Vertex-oriented Hamilton cycles in directed graphs

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Abstract

Let D be a directed graph of order n. An *anti-directed Hamilton cycle* H in D is a Hamilton cycle in the graph underlying D such that no pair of consecutive arcs in H form a directed path in D. We prove that if D is a directed graph with even order n and if the indegree and the outdegree of each vertex of D is at least $\frac{2}{3}n$ then D contains an anti-directed Hamilton cycle. This improves a bound of Grant [7]. Let $V(D) = P \cup Q$ be a partition of $V(D)$. A (P, Q) *vertex-oriented Hamilton cycle* in D is a Hamilton cycle H in the graph underlying D such that for each $v \in P$, consecutive arcs of H incident on v do not form a directed path in D, and, for each $v \in Q$, consecutive arcs of H incident on v form a directed path in D. We give sufficient conditions for the existence of a (P, Q) vertex-oriented Hamilton cycle in D for the cases when $|P| \geqslant \frac{2}{3}n$ and when $\frac{1}{3}n \leqslant |P| \leqslant \frac{2}{3}n$. This sharpens a bound given by Badheka et al. in [1].

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the degree of v in G, denoted by $\deg(v, G)$ is the number of edges of G incident on v. Let $\delta(G) = \min_{v \in V(G)}\{\deg(v, G)\}$. Let D be a directed graph with vertex set $V(D)$ and arc set $A(D)$. For a vertex $v \in V(D)$, the *outdegree* (respectively, *indegree*) of v in D denoted by $d^+(v, D)$ (respectively, $d^-(v, D)$) is the number of arcs of D directed out of v (respectively, directed into v). Let $\delta^0(D) = \min_{v \in V(D)}\{\min\{d^+(v, D), d^-(v, D)\}\}$. The *graph underlying* D is the graph obtained from D by ignoring the directions of the arcs of D. A *directed Hamilton cycle* H in D is a Hamilton cycle in the graph underlying D such that all pairs of consecutive arcs in H form a directed path in D. An *anti-directed Hamilton cycle* H in D is a Hamilton cycle in the graph underlying D such that no pair of consecutive arcs in H

form a directed path in D. Note that if D contains an anti-directed Hamilton cycle then $|V(D)|$ must be even. Let D be a directed graph, and let $V(D) = P \cup Q$ be a partition of $V(D)$. A (P, Q) *vertex-oriented Hamilton cycle* in D is a Hamilton cycle H in the graph underlying D such that for each $v \in P$, consecutive arcs of H incident on v do not form a directed path in D, and, for each $v \in Q$, consecutive arcs of H incident on v form a directed path in D. Note that if D contains a (P, Q) vertex-oriented Hamilton cycle then $|P|$ must be even. The idea of a (P, Q) vertex-oriented Hamilton cycle generalizes the ideas of a directed Hamilton cycle and an an anti-directed Hamilton cycle, because a directed Hamilton cycle in D is a $(\emptyset, V(D))$ vertex-oriented Hamilton cycle in D and an anti-directed Hamilton cycle in D is a $(V(D), \emptyset)$ vertex-oriented Hamilton cycle in D. We refer the reader to ([1,2,5]) for all terminology and notation that is not defined in this paper.

The following classical theorems by Dirac [3] and Ghouila-Houri [6] give sufficient conditions for the existence of a Hamilton cycle in a graph G and for the existence of a directed Hamilton cycle in a directed graph D respectively.

Theorem 1 [3] *If G is a graph of order $n \geqslant 3$ and $\delta(G) \geqslant \frac{n}{2}$, then G contains a Hamilton cycle.*

Theorem 2 [6] *If D is a directed graph of order n and $\delta^0(D) \geqslant \frac{n}{2}$, then D contains a directed Hamilton cycle.*

The following theorem by Grant [7] gives a sufficient condition for the existence of an anti-directed Hamilton cycle in a directed graph D.

Theorem 3 [7] *If D is a directed graph with even order n and if $\delta^0(D) \geqslant \frac{2}{3}n + \sqrt{n\log(n)}$ then D contains an anti-directed Hamilton cycle.*

In his paper Grant [7] conjectured that the theorem above can be strengthened to assert that if D is a directed graph with even order n and if $\delta^0(D) \geqslant \frac{1}{2}n$ then D contains an anti-directed Hamilton cycle. Mao-cheng Cai [10] gave a counter-example to this conjecture. However, the following theorem by Häggkvist and Thomason [8] proves that Grant's conjecture is asymptotically true.

Theorem 4 [8] *There exists an integer N such that if D is a directed graph of order $n \geqslant N$ and $\delta^0(D) \geqslant (\frac{1}{2} + n^{-\frac{1}{6}})n$ then D contains an n-cycle with arbitrary orientation.*

We point out here that if D is an oriented graph (i.e. a digraph for which at most one of the arcs (u, v) and (v, u) can be in $A(D)$) Häggkvist and Thomason [9] have obtained the following result.

Theorem 5 [9] *For every $\epsilon > 0$, there exists $N(\epsilon)$ such that if D is an oriented graph of order $n \geqslant N(\epsilon)$ and $\delta^0(D) \geqslant (\frac{5}{12} + \epsilon)n$ then D contains an n-cycle with arbitrary orientation.*

In Section 2 of this paper we prove the following improvement of Theorem 3 by Grant [7].

Theorem 6 *If D is a directed graph with even order n and if $\delta^0(D) \geqslant \frac{2}{3}n$ then D contains an anti-directed Hamilton cycle.*

In Section 3 of this paper we turn our attention to (P, Q) vertex-oriented Hamilton cycles. In [1] the following theorem giving a sufficient condition for the existence of a (P, Q) vertex-oriented Hamilton cycle was proved. For the sake of completeness we include the proof of this theorem in Section 3.

Theorem 7 [1] *Let D be a directed graph of order n and let $V(D) = P \cup Q$ be a partition of $V(D)$. If $|P| = 2j$ for some integer $j \geqslant 0$, and $\delta^0(D) \geqslant \frac{n}{2} + j$, then D contains a (P, Q) vertex-oriented Hamilton cycle.*

Let D be a directed graph and let D' be the spanning directed subgraph of D consisting of all arcs $uv \in A(D)$ for which $vu \in A(D)$. Let G' be the graph underlying D'. We note that if $\delta^0(D) \geqslant \frac{3}{4}n$, then $\delta(G') \geqslant \frac{n}{2}$, and hence Theorem 1 implies that G' contains a Hamilton cycle. Thus, if $\delta^0(D) \geqslant \frac{3}{4}n$ and $|P|$ is even, then D trivially contains a (P, Q) vertex-oriented Hamilton cycle for any partition $V(D) = P \cup Q$ of $V(D)$.

In Section 3 of this paper we prove the following two theorems that give sufficient conditions for the existence of a (P, Q) vertex-oriented Hamilton cycle that are sharper than the one given in Theorem 7 for the cases when $|P| \geqslant \frac{2}{3}n$ and when $\frac{1}{3}n \leqslant |P| \leqslant \frac{2}{3}n$.

Theorem 8 *Let D be a directed graph of order $n \geqslant 4$ and let $V(D) = P \cup Q$ be a partition of $V(D)$. If $|P| = 2j \geqslant \frac{2}{3}n$ for some integer $j \geqslant 0$, and $\delta^0(D) \geqslant \frac{n}{2} + \frac{j}{2}$, then D contains a (P, Q) vertex-oriented Hamilton cycle.*

Theorem 9 *Let D be a directed graph of order $n \geqslant 4$ and let $V(D) = P \cup Q$ be a partition of $V(D)$. If $|P| = 2j$ for some integer $j \geqslant 0$ with $\frac{1}{3}n \leqslant 2j \leqslant \frac{2}{3}n$ and $\delta^0(D) \geqslant \frac{2}{3}n$, then D contains a (P, Q) vertex-oriented Hamilton cycle.*

2 Proof of Theorem 6

A partition of a set S with $|S|$ being even into $S = X \cup Y$ is an *equipartition* of S if $|X| = |Y| = \frac{|S|}{2}$. We will use the following theorem by Moon and Moser [11].

Theorem 10 [11] *Let G be a bipartite graph of even order n, with equipartition $V(G) = X \cup Y$. If $x \in X$, $y \in Y$, $xy \notin E(G)$, and, $\deg(x) + \deg(y) > \frac{n}{2}$, then G contains a Hamilton cycle if and only if $G + xy$ contains a Hamilton cycle.*

For a bipartite graph G of order n, with partition $V(G) = X \cup Y$, the *closure* of G is defined as the supergraph of G obtained by iteratively adding edges between pairs of nonadjacent vertices $x \in X$ and $y \in Y$ whose degree sum is greater than $\frac{n}{2}$.
For an equipartition of $V(D)$ into $V(D) = X \cup Y$, let $B(X \to Y)$ be the bipartite directed graph with vertex set $V(D)$, equipartition $V(D) = X \cup Y$, and with $(x, y) \in A(B(X \to Y))$ if and only if $x \in X$, $y \in Y$, and, $(x, y) \in A(D)$. Let $B(X, Y)$ denote the bipartite graph underlying $B(X \to Y)$. It is clear that $B(X, Y)$ contains a Hamilton cycle if and only if $B(X \to Y)$ contains an anti-directed Hamilton cycle. The following lemma will imply Theorem 6.

Lemma 1 *If D is a directed graph with even order n and if $\delta^0(D) \geqslant \frac{2}{3}n$ then there exists an equipartition of $V(D)$ into $V(D) = X \cup Y$, such that $|\{v \in V(D) : \deg(v, B(X,Y)) \geqslant \frac{1}{3}n\}| > \frac{n}{2}$.*

Proof. For a vertex $v \in V(D)$, let $n_1(v)$ be the number of equipartitions of $V(D)$ into $V(D) = X \cup Y$ for which $\deg(v, B(X,Y)) \geqslant \frac{1}{3}n$ and let $n_2(v)$ be the number of equipartitions of $V(D)$ for which $\deg(v, B(X,Y)) < \frac{1}{3}n$. We will show that $n_1(v) > n_2(v)$ for each $v \in V(D)$ which in turn clearly implies the conclusion in the lemma.

Since n is even, we have that $n \equiv 0 \bmod 6$ or $n \equiv 2 \bmod 6$ or $n \equiv 4 \bmod 6$. We give the proof for the case in which $n \equiv 2 \bmod 6$; the other cases can be proved similarly.

Hence, assume that $|V(D)| = n = 6k+2$ for some positive integer k. Let v be a vertex in $V(D)$. Now, $\delta^0(D) \geqslant \frac{2}{3}n$ implies that $d^+(v, D) \geqslant 4k+2$, and since we wish to argue that $n_1(v) > n_2(v)$, we can assume that $d^+(v, D) = 4k+2$. Note that this implies that $\deg(v, B(X,Y)) \geqslant k+2$ for every equipartition of $V(D)$ into $V(D) = X \cup Y$. Now, $n_1(v)$ is the number of equipartitions of $V(D)$ into $V(D) = X \cup Y$ for which $2k+2 \leqslant \deg(v, B(X,Y)) \leqslant 3k+1$, and, $n_2(v)$ is the number of equipartitions of $V(D)$ into $V(D) = X \cup Y$ for which $k+2 \leqslant \deg(v, B(X,Y)) < 2k+1$. Hence, because v may be in X or Y, we have that

$$n_1(v) = 2\sum_{i=1}^{k} \binom{4k+2}{2k+i+1}\binom{2k-1}{k-i},$$

and that,

$$n_2(v) = 2\sum_{i=1}^{k} \binom{4k+2}{2k+2-i}\binom{2k-1}{k+i-1}.$$

Since $\binom{4k+2}{2k+i+1}\binom{2k-1}{k-i} > \binom{4k+2}{2k+2-i}\binom{2k-1}{k+i-1}$ for each $i = 1, 2, \ldots, k$, we have that $n_1(v) > n_2(v)$ and this completes the proof of the lemma. ■

Proof of Theorem 6. As given by Lemma 1, consider an equipartition of $V(D)$ into $V(D) = X \cup Y$ such that $|\{v \in V(D) : \deg(v, B(X,Y)) \geqslant \frac{1}{3}n\}| > \frac{n}{2}$. Let $Z = \{v \in V(D) : \deg(v, B(X,Y)) \geqslant \frac{1}{3}n\}$ and let $X^* = X \cap Z$ with $|X^*| = k > 0$, and let $Y^* = Y \cap Z$ with $|Y^*| \geqslant \frac{n}{2} - k + 1$. Let $B^+(X,Y)$ denote the closure of $B(X,Y)$. Note that since $\delta^0(D) \geqslant \frac{2}{3}n$, we have that $\deg(v, B(X,Y)) > \frac{n}{6}$ for each vertex v. Hence, $\deg(v, B^+(X,Y)) = \frac{n}{2}$ for each $v \in X^* \cup Y^*$. Therefore, $\deg(v, B^+(X,Y)) \geqslant \frac{n}{2} - k + 1$ for each $v \in X$ and $\deg(v, B^+(X,Y)) \geqslant k$ for each $v \in Y$. Now, Theorem 10 implies that $B^+(X,Y)$ contains a Hamilton cycle and hence $B(X,Y)$ contains a Hamilton cycle. This in turn implies that D contains an anti-directed Hamilton cycle. ■

3 Proofs of Theorems 7, 8 and 9

In [1] the following Type 1 reduction was used to prove Theorem 7.

Type 1 reduction. Let D be a directed graph and let $V(D) = P \cup Q$ be a partition of $V(D)$. Let p and p' be distinct vertices in P and let $q \in Q$ such that $pq \in A(D)$ and $qp' \in A(D)$. A *Type 1 reduction* applied to D with respect to the vertices p, q, and p' produces a directed graph D_1 from D with $V(D_1) = (V(D) - \{p, q, p'\}) \cup \{q_1\}$ and with $E(D_1)$ obtained from $A(D)$ as follows: Delete arcs $vp \in A(D)$ for each $v \in V(D)$, delete arcs $p'v \in A(D)$ for each $v \in V(D)$, delete all arcs incident on q, replace arc $pv \in A(D)$ by an arc q_1v for each $v \in V(D)$, and, replace arc $vp' \in A(D)$ by an arc vq_1 for each $v \in V(D)$. Let $P_1 = P - \{p, p'\}$ and $Q_1 = (Q - \{q\}) \cup \{q_1\}$. Clearly, if D_1 contains a (P_1, Q_1) vertex-oriented Hamilton cycle then D contains a (P, Q) vertex-oriented Hamilton cycle that includes the arcs pq and qp'.

For the sake of completeness we include the proof of Theorem 7 here.

Proof of Theorem 7. If $j = 0$, then $P = \emptyset$ and $\delta^0(D) \geqslant \frac{n}{2}$. Theorem 2 implies that D contains a directed Hamilton cycle which is a $(\emptyset, V(D))$ vertex-oriented Hamilton cycle in D. Now suppose that $j \geqslant 1$. Let p and p' be distinct vertices in P. It is easy to see that there exists $q \in Q$ such that $pq \in A(D)$ and $qp' \in A(D)$. We now apply a Type 1 reduction to D with respect to the vertices p, q, and p' to obtain the directed graph D_1 with partition of $V(D_1)$ into $V(D_1) = P_1 \cup Q_1$, where $P_1 = P - \{p, p'\}$ and $Q_1 = (Q - \{q\}) \cup \{q_1\}$. Now, $|V(D_1)| = n - 2$, $|P_1| = 2j - 2$, and since $\delta^0(D) \geqslant \frac{n}{2} + j$ we have that $\delta^0(D_1) \geqslant (\frac{n}{2} + j) - 2 = \frac{n-2}{2} + \frac{2j-2}{2}$. So, we can apply a Type 1 reduction to D_1 to get the directed graph D_2 with partition $V(D_2)$ into $V(D_2) = P_2 \cup Q_2$, where P_2 and Q_2 are obtained from P_1 and Q_1 in a manner similar to the one by which P_1 and Q_1 were obtained from P and Q. Iterating this procedure a total of j times yields a directed graph D_j with $P_j = \emptyset$ and $Q_j = V(D_j)$ with $|V(D_j)| = n - 2j$ and $\delta^0(D_j) \geqslant \frac{n}{2} + j - 2j = \frac{n-2j}{2}$. Now, Theorem 2 implies that D_j contains a directed Hamilton cycle which in turn implies that D contains a (P, Q) vertex-oriented Hamilton cycle. ■

To prove Theorems 8 and 9 we will use the following Type 2 reduction.

Type 2 reduction. Let D be a directed graph and let $V(D) = P \cup Q$ be a partition of $V(D)$. Let p and p' be distinct vertices in P with $pp' \in A(D)$. A *Type 2 reduction* applied to D with respect to the vertices p and p' produces a directed graph D_2 from D with $V(D_2) = (V(D) - \{p, p'\}) \cup \{q_2\}$ and with $E(D_2)$ obtained from $A(D)$ as follows: Delete arcs $vp \in A(D)$ for each $v \in V(D)$, delete arcs $p'v \in A(D)$ for each $v \in V(D)$, replace arc $pv \in A(D)$ by an arc q_2v for each $v \in V(D)$, and, replace arc $vp' \in A(D)$ by an arc vq_2 for each $v \in V(D)$. Let $P_2 = P - \{p, p'\}$ and $Q_2 = Q \cup \{q_2\}$. Clearly, if D_2 contains a (P_2, Q_2) vertex-oriented Hamilton cycle then D contains a (P, Q) vertex-oriented Hamilton cycle that includes the arc pp'.

Proof of Theorem 8. Let D be a directed graph of order n. Let $V(D) = P \cup Q$ be a partition of $V(D)$ with $|P| = 2j \geqslant \frac{2}{3}n$ for some integer $j \geqslant 0$. Let $D[P]$ be the directed subgraph of D induced by vertices in P, and let $G(P)$ be the simple graph underlying $D[P]$. Since $\delta^0(D) \geqslant \frac{n}{2} + \frac{j}{2}$, $2j \geqslant \frac{2}{3}n$, and, $|Q| = n - 2j$, we have that $\delta(G(P)) \geqslant (\frac{n}{2} + \frac{j}{2}) - (n - 2j) \geqslant j$. Hence, Theorem 1 implies that $G(P)$ contains a Hamilton cycle and hence a perfect matching M. Let $(p_i, p_i'), i = 1, 2, \ldots, j$ be the j arcs in $D[P]$ corresponding to the edges in M. We now successively apply j Type 2 reductions to D with respect to the vertices p_i and p_i' for $i = 1, 2, \ldots, j$. Let D^* be the directed graph obtained from D after these j Type 2 reductions. Then, $|V(D^*| = n - j$ and since $\delta^0(D) \geqslant \frac{n}{2} + \frac{j}{2}$, we have that $\delta^0(D^*) \geqslant (\frac{n}{2} + \frac{j}{2}) - j = \frac{n-j}{2}$. Now, Theorem 2 implies that D^* contains a directed Hamilton cycle which in turn implies that D contains a (P, Q) vertex-oriented Hamilton cycle. ■

We will need the following Lemma [4] in the proof of Theorem 9.

Lemma 2 [4] *Let G be a graph of order n and let $\beta(G)$ be the maximum cardinality of a matching in G. Then $\beta(G) \geqslant \min\{\delta(G), \lfloor \frac{n}{2} \rfloor\}$.*

Proof of Theorem 9. Let D be a directed graph of order n. Let $V(D) = P \cup Q$ be a partition of $V(D)$ with $|P| = 2j$ for some integer $j \geqslant 0$ and with $\frac{1}{3}n \leqslant 2j \leqslant \frac{2}{3}n$. Let $2j = \frac{1}{3}n + k$, $0 \leqslant k \leqslant \frac{1}{3}n$. Let $D[P]$ be the directed subgraph of D induced by vertices in P, and let $G(P)$ be the simple graph underlying $D[P]$. Since $\delta^0(D) \geqslant \frac{2}{3}n$ and $|Q| = n - 2j$, we have that $\delta(G(P)) \geqslant \frac{2}{3}n - (n - 2j) = 2j - \frac{1}{3}n = k$. Since $2j \leqslant \frac{2}{3}n$, we have that $k = 2j - \frac{1}{3}n \leqslant j = \frac{|V(G(P))|}{2}$. Lemma 2 implies that $G(P)$ contains a matching M with $|M| = \lceil k \rceil$. Let $(p_i, p_i'), i = 1, 2, \ldots, \lceil k \rceil$ be the $\lceil k \rceil$ arcs in $D[P]$ corresponding to the edges in M. We now successively apply $\lceil k \rceil$ Type 2 reductions to D with respect to the vertices p_i and p_i' for $i = 1, 2, \ldots, \lceil k \rceil$. Let D^* be the directed graph obtained from D after these $\lceil k \rceil$ Type 2 reductions. Then, $|V(D^*| = n - \lceil k \rceil$ and since $\delta^0(D) \geqslant \frac{2}{3}n$, we have that $\delta(D^*) \geqslant \frac{2}{3}n - \lceil k \rceil$. Let $P^* = P - \cup_{i=1}^{\lceil k \rceil}\{p_i\} - \cup_{i=1}^{\lceil k \rceil}\{p_i'\}$ and let $Q^* = V(D^*) - P^*$. We have that $|P^*| = 2j - 2\lceil k \rceil = \frac{1}{3}n + k - 2\lceil k \rceil$. Hence, $\delta(D^*) \geqslant \frac{2}{3}n - \lceil k \rceil \geqslant \frac{1}{2}|V(D^*)| + \frac{1}{2}|P^*|$. Now, Theorem 7 implies that D^* contains a (P^*, Q^*) vertex-oriented Hamilton cycle which in turn implies that D contains a (P, Q) vertex-oriented Hamilton cycle. ■

4 Conclusion

We summarize the results given in this paper as follows. Let D be a directed graph of order n and let $V(D) = P \cup Q$ be a partition of $V(D)$ with $|P| = p$, and p being even. By Theorems 7, 8, and 9, with $f(n, p)$ as defined below, if $\delta^0(D) \geqslant f(n, p)$ then D contains a (P, Q) vertex-oriented Hamilton cycle.

$$f(n,p) = \begin{cases} \frac{1}{2}n + \frac{1}{2}p, & \text{if } 0 \leqslant p \leqslant \frac{1}{3}n \\ \frac{2}{3}n, & \text{if } \frac{1}{3}n \leqslant p \leqslant \frac{2}{3}n \\ \frac{1}{2}n + \frac{1}{4}p, & \text{if } \frac{2}{3}n \leqslant p \leqslant n. \end{cases}$$

In the case when $p = n$, we can do better than the previous statement promises. Theorem 6 gives us that $f(n, p) = \frac{2}{3}n$ if $p = n$, thus, it is natural to expect that the lower bounds on $\delta^0(D)$ that guarantee a (P, Q) vertex-oriented Hamilton cycle can be significantly improved when p is relatively large.

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