } c \equiv 3 \pmod 4 \end{cases}$$

Corollary 2 guarantees the interval $[1, m]$ is monochromatic, say red. Lemma 5 ensures the interval $[M, N]$, where $M = 3 - c - m$, is monochromatic of the opposite color, blue.

We consider four cases according to the residue of c modulo 4.

Case 1: $c \equiv 0 \pmod 4$.

Consider the elements $1, N, N-1, N-2$. Now

$$\left(\frac{12-5c}{4}\right) + \left(\frac{12-5c}{4}\right) + c = 3\left(2 - \frac{c}{2}\right) \Rightarrow 2 - \frac{c}{2} \text{ is red,}$$

$$\left(\frac{12-5c}{4} - 1\right) + \left(\frac{12-5c}{4} - 2\right) + c = 3\left(1 - \frac{c}{2}\right) \Rightarrow 1 - \frac{c}{2} \text{ is red,}$$

$$\text{so } \left(2 - \frac{c}{2}\right) + \left(1 - \frac{c}{2}\right) + c = 3 \cdot 1 \text{ is all red.}$$

We need to verify that $M \leqslant N-2$. We have $M = 3-c-m$ and, by Lemma 4, $m \geqslant \frac{5-c}{8}$, so $M = 3-c-m \leqslant 3-c+\frac{c-5}{8} = \frac{19-7c}{8}$. Now $\frac{19-7c}{8} \leqslant 1-\frac{5c}{4} = N-2$ if and only if $c \leqslant -\frac{11}{3}$, so $M \leqslant N-2$.

Case 2: $c \equiv 1 \pmod 4$.

We need only look at 1 and N:

$$\left(\frac{9-5c}{4}\right)+\left(\frac{9-5c}{4}\right)+c = 3\left(\frac{3-c}{2}\right) \Rightarrow \left(\frac{3-c}{2}\right) \text{ is red, and}$$

$$\left(\frac{3-c}{2}\right)+\left(\frac{3-c}{2}\right)+c = 3\cdot 1 \text{ is all red.}$$

Case 3: $c \equiv 2 \pmod 4$.

Consider the red element 1 and blue elements $N, N-1, N-3$. Then

$$\left(\frac{14-5c}{4}-1\right)+\left(\frac{14-5c}{4}-3\right)+c = 3\left(1-\frac{c}{2}\right) \Rightarrow 1-\frac{c}{2} \text{ is red,}$$

$$\left(\frac{14-5c}{4}\right)+\left(\frac{14-5c}{4}-1\right)+c = 3\left(2-\frac{c}{2}\right) \Rightarrow 2-\frac{c}{2} \text{ is red, and so}$$

$$\left(1-\frac{c}{2}\right)+\left(2-\frac{c}{2}\right)+c = 3\cdot 1 \text{ is all red.}$$

It is easily verified in a manner similar to Case 1 that $M \leqslant N-3$.

Case 4: $c \equiv 3 \pmod 4$.

Consider the red element of 1 and blue elements $N, N-1$. We have

$$\left(\frac{11-5c}{4}\right)+\left(\frac{11-5c}{4}-1\right)+c = 3\left(\frac{3-c}{2}\right) \Rightarrow \frac{3-c}{2} \text{ is red, and}$$

$$\left(\frac{3-c}{2}\right)+\left(\frac{3-c}{2}\right)+c = 3\cdot 1 \text{ is all red.}$$

Again, it is easy to verify in a manner similar to Case 1 that $M \leqslant N-1$. ◇

Section 4. $x_1 + x_2 + c = kx_3$

In this section we briefly address the function $r(c,k)$ which is defined (for every positive integer k and every integer c) to equal the smallest integer n, provided that it exists, such that every 2-coloring of $[1,n]$ has a monochromatic solution to $x_1+x_2+c = kx_3$. Martinelli and Schaal prove the lower bound

$$r(c,k) \geqslant \left\lceil \frac{2\lceil\frac{2+c}{k}\rceil + c}{k} \right\rceil, \text{ for all } c,k>0. \tag{4}$$

This lower bound is achieved for infinitely many values of c and k as the next proposition shows.

Proposition 1 *If m is a positive integer, $k = 2m + 1$ and $c = m(2m + 1)^2$, then*

$$\begin{aligned} r(c,k) &= \left\lceil \frac{2\lceil \frac{2+c}{k} \rceil + c}{k} \right\rceil \\ &= (m+1)(2m+1). \end{aligned}$$

Proof. Let $k = 2m + 1$, $c = m(2m + 1)^2$ and $r = (m + 1)(2m + 1)$. Because of the lower bound (4), it suffices to prove that every 2-coloring of $[1, r]$, using colors red and blue say, results in a monochromatic solution to $x_1 + x_2 + c = kx_3$. Without loss of generality, r is red. We now prove by induction on j, for $j = 0, \ldots, m$ that if $r - j$ is red, then $r - (j + 1)k$ is blue and $r - (j + 1)$ is red. If $r - j$ is red, then for these values of k, c, and r:

$$\begin{aligned} (r - (j+1)k) + r + c = k(r - j) \quad &\Rightarrow \quad (r - (j+1)k) \text{ is blue.} \\ (r - (j+1)k) + (r - k) + c = k(r - (j+1)) \quad &\Rightarrow \quad (r - (j+1)) \text{ is red.} \end{aligned}$$

It follows that $r - m$ and $r - (m + 1)$ are both red. Therefore, we have a monochromatic solution to $x_1 + x_2 + c = kx_3$:

$$(r - m) + (r - (m + 1)) + c = kr.$$

◇

Finally we illustrate an infinite number of values of c and k for which the bound (4) is not sharp.

Proposition 2 *If $m \geqslant 2$ is a positive integer, $k = 2m + 1$ and $c = m(2m + 1)^2 + 1$, then*

$$\begin{aligned} r(c,k) &> \left\lceil \frac{2\lceil \frac{2+c}{k} \rceil + c}{k} \right\rceil \\ &= (m+1)(2m+1). \end{aligned}$$

Proof. Let $k = 2m+1$, $c = m(2m+1)^2+1 = mk^2+1$ and $r = k(m+1) = 2m^2+3m+1$. Consider this 2-coloring of $[1, r]$ into red (R) and blue (B):

$$\begin{aligned} R &= \{1, \ldots, 2m^2 + m - 2\} \cup \{2m^2 + m\} \cup \{2m^2 + 3m + 1\} \\ B &= \{2m^2 + m - 1\} \cup \{2m^2 + m + 1, \ldots, 2m^2 + 3m\}. \end{aligned}$$

We must prove that there are no monochromatic $x_1, x_2, x_3 \in [1, r]$ that satisfy

$$x_1 + x_2 + m(2m + 1)^2 + 1 = (2m + 1)x_3. \tag{5}$$

If $x_3 \leqslant 2m^2+m$, then $kx_3 \leqslant c$ and therefore $x_1+x_2 < 0$, which clearly has no solution in $[1, r]$. So we may assume that $x_3 > 2m^2 + m$.

Case 1: $x_3 \in R$

Because $x_3 > 2m^2 + m$, we have $x_3 = 2m^2 + 3m + 1$, so from (5) we find $x_1 + x_2 = 4m^2 + 4m$ which has no solution in R.

Case 2: $x_3 \in B$

Since $x_3 \leqslant 2m^2 + 3m$, from (5) we find $x_1 + x_2 \leqslant 4m^2 + 2m - 1$ which implies, if $x_1, x_2 \in B$, that $x_1 = x_2 = 2m^2 + m - 1$. But these values for x_1 and x_2 do not produce, from (5), a value of x_3 in B. ◇

References

[1] Bialostocki, A. and Schaal, D. On a variation of Schur numbers, *Graphs Combin.* 16 (2000), no. 2, 139–147.

[2] Harborth, H. and Maasberg, S., All two-color Rado numbers for $a(x+y) = bz$, *Discrete Math.* **197/198** (1999), 397–407.

[3] Martinelli, B. and Schaal, D., On generalized Schur numbers for $x_1 + x_2 + c = kx_3$, *Ars Combinatoria* **85** (2007), 33–42.

[4] Sabo, D., Schaal, D. and Tokaz, J., Disjunctive Rado numbers for $x_1 + x_2 + c = x_3$, *Integers* 7 (2007), A29, 5 pp. (electronic).

[5] Schur, I., Über die Kongruenz $x^m + y^m \equiv z^m \pmod p$, Jahresber. Deutsch. Math. Verein. **25** (1916), 114-117.