Isosceles Sets

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Abstract

In 1946, Paul Erdős posed a problem of determining the largest possible cardinality of an isosceles set, i.e., a set of points in plane or in space, any three of which form an isosceles triangle. Such a question can be asked for any metric space, and an upper bound $\binom{n+2}{2}$ for the Euclidean space $\mathbb{E}^n$ was found by Blokhuis [3]. This upper bound is known to be sharp for $n = 1$, 2, 6, and 8. We will consider Erdős' question for the binary Hamming space H_n and obtain the following upper bounds on the cardinality of an isosceles subset S of H_n: if there are at most two distinct nonzero distances between points of S, then $|S| \leqslant \binom{n+1}{2} + 1$; if, furthermore, $n \geqslant 4$, $n \neq 6$, and, as a set of vertices of the n-cube, S is contained in a hyperplane, then $|S| \leqslant \binom{n}{2}$; if there are more than two distinct nonzero distances between points of S, then $|S| \leqslant \binom{n}{2} + 1$. The first bound is sharp if and only if $n = 2$ or $n = 5$; the other two bounds are sharp for all relevant values of n, except the third bound for $n = 6$, when the sharp upper bound is 12. We also give the exact answer to the Erdős problem for $\mathbb{E}^n$ with $n \leqslant 7$ and describe all isosceles sets of the largest cardinality in these dimensions.

1 Introduction

In 1946, Paul Erdős [9] asked the following question in the problem section of The American Mathematical Monthly:

Six points can be arranged in the plane so that all triangles formed by triples of these points are isosceles. Show that seven points in the plane cannot be so arranged. What is the least number of points in the space which cannot be so arranged?

Erdős' question can be generalized to any metric space.

Definition 1.1. A nonempty subset S of a metric space M is called *isosceles* if, for all $x, y, z \in S$, at least two of the distances between x and y, y and z, z and x are equal.

In 1947, Kelly [11] showed that there is no isosceles set of cardinality 7 in $\mathbb{E}^2$, and the only (up to similarity) isosceles set of cardinality 6 is the set consisting of the vertices and center of a regular pentagon. He also gave an example of an isosceles set of cardinality 8 in $\mathbb{E}^3$. In 1962, Croft [6] showed that there is no isosceles set of cardinality 9 in $\mathbb{E}^3$. In 2006, Kido [14] showed that Kelly's example presents a unique (up to similarity) isosceles set of cardinality 8 in $\mathbb{E}^3$. A short proof of this result is given in Section 5.

The best known upper bound for the cardinality of an isosceles set S in $\mathbb{E}^n$ is due to Blokhuis [3]: $|S| \leqslant \binom{n+2}{2}$. He also showed that the problem of finding the biggest isosceles sets can be in large part reduced to determining the biggest 2-*distance sets*. (See Theorem 2.15 below.)

Definition 1.2. A nonempty subset S of a metric space M is called an s-*distance set* if there are at most s nonzero distances between points of S.

Bannai, Bannai and Stanton [2] and Blokhuis [3] showed independently that the cardinality of a s-distance set in $\mathbb{E}^n$ does not exceed $\binom{n+s}{s}$, so we have the same upper bound for the cardinalities of both isosceles sets and 2-distance sets.

In 1997, Lisŏnek [15] determined the actual maximum size of 2-distance sets in $\mathbb{E}^n$ for $n \leqslant 8$ and found all maximum size 2-distance sets for $n \leqslant 7$. Lisŏnek's results and Blokhuis' Theorem (Theorem 2.15) give a good tool for determining maximum size isosceles sets. In the table below, the second and third row give the maximum size and the number of nonsimilar 2-distance sets of the maximum size in $\mathbb{E}^n$, and the last two rows give the same information for isosceles sets. Since the latter information does not seem to be known for $n > 3$, we will justify it in Section 5.

Maximum size of 2-distance and isosceles sets in $\mathbb{E}^n$

n	1	2	3	4	5	6	7	8
Max cardinality of 2-distance sets	3	5	6	10	16	27	29	45
Number of sets of max cardinality	1	1	6	1	1	1	1	$\geqslant 1$
Max cardinality of isosceles sets	3	6	8	11	17	28	30	45
Number of sets of max cardinality	1	1	1	2	1	1	1	$\geqslant 1$

As this table shows, Blokhuis' upper bound is attained in dimensions 1 and 8 for 2-distance sets and in dimensions 1, 2, 6, and 8 for isosceles sets.

The binary Hamming space H_n is the set of all binary words $(a_1, a_2, \ldots, a_n)$ of length n with the distance between two words being the number of positions in which they differ. The words can be interpreted as vertices of the n-dimensional unit cube (the Hamming distance between two vertices is just the square of the Euclidean distance) or as subsets of the set $[n] = \{1, 2, \dots, n\}$ (and then the Hamming distance between two sets is the cardinality of their symmetric difference).

It follows from Delsarte [7, 8] and Noda [17] that the cardinality of a 2-distance subset of H_n does not exceed $1 + \frac{n(n+1)}{2}$, and the only 2-distance subsets attaining this bound are the entire H_2, the set of all words of even weight in H_5, and the set of all words of odd weight in H_5.

This result is somewhat disappointing because it shows that we in fact do not know the maximum size of a 2-distance subset of H_n for $n > 5$. And it is not surprising. For a seemingly easier case of 1-distance sets, while the upper bound $n + 1$ in $\mathbb{E}^n$ is attained for every n, a 1-distance set of cardinality $n + 1$ in H_n exists if and only if there exists a Hadamard matrix of order $n + 1$ (see, for instance, [13], Theorem 1.4.6). Thus, there are infinitely many values of n for which the maximum size of a 1-distance set in H_n is not known.

However, the maximum size of a 1-distance subset S of H_n with $\dim S = n - 1$ is n, and it is attained for every n. (Take the intersection of H_n and the hyperplane $x_1 + x_2 + \dots + x_n = 1$.) In Section 3 we obtain an upper bound, similar to Delsarte's, for the cardinality of an s-distance set of dimension $m < n$ in H_n and then determine, for every n, the maximum size of a 2-distance subset S of H_n with $\dim S = n - 1$. (Here $\dim S$ is the dimension of the affine subspace of $\mathbb{E}^n$ generated by S.)

In Section 4 we will show that the cardinality of an isosceles subset S of H_n with more than two distances between points of S does not exceed $\binom{n}{2} + 1$. This bound is sharp for $n = 5$ and for every $n \geqslant 7$. For $n = 6$, the maximum size of S is 12, and there is no such a subset S for $n \leqslant 4$.

2 Preliminaries

Throughout the paper, for any positive integer n, $[n]$ denotes the set $\{1, 2, \dots, n\}$ and H_n denotes the set of all points $(a_1, a_2, \dots, a_n)$ in the Euclidean space $\mathbb{E}^n$ with each coordinate a_i equal 0 or 1. We will reserve letter O for the point with all coordinates equal 0. For $A = (a_1, a_2, \dots, a_n)$ and $B = (b_1, b_2, \dots, b_n)$ in H_n, the *Hamming distance* $d(A, B)$ between A and B is defined as the number of indices $i \in [n]$ such that $a_i \neq b_i$. Then $AB = \sqrt{d(A, B)}$ is the Euclidean distance between A and B. With each $A \in H_n$, we associate the subset $A = \{i \in [n] \colon a_i = 1\}$ of $[n]$ (denoted by the same letter A). If $A, B \in H_n$ are regarded as subsets of $[n]$, then $d(A, B) = |A \triangle B|$ and $|A| = d(A, \varnothing) = \sum_{i=1}^{n} a_i$. This immediately implies that

$$|A| + |B| \equiv d(A, B) \pmod 2. \tag{1}$$

Since H_n is a finite set, every function $f \colon H_n \to \mathbb{R}$ can be represented by a polynomial in variables $x_1, x_2, \dots, x_n$. We will denote as $Pol(n, s)$ the set of all functions $f \colon H_n \to \mathbb{R}$ that can be represented by polynomials of degree at most s. We will regard $Pol(n, s)$ as a linear space over $\mathbb{R}$. For any $I \subseteq [n]$, let $x_I = \prod_{i \in I} x_i$ (so $x_\varnothing = 1$). Since polynomials x_i and x_i^k, where k is a positive integer, represent the same function on H_n, the set

$\{x_I \colon I \subseteq [n], 0 \leqslant |I| \leqslant s\}$ is a basis of $Pol(n,s)$ and therefore

$$\dim Pol(n,s) = \sum_{i=0}^{s} \binom{n}{i}.$$

For $s \geqslant 1$, with each $A = (a_1, a_2, \ldots, a_n) \in H_n$ we associate the following function $f_A \in Pol(n,s)$:

$$f_A(x_1, x_2, .., x_n) = \sum_{i=1}^{n} (1 - 2a_i)x_i + \sum_{i=1}^{n} a_i.$$

If A is regarded as a subset of $[n]$, then

$$f_A(x_1, x_2, .., x_n) = \sum_{i \notin A} x_i - \sum_{i \in A} x_i + |A|.$$

Observe that for $A, B \in H_n$

$$d(A,B) = f_A(B). \tag{2}$$

The next definition will be often applied to subsets of H_n regarded as subsets of $\mathbb{E}^n$.

Definition 2.1. For any nonempty set X in $\mathbb{E}^n$, $\dim X$ is the dimension of the smallest affine subspace of $\mathbb{E}^n$ containing X. If $X = \varnothing$, then $\dim X = -1$.

Thus, $\dim X = 0$ if and only if $|X| = 1$. If $S \subseteq H_n$, then $\dim S = 1$ if and only if $|S| = 2$.

If a hyperplane π in $\mathbb{E}^n$ is given by an equation $\alpha_0 + \sum_{i=1}^{n} \alpha_i x_i = 0$, we will write $\pi = \{\alpha_0 + \sum_{i=1}^{n} \alpha_i x_i = 0\}$. The next two lemmas are straightforward.

Lemma 2.2. *If π is an m-dimensional affine subspace of $\mathbb{E}^n$, then $|H_n \cap \pi| \leqslant 2^m$.*

Lemma 2.3. *Let nonzero functions $\varphi_1, \varphi_2 \in Pol(n,1)$ be such that $\varphi_1 \varphi_2 = 0$. Then there exist $c_1, c_2 \neq 0$ such that φ_1 and φ_2 are either $c_1 x_i$ and $c_2(x_i - 1)$ for some $i \in [n]$ or $c_1(x_i - x_j)$ and $c_2(x_i + x_j - 1)$ for some distinct $i, j \in [n]$. Equivalently, if π_1 and π_2 are hyperplanes in $\mathbb{E}^n$ such that $H_n \subset \pi_1 \cup \pi_2$, then π_1 and π_2 are either $\{x_i = 0\}$ and $\{x_i = 1\}$ or $\{x_i - x_j = 0\}$ and $\{x_i + x_j = 1\}$.*

The next two lemmas provide useful restrictions on distances in 2-distance and isosceles subsets of H_n.

Lemma 2.4. *Let S be a 2-distance subset of H_n. If $|S| \geqslant 2n + 3$, then all distances between points of S are even.*

Proof. We obtain from (1) that, since $|S| \geqslant 3$, at least one nonzero distance in S is even. Suppose the other nonzero distance is odd. For $i = 0, 1$, let $S_i = \{A \in S \colon |A| \equiv i \pmod 2\}$. Then (1) implies that S_0 and S_1 are 1-distance sets. The largest 1-distance set in $\mathbb{E}^n$ is the set of $n+1$ vertices of a regular n-simplex. Therefore, $|S| = |S_0| + |S_1| \leqslant 2n+2$, a contradiction. ∎

Lemma 2.5. *Let S be an isosceles subset of H_n. Then at most one distance between points of S is odd.*

Proof. Suppose S has two distinct odd distances, d_1 and d_2, and let $d(A,B) = d_1$ and $d(C,D) = d_2$. We apply (1) and assume, without loss of generality, that $|A|$ and $|C|$ are even, while $|B|$ and $|D|$ are odd. Then $d(A,C)$ is even and therefore, $d(B,C) = d(A,B) = d_1$. Now $\triangle BCD$ is not isosceles, because it has an even side $d(B,D)$ and two distinct odd sides, a contradiction. ☐

Proposition 2.6. *For $n \leqslant 4$, every isosceles subset of H_n is a 2-distance set.*

Proof. If $n = 1$ or 2, then there are at most two nonzero distances in H_n. Lemma 2.5 implies that there is no isosceles set in H_3 with distances 1, 2, and 3. Suppose S is an isosceles set in H_4 with more than two nonzero distances. Then, by Lemma 2.5, the distances are 1, 2, and 4 or 2, 3, and 4. Let $d(A,B) = 4$ for $A, B \in S$. Without loss of generality, we assume that $A = (0,0,0,0)$ and $B = (1,1,1,1)$. Then $|C| = 2$ for every other $C \in S$, and therefore there is no odd distance between points of S, a contradiction. ☐

Theorem 2.15 below indicates that spheres may play an important role in investigating isosceles sets.

Definition 2.7. Let $C \in H_n$ and let r be a positive integer. The sphere with center C and radius r in H_n is the set $Sp(C,r) = \{X \in H_n : d(C,X) = r\}$.

Lemma 2.8. *Let $C = (c_1, c_2, \dots, c_n) \in H_n$ and let r be a positive integer. Then*

$$Sp(C,r) = H_n \cap \{(1-2c_1)x_1 + (1-2c_2)x_2 + \cdots + (1-2c_n)x_n = r - |C|\}.$$

Furthermore, spheres $Sp(C_1,r_1)$ and $Sp(C_2,r_2)$ of dimension $n-1$ with distinct centers C_1 and C_2 are equal (as sets) if and only if $d(C_1,C_2) = r_1 + r_2 = n$.

Proof. The first statement of the lemma follows immediately from (2).

Let $C_i = (c_{i1}, c_{i2}, \dots, c_{in}) \in H_n$, $i = 1,2$, and let r_1 and r_2 be positive integers. Suppose $\dim Sp(C_1,r_1) = \dim Sp(C_2,r_2) = n-1$. Then $Sp(C_1,r_1) = Sp(C_2,r_2)$ if and only if

$$\left\{\sum_{j=1}^{n}(1-2c_{1j})x_j = r_i - |C_1|\right\} = \left\{\sum_{j=1}^{n}(1-2c_{2j})x_j = r_2 - |C_2|\right\}.$$

Since each coordinate of normal vectors $(1-2c_{i1}, 1-2c_{i2}, \dots, 1-2c_{in})$, $i = 1,2$, equals ± 1 and since $C_1 \neq C_2$, we obtain that $Sp(C_1,r_1) = Sp(C_2,r_2)$ if and only if $1-2c_{1j} = 2c_{2j}-1$ for $j = 1,2,\dots,n$ and $r_1 - |C_1| = |C_2| - r_2$, i.e., $d(C_1,C_2) = |C_1| + |C_2| = r_1 + r_2 = n$. ☐

Corollary 2.9. *If a subset S of H_n is contained in two distinct spheres, then* $\dim S \leqslant n-2$.

Lemma 2.10. *If a subset S of H_n is contained in three distinct spheres, then* $\dim S \leqslant n-3$.

Proof. If three distinct spheres have a nonempty intersection, they have distinct centers C_1, C_2, and C_3. Let $C_i = (c_{i1}, c_{i2}, \ldots, c_{in})$, $i = 1, 2, 3$, and let $n_{ij} = 1 - 2c_{ij}$. It suffices to show that the rank of $3 \times n$ matrix $N = [n_{ij}]$ equals 3.

Suppose that there are $\alpha, \beta \neq 0$ such that $n_{3j} = \alpha n_{1j} + \beta n_{2j}$ for $j = 1, 2, \ldots, n$. Since the spheres are distinct and have distinct centers, normal vectors of the corresponding hyperplanes are neither equal, nor opposite. Since each n_{ij} is equal to 1 or -1, we can find indices j and h such that $n_{1j} = n_{2j}$ and $n_{1h} = -n_{2h}$. This implies that both $\alpha + \beta$ and $\alpha - \beta$ must be equal to 1 or -1, which is not possible for nonzero α and β. Thus, $rank(N) = 3$. □

We will now state four powerful theorems that will be used in subsequent sections. For the first two theorems we need the notion of an orthogonal array.

Definition 2.11. An $N \times n$ array M with entries from $\{0, 1\}$ is called a *binary orthogonal array of strength* t (for some t in the range $1 \leqslant t \leqslant n$) if every $N \times t$ subarray of M contains each binary t-tuple the same number of times.

Theorem 2.12 (Delsarte [7, 8, 10])**.** *If S is an s-distance subset of H_n, then $|S| \leqslant N = \sum_{i=0}^{s} \binom{n}{i}$. Furthermore, if $n \geqslant 2s$ and $|S| = N$, then the words of S form an $N \times n$ binary orthogonal array of strength $2s$.*

Theorem 2.13 (Rao, Noda [18, 17, 12])**.** *If M is an $N \times n$ binary orthogonal array of even strength $2s$, then $N \geqslant \sum_{i=0}^{s} \binom{n}{i}$. Furthermore, if $s = 2$ and $N = 1 + \frac{n(n+1)}{2}$, then either $n = 2$ and the rows of M are all words of H_2 or $n = 5$ and the rows of M are all words of even weight in H_5 or all words of odd weight in H_5.*

The next theorem combines results of several important papers. For references see Cameron and van Lint [4], Theorems 1.52 and 1.54. Note that these theorems provide a much stronger result than the one below.

Theorem 2.14. *Let S be a set of subsets of $[n]$ such that $|A| = |B|$ for all $A, B \in S$, $|S| = \binom{n}{2}$, and $|\{|A \cap B| \colon A, B \in S, A \neq B\}| = 2$ .Then at least one of the following is true:*

(i) *S is the set of all 2-subsets of $[n]$;*
(ii) *S is the set of all $(n-2)$-subsets of $[n]$;*
(iii) *$n = 23$.*

The next theorem was originally stated for the Euclidean space but its proof in [3] works in any metric space.

Theorem 2.15 (Blokhuis [3])**.** *Let S be a finite isosceles set in a metric space M. If there are more than two distinct nonzero distances between points of S, then there exist subsets X and Y of S such that the following conditions are satisfied:*

(i) *$S = X \cup Y$ and $X \cap Y = \varnothing$;*
(ii) *$|X| \geqslant 2$ and $|Y| \geqslant 1$;*
(iii) *every $y \in Y$ is the center of a sphere containing the entire set X.*

Furthermore, if M is the Euclidean space $\mathbb{E}^n$, then the affine subspaces generated by X and Y are orthogonal, and therefore, $\dim S \geqslant \dim X + \dim Y$.

3 s-distance sets in H_n

Throughout this section, S is a subset of H_n, $|S| \geqslant 2$, and $d_1, d_2, \ldots, d_s$ are all distinct nonzero distances between points of S.

For each $A \in S$, consider the following function $F_A \in Pol(n, s)$:

$$F_A(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{s} (f_A(x_1, x_2, \ldots, x_n) - d_i).$$

From (2),

$$F_A(B) = \begin{cases} 0 & \text{if } A \neq B, \\ (-1)^s d_1 d_2 \cdots d_s & \text{if } A = B, \end{cases}$$

for all $A, B \in S$.

This implies that the subset $\{F_A \colon A \in S\}$ of $Pol(n, s)$ is linearly independent. (If $\sum_{A \in S} \alpha_A F_A = 0$ for some real numbers α_A, then, for any $B \in S$, $\sum_{A \in S} \alpha_A F_A(B) = 0$, so $\alpha_B d_1 d_2 \cdots d_s = 0$, and then $\alpha_B = 0$.) Therefore, the cardinality of S does not exceed the dimension of $Pol(n, s)$. This proof of *Delsarte's Inequality* for binary codes [7, 8] is similar to the one given in [1].

Theorem 3.1. *If S is an s-distance subset of H_n, then*

$$|S| \leqslant \sum_{i=0}^{s} \binom{n}{i}.$$

Example 3.2. Let $n = 2s + 1$ and let S be a set of 2^{2s} vertices of the n-dimensional unit cube, no two of which are adjacent. (There are two such sets of vertices: one consists of all vertices with even sum of coordinates, the other consists of all vertices with odd sum of coordinates.) Then the nonzero distances in S are $2, 4, \ldots, 2s$ and S attains the Delsarte bound.

For $s = 3$ and $n = 23$, there is another subset of H_n attaining the Delsarte bound.

Example 3.3. Consider the binary Golay code G_{23}. The words of the dual code form a 3-distance subset of H_{23} of cardinality $2^{11} = \binom{23}{0} + \binom{23}{1} + \binom{23}{2} + \binom{23}{3}$. [16]

If an s-distance subset of H_n has dimension less than n, a stronger inequality can be obtained.

Theorem 3.4. *Let S be an s-distance subset of H_n. If $\dim S = m$, then*

$$|S| \leqslant \sum_{i=0}^{s} \binom{m}{i}.$$

Proof. We may assume that $m < n$ and that S has exactly s nonzero distances and let them be $d_1, d_2, \ldots, d_s$. The affine subspace U of $\mathbb{E}^n$ generated by S can be regarded as the solution set of a system of linear equations of rank $n-m$ in variables $x_1, x_2, \dots, x_n$. Without loss of generality, we assume that there exist linear polynomials $\varphi_{m+1}, \varphi_{m+2}, \dots, \varphi_n$ in variables $x_1, x_2, \dots, x_m$ such that

$$U = \{(\alpha_1, \dots, \alpha_m, \varphi_{m+1}(\alpha_1, \dots, \alpha_m), \dots, \varphi_n(\alpha_1, \dots, \alpha_m)) \colon \alpha_1, \dots, \alpha_m \in \mathbb{R}\}.$$

For each $A = (a_1, a_2, \dots, a_n) \in S$, let $\overline{A} = (a_1, a_2, \dots, a_m)$ and

$$\overline{F}_A(x_1, x_2, \dots, x_m) = F_A(x_1, \dots, x_m, \varphi_{m+1}(x_1, \dots, x_m), \dots, \varphi_n(x_1, \dots, x_m)).$$

If $A, B \in S$, then $A, B \in U$ and therefore

$$\overline{F}_A(\overline{B}) = F_A(B) = \begin{cases} 0 & \text{if } B \neq A, \\ (-1)^s d_1 d_2 \cdots d_s & \text{if } B = A. \end{cases}$$

Hence, $\{\overline{F}_A \colon A \in S\}$ is a linearly independent subset of $Pol(m, s)$. Therefore,

$$|S| \leqslant \dim Pol(m, s) = \sum_{i=0}^{s} \binom{m}{i}.$$

□

For $s = 2$ and $m = n-1$, Theorem 3.4 gives $|S| \leqslant \binom{n}{2} + 1$. The next theorem strengthens this result. First we need a lemma.

Lemma 3.5. *Let S be a 2-distance set of cardinality $\binom{n}{2} + 1$ in H_n with $n \geqslant 3$. Then S is not contained in a hyperplane $\{x_i - x_j = 0\}$.*

Proof. Suppose, without loss of generality, that $S \subset \{x_{n-1} - x_n = 0\}$. Consider the following subset $\mathcal{B}$ of $Pol(n, 2)$:

$$\mathcal{B} = \{F_A \colon A \in S\} \cup \{x_{n-1} - x_n\} \cup \{x_j(x_{n-1} - x_n) \colon 1 \leqslant j \leqslant n-1\}.$$

Claim. *$\mathcal{B}$ is linearly independent.*

Suppose

$$\sum_{A \in S} \alpha_A F_A + \beta_0(x_{n-1} - x_n) + \sum_{j=1}^{n-1} \beta_j x_j (x_{n-1} - x_n) = 0.$$

Applying both sides to $B \in S$ yields $\alpha_B = 0$, so

$$(\beta_0 + \beta_1 x_1 + \cdots + \beta_{n-1} x_{n-1})(x_{n-1} - x_n) = 0.$$

Now Lemma 2.3 implies that $\beta_j = 0$ for $0 \leqslant j \leqslant n-1$.

Since $\mathcal{B}$ is linearly independent and $|\mathcal{B}| = |S| + n = \dim Pol(n,s)$, $\mathcal{B}$ is a basis of $Pol(n,s)$. We will expand in this basis monomials d_1d_2 and $d_1d_2x_j$, $1 \leqslant j \leqslant n-1$. Applying both sides of each expansion to $B \in S$ will yield the following equations:

$$\sum_{A\in S} F_A(x_1,x_2,\ldots,x_n) + \varphi_0(x_1,x_2,\ldots,x_{n-1})(x_{n-1}-x_n) = d_1d_2; \tag{3}$$

$$\sum_{A\in S, A\ni j} F_A(x_1,x_2,\ldots,x_n) + \varphi_j(x_1,x_2,\ldots,x_{n-1})(x_{n-1}-x_n) = d_1d_2x_j. \tag{4}$$

In these equations, $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ are linear polynomials in $x_1, x_2, \ldots, x_{n-1}$.

Since $S \subset \{x_{n-1} - x_n = 0\}$, each $A \in S$ either contains $\{n-1, n\}$ or is disjoint from this 2-set. Therefore, F_A has the same coefficient of x_ix_{n-1} as of x_ix_n, $1 \leqslant i \leqslant n-2$, F_A has the same coefficient of x_{n-1} as of x_n, and the coefficient of $x_{n-1}x_n$ in F_A equals 2. Since each product $\varphi_j \cdot (x_{n-1} - x_n)$, $0 \leqslant j \leqslant n-1$, has opposite coefficients of x_ix_{n-1} and x_ix_n, $1 \leqslant i \leqslant n-2$, we conclude that the functions φ_j can be written as follows:

$$\varphi_j(x_1,x_2,\ldots,x_{n-1}) = \varepsilon_j + \zeta_j x_{n-1}, \quad 0 \leqslant j \leqslant n-1.$$

For any $K \subseteq [n]$, let $\lambda(K)$ denote the number of $A \in S$ such that $K \subseteq A$. Comparing the coefficients of $x_{n-1}x_n$ in both sides of equations (3) and (4) implies that $\zeta_0 = -2|S|$ and $\zeta_j = -2\lambda(j)$ for $j = 1, 2, \ldots, n-1$. Comparing the coefficient of x_{n-1} to the coefficient of x_n in these equations implies that $\varepsilon_j + \zeta_j = -\varepsilon_j$ for $0 \leqslant j \leqslant n-2$ and $\varepsilon_{n-1} + \zeta_{n-1} - d_1d_2 = -\varepsilon_{n-1}$, so equations (3) and (4) can be rewritten as

$$\sum_{A\in S} F_A = |S|(2x_{n-1}x_n - x_{n-1} - x_n) + d_1d_2; \tag{5}$$

$$\sum_{A\in S, A\ni j} F_A = \lambda(j)(2x_{n-1}x_n - x_{n-1} - x_n) + d_1d_2x_j, \quad 1 \leqslant j \leqslant n-2; \tag{6}$$

$$\sum_{A\in S, A\ni n-1} F_A = 2\lambda(n-1)x_{n-1}x_n + (d_1d_2/2 + \lambda(n-1))(x_{n-1} + x_n). \tag{7}$$

For distinct $j, k \in [n-2]$, comparing the coefficients of x_jx_k in both sides of (6) yields $\lambda(j) = 2\lambda(j,k)$. Therefore, $\lambda(k) = 2\lambda(k,j) = \lambda(j)$. Thus, $|S \cap \{x_j = 1\}| = |S \cap \{x_k = 1\}|$ for any distinct $j, k \in [n-2]$. Fix j and k and consider the isometry Φ of $\mathbb{E}^n$ given by $\Phi(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$ where

$$y_i = \begin{cases} x_i & \text{if } i \neq j, \\ 1 - x_j & \text{if } i = j. \end{cases}$$

Then Φ is also an isometry of H_n, and therefore $\Phi(S)$ is a 2-distance subset of $H_n \cap \{x_{n-1} - x_n = 0\}$ of cardinality $\binom{n}{2} + 1$. This implies that $|\Phi(S) \cap \{x_j = 1\}| = |\Phi(S) \cap \{x_k = 1\}|$ and therefore $|S \cap \{x_j = 0\}| = |S \cap \{x_k = 1\}|$. Then $|S \cap \{x_j = 0\}| = |S \cap \{x_j = 1\}| = |S|/2$. Thus, for $1 \leqslant j < k \leqslant n-2$, $\lambda(j) = |S|/2$ and $\lambda(j,k) = |S|/4$.

For $1 \leqslant j \leqslant n-2$, comparing the coefficients of $x_j x_{n-1}$ in (6) yields $\lambda(n-1,j) = \frac{1}{2}\lambda(j) = |S|/4$ and then comparing the coefficients of $x_j x_{n-1}$ in (7) yields $\lambda(n-1) = 2\lambda(j,n-1) = |S|/2$. Thus, $\lambda(j) = |S|/2$ for all $j \in [n]$. This implies

$$\sum_{A\in S} |A| = \sum_{j=1}^{n} \lambda(j) = \frac{n|S|}{2}. \tag{8}$$

Compare the coefficients of x_{n-1} in (5):

$$\sum_{A\in S, A\ni n-1} (1+d_1+d_2-2|A|) + \sum_{A\in S, A\not\ni n-1} (1-d_1-d_2+2|A|) = -|S|;$$

$$\sum_{A\in S, A\ni n-1} |A| - \sum_{A\in S, A\not\ni n-1} |A| = |S|.$$

The last equation and (8) imply that

$$\sum_{A\in S, A\ni n-1} |A| = \frac{(n+2)|S|}{4}. \tag{9}$$

Compare now the coefficients of x_{n-1} in (7):

$$\sum_{A\in S, A\ni n-1} (1+d_1+d_2-2|A|) = \frac{d_1 d_2}{2} - \frac{|S|}{2};$$

$$\sum_{A\in S, A\ni n-1} |A| = \frac{(d_1+d_2)|S|}{4} - \frac{d_1 d_2}{4}.$$

The last equation and (9) imply that

$$(n^2-n+2)(d_1+d_2-n-2) = 2d_1 d_2.$$

Therefore, $d_1+d_2-n-2 > 0$. Besides, since d_1 and d_2 are distinct distances in H_n, we may assume that $d_2 \leqslant n$ and $d_1 \leqslant n-1$, and then

$$d_1+d_2-n-2 \leqslant \frac{2n(n-1)}{n^2-n+2} < 2.$$

Thus, $d_1+d_2-n-2 = 1$, so d_1 and d_2 satisfy equations $d_1+d_2 = n+3$ and $2d_1d_2 = n^2-n+2$. However, this system of equations has no solution in integers, a contradiction. □

Theorem 3.6. *Let $n \geqslant 3$ and let S be a 2-distance subset of $H_n \cap \pi$ where π is a hyperplane in $\mathbb{E}^n$. Then $|S| \leqslant \binom{n}{2}$, unless the following conditions are satisfied:* (i) $n = 3$ or 6 and (ii) π is $\{x_i = 0\}$ or $\{x_i = 1\}$.

Proof. Let d_1 and d_2 be distinct distances in S. We first assume that $O \in \pi$. Then we can write $\pi = \{\varphi_0(x_1, x_2, \ldots, x_n) = 0\}$ where $\varphi_0(x_1, x_2, \ldots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$. For $j = 1, 2, \ldots, n$, let $\varphi_j(x_1, x_2, \ldots, x_n) = x_j \varphi_0(x_1, x_2, \ldots, x_n)$. Then, for $0 \leqslant j \leqslant n$ and for all $A \in S$, $\varphi_j(A) = 0$.

By Theorem 3.4, $|S| \leqslant \binom{n}{2} + 1$. Suppose $|S| = \binom{n}{2} + 1$. Then

$$\{F_A \colon A \in S\} \cup \{\varphi_j \colon 0 \leqslant j \leqslant n\}$$

is a linearly dependent subset of $Pol(n, s)$. Let

$$\sum_{A \in S} \gamma_A F_A + \sum_{j=0}^{n} \beta_j \varphi_j = 0$$

where not all the coefficients γ_A, β_j equal 0. Applying both sides of this equation to $B \in S$ yields $\gamma_B = 0$, and we obtain that

$$(\beta_0 + \beta_1 x_1 + \cdots + \beta_n x_n)(\alpha_1 x_1 + \cdots + \alpha_n x_n) = 0. \tag{10}$$

Lemmas 2.3 and 3.5 now imply that $\pi = \{x_i = 0\}$.

Suppose now that $O \notin \pi$. Choose $A = (a_1, a_2, \ldots, a_n) \in S$ and consider the following isometry Φ of $\mathbb{E}^n$: $\Phi(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$ where

$$y_i = \begin{cases} 1 - x_i & \text{if } a_i = 1, \\ x_i & \text{if } a_i = 0. \end{cases}$$

Then $\Phi(S)$ is a 2-distance subset of $H_n \cap \Phi(\pi)$ and $\Phi(A) = O$, so $O \in \Phi(\pi)$. Therefore, $\Phi(\pi) = \{x_i = 0\}$ for some $i \in [n]$, and then π is $\{x_i = 1\}$ or $\{x_i = 0\}$.

In either case, the set S can be regarded as a 2-distance subset of an $(n-1)$-dimensional cube. By Theorem 3.1, $|S| = \binom{n}{2}+1$ only if $n = 3$ or $n = 6$. The proof is now complete. □

Example 3.7. Let S be the set of all 2-subsets of $[n]$. Then S is a 2-distance set of cardinality $\binom{n}{2}$ in the intersection of H_n and $\{x_1 + x_2 + \cdots + x_n = 2\}$. Thus, the bound obtained in Theorem 3.6 is sharp for every $n \geqslant 2$.

Example 3.8. Let S be the set of blocks of the unique 4-$(23, 7, 1)$ design. Then S is a 2-distance set of cardinality $\binom{23}{2}$ in the intersection of H_{23} and $\{x_1 + x_2 + \cdots + x_n = 7\}$.

Example 3.9. The following 10 points in $H_5 \cap \{x_1 + x_2 + x_3 = x_4 + x_5\}$ form a 2-distance set: (00000), (10010), (10001), (01010), (01001), (00110), (00101), (11011), (10111), (01111).

The following generalization of Theorem 3.6 can be obtained in a similar manner.

Theorem 3.10. *Let $n \geqslant s$ and let S be an s-distance subset of $H_n \cap \pi$ where π is a hyperplane of $\mathbb{E}^n$. If there exists $A \in H_n$ such that $d(A, X) \geqslant s$ for all $X \in H_n \cap \pi$, then $|S| \leqslant \binom{n}{s}$.*

The following corollary is well known [19].

Corollary 3.11. *For $k \geqslant s$, if S is a set of k-subsets of $[n]$, and $|\{|A \cap B| \colon A, B \in S, A \neq B\}| = s$, then $|S| \leqslant \binom{n}{s}$.*

Proof. The set S is an s-distance subset of H_n lying in the hyperplane $\pi = \{x_1 + x_2 + \cdots + x_n = k\}$ and $d(O, X) = k$ for all $X \in H_n \cap \pi$. □

4 Isosceles sets in H_n with more than two distances

The main tool in this section is the following extension of Theorem 2.15.

Definition 4.1. Let S be a nonempty subset of a metric space. A partition $(S_1, S_2, \ldots, S_k)$ of S is said to be a *complete decomposition of* S if it satisfies the following conditions:
(i) for $1 \leqslant i \leqslant k$, S_i is a 2-distance set;
(ii) for $1 \leqslant i \leqslant k-1$, $|S_i| \geqslant 2$; $|S_k| \geqslant 1$;
(iii) for $1 \leqslant i < j \leqslant k$, each $A \in S_j$ is the center of a sphere containing S_i.

Proposition 4.2. *Any finite isosceles set S in a metric space M admits a complete decomposition. Furthermore, if $M = \mathbb{E}^n$ and $(S_1, S_2, \ldots, S_k)$ is a complete decomposition of S, then* $\dim S \geqslant \dim S_1 + \dim S_2 + \cdots + \dim S_k$.

Proof. Let S be a finite isosceles set of cardinality N. We will prove the theorem by induction on N. The statement is trivial if $N = 1$ and also if S is a 2-distance set. Suppose S is an isosceles set of cardinality N with more than two distances and assume that both statements of the proposition are true for isosceles sets of cardinality less than N.

Let X and Y be subsets of S provided by Theorem 2.15 with the least possible cardinality of X. Then X is a 2-distance set. Indeed, if X has more than two nonzero distances, then Theorem 2.15 can be applied to X: $X = X_1 \cup Y_1$, and then $S = X_1 \cup (Y_1 \cup Y)$ with sets X_1 and $Y_1 \cup Y$ satisfying Theorem 2.15 and with $|X_1| < |X|$.

If $M = \mathbb{E}^n$, then $\dim S \geqslant \dim X + \dim Y$.

Since $|Y| < N$, we apply the induction hypothesis to Y. If $(S_2, \ldots, S_k)$ is a complete decomposition of Y, we put $S_1 = X$ and obtain a complete decomposition $(S_1, S_2, \ldots, S_k)$ of S.

If $M = \mathbb{E}^n$, then $\dim S \geqslant \dim S_1 + \dim Y \geqslant \sum_{i=1}^{k} \dim S_i$. □

By Lemma 2.6, for $n \leqslant 4$, there is no isosceles set in H_n with more than two distances. For every $n \geqslant 5$, the set S consisting of $[n]$ and all 2-subsets of $[n]$ is an isosceles subset of H_n of cardinality $\binom{n}{2} + 1$. If $n \neq 6$, this subset has three distinct nonzero distances: 2, 4, and $n-2$. For $n = 6$, the set S consisting of the empty set, the set $\{1, 2, 3, 4, 5, 6\}$, and of all 3-subsets of $\{1, 2, 3, 4, 5, 6\}$, containing 1, is an isosceles set of cardinality 12 with three distinct nonzero distances: 2, 3, and 4. As the next two theorems show, these are examples of isosceles sets of maximum size with more than two distances. But first we need the following lemma.

Lemma 4.3. *Let S be a 2-distance set of cardinality $\binom{n}{2} + 1$ in H_n, $n \geqslant 3$. Let $O \in S$, and let $S^* = S \setminus \{O\}$. Then S^* is not contained in a hyperplane $\{x_i + x_j = 1\}$.*

Proof. The statement is true for $n = 3$. If $n \geqslant 4$, then $|S| > n+1$, so S is not a 1-distance set. Let d_1 and d_2 be the nonzero distances in S, $d_1 < d_2$.

Suppose, without loss of generality, that $S^* \subset \{x_{n-1} + x_n = 1\}$. Then $|A \cap \{n-1, n\}| = 1$ and $|A \cap [n-2]| \in \{d_1 - 1, d_2 - 1\}$ for all $A \in S^*$. Consider polynomials $\varphi_0 = x_{n-1} + x_n - 1$,

$\varphi_j = x_j\varphi_0$ $(1 \leqslant j \leqslant n-2)$, $\varphi_{n-1} = x_{n-1}x_n$, and

$$\varphi_n = \left(\sum_{i=0}^{n-2} x_i - d_1 + 1\right)\left(\sum_{i=0}^{n-2} x_i - d_2 + 1\right).$$

Then $\varphi_j(A) = 0$ for $0 \leqslant j \leqslant n$ and for all $A \in S^*$.

Consider the following subset $\mathcal{B}$ of $Pol(n,2)$:

$$\mathcal{B} = \{F_A \colon A \in S^*\} \cup \{\varphi_j \colon 0 \leqslant j \leqslant n\}.$$

Claim. *$\mathcal{B}$ is linearly independent.*

Suppose

$$\sum_{A \in S^*} \alpha_A F_A + \sum_{j=0}^{n} \beta_j \varphi_j = 0.$$

Applying both sides to $B \in S^*$ yields $\alpha_B = 0$, so $\sum_{j=0}^{n} \beta_j\varphi_j = 0$. For $1 \leqslant i \leqslant n-1$, comparing the coefficients of $x_i x_n$ in both sides of this equation implies that $\beta_i = 0$. Therefore, $\beta_0\varphi_0 + \beta_n\varphi_n = 0$. Comparing the coefficients of x_n in this equation yields $\beta_0 = 0$ and then $\beta_n = 0$, so $\mathcal{B}$ is linearly independent.

Since $|\mathcal{B}| = |S^*| + n + 1 = \dim Pol(n,2)$, $\mathcal{B}$ is a basis of $Pol(n,2)$. For $1 \leqslant i \leqslant n$, we will expand $d_1 d_2 x_i$ in this basis. Applying both sides of this expansion to $B \in S^*$ would show that the coefficient of F_B in this expansion equals 1 if $i \in B$ and it equals 0 if $i \notin B$. Therefore,

$$d_1 d_2 x_i = \sum_{A \in S^*, A \ni i} F_A + \sum_{j=0}^{n} \gamma_{ij}\varphi_j.$$

Let $1 \leqslant k \leqslant n-2$. Since each $A \in S^*$ contains exactly one element of $\{n-1, n\}$, the coefficients of $x_k x_{n-1}$ and $x_k x_n$ in each F_A add up to 0. However, these monomials occur neither in $d_1 d_2 x_i$, nor in φ_j with $j \neq k$, and they occur in φ_k with the same coefficient γ_{ik}. Therefore, $\gamma_{ik} = 0$ for $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant n-2$, and we have

$$d_1 d_2 x_i = \sum_{A \in S^*, A \ni i} F_A + \rho_i\varphi_0 + \sigma_i\varphi_{n-1} + \tau_i\varphi_n. \tag{11}$$

For any $K \subseteq [n]$, let $\lambda(K)$ denotes the number of sets $A \in S$ containing K.

For $i = 1, 2, \ldots, n-2$, we compare the coefficients of $x_i x_{n-1}$ in both sides of (11): $0 = 2\lambda(i, n-1) - 2(\lambda(i) - \lambda(i, n-1))$, so

$$\lambda(i) = 2\lambda(i, n-1) \quad \text{and, similarly,} \quad \lambda(i) = 2\lambda(i, n). \tag{12}$$

For $i = n-1$ and for $i = n$, we compare the coefficients of $x_1 x_{n-1}$ and $x_1 x_n$, respectively, in both sides of (11) to obtain

$$\lambda(n-1) = 2\lambda(1, n-1) = \lambda(1), \quad \lambda(n) = 2\lambda(1, n) = \lambda(1). \tag{13}$$

For $i = 1, 2, \ldots, n-2$, we expand $d_1 d_2 x_i x_n$ in the basis $\mathcal{B}$. Applying both sides of this expansion to $B \in S^*$ would show that the coefficient of F_B equals 0 or 1, and it equals 1 if and only if $\{i, n\} \subseteq B$. Therefore,

$$d_1 d_2 x_i x_n = \sum_{A \in S^*, A \ni i, n} F_A + \sum_{j=0}^{n} \varepsilon_{ij} \varphi_j. \tag{14}$$

If $i, n \in A$, then $x_i x_n$ occurs in F_A with coefficient 2 and, since $n-1 \notin A$, $x_i x_{n-1}$ occurs in F_A with coefficient -2. Comparing the coefficients of $x_i x_{n-1}$ and also the coefficients of $x_i x_n$ in both sides of (14) yields

$$0 = -2\lambda(i, n) + \varepsilon_{ii}, \quad d_1 d_2 = 2\lambda(i, n) + \varepsilon_{ii}.$$

Therefore, $\lambda(i, n) = d_1 d_2 / 4$, and then (12) and (13) imply that $\lambda(i) = d_1 d_2 / 2$ for $i = 1, 2, \ldots, n$.

For $i = 1, 2$, let $N_i = |\{A \in S^* \colon |A| = d_i\}|$. Then $N_1 + N_2 = \binom{n}{2}$, and counting in two ways pairs (A, j) with $A \in S^*$ and $j \in A$ yields another equation: $N_1 d_1 + N_2 d_2 = n d_1 d_2 / 2$. From these equations, we find

$$N_1 = \frac{n d_2 (n - d_1 - 1)}{2(d_2 - d_1)}, \quad N_2 = \frac{n d_1 (d_2 + 1 - n)}{2(d_2 - d_1)}.$$

Therefore, $d_2 \geqslant n-1$. Since no set $A \in S^*$ contains $\{n-1, n\}$, we have $d_2 \neq n$, so $d_2 = n-1$. Then $N_2 = 0$ and therefore S^* lies in the hyperplane $\pi = \{x_1 + x_2 + \cdots + x_n = d_1\}$. Since $n \geqslant 3$, we have $\pi \neq \{x_{n-1} + x_n = 1\}$. This implies that $\dim S^* \leqslant n-2$. Then, by Theorem 3.4, $\binom{n}{2} = |S^*| \leqslant \binom{n-1}{2} + 1$, a contradiction. □

Theorem 4.4. *Let S be an isosceles subset of H_n with more than two nonzero distances. Then $|S| \leqslant \binom{n}{2} + 1$.*

Proof. Due to Lemma 2.6, we have $n \geqslant 5$. Let $(S_1, S_2, \ldots, S_k)$ be a complete decomposition of S. Since S has more than two distances, $k \geqslant 2$. For $i = 1, 2, \ldots, k$, let $m_i = \dim S_i$ and let $m = \sum_{i=1}^{k} m_i$. Then $m \leqslant \dim S \leqslant n$. Let $\mathbf{m} = (m_1, m_2, \ldots, m_k)$.

Note that $m_1, m_2, \ldots, m_{k-1}$ are positive while m_k is nonnegative. Since each S_i is a 2-distance set, Theorem 3.4 implies that

$$|S| \leqslant k + \sum_{i=1}^{k} \binom{m_i + 1}{2}.$$

Thus, it suffices to prove that

$$k + \sum_{i=1}^{k} \binom{m_i + 1}{2} \leqslant \binom{n}{2}. \tag{15}$$

In each of the following six cases, we either prove (15) or prove directly that $|S| \leqslant \binom{n}{2}$.

Case 1. $k \geqslant 5$ and $m_k \geqslant 1$.

Let $X_1, X_2, \ldots, X_k$ be pairwise disjoint sets with $|X_i| = m_i$ and let X be the union of these sets. Let $X_0 = X_k$. For $i = 1, 2, \ldots, k$, choose $a_i \in X_i$ and let E_i be the set of all 2-subsets of $X_{i-1} \cup \{a_i\}$. Let

$$F = \{\{a_i, a_j\} \colon 1 \leqslant i < j \leqslant k, j - i \not\equiv \pm 1 \pmod{k}\}.$$

The sets $E_1, E_2, \ldots, E_k, F$ are pairwise disjoint subsets of the set of all 2-subsets of X. Since $|E_i| = \binom{m_i+1}{2}$ and $|F| = k(k-3)/2 \geqslant k$, (15) follows.

Case 2. $k = 4$ and $m_k \geqslant 1$.

Let X_i, a_i, E_i, and X be the same as in Case 1. Let $F_1 = \{\{x, y\} \colon x \in X_1, y \in X_3\}$ and $F_2 = \{\{x, y\} \colon x \in X_2, y \in X_4\}$. Since $E_1, E_2, \ldots, E_k, F_1, F_2$ are pairwise disjoint,

$$m_1 m_3 + m_2 m_4 + \sum_{i=1}^{k} \binom{m_i + 1}{2} \leqslant \binom{m}{2}.$$

If $m \leqslant n - 1$, then $2 + \binom{m}{2} \leqslant (n^2 - 3n + 6)/2 \leqslant \binom{n}{2}$. Since $m_1 m_3 + m_2 m_4 \geqslant 2$, (15) follows. If $m = n \geqslant 6$, then $m_1 m_3 + m_2 m_4 \geqslant 4$ and (15) follows. If $m = n = 5$, then either side of (15) equals 10.

Case 3. $k = 3$ and $m_k \geqslant 1$.

Let X_i, a_i, E_i, and X be the same as in Case 1 and let $a_0 = a_3$. For $i = 1, 2, 3$, let $F_i = \{\{x, a_{i-1}\} \colon x \in X_i, x \neq a_i\}$, so $|F_i| = m_i - 1$. We obtain

$$m - 3 + \sum_{i=1}^{3} \binom{m_i + 1}{2} \leqslant \binom{m}{2}.$$

If $m \geqslant 6$, then (15) follows. If $m \leqslant n - 1$, then $\binom{m}{2} - m + 6 \leqslant \binom{n-1}{2} + 3 \leqslant \binom{n}{2}$, so again (15) follows.

Suppose $m = n = 5$. If m_1, m_2, and m_3 are 1, 2, and 2 (in any order), then (15) holds. Suppose m_1, m_2, and m_3 are 1, 1, and 3. Since the maximum size of a 2-distance set in $\mathbb{E}^3$ is 6 (see [6]) and no line meets H_n in more than two points, we obtain that $|S| \leqslant 2 + 2 + 6 = \binom{5}{2}$.

Case 4. $k = 2$, $m_k \geqslant 1$, and $m \leqslant n - 1$.

We have

$$\binom{m_1 + 1}{2} + \binom{m_2 + 1}{2} = \binom{m + 1}{2} - m_1 m_2.$$

If $m \leqslant n-2$, then $2+\binom{m+1}{2}-m_1m_2 \leqslant \binom{n-1}{2}+1 \leqslant \binom{n}{2}$. If $m=n-1$, then $2+\binom{m+1}{2}-m_1m_2 \leqslant \binom{m+1}{2}=\binom{n}{2}$. In either case, (15) follows.

Case 5. $k=2$, $m=n$, $m_1 \geqslant 2$, and $m_2 \geqslant 2$.

We have

$$2+\binom{m_1+1}{2}+\binom{m_2+1}{2}=2+\binom{n+1}{2}-m_1m_2 \leqslant 2+\binom{n+1}{2}-2(n-2).$$

If $n \geqslant 6$, then $2+\binom{n+1}{2}-2(n-2) \leqslant \binom{n}{2}$. If $n=5$, then m_1 and m_2 are 2 and 3. The maximum size of a 2-distance set in $\mathbb{E}^3$ is 6. Since the maximum size of a 2-dimensional subset of H_n is 4 (Lemma 2.2), we have $|S| \leqslant 10=\binom{5}{2}$.

Case 6. $k=2$, $m_1 \leqslant n-1$, and $m_2=0$.

Then the left hand side of (15) does not exceed $2+\binom{n-1}{2}<\binom{n}{2}$.

If $k \geqslant 3$ and $m_k=0$, we let $S'=S_1 \cup S_2 \cup \ldots \cup S_{k-1}$. Since $|S|=|S'|+1$, the inequality $|S| \leqslant \binom{n}{2}+1$ will follow, whenever one of Cases 1–5 applies to S or to S'. This leaves the following five cases open: $\mathbf{m}=(n-1,1,0)$, $\mathbf{m}=(1,n-1,0)$, $\mathbf{m}=(n,0)$, $\mathbf{m}=(n-1,1)$, and $\mathbf{m}=(1,n-1)$.

Case 7. $\mathbf{m}=(n-1,1,0)$.

Let $S_2=\{C_1,C_2\}$ and $S_3=\{C_3\}$. Then S_1 is contained in spheres with centers C_1, C_2, and C_3. Lemma 2.8 implies that at least two of these spheres are not equal and then Corollary 2.9 implies that $m_1 \leqslant n-2$, a contradiction.

Case 8. $\mathbf{m}=(1,n-1,0)$.

Let $S_1=\{C_1,C_2\}$ and $S_3=\{C_3\}$. For $1 \leqslant j \leqslant 3$, let $C_j=(c_{j1},\ldots,c_{jn})$. Since every point of S_2 is equidistant from C_1 and C_2, S_2 is contained in the perpendicular bisector π_1 of segment C_1C_2 in $\mathbb{E}^n$. On the other hand, S_2 is contained in a sphere of H_n with center C_3. Since $\dim S_2=n-1$, this sphere is contained in a unique hyperplane π_2. Hyperplanes π_1 and π_2 have normal vectors with coordinates $c_{1i}-c_{2i}$ and $1-2c_{3i}$, respectively. Since $c_{1i},c_{2i},c_{3i} \in \{0,1\}$, these normal vectors are collinear if and only if $C_3=C_1$ or $C_3=C_2$. Thus π_1 and π_2 are distinct hyperplanes, and then $m_2 \leqslant n-2$, a contradiction.

Case 9. $\mathbf{m}=(n,0)$.

Then S_1 lies in a hyperplane (other that $x_i=0$ or $x_i=1$), and Theorem 3.6 implies that $|S_1| \leqslant \binom{n}{2}$, so $|S| \leqslant \binom{n}{2}+1$.

Case 10. $\mathbf{m} = (n-1, 1)$.

Let $S_2 = \{C_1, C_2\}$. Then S_1 is contained in spheres with distinct centers C_1 and C_2. Since $\dim S_1 = n-1$, these spheres lie in the same hyperplane and then Lemma 2.8 implies that $d(C_1, C_2) = n$. Without loss of generality, we assume that $C_1 = O$ and C_2 is the point with all coordinates equal 1. If $A \in S_1$, then $d(A, C_1) < n$ and $d(A, C_2) < n$. Since $\triangle C_1 A C_2$ is isosceles, we have $d(A, C_1) = d(A, C_2) = n/2$. Therefore, n is even and $|A| = \frac{n}{2}$ for all $A \in S_1$. If d_1 and d_2 are distinct nonzero distances in S_1, then $|A \cap B| \in \{(n-d_1)/2, (n-d_2)/2\}$ for all distinct $A, B \in S_1$. By Corollary 3.11, $|S_1| \leqslant \binom{n}{2}$. If $|S_1| = \binom{n}{2}$, then S_1 has to satisfy (i), (ii), or (iii) of Theorem 2.14. However, S_1 does not satisfy (i) or (ii), because $n \geqslant 5$, and S_1 does not satisfy (iii), because n is even. Therefore, $|S_1| \leqslant \binom{n}{2} - 1$, and then $|S| \leqslant \binom{n}{2} + 1$.

Case 11. $\mathbf{m} = (1, n-1)$.

Without loss of generality, we assume that $S_1 = \{O, C\}$ where C is a point with the first k coordinates equal 1 and the remaining coordinates equal 0. Then S_2 lies in the perpendicular bisector π of segment OC of $\mathbb{E}^n$ and $\pi = \{x_1 + x_2 + \cdots + x_k = k/2\}$. Therefore, k is even.

If $k = n$, then $|A| = n/2$ for all $A \in S_2$ and, as in the previous case, we apply Corollary 3.11 and Theorem 2.14 to obtain that $|S_2| \leqslant \binom{n}{2} - 1$. Thus, if $k = n$, then $|S| \leqslant \binom{n}{2} + 1$.

Since k is even, we now assume that $2 \leqslant k \leqslant n-1$. Since $S_2 \subset \pi$, Theorem 3.6 implies that $|S_2| \leqslant \binom{n}{2}$. If $|S_2| \leqslant \binom{n}{2} - 1$, then $|S| \leqslant \binom{n}{2} + 1$, so we assume that $|S_2| = \binom{n}{2}$.

Let d_1 and d_2 be distinct nonzero distances in S_2 and let

$$T_1 = \{A \in S_2 \colon |A| \notin \{d_1, d_2\}\}, \quad T_2 = \{B \in S_2 \colon |B| \in \{d_1, d_2\}\}.$$

If $A \in T_1$ and $B \in T_2$, then $d(O, A) \neq d(O, B)$ and $d(O, A) \neq d(A, B)$. Therefore, $d(A, B) = d(O, B) = |B|$. Thus, each $B \in T_2$ is the center of a sphere containing $S_1 \cup T_1$.

If $A_1, A_2 \in T_1$, then $d(O, A_1) \neq d(A_1, A_2)$ and $d(O, A_2) \neq d(A_1, A_2)$. Therefore, $d(O, A_1) = d(O, A_2)$. Then $d(C, A_1) = d(C, A_2)$, so each point of S_1 is the center of a sphere containing T_1. If $|T_1| \geqslant 2$ and $|T_2| \geqslant 1$, then (T_1, S_1, T_2) is a complete decomposition of S. Since it consists of three sets, we apply one of the previous cases to obtain that $|S| \leqslant \binom{n}{2} + 1$. If $T_2 = \varnothing$, then (T_1, S_1) is a complete decomposition of S, and we refer to Case 10.

If $|T_1| = 1$, then $S_1 \cup T_1$ is a 2-distance set and therefore, $(S_1 \cup T_1, T_2)$ is a complete decomposition of S. Since $\dim(S_1 \cup T_1) = 2$, we again refer to previous cases.

Suppose now that $T_1 = \varnothing$, i.e., $|A| \in \{d_1, d_2\}$ for all $A \in S_2$. For $A \in S$, we consider, as before, polynomials $F_A = (f_A - d_1)(f_A - d_2)$. Note that $f_O = \sum_{i=1}^{n} x_i$. Let $\varphi_0 = \sum_{i=1}^{k} x_i - k/2$ and, for $j = 1, 2, \ldots, n$, let $\varphi_j(x_1, x_2, \ldots, x_n) = x_j \varphi_0(x_1, x_2, .., x_n)$. Then $\varphi_j(A) = 0$ for $0 \leqslant j \leqslant n$ and for all $A \in S_2$. Let

$$\mathcal{B} = \{F_A \colon A \in S_2 \cup \{O\}\} \cup \{\varphi_j \colon 0 \leqslant j \leqslant n\}.$$

Since $|\mathcal{B}| > \dim Pol(n,2)$, the set $\mathcal{B}$ is linearly dependent. Let

$$\sum_{A\in S_2\cup\{O\}} \alpha_A F_A + \sum_{j=0}^{n} \beta_j \varphi_j = 0$$

with not all α_A, β_j equal 0. Applying both sides of this equality to $B \in S_2$ yields $\alpha_B = 0$, so we have

$$\alpha_O \left(\sum_{i=0}^{n} x_i - d_1\right)\left(\sum_{i=0}^{n} x_i - d_2\right) + \left(\beta_0 + \sum_{j=1}^{n} \beta_j x_j\right)\left(\sum_{i=1}^{k} x_i - \frac{k}{2}\right) = 0. \qquad (16)$$

Suppose $k < n-1$. Then comparing the coefficients of $x_{n-1}x_n$ in both sides of (16) yields $\alpha_O = 0$. Therefore, H_n is contained in the union of hyperplanes $\{\beta_0 + \sum_{j=1}^{n} \beta_j x_j = 0\}$ and $\{\sum_{i=1}^{k} x_i = k/2\}$. Lemma 2.3 implies that $k = 2$, i.e., $\pi = \{x_1 + x_2 = 1\}$. Since $S_2 \cup \{O\}$ is a 2-distance set of cardinality $\binom{n}{2} + 1$, this contradicts Lemma 4.3.

Thus, $k = n-1$. Then n is odd. Comparing the coefficients of x_n in (16) yields $\alpha_O(1 - d_1 - d_2) - \beta_n k/2 = 0$ and comparing the coefficients of $x_{n-1}x_n$ yields $2\alpha_O + \beta_n = 0$. From these two equations, $\alpha_O(n - d_1 - d_2) = 0$.

Suppose $d_1 + d_2 \neq n$. Then $\alpha_O = 0$, so

$$\left(\beta_0 + \sum_{j=1}^{n} \beta_j x_j\right)\left(\sum_{i=1}^{n-1} x_i - \frac{n-1}{2}\right) = 0,$$

and then Lemma 2.3 implies that all β_j equal 0.

Thus, $d_1 + d_2 = n$. Since n is odd, Lemma 2.4 implies that $|S_2| \leqslant 2n+2$, so $\binom{n}{2} \leqslant 2n+2$, $n = 5$. Since $d(O,C) = 4$, we have $d(O,A) \geqslant 2$ for all $A \in S_2$. Therefore, d_1 and d_2 are 2 and 3. Since the distance between two sets of the same cardinality is even, S_2 contains at most four 2-subsets of $\{1,2,3,4,5\}$ and at most four 3-subsets. Then $|S| < \binom{5}{2}$, a contradiction. □

For every $n \geqslant 2$, the set $S = \{A \in H_n : |A| = 2 \text{ or } 0\}$ is a 2-distance set of cardinality $\binom{n}{2} + 1$. This implies the following result.

Corollary 4.5. *For any isosceles subset S of H_n there exists a 2-distance subset T of H_n such that $|S| \leqslant |T|$.*

As the table in Section 1 shows, a similar result for $\mathbb{E}^n$ is not true.

Theorem 4.6. *Let S be an isosceles subset of H_6 and let there be at least three distinct nonzero distances between points of S. Then $|S| \leqslant 12$.*

Proof. Suppose there are points in S at distance 6. Without loss of generality, let O and $Z = (1,1,1,1,1,1)$ be in S. Then $|A| = 3$ for every $A \in S \setminus \{O, Z\}$. The set $\{X \in H_6 : |X| = 3\}$ of cardinality 20 consists of 10 pairs with distance 6 in each pair. If A and B from the same pair are in S, then every other $C \in S \setminus \{O, Z\}$ has to be at distance

3 from A and B. However, (1) would imply that $d(A, C)$ is even. Therefore, $S \setminus \{O, Z\}$ contains at most one element from each pair and $|S| \leqslant 12$.

Suppose the maximum distance between points of S is 5. Without loss of generality, let O and $Y = (1,1,1,1,1,0)$ be in S. Since no point of H_n is equidistant from O and Y, every point of S has to be at distance 5 from O or from Y. This implies that $|S| \leqslant 12$.

Suppose there are points in S at distance 1. Without loss of generality, let $O \in S$ and $X = (1,0,0,0,0,0) \in S$. Then every point of S has to be at distance 1 from O or from X. This implies that $|S| \leqslant 12$.

Suppose now that the nonzero distances between points of S are 2, 3, and 4. For $i = 0, 1$, let $S_i = \{A \in S \colon |A| \equiv i \pmod 2\}$. Then (1) implies that S_0 and S_1 are 2-distance sets and $d(A, B) = 3$ whenever $A \in S_0$ and $B \in S_1$.

Since $|A| + |B| \neq 6$ for any $A, B \in S$, Lemma 2.8 implies that spheres $Sp(A, 3)$ and $Sp(B, 3)$ (with distinct A and B) are not equal. Therefore, if $|S_i| \geqslant 3$ for $i = 0$ and for $i = 1$, then each S_i is contained in the intersection of three distinct spheres. Now Lemma 2.10 implies that $\dim S_i \leqslant 3$. Since the cardinality of a 2-distance set in $\mathbb{E}^3$ does not exceed 6 (see [6]), we obtain that $|S| \leqslant 12$.

If $|S_i| = 1$ for $i = 0$ or for $i = 1$, then S_{1-i} is contained in a sphere of radius 3. Such a sphere is isometric to $Sp(O, 3)$ and therefore consists of 10 pairs of points with distance 6 between the points of each pair. Therefore, S_{1-i} contains at most one point from each pair and we obtain that $|S| \leqslant 11$.

Suppose $|S_0| = 2$. Since S_1 consists of 3-subsets of $\{1,2,3,4,5,6\}$ no two of which are complementary, we have $|S_1| \leqslant 10$ and therefore $|S| \leqslant 12$.

Suppose $|S_1| = 2$. If we replace each element of cardinality 4 in S_0 by its complement, we obtain a set S_0' of 2-subsets of $\{1,2,3,4,5,6\}$, each at distance 3 from both elements of S_1. Without loss of generality, we assume that the elements of S_1 are (i) $\{1,2,3\}$ and $\{1,2,4\}$ or (ii) $\{1,2,3\}$ and $\{1,4,5\}$. In either case, there are only five 2-subsets of $\{1,2,3,4,5,6\}$ at distance 3 from both elements of S_1, so $|S| \leqslant 7$.

Thus, in all cases, $|S| \leqslant 12$. □

5 Isosceles sets in $\mathbb{E}^n$ for $n \leqslant 8$

In this section we determine the maximum size of an isosceles subset of $\mathbb{E}^n$ for $n \leqslant 8$ and describe all isosceles sets of the maximum size for $n \leqslant 7$. Throughout the section, we will use the maximum size of 2-distance sets given in the second row of the table in Section 1. For $A, B \in \mathbb{E}^n$, AB denotes the distance between A and B.

Obviously, any maximum size isosceles set in $\mathbb{E}^1$ consists of the endpoints and the midpoint of a segment.

For $n = 2$, we refer to Kelly [11] who proved that there is no isosceles set of cardinality 7 in $\mathbb{E}^2$ and that any isosceles set of cardinality 6 consists of the vertices and center of a regular pentagon.

In cases $n = 3, 4$, and 7 we will use the following lemma.

Lemma 5.1. *Let S be a 2-distance subset of $\mathbb{E}^n$ and let d be a nonzero distance between points of S. Let Γ be the graph whose vertex set is S and two vertices form an edge if and only if the distance between the vertices equals d. Suppose Γ is a strongly regular graph with parameters (v, k, λ, μ) such that $v \geqslant 2k+1$. Suppose further that* $\dim S = n-1$, $\dim(S \setminus \{A\}) = n-1$ *for all $A \in S$, and at least one of the following three conditions is satisfied:*

(i) $v \leqslant 3k - 2\lambda$ *and* $v \leqslant 3k - 2\mu + 2$;

(ii) $v \leqslant 3k - 2\lambda$ *and* $2k \geqslant 2\mu + n - 1$;

(iii) $v \leqslant 3k - 2\mu + 2$ *and* $2k \geqslant 2\lambda + n + 1$.

Let $P \in \mathbb{E}^n$ be such that $S \cup \{P\}$ is an isosceles set of dimension n. Then either $S \cup \{P\}$ is a 2-distance set or S lies on a sphere centered at P.

Proof. Suppose $S \cup \{P\}$ is not a 2-distance set and let $S \cup \{P\} = X \cup Y$ with X and Y satisfying Theorem 2.15. If $Y = \{P\}$, then S lies on a sphere centered at P. Suppose $Y \neq \{P\}$.

Since $v \geqslant 2k+1$, any sphere, whose center is in S, contains at most $v-k-1$ points of S, so $|S \cap X| \leqslant v - k - 1$. Let $A, B \in S \cap X$, $A \neq B$. If $\{A, B\}$ is an edge of Γ, then $|S \cap Y| \leqslant \lambda + (v - 2k + \lambda)$ and we have $v = |S| \leqslant 2v - 3k + 2\lambda - 1$. Thus, $v \geqslant 3k - 2\lambda + 1$. If $\{A, B\}$ is not an edge, then $|S \cap Y| \leqslant \mu + (v - 2k + \mu - 2)$. Therefore $v = |S| \leqslant 2v - 3k + 2\mu - 3$, and we have $v \geqslant 3k - 2\mu + 3$.

If (ii) is satisfied, then $\{A, B\}$ is not an edge of Γ for all $A, B \in S \cap X$, so $S \cap X$ is a 1-distance set. Therefore, $|S \cap X| \leqslant \dim(S \cap X) + 1 \leqslant n$, $|S \cap Y| \geqslant v - n$, and we obtain that $v - 2k + 2\mu - 2 \geqslant v - n$, a contradiction.

If (iii) is satisfied, then $\{A, B\}$ is an edge of Γ for all distinct $A, B \in S \cap X$, so $S \cap X$ is again a 1-distance set, and we obtain that $v - 2k + 2\lambda \geqslant v - n$, a contradiction.

If (i) is satisfied, then $|S \cap X| = 1$, i.e., $X = \{A, P\}$ with $A \in S$. Let π be the hyperplane containing S. Since $S \setminus \{A\}$ generates π and since each point of $S \setminus \{A\}$ has to be equidistant from A and P, the hyperplane π passes through the midpoint of the segment PA. However, this is impossible because $P \notin \pi$ and $A \in \pi$. □

Corollary 5.2. *Let S be the set of vertices of a regular pentagon in a plane π and let a point $P \notin \pi$ be such that $S \cup \{P\}$ is an isosceles set. Then the orthogonal projection of P onto π is the center of the pentagon.*

Proof. Let d being the smaller of the two distances in S. Then the graph Γ is strongly regular with parameters $(5, 2, 0, 1)$. Therefore, P is equidistant from at least three points of T, so the orthogonal projection of P onto π is the center of T. □

For $n \geqslant 3$, let S be an isosceles set in $\mathbb{E}^n$. If there are more than two nonzero distances between points of S, we have $S = X \cup Y$ with X and Y satisfying Theorem 2.15. We will always choose $|X|$ as small as possible, and then X is a 2-distance set (see the proof of Proposition 4.2). If $\dim X = 1$, then X lies on a line and on a sphere, and therefore $|X| = 2$.

Let $n = 3$ and let $|S| \geqslant 8$. Then S is not a 2-distance set. Since X is 2-distance set, we have $|X| \leqslant 6$, and therefore $\dim Y \neq 0$.

Suppose $\dim Y = 1$. Then $\dim X \leqslant 2$, so $|Y| \leqslant 3$ and $|X| \leqslant 5$. Therefore, $|S| = 8$, $|X| = 5$, and $|Y| = 3$. Thus, X is the set of vertices of a regular pentagon in a plane π and Y consists of the endpoints P and Q and the midpoint M of a segment of a line $l \perp \pi$. Let O be the center of X. Since P and Q are equidistant from the vertices of the pentagon, $l \cap \pi = \{O\}$. Let $OP \geqslant OQ$. Then, for $A \in X$, we derive from an isosceles $\triangle PAQ$ that $AM < AP$ and $MP < AP$. Therefore, $AM = MP$. This implies $M = O$, and then the remaining 7 points of S lie on a sphere with center O.

Let $\dim Y = 2$ and let π be the plane containing Y. Then $|Y| \leqslant 6$, $\dim X = 1$, and therefore $|X| = 2$. Thus, $|Y| = 6$ and Y consists of the vertices of a regular pentagon and its center O. Corollary 5.2 implies that the orthogonal projection of either point of X onto π is O, so we have the same configuration of 8 points as in the previous paragraph.

Let $n = 4$. The cardinality of a 2-distance set in $\mathbb{E}^4$ does not exceed 10, and it equals 10 only for the set of the midpoints of the edges of a regular 4-dimensional simplex [15]. Adjoining the center of the simplex, we obtain an isosceles set of cardinality 11.

Let S be an isosceles set in $\mathbb{E}^4$ with more than two distances and let $|S| \geqslant 11$. If $\dim Y = 0$, then Y consists of the center of a sphere containing X and, since X is a 2-distance set of cardinality 10, the set S of cardinality 11 has just been described.

If $\dim Y = 1$, then $\dim X \leqslant 3$, so $|X| \leqslant 6$, $|Y| \leqslant 3$, and $|S| < 11$.

If $\dim Y = 3$, then $|Y| \leqslant 8$. Since $\dim X = 1$, we have $|X| = 2$, so $|S| < 11$.

Let $\dim Y = 2$. Then $\dim X \leqslant 2$, so $|X| \leqslant 5$, $|Y| \leqslant 6$. Therefore, $|S| = 11$, X is the set of vertices of a regular pentagon and Y consists of the vertices and center of a regular pentagon. Let O be the intersection point of orthogonal planes π_1 and π_2 generated by X and Y, respectively. Corollary 5.2 implies that O is the center of both pentagons. Therefore, $\triangle AOB$ with $A \in X$ and $B \in Y \setminus \{O\}$ is isosceles, and then $OA = OB$. Thus, S consists of the vertices of two congruent regular pentagons, lying in orthogonal planes, and their common center. This is the second example of an isosceles set of cardinality 11 in $\mathbb{E}^4$.

Let $n = 5$. If S is a 2-distance set, then $|S| \leqslant 16$; furthermore, if $|S| = 16$, then S is a set of vertices of a 5-dimensional cube, no two of which are adjacent [15]. There are two such set for the given 5-cube, but they are symmetric with respect to the center of the cube. (See Example 3.2.) Let S have more than two distances. If $\dim Y = 4$, then $|S| \leqslant 2+11 = 13$; if $\dim Y = 3$, then $|S| \leqslant 5+8 = 13$; if $\dim Y = 2$, then $|S| \leqslant 6+6 = 12$; if $\dim Y = 1$, then $|S| \leqslant 10 + 3 = 13$. If Y is a singleton, then it is the center of a cube and X consists of 16 vertices of that cube, no two of which are adjacent, so $|S| = 17$, and this is the only (up to similarity) isosceles set of this size in $\mathbb{E}^5$.

Let $n = 6$ and let S have more than two distances. If $\dim Y = 0, 1, 2, 3, 4, 5$, then $|S|$ is bounded by $27+1 = 28$, $16+3 = 19$, $10+6 = 16$, $6+8 = 14$, $5+11 = 16$, or $2+17 = 19$, respectively. Since a unique 2-distance set of 27 points lies on a sphere ([15]), adjoining the center of this sphere yields a unique isosceles set of cardinality 28. Due to Coxeter

[5], this set can be described as the 2-distance subset T of $\mathbb{E}^8$ below adjoined by a point $Q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1, 1)$, the center of a sphere, containing T.

$$T = \{A_i, B_i \colon 1 \leqslant i \leqslant 6\} \cup \{C_{ij} \colon 1 \leqslant i < j \leqslant 6\},$$

where A_i has i^{th} and 7^{th} coordinate equal 2 and the other six coordinates equal 0, B_i has i^{th} and 8^{th} coordinate equal 2 and the other six coordinates equal 0, and C_{ij} has i^{th} and j^{th} coordinate equal -1 and the other six coordinates equal 1. Thus, $|T| = 27$ and the distance between any two distinct points of T is 4 or $\sqrt{8}$. Since all 28 points lie in hyperplanes $\sum_{i=1}^{6} x_i = 2$ and $x_7 + x_8 = 2$, we have $\dim(T \cup \{Q\}) = 6$.

Let $n = 7$. The only 2-distance set of cardinality 29 in $\mathbb{E}^7$ consists of 28 points lying on a sphere and one point off this sphere [15]. Let S have more than two distances. If $\dim Y = 0$, then, since X is a 2-distance set in $\mathbb{E}^7$, lying on a sphere, we have $|X| \leqslant 28$, so $|S| \leqslant 29$. For $\dim Y = 1, 2, 3, 4, 5, 6$, the cardinality of S is bounded by $27 + 3 = 30$, $16 + 6 = 22$, $10 + 8 = 18$, $6 + 11 = 17$, $5 + 17 = 22$, or $2 + 28 = 30$, respectively. Thus, the cardinality of an isosceles set in $\mathbb{E}^7$ does not exceed 30.

Suppose $|X| = 27$ and $|Y| = 3$. Then X is a unique 2-distance set lying in a hyperplane π and Y consists of the endpoints P and P' and the midpoint of a segment of a line $l \perp \pi$. As in the case $n = 3$, one can show that l intersects π at the center Q of the sphere (in π) containing X and that $PQ = P'Q$ is the radius of the sphere.

Suppose now that $|X| = 2$ and $|Y| = 28$. Let $X = \{P, P'\}$. We assume that S is embedded in $\mathbb{E}^8$ and that $Y = T \cup \{Q\}$ with the 2-distance set T and its center Q described in the case $n = 6$. We apply Lemma 5.1 to T and P with d being the larger of the two distances in T. The graph Γ is strongly regular with parameters $(27, 10, 1, 5)$, so the conditions of the lemma are satisfied.

If P is the center of a sphere containing T, then the orthogonal projection of P onto the 6-flat containing T is Q. Since all triangles PQR with $R \in T$ are isosceles, we derive that $PQ = QR$, so we have obtained the same configuration of 30 points as above.

Suppose $T \cup \{P\}$ is a 2-distance set. It suffices to show that P is equidistant from a set of points of T which generates the 6-flat π containing T. Note that since the cardinality of a 2-subset of $\mathbb{E}^5$ does not exceed 16, any 17 points of T generate π. (However, the 16 points at distance $\sqrt{8}$ from a point of T generate a 5-flat.)

Let $P = (p_1, p_2, ..., p_8)$ and $p = \{p_1, p_2, ..., p_8\}$. For distinct $i, j, k, l \in \{1, 2, 3, 4, 5, 6\}$,

$$PA_i = PA_j \;\Leftrightarrow\; PB_i = PB_j \;\Leftrightarrow\; p_i = p_j, \qquad (17)$$
$$PA_i < PA_j \;\Leftrightarrow\; PB_i < PB_j \;\Leftrightarrow\; p_i = p_j + 2;$$

$$PA_i = PB_i \;\Leftrightarrow\; p_7 = p_8, \qquad (18)$$
$$PA_i < PB_i \;\Leftrightarrow\; p_7 = p_8 + 2,$$
$$PA_i > PB_i \;\Leftrightarrow\; p_7 = p_8 - 2;$$

$$PC_{ij} = PC_{ik} \;\Leftrightarrow\; p_j = p_k, \qquad (19)$$
$$PC_{ij} > PC_{ik} \;\Leftrightarrow\; p_j = p_k + 2;$$

$$PC_{ij} = PC_{kl} \;\Leftrightarrow\; p_i + p_j = p_k + p_l, \qquad (20)$$
$$PC_{ij} > PC_{kl} \;\Leftrightarrow\; p_i + p_j = p_k + p_l + 2.$$

From (17), $|p| = 1$ or 2. If $|p| = 1$, then (17) and (19) imply that T contains at least 21 points at the same distance from P. These 21 points generate the 6-flat π.

Suppose $|p| = 2$. Then (18) implies that $p_7 = p_8$. Now (20) implies that one of the elements of p occurs only once among the coordinates p_i, $1 \leqslant i \leqslant 6$, so let it be p_1. Then P is equidistant from the 10 points A_i, B_i, $2 \leqslant i \leqslant 6$, and P is equidistant from the 10 points C_{ij}, $2 \leqslant i < j \leqslant 6$. Observe that points A_2, A_3, A_4, A_5, A_6, and B_6 generate a 5-flat, and this 5-flat is the intersection of hyperplanes $\{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 2\}$, $\{x_7 + x_8 = 2\}$, and $\{x_1 = 0\}$. Since none of the points C_{ij} lies in $\{x_1 = 0\}$, the 10 points A_i, B_i, $2 \leqslant i \leqslant 6$, and any one point C_{jk} generate the 6-flat π.

Thus, $\mathbb{E}^7$ contains a unique (up to similarity) isosceles set of cardinality 30.

If $n = 8$, then there exists a 2-distance set of cardinality $\binom{10}{2} = 45$ (see [15]). It is an isosceles set meeting Blokhuis' bound. Theorem 2.15 implies that any isosceles set of cardinality 45 in $\mathbb{E}^8$ has to be a 2-distance set.

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