# Permutations with Kazhdan-Lusztig polynomial $P_{id,w}(q) = 1 + q^h$


Alexander Woo*

Department of Mathematics, Statistics, and Computer Science
Saint Olaf College
1520 Saint Olaf Avenue
Northfield, MN 55057

(Appendix by Sara Billey† and Jonathan Weed‡)



Submitted: Sep 23, 2008; Accepted: May 4, 2009; Published: May 12, 2009
Mathematics Subject Classifications: 14M15; 05E15, 20F55



**Abstract**

Using resolutions of singularities introduced by Cortez and a method for calculating Kazhdan-Lusztig polynomials due to Polo, we prove the conjecture of Billey and Braden characterizing permutations $w$ with Kazhdan-Lusztig polynomial $P_{id,w}(q) = 1 + q^h$ for some $h$.


## Contents




*AW gratefully acknowledges support from NSF VIGRE grant DMS-0135345.
†SB gratefully acknowledges support from NSF grant DMS-0800978.
‡JW gratefully acknowledges support from NSF REU grant DMS-0754486.

# 1 Introduction

Kazhdan-Lusztig polynomials are polynomials $P_{u,w}(q)$ in one variable associated to each pair of elements $u$ and $w$ in the symmetric group $S_n$ (or more generally in any Coxeter group). They have an elementary definition in terms of the Hecke algebra [24, 21, 9] and numerous applications in representation theory, most notably in [24, 1, 13], and the geometry of homogeneous spaces [25, 17]. While their definition makes it fairly easy to compute any particular Kazhdan-Lusztig polynomial, on the whole they are poorly understood. General closed formulas are known [5, 12], but they are fairly complicated; furthermore, although they are known to be positive (for $S_n$ and other Weyl groups), these formulas have negative signs. For $S_n$, positive formulas are known only for 3412 avoiding permutations [27, 28], 321-hexagon avoiding permutations [7], and some isolated cases related to the generic singularities of Schubert varieties [8, 31, 16, 34].

One important interpretation of Kazhdan-Lusztig polynomials is as local intersection homology Poincaré polynomials for Schubert varieties. This interpretation, originally established by Kazhdan and Lusztig [25], shows, in an entirely non-constructive manner, that Kazhdan-Lusztig polynomials have nonnegative integer coefficients and constant term 1. Furthermore, as shown by Deodhar [17], $P_{id,w}(q) = 1$ (for $S_n$) if and only if the Schubert variety $X_w$ is smooth, and, more generally, $P_{u,w}(q) = 1$ if and only if $X_w$ is smooth over the Schubert cell $X_u^\circ$.

The purpose of this paper is to prove Theorem 1.1, for which we require one preliminary definition. A 3412 **embedding** is a sequence of indices $i_1 < i_2 < i_3 < i_4$ such that $w(i_3) < w(i_4) < w(i_1) < w(i_2)$, and the **height** of a 3412 embedding is $w(i_1) - w(i_4)$.

**Theorem 1.1.** *The Kazhdan-Lusztig polynomial for $w$ satisfies $P_{id,w}(1) = 2$ if and only if the following two conditions are both satisfied:*

- *The singular locus of $X_w$ has exactly one irreducible component.*
- *The permutation $w$ avoids the patterns* 653421*,* 632541*,* 463152*,* 526413*,* 546213*, and* 465132*.*

*More precisely, when these conditions are satisfied, $P_{id,w}(q) = 1 + q^h$ where $h$ is the minimum height of a* 3412 *embedding, with $h = 1$ if no such embedding exists.*

Given the first part of the theorem, the second part can be immediately deduced from the unimodality of Kazhdan-Lusztig polynomials [22, 11] and the calculation of the Kazhdan-Lusztig polynomial at the unique generic singularity [8, 31, 16]. Indeed, unimodality and this calculation imply the following corollary.

**Corollary 1.2.** *Suppose $w$ satisfies both conditions in Theorem 1.1. Let $X_v$ be the singular locus of $X_w$. Then $P_{u,w}(q) = 1 + q^h$ (with $h$ as in Theorem 1.1) if $u \leq v$ in Bruhat order, and $P_{u,w}(q) = 1$ otherwise.*

The permutation $v$ and the singular locus in general has a combinatorial description given in Theorem 2.1, which was originally proved independently in [8, 16, 23, 30].

Theorem 1.1 was conjectured by Billey and Braden [6]. They claim in their paper to have a proof that $P_{id,w}(1) = 2$ implies the given conditions. An outline of this proof is as follows. If $P_{id,w}(1) = 1$ then $X_w$ is nonsingular [17]. The methods for calculating Kazhdan-Lusztig polynomials due to Braden and MacPherson [11] show that whenever $P_{id,w}(1) \leq 2$ the singular locus of $X_w$ has at most one component. That $P_{id,w}(1) \leq 2$ implies the pattern avoidance conditions follows from [6, Thm. 1] and the computation of Kazhdan-Lusztig polynomials for the six pattern permutations.

While this paper was being written, Billey and Weed found an alternative formulation of Theorem 1.1 purely in terms of pattern avoidance, replacing the condition that the singular locus of $X_w$ have only one component with sixty patterns. They have graciously agreed to allow their result, Theorem A.1, to be included in an appendix to this paper. Theorem A.1 also provides an alternate method for proving that $P_{id,w}(2) = 1$ implies the given conditions using only [6, Thm. 1] and bypassing the methods of [11].

To prove Theorem 1.1, we study resolutions of singularities for Schubert varieties that were introduced by Cortez [15, 16] and use an interpretation of the Decomposition Theorem [2] given by Polo [32] which allows computation of Kazhdan-Lusztig polynomials $P_{v,w}$ (and more generally local intersection homology Poincaré polynomials for appropriate varieties) from information about the fibers of a resolution of singularities. In the 3412-avoiding case, we use a resolution of singularities from [15] and a second resolution of singularities which is closely related. An alternative approach which we do not take here would be to analyze the algorithm of Lascoux [27] for calculating these Kazhdan-Lusztig polynomials. For permutations containing 3412, we use one of the partial resolutions introduced in [16] for the purpose of determining the singular locus of $X_w$. Under the conditions described above, this partial resolution is actually a resolution of singularities, and we use Polo's methods on it.

Though we have used purely geometric arguments, it is possible to combinatorialize the calculation of Kazhdan-Lusztig polynomials from resolutions of singularities using a Bialynicki-Birula decomposition [3, 4, 14] of the resolution. See Remark 4.7 for details.

Corollary 1.2 suggests the problem of describing all pairs $u$ and $w$ for which $P_{u,w}(1) = 2$. It seems possible to extend the methods of this paper to characterize such pairs; presumably $X_u$ would need to lie in no more than one component of the singular locus of $X_w$, and $[u, w]$ would need to avoid certain intervals (see Section 2.3). Any further extension to characterize $w$ for which $P_{id,w}(1) = 3$ is likely to be extremely combinatorially intricate. An extension to other Weyl groups would also be interesting, not only for its intrinsic value, but because methods for proving such a result may suggest methods for proving any (currently nonexistent) conjecture combinatorially describing the singular loci of Schubert varieties for these other Weyl groups.

I wish to thank Eric Babson for encouraging conversations and Sara Billey for helpful comments and suggestions on earlier drafts. I used Greg Warrington's software [33] for computing Kazhdan-Lusztig polynomials in explorations leading to this work.

# 2 Preliminaries

## 2.1 The symmetric group and Bruhat order

We begin by setting notation and basic definitions. We let $S_n$ denote the symmetric group on $n$ letters. We let $s_i \in S_n$ denote the adjacent transposition which switches $i$ and $i+1$; the elements $s_i$ for $i = 1, \ldots, n-1$ generate $S_n$. Given an element $w \in S_n$, its **length**, denoted $\ell(w)$, is the minimal number of generators such that $w$ can be written as $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$. An **inversion** in $w$ is a pair of indices $i < j$ such that $w(i) > w(j)$. The length of a permutation $w$ is equal to the number of inversions it has.

Unless otherwise stated, permutations are written in one-line notation, so that $w = 3142$ is the permutation such that $w(1) = 3$, $w(2) = 1$, $w(3) = 4$, and $w(4) = 2$.

Given a permutation $w \in S_n$, the **graph** of $w$ is the set of points $(i, w(i))$ for $i \in \{1, \ldots, n\}$. We will draw graphs according to the Cartesian convention, so that $(0,0)$ is at the bottom left and $(n, 0)$ the bottom right.

The **rank function** $r_w$ is defined by

$$r_w(p,q) = \#\{i \mid 1 \leq i \leq p, 1 \leq w(i) \leq q\}$$

for any $p, q \in \{1, \ldots, n\}$. We can visualize $r_w(p,q)$ as the number of points of the graph of $w$ in the rectangle defined by $(1,1)$ and $(p,q)$. There is a partial order on $S_n$, known as **Bruhat order**, which can be defined as the reverse of the natural partial order on the rank function; explicitly, $u \leq w$ if $r_u(p,q) \geq r_w(p,q)$ for all $p, q \in \{1, \ldots, n\}$. The Bruhat order and the length function are closely related. If $u < w$, then $\ell(u) < \ell(w)$; moreover, if $u < w$ and $j = \ell(w) - \ell(u)$, then there exist (not necessarily adjacent) transpositions $t_1, \ldots, t_j$ such that $u = t_j \cdots t_1 w$ and $\ell(t_{i+1} \cdots t_1 w) = \ell(t_i \cdots t_1 w) - 1$ for all $i$, $1 \leq i < j$.

## 2.2 Schubert varieties

Now we briefly define Schubert varieties. A **(complete) flag** $F_\bullet$ in $\mathbb{C}^n$ is a sequence of subspaces $\{0\} \subseteq F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^n$, with $\dim F_i = i$. As a set, the **flag variety** $\mathcal{F}_n$ has one point for every flag in $\mathbb{C}^n$. The flag variety $\mathcal{F}_n$ has a geometric structure as $GL(n)/B$, where $B$ is the group of invertible upper triangular matrices, as follows. Given a matrix $g \in GL(n)$, we can associate to it the flag $F_\bullet$ with $F_i$ being the span of the first $i$ columns of $g$. Two matrices $g$ and $g'$ represent the same flag if and only if $g' = gb$ for some $b \in B$, so complete flags are in one-to-one correspondence with left $B$-cosets of $GL(n)$.

Fix an ordered basis $e_1, \ldots, e_n$ for $\mathbb{C}^n$, and let $E_\bullet$ be the flag where $E_i$ is the span of the first $i$ basis vectors. Given a permutation $w \in S_n$, the **Schubert cell** associated to $w$, denoted $X_w^\circ$, is the subset of $\mathcal{F}_n$ corresponding to the set of flags

$$\{F_\bullet \mid \dim(F_p \cap E_q) = r_w(p,q)\ \forall p,q\}. \tag{2.1}$$

The conditions in 2.1 are called **rank conditions**. The **Schubert variety** $X_w$ is the closure of the Schubert cell $X_w^\circ$; its points correspond to the flags

$$\{F_\bullet \mid \dim(F_p \cap E_q) \geq r_w(p,q)\ \forall p,q\}.$$

Bruhat order has an alternative definition in terms of Schubert varieties; the Schubert variety $X_w$ is a union of Schubert cells, and $u \leq w$ if and only if $X_u^\circ \subset X_w$. In each Schubert cell $X_w^\circ$ there is a **Schubert point** $e_w$, which is the point associated to the permutation matrix $w$; in terms of flags, the flag $E_\bullet^{(w)}$ corresponding to $e_w$ is defined by $E_i^{(w)} = \mathbb{C}\{e_{w(1)}, \ldots, e_{w(i)}\}$. The Schubert cell $X_w^\circ$ is the orbit of $e_w$ under the left action of the group $B$.

Many of the rank conditions in (2.1) are actually redundant. Fulton [20] showed that for any $w$ there is a minimal set, called the **coessential set**[1], of rank conditions which suffice to define $X_w$. To be precise, the coessential set is given by

$$\mathrm{Coess}(w) = \{(p,q) \mid w(p) \leq q < w(p+1), w^{-1}(q) \leq p < w^{-1}(q+1)\},$$

and a flag $F_\bullet$ corresponds to a point in $X_w$ if and only if $\dim(F_p \cap E_q) \geq r_w(p,q)$ for all $(p,q) \in \mathrm{Coess}(w)$.

While we have distinguished between points in flag and Schubert varieties and the flags they correspond to here, we will freely ignore this distinction in the rest of the paper.

## 2.3 Pattern avoidance and interval pattern avoidance

Let $v \in S_m$ and $w \in S_n$, with $m \leq n$. A **(pattern) embedding** of $v$ into $w$ is a set of indices $i_1 < \cdots < i_m$ such that the entries of $w$ in those indices are in the same relative order as the entries of $v$. Stated precisely, this means that, for all $j, k \in \{1, \ldots, m\}$, $v(j) < v(k)$ if and only if $w(i_j) < w(i_k)$. A permutation $w$ is said to **avoid** $v$ if there are no embeddings of $v$ into $w$.

Now let $[x,v] \subseteq S_m$ and $[u,w] \subseteq S_n$ be two intervals in Bruhat order. An **(interval) (pattern) embedding** of $[x,v]$ into $[u,w]$ is a simultaneous pattern embedding of $x$ into $u$ and $v$ into $w$ using the same set of indices $i_1 < \cdots < i_m$, with the additional property

[1]Fulton [20] indexes Schubert varieties in a manner reversed from our indexing as it is more convenient in his context. As a result, his Schubert varieties are defined by inequalities in the opposite direction, and he defines the **essential set** with inequalities reversed from ours. Our conventions also differ from those of Cortez [15] in replacing her $p-1$ with $p$.

that $[x, v]$ and $[u, w]$ are isomorphic as posets. For the last condition, it suffices to check that $\ell(v) - \ell(x) = \ell(w) - \ell(u)$ [35, Lemma 2.1].

Note that given the embedding indices $i_1 < \cdots < i_m$, any three of the four permutations $x$, $v$, $u$, and $w$ determine the fourth. Therefore, for convenience, we sometimes drop $u$ from the terminology and discuss embeddings of $[x, v]$ in $w$, with $u$ implied. We also say that $w$ **(interval) (pattern) avoids** $[x, v]$ if there are no interval pattern embeddings of $[x, v]$ into $[u, w]$ for any $u \leq w$.

## 2.4 Singular locus of Schubert varieties

Now we describe combinatorially the singular loci of Schubert varieties. The results of this section are due independently to Billey and Warrington [8], Cortez [15, 16], Kassel, Lascoux, and Reutenauer [23], and Manivel [30].

Stated in terms of interval pattern embeddings as in [35, Thm. 6.1], the theorem is as follows. Permutations are given in 1-line notation. We use the convention that the segment "$j \cdots i$" means $j, j-1, j-2, \ldots, i+1, i$. In particular, if $j < i$ then the segment is empty.

**Theorem 2.1.** *The Schubert variety $X_w$ is singular at $e_{u'}$ if and only if there exists $u$ with $u' \leq u < w$ such that one of the following (infinitely many) intervals embeds in $[u, w]$:*

*I:* $\big[(y+1)z\cdots 1(y+z+2)\cdots(y+2), \quad (y+z+2)(y+1)y\cdots 2(y+z+1)\cdots(y+2)1\big]$ *for some integers $y, z > 0$.*

*IIA:* $\big[(y+1)\cdots 1(y+3)(y+2)(y+z+4)\cdots(y+4), \quad (y+3)(y+1)\cdots 2(y+z+4)1(y+z+3)\cdots(y+4)(y+2)\big]$ *for some integers $y, z \geq 0$.*

*IIB:* $\big[1(y+3)\cdots 2(y+4), \quad (y+3)(y+4)(y+2)\cdots 312\big]$ *for some integer $y > 1$.*

*Equivalently, the irreducible components of the singular locus of $X_w$ are the subvarieties $X_u$ for which one of these intervals embeds in $[u, w]$.*

We call irreducible components of the singular locus of $X_w$ type I or type II (or IIA or IIB) depending on the interval which embeds in $[u, w]$, as labelled above.

We also wish to restate this theorem in terms of the graph of $w$, which is closer in spirit to the original statements [8, 16, 23, 30].

A type I component of the singular locus of $X_w$ is associated to an embedding of $(y+z+2)(y+1)y\cdots 2(y+z+1)\cdots(y+2)1$ into $w$. If we label the embedding by $i = j_0 < j_1 < \cdots < j_y < k_1 < \cdots < k_z < m = k_{z+1}$, the requirement that these positions give the appropriate interval embedding is equivalent to the requirement that the regions $\{(p, q) \mid j_{r-1} < p < j_r, w(j_r) < q < w(i)\}$, $\{(p, q) \mid k_s < p < k_{s+1}, w(m) < q < w(k_s)\}$, and $\{(p, q) \mid j_y < p < k_1, w(m) < q < w(i)\}$ contain no point $(p, w(p))$ in the graph of $w$ for all $r$, $1 \leq r \leq y$, and for all $s$, $1 \leq s \leq z$. This is illustrated in Figure 1. We will usually say that the type I component given by this embedding is defined by $i$, the set $\{j_1, \ldots, j_y\}$, the set $\{k_1, \ldots, k_z\}$, and $m$.

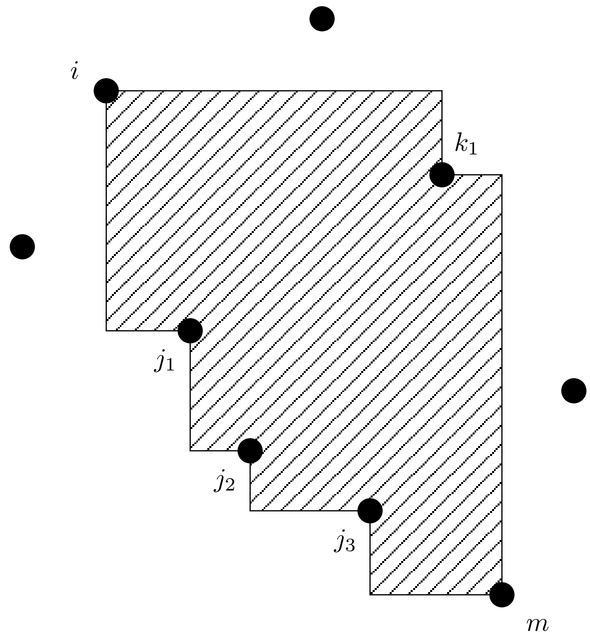


Figure 1: A type I embedding with $y = 3$, $z = 1$, defining a component of the singular locus for $w = 685392714$. The shaded region is not allowed have points in the graph of $w$.

Every type II component of the singular locus $X_w$ is defined by four indices $i < j < k < m$ which gives an embedding of 3412 into $w$. The interval pattern embedding requirement forces the regions $\{(p, q) \mid i < p < j, w(m) < q < w(i)\}$, $\{(p, q) \mid j < p < k, w(i) < q < w(j)\}$, $\{(p, q) \mid k < p < m, w(m) < q < w(i)\}$, and $\{(p, q) \mid j < p < k, w(k) < q < w(m)\}$ to have no points in the graph of $w$. We call these regions the **critical regions** of the 3412 embedding, and if they are empty, we call $i < j < k < m$ a **critical** 3412 **embedding** whether or not they are part of a type II component.

Given a critical 3412 embedding $i < j < k < m$, let $B = \{p \mid j < p < k, w(m) < w(p) < w(i)\}$, $A_1 = \{p \mid i < p < j, w(k) < w(p) < w(m)\}$, $A_2 = \{p \mid k < p < m, w(i) < w(p) < w(j)\}$, and $A = A_1 \cup A_2$. We call these regions the $A$, $A_1$, $A_2$, and $B$ regions associated to our critical 3412 embedding. This is illustrated in Figure 2. If $w(b_1) > w(b_2)$ for all $b_1 < b_2 \in B$, we say our critical 3412 embedding is **reduced**. If a critical embedding is not reduced, there will necessarily be at least one critical 3412 embedding involving $i$, $j$, and two indices in $B$, and one involving two indices in $B$, $k$, and $m$; by induction each will include at least one reduced critical 3412 embedding.

We associate one or two irreducible components of the singular locus of $X_w$ to every reduced critical 3412 embedding. If $B$ is empty, then the embedding is part of a component of type IIA. If $A$ is empty, then the embedding is part of a component of type IIB. Note that any type II component of the singular locus is associated to exactly one reduced critical 3412 embedding. However, if both $A$ and $B$ are nonempty, then we do not have a type II component. In this case, we can associate a type I component of the singular locus to our reduced critical 3412 embedding $i < j < k < m$. When both $A_1$ and $B$

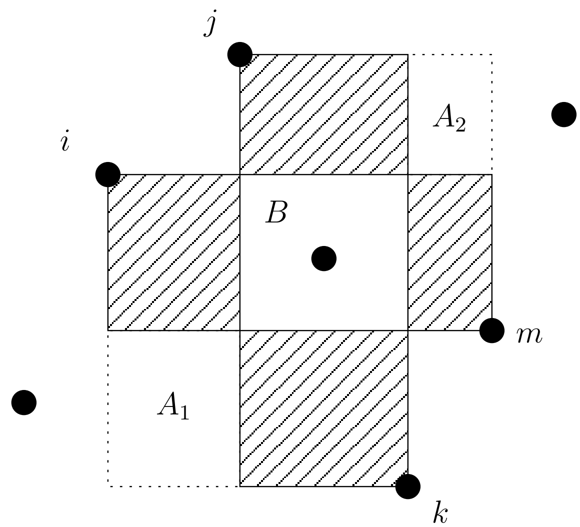


Figure 2: A critical 3412 embedding in $w = 2574136$. The shaded regions are the critical regions of the embedding.

are nonempty, then $i$, a nonempty subset of $A_1$, $B$, and $k$ define a type I component; in this case $w$ has an embedding of 526413. When both $A_2$ and $B$ are nonempty, then $j$, $B$, a nonempty subset of $A_2$, and $m$ define a type I component; in this case $w$ has an embedding of 463152. When $A_1$, $A_2$, and $B$ are all nonempty, we have two distinct type I components associated to our 3412 embedding. Note that it is possible for a type I component to be associated to more than one reduced critical 3412 embedding, as in the permutation 47318625.

# 3 Necessity in the covexillary case

We begin with the case where $w$ avoids 3412; such a permutation is commonly called **covexillary**. We show here that, if $w$ is covexillary, the singular locus of $X_w$ has only one component, and $w$ avoids 653421 and 632541, then $P_{id,w}(q) = 1 + q$. Throughout this section $w$ is assumed to be covexillary unless otherwise noted.

## 3.1 The Cortez-Zelevinsky resolution

For a covexillary permutation, the coessential set has the special property that, for any $(p, q), (p', q') \in \text{Coess}(w)$ with $p \leq p'$, we also have $q \leq q'$. Therefore have a natural total order on the coessential set, and we label its elements $(p_1, q_1), \ldots, (p_k, q_k)$ in order. We let $r_i = r_w(p_i, q_i)$; note that, by the definition of $r_w$ and the minimality of the coessential set, $r_i < r_j$ when $i < j$. When $r_i = \min\{p_i, q_i\}$, we call $(p_i, q_i)$ an **inclusion element** of the coessential set, since the condition it implies for $X_w$ will either be $E_{q_i} \subseteq F_{p_i}$ (if $r_i = q_i$) or $F_{p_i} \subseteq E_{q_i}$ (if $r_i = p_i$).

Zelevinsky [36] described some resolutions of singularities of $X_w$ in the case where $w$ has at most one ascent (meaning that $w(i) < w(i+1)$ for at most one index $i$), explaining a formula of Lascoux and Schützenberger [28] for Kazhdan-Lusztig polynomials $P_{v,w}(q)$ in that case. Following a generalization by Lascoux [27] of this formula to covexillary permutations, Cortez [15] generalized the Zelevinsky resolution to this case.

Let $\mathcal{F}_{i_1,\ldots,i_k}$ denote the partial flag manifold whose points correspond to flags whose component subspaces have dimensions $i_1 < \cdots < i_k$. Define the configuration variety $Z_w$ by

$$Z_w := \{(G_\bullet, F_\bullet) \in \mathcal{F}_{r_1,\ldots,r_k}(\mathbb{C}^n) \times X_w \mid G_{r_i} \subseteq (F_{p_i} \cap E_{q_i}) \ \forall i\}.$$

Cortez shows that the projection $\pi_2 : Z_w \to X_w$ is a resolution of singularities. She furthermore shows that the exceptional locus of $\pi_2$ is precisely the singular locus of $X_w$, and describes a one-to-one correspondence between components of the singular locus of $X_w$ and elements of the coessential set which are not inclusion elements. (This last fact about the singular locus was implicit in Lascoux's formula [27] for covexillary Kazhdan-Lusztig polynomials.)

We now have the following lemma, whose proof is deferred to Section 5.

**Lemma 3.1.** *Suppose the singular locus of $X_w$ has only one component. If $w$ contains both* 53241 *and* 52431*, then $w$ contains* 632541*.*

This lemma allows us to treat separately the two cases where $w$ avoids 53241 and where $w$ avoids 52431. We treat first the case where $w$ avoids 53241, for which we use the resolution of singularities just described. The case where $w$ avoids 52431 requires the use of a resolution of singularities which is dual (in the sense of dual vector spaces) to the one just described; we will describe this resolution at the end of this section.

## 3.2 The 53241-avoiding case

In this subsection we show that $P_{id,w}(q) = 1+q$ when the singular locus of $X_w$ has exactly one component and $w$ avoids 653421 and 53241. To maintain the flow of the argument, proofs of lemmas are deferred to Section 5.

When $(p_j, q_j)$ is an inclusion element, then $\dim(F_{p_j} \cap E_{q_j}) = r_j$ for any flag $F_\bullet$ in $X_w$ and not merely generic flags in $X_w$. Therefore, given any $F_\bullet$ we will have only one choice for $G_{r_j}$, namely $F_{p_j} \cap E_{q_j}$, in the fiber $\pi_2^{-1}(F_\bullet)$. In particular, for the flag $E_\bullet$, any $G_\bullet$ in the fiber $\pi^{-1}(E_\bullet)$ will have $G_{r_j} = E_{r_j}$. Now let $i$ be the unique index such that $(p_i, q_i)$ is not an inclusion element; there is only one such index since the singular locus of $X_w$ has only one irreducible component. For convenience, we let $p = p_i$, $q = q_i$, and $r = r_i$. Now we have the following lemmas. (In the case where $i = 1$, we define $p_0 = q_0 = r_0$.)

**Lemma 3.2.** *Suppose $w$ avoids* 653421 *(and* 3412*). Then* $\min\{p, q\} = r + 1$.

**Lemma 3.3.** *Suppose $w$ avoids* 53241 *(and* 3412*). Then* $r_{i-1} = r - 1$.

By definition, $G_r \supseteq G_{r_{i-1}}$. Therefore, the fiber $\pi_2^{-1}(e_{id}) = \pi_2^{-1}(E_\bullet)$ is precisely

$$\{(G_\bullet, E_\bullet) \mid G_{r_j} = E_{r_j} \text{ for } j \neq i \text{ and } E_{r-1} = E_{r_{i-1}} \subseteq G_r \subseteq (E_p \cap E_q) = E_{r+1}\}.$$

This fiber is clearly isomorphic to $\mathbb{P}^1$.

By Polo's interpretation [32] of the Decomposition Theorem [2],

$$H_{z,\pi_2}(q) = P_{z,w}(q) + \sum_{z \leq v < w} q^{\ell(w)-\ell(v)} E_v(q) P_{z,v}(q),$$

where

$$H_{z,\pi_2}(q) = \sum_{i \geq 0} q^i \dim H^{2i}(\pi_2^{-1}(e_z)),$$

and the $E_v(q)$ are some Laurent polynomials in $q^{\frac{1}{2}}$, depending only on $v$ and $\pi_2$ and not on $z$, which have with positive integer coefficients and satisfy the identity $E_v(q) = E_v(q^{-1})$. Since the fiber of $\pi_2$ at $e_{id}$ is $\mathbb{P}^1$, it follows that $H_{id,\pi_2}(q) = 1 + q$. As $P_{id,w}(q) \neq 1$ (since by assumption $X_w$ is singular), and all coefficients of all polynomials involved must be nonnegative integers, $E_v(q) = 0$ for all $v$ and

$$P_{id,w}(q) = 1 + q.$$

### 3.3 The 52431-avoiding case

When $w$ avoids 52431 instead, we use the resolution

$$Z'_w := \{(G_\bullet, F_\bullet) \in \mathcal{F}_{r'_1,\ldots,r'_k}(\mathbb{C}^n) \times X_w \mid G_{r'_i} \supseteq (F_{p_i} + E_{q_i}) \ \forall i\},$$

where $r'_i := p_i + q_i - r_i$. Arguments similar to the above show that, if we let $i$ be the index so that $(p_i, q_i)$ does not give an inclusion element, the fiber $\pi_2^{-1}(e_{id})$ is

$$\{(G_\bullet, E_\bullet) \mid G_{r'_j} = E_{r'_j} \text{ for } j \neq i \text{ and } E_{r'_i-1} \subseteq G_{r'_i} \subseteq E_{r'_i+1}\}.$$

Hence the fiber over $e_{id}$ is isomorphic to $\mathbb{P}^1$ and $P_{id,w}(q) = 1+q$ by the same argument as above.

## 4 Necessity in the 3412 containing case

In this section we treat the case where $w$ contains a 3412 pattern. Our strategy in this case is to use another resolution of singularities given by Cortez [16]. We will again apply the Decomposition Theorem [2] to this resolution, but in this case the calculation is more complicated as the fiber at $e_{id}$ will no longer always be isomorphic to $\mathbb{P}^1$. When the fiber at $e_{id}$ is not $\mathbb{P}^1$, we will need to identify the image of the exceptional locus, which turns out to be irreducible, and calculate the generic fiber over the image of the exceptional locus as well as the fiber over $e_{id}$. We then follow Polo's strategy in [32] to calculate that $P_{id,w}(q) = 1 + q^h$, where $h$ is the minimum height of a 3412 embedding as defined below.

## 4.1 Cortez's resolution

We begin with some definitions necessary for defining a variety $Z$ and a map $\pi_2 : Z \to X_w$ which we will show is our resolution of singularities. Our notation and terminology generally follows that of Cortez [16]. Given an embedding $i_1 < i_2 < i_3 < i_4$ of 3412 into $w$, we call $w(i_1) - w(i_4)$ its **height** (*hauteur*), and $w(i_2) - w(i_3)$ its **amplitude**. Among all embeddings of 3412 in $w$, we take the ones with minimum height, and among embeddings of minimum height, we choose one with minimum amplitude. As we will be continually referring this particular embedding, we denote the indices of this embedding by $a < b < c < d$ and entries of $w$ at these indices by $\alpha = w(a)$, $\beta = w(b)$, $\gamma = w(c)$, and $\delta = w(d)$. We let $h = \alpha - \delta$ be the height of this embedding.

Let $\alpha'$ be the largest number such that $w^{-1}(\alpha') < w^{-1}(\alpha' - 1) < \cdots < w^{-1}(\alpha + 1) < w^{-1}(\alpha)$ and $\delta'$ the smallest number such that $w^{-1}(\delta) < w^{-1}(\delta - 1) < \cdots < w^{-1}(\delta')$. Also let $a' = w^{-1}(\alpha')$ and $d' = w^{-1}(\delta')$. Now let $\kappa = \delta' + \alpha' - \alpha$, let $I$ denote the set of simple transpositions $\{s_{\delta'}, \cdots, s_{\alpha'-1}\}$, and let $J$ be $I \setminus \{s_\kappa\}$. Furthermore, let $v = w_0^J w_0^I w$, where $w_0^J$ and $w_0^I$ denote the longest permutations in the parabolic subgroups of $S_n$ generated by $J$ and $I$ respectively.

As an example, let $w = 817396254 \in S_9$; its graph is in figure 3. Then $a = 3$, $b = 5$, $c = 7$, and $d = 8$, while $\alpha = 7$, $\beta = 9$, $\gamma = 2$, and $\delta = 5$. We also have $h = 2$, $\alpha' = 8$ and $\delta' = 4$. Hence $\kappa = 5$ and $v = 514398276$.

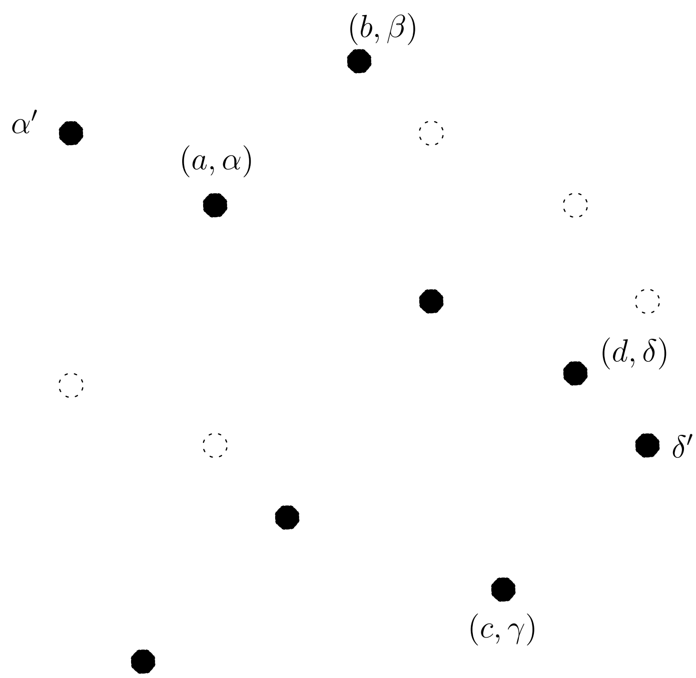


Figure 3: The graph of $w = 817396254$ in black, labelled. The points of the graph of $v = 514398276$ which are different from $w$ are in clear circles.

Now consider the variety $Z = P_I \times^{P_J} X_v$. By definition, $Z$ is a quotient of $P_I \times X_v$ under the free action of $P_J$ where $q\cdot(p,x) = (pq^{-1}, q\cdot x)$ for any $q \in P_J$, $p \in P_I$, and $x \in X_v$. In the spirit of Magyar's realization [29] of full Bott-Samelson varieties as configuration varieties, we can also consider $Z$ as the configuration variety

$$\{(G, F_\bullet) \in Gr_\kappa(\mathbb{C}^n) \times X_w \mid E_{\delta'-1} \subseteq G \subseteq E_{\alpha'} \text{ and } \dim(F_i \cap G) \geq r_v(i,\kappa)\}.^2$$

By the definition of $v$, $r_v(i,\kappa) = r_w(i,\alpha')$ for $i < w^{-1}(\alpha-1)$, $r_v(i,\kappa) = r_w(i,\alpha') - j$ when $w^{-1}(\alpha-j) \leq i < w^{-1}(\alpha-j-1)$, and $r_v(i,\kappa) = r_w(i,\alpha') - \alpha' + \kappa$ when $i \geq d'$. The last condition is automatically satisfied since, as $G \subseteq E_{\alpha'}$, we always have $\dim(G \cap F_i) \geq \dim(E_{\alpha'} \cap F_i) - (\alpha' - \kappa) \geq r_w(i,\alpha') - \alpha' + \kappa$.

Cortez [16] introduced the variety $Z$ along with several other varieties (constructed by defining $\kappa = \delta' + \alpha' - \alpha + i - 1$ for $i = 1, \dots, h$) to help in describing the singular locus of Schubert varieties[3]. A virtually identical proof would follow from analyzing the resolution given by $i = h$ instead of $i = 1$ as we are doing, but the other choices of $i$ give maps which are harder to analyze as they have more complicated fibers.

The variety $Z$ has maps $\pi_1 : Z \to P_I/P_J \cong Gr_{\alpha'-\alpha+1}(\mathbb{C}^{\alpha'-\delta'+1})$ sending the orbit of $(p,x)$ to the class of $p$ under the right action of $P_J$ and $\pi_2 : Z \to X_w$ sending the orbit of $(p,x)$ to $p \cdot x$. Under the configuration space description, $\pi_1$ sends $(G, F_\bullet)$ to the point in $Gr_{\alpha'-\alpha+1}(\mathbb{C}^{\alpha'-\delta'+1})$ corresponding to the plane $G/E_{\delta'-1} \subseteq E_{\alpha'}/E_{\delta'-1}$, and $\pi_2$ sends $(G, F_\bullet)$ to $F_\bullet$. The map $\pi_1$ is a fiber bundle with fiber $X_v$, and, by [16, Prop. 4.4], the map $\pi_2$ is surjective and birational. (In our case where the singular locus of $X_w$ has only one component, the latter statement is also a consequence our proof of Lemma 4.5.)

In general $Z$ is not smooth; hence $\pi_2$ is only a partial resolution of singularities. However, we show in Section 5 the following.

**Lemma 4.1.** *Suppose $w$ avoids 463152 and the singular locus of $X_w$ has only one irreducible component. Then $Z$ is smooth.*

## 4.2 Fibers of the resolution

We now describe of the fibers of $\pi_2$. To highlight the main flow of the argument, proofs of individual lemmas will be deferred to Section 5. Define $M = \max\{p \mid p < c, w(p) < \delta'\} \cup \{a\}$ and $N = \max\{p \mid w(p) < \delta'\}$.

**Lemma 4.2.** *The fiber of $\pi_2$ over a flag $F_\bullet$ is*

$$\{G \in Gr_\kappa(\mathbb{C}^n) \mid E_{\delta'-1} + F_M \subseteq G \subseteq E_{\alpha'} \cap F_N\}.$$

Now we focus on the fiber at the identity, and show that it is isomorphic to $\mathbb{P}^h$. Since the flag corresponding to the identity is $E_\bullet$, it suffices by the previous lemma to show that $\dim(E_{\delta'-1} + E_M) = \kappa - 1$ and $\dim(E_{\alpha'} \cap E_N) = \kappa + h$.

[2] The statement of this geometric description in [16] has a typographical error.

[3] Cortez's choice of 3412 embedding in [16] is slightly different from ours. For technical reasons she chooses one of minimum amplitude among those satisfying a condition she calls "well-filled" (*bien remplie*). As she notes, 3412 embeddings of minimum height are automatically "well-filled".

**Lemma 4.3.** *Suppose that the singular locus of $X_w$ has only one component and $w$ avoids* 546213*. Then* $\dim(E_{\delta'-1} + E_M) = \kappa - 1$.

**Lemma 4.4.** *Suppose that the singular locus of $X_w$ has only one component and $w$ avoids* 465132*. Then* $\dim(E_{\alpha'} \cap E_N) = \kappa + h$.

In the case where $h = 1$, these are all the geometric facts we need. When $h > 1$, we identify the image of the exceptional locus as $X_u$ for a particular permutation $u$ of length $\ell(u) = \ell(w) - h$. We then show that the fiber over a generic point of $X_u$ is isomorphic to $\mathbb{P}^{h-1}$.

First we describe the image of the exceptional locus geometrically.

**Lemma 4.5.** *Suppose the singular locus of $X_w$ has only one component, and $h > 1$. Then the image of the exceptional locus of $\pi_2$ is* $\{F_\bullet \mid \dim(E_{\delta'-1} \cap F_M) > r_w(M, \delta' - 1)\}$.

Now let $\sigma \in S_n$ be the cycle $(\gamma, \delta + 1, \delta + 2, \dots, \alpha = \delta + h)$, and let $u = \sigma w$. We show the following.

**Lemma 4.6.** *Assume that the singular locus of $X_w$ has only one component, that $h > 1$, and that $w$ avoids* 526413*. Then the image of the exceptional locus of $\pi_2$ is $X_u$, $\ell(w) - \ell(u) = h$, and the generic fiber over $X_u$ is isomorphic to $\mathbb{P}^{h-1}$.*

## 4.3 Calculation of $P_{id,w}(q)$

We now have all the geometric information we need to calculate $P_{id,w}(q)$, following the methods of Polo [32]. The Decomposition Theorem [2] shows that

$$H_{z,\pi_2}(q) = P_{z,w}(q) + \sum_{z \leq v < w} q^{\ell(w)-\ell(v)} E_v(q) P_{z,v}(q),$$

where

$$H_{z,\pi_2}(q) = \sum_{i \geq 0} q^i \dim H^{2i}(\pi_2^{-1}(e_z)),$$

and $E_v(q)$ are some Laurent polynomials in $q^{\frac{1}{2}}$, depending on $v$ and $\pi_2$ but not $z$, which have positive integer coefficients and satisfy the identity $E_v(q) = E_v(q^{-1})$. [4]

When $h = 1$, the fiber of $\pi_2$ at $e_{id}$ is isomorphic to $\mathbb{P}^1$, and so by same argument as in Section 3.2, $P_{id,w}(q) = 1 + q$.

For $h > 1$, let $u$ be the permutation specified above. For any $x$ with $x \leq w$ and $x \not\leq u$, $\pi_2^{-1}(e_x)$ is a point, so $X_w$ is smooth at $e_x$, and $H_{x,w}(q) = 1 = P_{x,w}(q)$. Therefore, by induction downwards from $w$, $E_x(q) = 0$ for any $x$ with $x \leq w$ and $x \not\leq u$.

Now we calculate $E_u(q)$. From the above it follows that $H_{u,\pi_2}(q) = P_{u,w}(q) + q^{\frac{h}{2}} E_u(q)$. Since $H_{u,\pi_2}(q) - P_{u,w}(q)$ has nonnegative coefficients and $\deg P_{u,w}(q) \leq (h-1)/2 < h-1$,

$$P_{u,w}(q) = 1 + \cdots + q^{s-1}$$

[4] For those readers familiar with the Decomposition Theorem: No local systems appear in the formula since $X_w$ has a stratification, compatible with $\pi_2$, into Schubert cells, all of which are simply connected.

for some $s$, $1 \leq s \leq h-1$. Then $q^{\frac{h}{2}} E_u(q) = q^s + \cdots + q^{h-1}$, so $E_u(q) = q^{s-\frac{h}{2}} + \cdots + q^{\frac{h}{2}-1}$. Since $E_u(q^{-1}) = E_u(q)$, $s = 1$, so

$$q^{\frac{h}{2}} E_u(q) = q + \cdots + q^{h-1}.$$

To calculate $P_{id,w}(q)$, note that $H_{id,\pi_2} = 1 + q + \cdots + q^h$, so

$$\begin{aligned} P_{id,w}(q) &= H_{id,\pi_2}(q) - \sum_{x \leq w} q^{\frac{\ell(w)-\ell(x)}{2}} E_x(q) P_{id,x}(q) \\ &= 1 + \cdots + q^h - (q + \cdots + q^{h-1}) P_{id,u}(q) + \sum_{x<u} q^{\frac{\ell(w)-\ell(x)}{2}} E_x(q) P_{id,x}(q). \end{aligned}$$

Evaluating at $q = 1$, we see that

$$P_{id,w}(1) = h + 1 - (h-1) P_{id,u}(1) - \sum_{x<u} E_x(1) P_{id,x}(1).$$

Since $P_{id,w}(1) \geq 2$, $P_{id,x}(1)$ is a positive integer for all $x$, and $E_x(1)$ is a nonnegative integer for all $x$, we must have that $P_{id,u}(1) = 1$ and $E_x(1) = 0$ for all $x < u$. Therefore, $P_{id,u}(q) = 1$ and $E_x(q) = 0$ for all $x < u$, and

$$P_{id,w}(q) = 1 + q^h.$$

Readers may note that the last computation is essentially identical to the one given by Polo in the proof of [32, Prop. 2.4(b)]. In fact, in this case the resolution we use, due to Cortez [16], is very similar to the one described by Polo.

**Remark 4.7.** We could have used a simultaneous Bialynicki-Birula cell decomposition [3, 4, 14] of the $Z$ and $X_w$, compatible with the map $\pi_2$, to combinatorialize the above computation, turning many geometrically stated lemmas into purely combinatorial ones. To be specific, for any $u$, the number $H_{u,\pi_2}(1)$ is the number of factorizations $u = \sigma\tau$ such that $\tau \leq v$, $\sigma \in W_I$, and $\sigma$ is maximal in its right $W_J$ coset. (The last condition can be replaced by any condition that forces us to pick at most one $\sigma$ from any $W_J$ coset.) This observation does not simplify the argument; the combinatorics required to determine which factorizations of the identity satisfy these conditions are exactly the same as the combinatorics used above to calculate the fiber of $\pi_2$ at the identity. It should also be possible to combinatorially calculate $H_{u,\pi_2}(q)$ by attaching the appropriate statistic to such a factorization. If $Z$ were the full Bott-Samelson resolution, the result would be Deodhar's approach [18] to calculating Kazhdan-Lusztig polynomials, and the aforementioned statistic would be his defect statistic. However, when $Z$ is some other resolution, even one "of Bott-Samelson type," no reasonable combinatorial description of the statistic appears to be known.

# 5 Lemmas

In this section we give proofs for the lemmas of Sections 3 and 4. We begin with Lemma 3.1.

**Lemma 3.1.** *Suppose the singular locus of $X_w$ has only one component. If $w$ contains both 53241 and 52431, then $w$ contains 632541.*

*Proof.* Let $a < b < c < d < e$ be an embedding of 53241, and $a' < b' < c' < d' < e'$ an embedding of 52431. Since $b < d$ and $w(b) < w(d)$, there must be an element $(p, q)$ of the coessential set such that $b < p < d$ and $w(b) < q < w(d)$. This cannot be an inclusion element since $a < p$ but $w(a) > q$, and $q < e$ but $w(e) > p$. We also have $c < d$ and $w(c) < w(d)$, also inducing an element of the coessential set which is not an inclusion element. Since the singular locus of $X_w$ has only one component, this element must also be $(p, q)$. The pairs $b' < c'$ and $b' < d'$ also each induce an element of the coessential set which is not an inclusion element; hence these must also be the same as $(p, q)$. Therefore, $b < c < p < c' < d'$, and $w(c) < w(b) < q < w(d') < w(c')$.

If $a' > b$ and $w(a) < w(c')$, then there must be an element $(p', q')$ of the coessential set with $a < b < p' < a' < c'$ and $w(b) < w(a) < q' < w(c') < w(a')$. We now have $p' < a' < p$ but $q < a \leq q'$, contradicting $w$ being covexillary. Therefore, $a' < b$ or $w(a) > w(c')$. Similarly, $e > d'$ or $w(e') < w(c)$. Let $a''$ be $a$ if $w(a) > w(c')$ and $a'$ if $a' < b$, and $e''$ be $e$ if $e > d'$ and $e'$ if $w(e') < w(c)$.

Now $a'' < b < c < c' < d' < e''$ is an embedding of 632541 in $w$. ☐

Recall that $(p, q) = (p_i, q_i)$ is the unique element of the coessential set which is not an inclusion element, and $r = r_i = r_w(p, q)$. Furthermore, $(p_{i-1}, q_{i-1})$ is the immediately preceeding element of the coessential set, and $r_{i-1} = r_w(p_{i-1}, q_{i-1} = \min(p_{i-1}, q_{i-1})$.

**Lemma 3.2.** *Suppose $w$ avoids 653421 (and 3412). Then* $\min\{p, q\} = r + 1$.

*Proof.* Suppose that $r \leq \min\{p, q\} - 2$; we show that in that case we have an embedding of 3412 or 653421. Since $r \leq p - 2$, there exist $a < b \leq p$ with $w(a), w(b) > q$. Note that $w(a) > w(b)$, as, otherwise, $a < b < p < w^{-1}(q + 1)$ would be an embedding of 3412. Similarly, since $r \leq q - 2$, there exist $d > c > p$ with $w(d), w(c) \leq q$, and we have $w(c) > w(d)$ since $w^{-1}(q) < p + 1 < c < d$ is an embedding of 3412 otherwise. Furthermore, if $b > w^{-1}(q)$, then $w(c) < w(p)$, as otherwise $w^{-1}(q) < b < p < c$ would be an embedding of 3412, and if $w(b) < w(p + 1)$, then $c > w^{-1}(q + 1)$ to avoid $b < p + 1 < c < w^{-1}(q + 1)$ being a similar embedding.

Now we have up to four potential cases depending on whether $b < w^{-1}(q)$ or $b > w^{-1}(q)$, and whether $w(b) > w(p + 1)$ or $w(b) < w(p + 1)$. In each case we produce an embedding of 653421. If $b < w^{-1}(q)$ and $w(b) > w(p + 1)$, then $a < b < w^{-1}(q) < p + 1 < c < d$ is such an embedding. If $b < w^{-1}(q)$ and $w(b) < w(p + 1)$, then we use $a < b < w^{-1}(q) < q^{-1}(q + 1) < c < d$. If $b > w^{-1}(q)$ and $w(b) > w(p + 1)$, then we use $a < b < p < p + 1 < c < d$. Finally, if $b > w^{-1}(q)$ and $w(b) < w(p + 1)$, $a < b < p < w^{-1}(q + 1) < c < d$ produces the desired embedding. ☐

**Lemma 3.3.** *Suppose $w$ avoids 53241 (and 3412). Then $r_{i-1} = r - 1$.*

*Proof.* We treat the two cases where $w(p) = q$ and $w(p) \neq q$ separately. First suppose $w(p) = q$. Suppose for contradiction that $r_{i-1} < r - 1$. Then there must exist an index $b \neq p$ which contributes to $r = r_w(p, q)$ but not to $r_{i-1} = r_w(p_{i-1}, q_{i-1})$. This happens when $b \leq p$ and $w(b) \leq q$, but $b > p_{i-1}$ or $w(b) > q_{i-1}$. Since $b < p$ and $w(b) < w(p) = q$, there must be an element $(p_j, q_j)$ of the coessential set such that $b \leq p_j < p$ and $w(b) \leq q_j < q$. But then we have that $p_j > p_{i-1}$ or $q_j > q_{i-1}$, contradicting the definition of $(p_{i-1}, q_{i-1})$ as the next element smaller than $(p_i, q_i)$ in our total ordering of the coessential set. Therefore, we must have $r_{i-1} = r_i - 1$.

Now suppose $w(p) \neq q$. Since $r < p$ and $r < q$, there exists $b < p$ with $w(b) > q$ and $c > p$ with $w(c) < q$. Note that we cannot have both $w(b) < w(p+1)$ and $c < w^{-1}(q+1)$, as, otherwise, $b < p + 1 < c < w^{-1}(q + 1)$ would be an embedding of 3412. It then follows that we cannot have both $b < w^{-1}(q)$ and $w(c) < w(p)$; when $w(b) > w(p + 1)$, $b < w^{-1}(q)$ and $w(c) < w(p)$ imply that $b < w^{-1}(q) < p < p + 1 < c$ is an embedding of 53241, and when $c > w^{-1}(q + 1)$, $b < w^{-1}(q)$ and $w(c) < w(p)$ imply that $b < w^{-1}(q) < p < w^{-1}(q+1) < c$ is an embedding of 53241. Therefore, $b > w^{-1}(q)$ or $w(c) > w(p)$, and we now treat these two cases separately.

Suppose $b > w^{-1}(q)$. We must have $w(c) < w(p)$ in this case, because otherwise $w^{-1}(q) < b < p < c$ would be an embedding of 3412. Let $a = \min\{b \mid w^{-1}(q) < b < p, w(b) > q\}$. We show that, for all $b'$ with $a \leq b' < p$, $w(b') > q$. First, we cannot have both $w(a) < w(p + 1)$ and $c < w^{-1}(q + 1)$, as $a < p + 1 < c < w^{-1}(q + 1)$ would be an embedding of 3412 otherwise. Now, if $w(b') < w(p)$, then $w^{-1}(q) < a < b' < p$ is an embedding of 3412, and if $w(p) < w(b') < q$, then either $a < b' < p < p + 1 < c$ or $a < b' < p < w^{-1}(q + 1) < c$ would be an embedding of 53241, depending on whether $w(a) > w(p + 1)$ or $c > w^{-1}(q + 1)$.

We have now established that there is an element of the coessential set at $(a - 1, q)$. Since this shares its second coordinate with $(p, q)$, and $w(b) > q$ for all $b$, $a < b < p$, there are no elements of the coessential set in between, and $(p_{i-1}, q_{i-1}) = (a - 1, q)$, so that $r_{i-1} = r_w(a - 1, q)$. Now, $r_w(a - 1, q) = r_w(p, q) - \#\{j \mid a - 1 < j \leq p, w(j) \leq q\}$. The latter list has just one element, namely $j = p$, so $r_{i-1} = r_i - 1$.

Now suppose $w(c) > w(p)$ instead. Then we let $s = \min\{t \mid w(p) < t < q, w^{-1}(s) > p\}$. By arguments symmetric with the above, for all $s'$ with $s \leq s' < q$, $s' > w(p)$. Therefore, there is an element of the coessential set at $(p, s-1)$, and this is the element immediately before $(p, q)$ in the total ordering. Furthermore, $r_w(p, s - 1) = r_w(p, q) - \#\{j \mid s - 1 < j < q, w^{-1}(j) \leq p\}$, and the latter list has one element, namely $j = q$, so $r_{i-1} = r_i - 1$. □

Before moving on to prove the lemmas of Section 4, we prove the following two lemmas which will be repeatedly used further. As in Section 4, $a < b < c < d$ is an embedding of 3412 of minimal amplitude among such embeddings of minimal height, and $\alpha$, $\beta$, $\gamma$, and $\delta$ respectively denote $w(a)$, $w(b)$, $w(c)$, and $w(d)$.

For Lemmas 4.1, 5.1, and 5.2, we use the description of the singular locus given in Section 2.4. It is worth noting that, since we only need to detect when the singular locus has more than one irreducible component, it is also possible to prove these lemmas using

Lemma A.2 (which was originally [8, Sect. 13]). Another alternate approach is first to directly prove Theorem A.1 by using the condition of avoiding all its patterns instead of the condition of having one component in the singular locus in the lemmas and then to prove Theorem 1.1 as a corollary. Neither approach appears to substantially reduce the need for detailed case-by-case analysis in the proof of these lemmas.

**Lemma 5.1.** *Suppose the singular locus of $X_w$ has only one component. Then the following sets are empty.*

*(i)* $\{p \mid p < a, w(p) > \beta\}$

*(ii)* $\{p \mid p < a, \alpha' < w(p) < \beta\}$

*(iii)* $\{p \mid a < p < b, \alpha < w(p) < \beta\}$

*(iv)* $\{p \mid b < p < c, \alpha < w(p) < \beta\}$

*(v)* $\{p \mid b < p < c, \beta < w(p)\}$

*(vi)* $\{p \mid p < b, \delta' < w(p) < \alpha\}$

*(vii)* $\{p \mid p > d, w(p) < \gamma\}$

*(viii)* $\{p \mid p > d, \gamma < w(p) < \delta'\}$

*(ix)* $\{p \mid c < p < d, \gamma < w(p) < \delta\}$

*(x)* $\{p \mid b < p < c, \gamma < w(p) < \delta\}$

*(xi)* $\{p \mid b < p < c, w(p) < \gamma\}$

*(xii)* $\{p \mid p > c, \delta < w(p) < \alpha'\}$

Most of this lemma and its proof is implicitly stated by Cortez, scattered as parts of the proofs of various lemmas in [16, Sect. 5]. The empty regions are illustrated in Figure 4.

*Proof.* If $p$ is in the set (vi), then $p < b < c < d$ is a 3412 embedding with height less than that of $a < b < c < d$. If $p$ is in (iii) or (iv), then $a < p < c < d$ is a 3412 embedding of the same height but smaller amplitude than $a < b < c < d$. Similar arguments apply to (ix), (x), and (xii).

Now we show that, if one of the other sets is nonempty, the singular locus of $X_w$ must have at least two components. Note that by the emptiness of (iv), (vi), (x), and (xii) $a < b < c < d$ is a critical 3412 embedding, and by the minimality of its height it must be reduced.

Suppose the set (v) is nonempty; let $p$ be the largest element of (v). Let $C = \{i \mid b < i < p, \delta < w(i) < \alpha\}$; if $C$ is nonempty, then $i < p < c < d$ is a 3412 embedding of smaller height than $a < b < c < d$ for any $i \in C$. Now suppose $C$ is empty. If the $A_2$ region

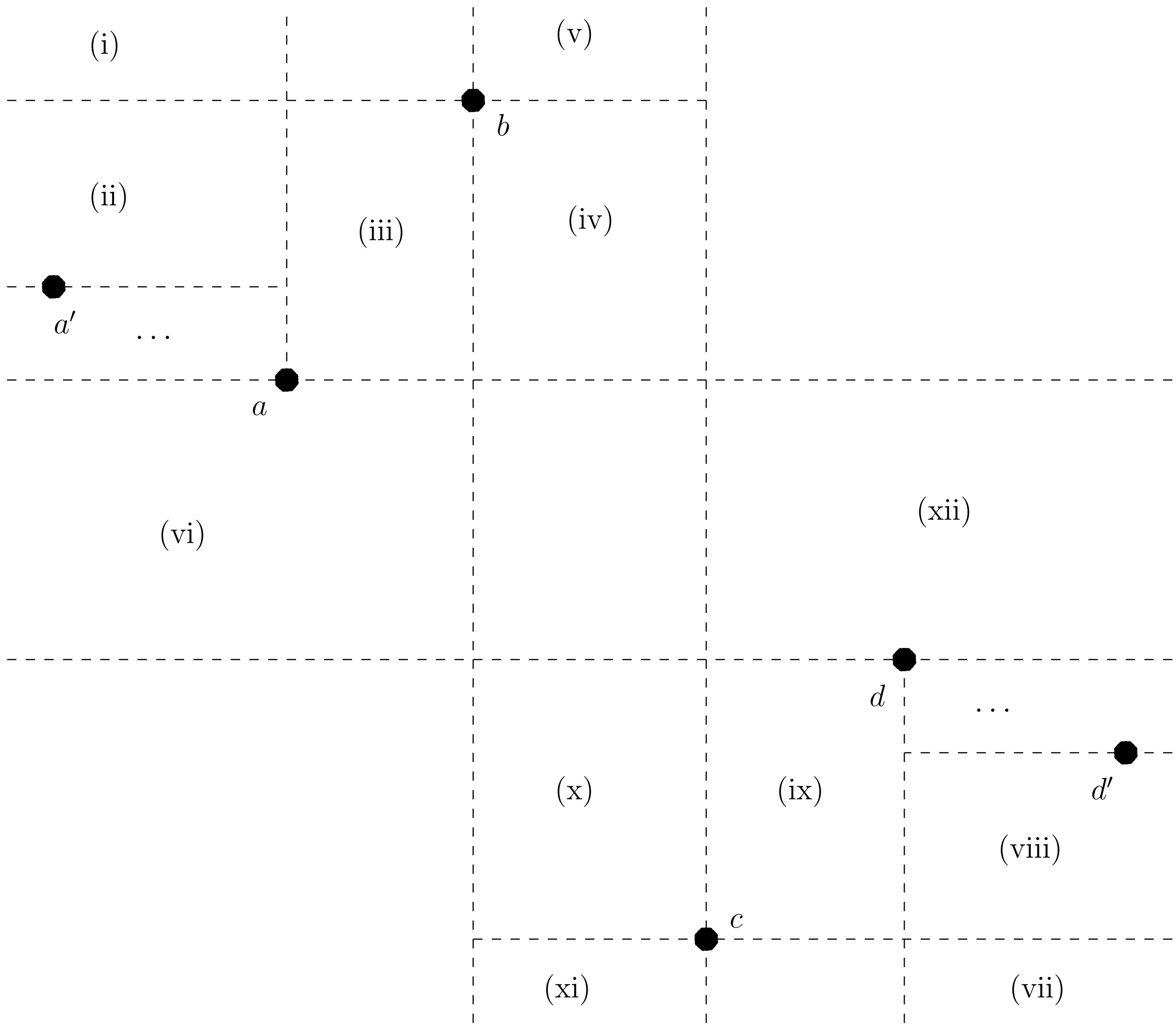


Figure 4: The regions forced to be empty by Lemma 5.1.

associated to $a < b < c < d$ is also empty, then $b < p < c < d$ is a reduced critical 3412 embedding. The top critical region is empty by our choice of $p$, the left critical region is empty by (iv) and the emptiness of $C$, the bottom critical region is empty by (x), and the right critical region is empty by (xii) and the emptiness of $A_2$; furthermore it is reduced since $a < b < c < d$ is reduced. Since $a \neq b$ and $b \neq p$, the components of the singular locus associated to these critical 3412 embeddings must be different, even if they are of type I. If $A_1$ or $B$ is empty, then $a < p < c < d$ is a reduced critical 3412 embedding. The critical regions are empty by the choice of $p$, the emptiness of $C$, and the emptiness of (iv), (vi), (x), and (xii). Since $b \neq p$, the only way the two critical 3412 embeddings gave rise to the same component is for the component to be a type I component using elements of both $A_1$ and $B$, but one of these sets is empty in this case. If $A_1$, $A_2$, and $B$ are all nonempty, then the singular locus of $X_w$ must already have more than one component.

Suppose (ii) is nonempty; let $e$ be the element of (ii) with the smallest value of $\epsilon = w(e)$. By the definition of $\alpha'$ and the emptiness of (iii) and (iv), either $w^{-1}(\epsilon-1) > c$, or $\epsilon = \alpha'+1$ and $a' < e < a$.

First we treat the case where $w^{-1}(\epsilon - 1) > c$. Let $f = w^{-1}(\epsilon - 1)$. If $h > 1$, then $e < b < w^{-1}(\alpha - 1) < f$ is a 3412 embedding of height 1 and amplitude smaller than that of $a < b < c < d$. If $f > d$, then the same holds for $e < b < d < f$. If $h = 1$ and $f < d$, then we have a type I component defined by $e$, $\{i \mid e < i \leq a, \alpha \leq w(i) \leq \alpha'\}$, which contains $a$, a subset of $\{j \mid c < j < d, \alpha' < w(j) < \epsilon\}$ that contains $f$, and $d$. This type I component cannot be the component of the singular locus of $X_w$ associated to $a < b < c < d$, since $b \neq e$.

Now we treat the case where $\epsilon = \alpha' + 1$ and $a' < e < a$. Let $i$ be the largest element of $\{i \mid a' \leq i < e, \alpha < w(i) \leq \alpha'\}$. Let $j$ be the smallest element of $\{j \mid e < j \leq c, \gamma \leq w(j) < \delta\}$, a set which contains $c$. Let $k$ be the smallest element of $\{k \mid j < k \leq d, w(j) < w(k) < w(i)\}$, a set which contains $d$. Then $i < e < j < k$ is a reduced critical 3412 embedding. The only portion of the critical region not directly guaranteed empty by the definitions of $i$, $e$, $j$, and $k$ is $\{m \mid e < m < j, \delta \leq w(m) < w(k)\}$; if $m$ is an element of this set then $m < k < c < d$ is a 3412 embedding of height smaller than $a < b < c < d$. Since $i \neq a$ and $e \neq b$, this must produce a second component of the singular locus of $X_w$. This shows (ii) must be empty.

Suppose (i) is nonempty; let $e$ be the largest element of (i). Then the singular locus of $X_w$ has a type I component defined by $e$, a set of which $a$ is the largest element, a set of which $b$ is the largest element, and $w^{-1}(\alpha - 1)$.

The proofs that (xi), (viii), and (vii) are empty are entirely analogous to those for (v), (ii), and (i) respectively. □

For the following lemma, recall the definitions $M = \max\{p \mid p < c, w(p) < \delta'\} \cup \{a\}$ and $N = \max\{p \mid w(p) < \delta'\}$, given in Section 4.

**Lemma 5.2.** *Suppose the singular component of $X_w$ has only one component. Then*

*(i)* $a \leq M < b$.

*(ii)* $\{p \mid a < p < M, w(p) > \alpha'\}$ *is empty.*

*(iii)* $c \leq N < d$.

*(iv)* $\{p \mid c < p < N, w(p) > \alpha'\}$ *is empty.*

This lemma is illustrated in Figure 5

*Proof.* We know that $a \leq M$ by definition, and $M < b$ by Lemma 5.1 (x) and (xi). Similarly, $c \leq N$ by definition, and $N < d$ by Lemma 5.1 (vii) and (viii).

Now, assume for contradiction that $\{p \mid a < p < M, w(p) > \alpha'\}$ is nonempty. Let $j = \max\{p \mid a < p < M, \alpha < w(p)\}$. By the definition of $j$ and Lemma 5.1 (vi),

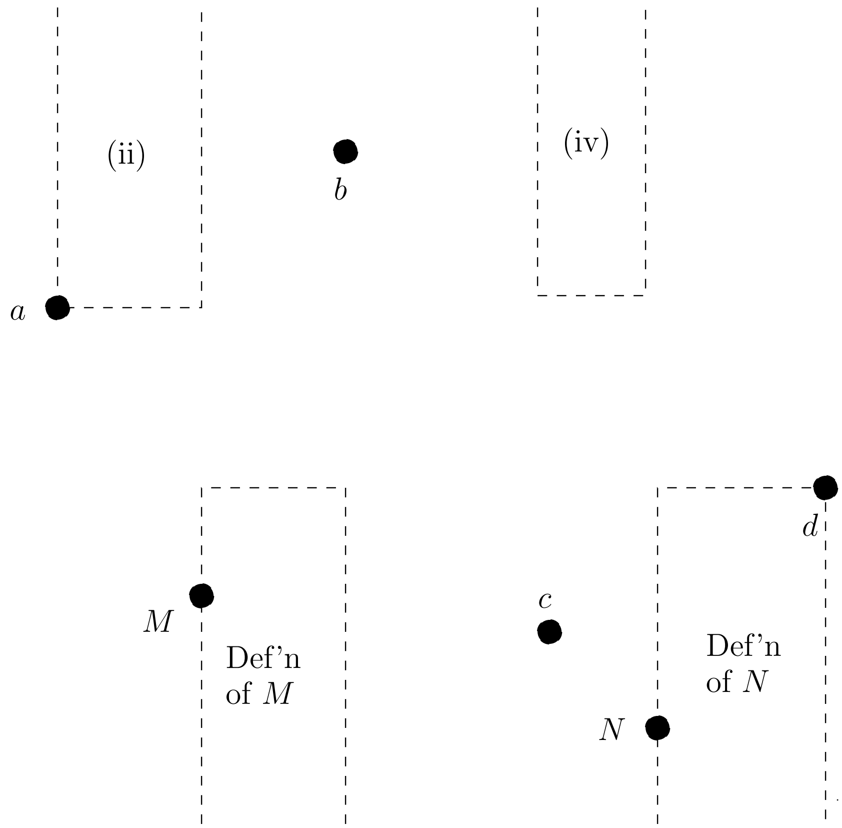


Figure 5: The regions forced to be empty by Lemma 5.2.

$w(j+1) < \delta'$. Then $a < j < j+1 < w^{-1}(\alpha-1)$ is a reduced critical 3412 embedding defining a component of the singular locus in addition to the one defined by $a < b < c < d$.

Similarly, suppose $\{p \mid c < p < N, w(p) > \alpha'\}$ is nonempty. Let $j = \max\{p \mid c < p < N, \alpha' < w(p)\}$. By the definition of $j$ and Lemma 5.1 (xi), $w(j+1) < \delta'$. Then $w^{-1}(\delta+1) < j < j+1 < d$ is a reduced critical 3412 embedding defining a component of the singular locus. □

We now proceed with the proof of the lemmas of Section 4, beginning with Lemma 4.1.

**Lemma 4.1.** *Suppose the singular locus of* $X_w$ *has only one component and* $w$ *avoids* 463152. *Let* $Z$ *be constructed as above; then* $Z$ *is smooth.*

*Proof.* Since $Z$ is a fibre bundle by the map $\pi_1$ over a smooth variety (the Grassmannian) with fibre $X_v$, it is smooth if and only if $X_v$ is.

We show the contrapositive of our stated lemma by showing that, if $X_v$ is not smooth and $w$ avoids 463152, then the singular locus of $X_w$ must have a component in addition to the one defined by the reduced critical 3412 embedding $a < b < c < d$.

Assume $X_v$ is singular. We choose a component of its singular locus. This component has a combinatorial description as in Section 2.4.

For convenience, we let $a_1 = a' = w^{-1}(\alpha')$, $a_2 = w^{-1}(\alpha'-1)$, and so on with $a_{\alpha'-\alpha+1} = w^{-1}(\alpha) = a$. Similarly, we let $d_1 = w^{-1}(\alpha - 1)$, $d_2 = w^{-1}(\alpha - 2)$, and so on with $d_h = d$ and $d_{h+\delta-\delta'} = d' = w^{-1}(\delta')$. We also let $\mathcal{A} = \{a_1, \ldots, a_{\alpha'-\alpha+1}\}$, $\mathcal{D}_1 = \{d_1, \ldots, d_{h-1}\}$, $\mathcal{D}_2 = \{d_h, \ldots, d_{h+\delta-\delta'}\}$, and $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$.

First we handle the case where our chosen component of the singular locus of $X_v$ is of type I. If no index of the embedding into $v$ defining the component is in $\mathcal{A}$ or $\mathcal{D}$, then the indices define an embedding of the same permutation into $w$, and the sets required to be empty by the interval condition remain in exactly the same positions. The horizontal boundaries of these regions are all above $\alpha'$ or below $\delta'$, so these regions remain empty in $w$. Therefore, the same embedding indices will define a type I component of the singular locus of $X_w$. This cannot be the same as the component associated to the critical 3412 embedding $a < b < c < d$; even if the component associated to $a < b < c < d$ is of type I, it still must involve at least either $a$ or $d$, whereas the component we just defined coming from the singular locus of $X_v$ involves neither. Therefore, the singular locus of $X_w$ has at least two components.

Now suppose our chosen type I component includes some index in $\mathcal{A}$ or $\mathcal{D}$. Let its defining embedding into $v$ be given by $i < j_1 < \cdots < j_y < k_1 < \cdots < k_z < m$. Define the sets $\mathcal{J}$ and $\mathcal{K}$ by $\mathcal{J} = \{j_1, \ldots, j_y\}$ and $\mathcal{K} = \{k_1, \ldots, k_z\}$. We first show that one of $\mathcal{A}$ and $\mathcal{D}$ contains no part of the embedding. If $a_r \in \mathcal{A}$ and $d_s \in \mathcal{D}$ are both in the embedding, then since $a_r < d_s$ and $v(a_r) < v(d_s)$, $a_r \in \mathcal{J}$ and $d_s \in \mathcal{K}$. Now we must have that $i < a_r$, and that $v(i) > \alpha'$, since, by definition, $v^{-1}(t) \in \mathcal{D}$ and hence $v^{-1}(t) > a_r$ whenever $d_s \leq t \leq \alpha'$. But then $i < a$ and $w(i) = v(i) > \alpha'$, which is forbidden by Lemma 5.1 (i) and (ii).

Therefore, we have two cases, one where $\mathcal{A}$ has some part of our type I embedding but $\mathcal{D}$ does not, and one where $\mathcal{D}$ has a part of our embedding but $\mathcal{A}$ does not. We first tackle the case where $\mathcal{A}$ contains a part of the embedding. In this case, $i \in \mathcal{A}$, since otherwise $i < a$ and $w(i) > \alpha'$, violating Lemma 5.1 (i) or (ii). Having $i \in \mathcal{A}$ then implies that $m \notin \mathcal{A}$ and $\mathcal{J} \cap \mathcal{A} = \emptyset$ as follows. First, we cannot have $m \in \mathcal{A}$ because, otherwise, any $r$ and $s$ satisfying $i < r < s < m$ would satisfy $v(r) > v(s)$, which contradicts $i$ and $m$ being the first and last indices of a type I embedding. Second, $\mathcal{J} \cap \mathcal{A}$ must be empty because, if $a_r \in \mathcal{A}$, $w(a_r) < w(k) < w(i)$ implies $i < k < a_r$ for any $k$, contradicting $a_r \in \mathcal{J}$ for any type I embedding starting with $i$.

We now have two subcases for the case where $\mathcal{A}$ has a part of our type I embedding, depending on whether $((\mathcal{K} \cup \{m\}) \setminus \mathcal{A}$ contains an index less than $b$. If it does, then either $m < b$ or $k_s < b$ and $w(k_s) < \delta'$ for some $s$. Either way, the forbidden region for the type I embedding does not intersect $\{(p, q) \mid b < p, \delta' < q < \alpha'\}$. Therefore $i < j_1 < \ldots < j_y < k_1 < \ldots < k_z < m$ defines a type I component of the singular locus of $X_w$ as well as $X_v$. The forbidden region may be a little larger in $w$, but it does not acquire any points in the graph of $w$. This cannot be the same as any type I component of the singular locus of $X_w$ associated to $a < b < c < d$ since both $\mathcal{J}$ and $\mathcal{K}$ contain indices outside of the region $B$ associated to $a < b < c < d$.

In the other case, since $m > b$, we must have $c \leq m < d$ by Lemma 5.1 (x), (xi), (vii), and (viii). One possibility is that $c = m$. In this case, taking the type I embedding in $v$

and adding $\mathcal{D}_1$ to $\mathcal{K}$ gives a type I component of the singular locus of $X_w$. Both $\mathcal{J}$ and $\mathcal{K}$ contain indices outside of $B$, so this will also be a second component of the singular locus of $X_w$.

If, on the other hand, $c \neq m$, then $c \in \mathcal{K}$ by the following argument. An example of this case is in Figure 6. By Lemma 5.1 (ix), $v(m) = w(m) < \gamma$. Furthermore, $j < c$ and $v(j) < \gamma$ for all $j \in \mathcal{J}$ as follows. If $j_r > c$ and $j_{r-1} < c$ (allowing for $r = 1$ in which case we define $j_0 = i$), the forbidden region $\{(p, q) \mid j_{r-1} < p < j_r, v(j_r) < q < v(i)\}$ for our type I embedding contains $(c, \gamma)$ as $v(j_r) < \gamma$ by Lemma 5.1 (ix). If $v(j) \geq \gamma$ for some $j \in \mathcal{J}$, then $v(k_z) > \gamma$, and hence $k_z < c$ by Lemma 5.1 (ix); now the forbidden region $\{(p, q) \mid k_z < p < m, v(m) < q < v(k_z)\}$ contains $(c, \gamma)$.

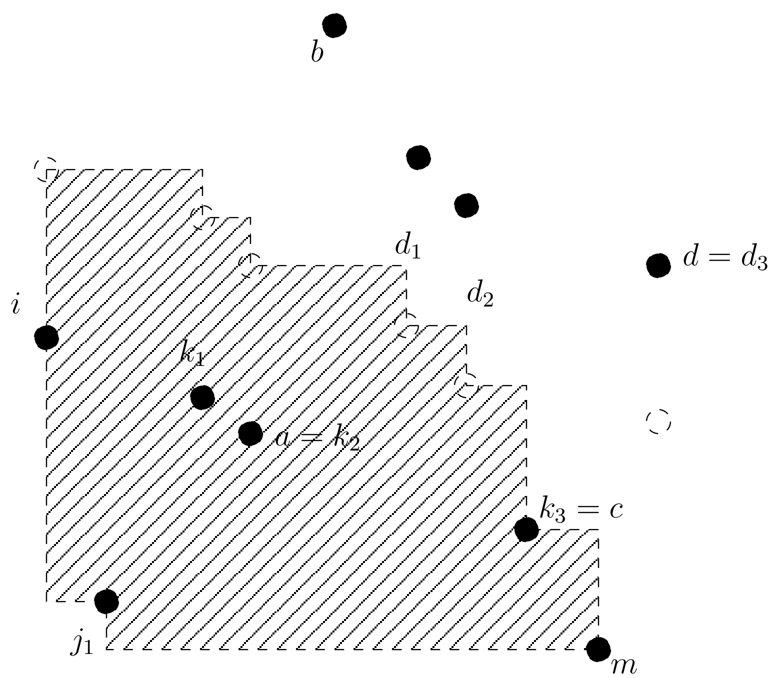


Figure 6: The case of a type I configuration in $v$, using points in $\mathcal{A}$, with $c < m < d$. The hollow points are in $w$, and the shaded region is the forbidden region of the associated configuration in $w$.

Recall that $(\mathcal{K} \setminus \mathcal{A})$ has no index less than $b$ in the case under consideration. Therefore, by Lemma 5.1 (xi), no index $k \in \mathcal{K}$ satisfies $k < c$, $v(k) < \gamma = v(c)$. As $i < c < m$, $v(m) < \gamma < v(i)$, and $j < c$ and $v(j) < \gamma$ for all $j \in \mathcal{J}$, we must have $c \in \mathcal{K}$ as otherwise $(c, \gamma)$ would be in a forbidden region. Therefore, taking the type I embedding in $v$ and adding $\mathcal{D}_1$ to $\mathcal{K}$ also gives a type I component of the singular locus of $X_w$ distinct from any associated to $a < b < c < d$.

Now suppose $\mathcal{D}$ contains some part of the embedding but $\mathcal{A}$ does not. If $i \notin \mathcal{D}$, then $w(i) = v(i) > \alpha'$, so by Lemma 5.1 (i) and (ii), $i > a$. If $i \in \mathcal{D}$, we also have $i > a$. (Actually, we cannot have $i \in \mathcal{D}$ but do not need this fact.) Therefore, $i < j_1 < \cdots < j_y < k_1 < \cdots < k_z < m$ also defines a type I component of the singular locus of $X_w$, since, as the forbidden region does not intersect $\{(p,q) \mid p \leq a, \delta' < q < \alpha'\}$, no points of the graph of $w$ move into the forbidden region. This type I component can be the same as one associated to the critical 3412 embedding $a < b < c < d$, but only if $w$ has an embedding of 463152.

We have completed the case where our chosen component of the singular locus of $X_v$ is of type I; now we move on to the case where it is of type II. Let $i < j < k < m$ be the reduced critical 3412 embedding associated to this component of the singular locus of $X_v$. If none of $i$, $j$, $k$, and $m$ are in $\mathcal{A}$ or $\mathcal{D}$, then the critical regions are in the same place in both $v$ and $w$, and they remain empty. Therefore, they produce a component of the singular locus of $X_w$ which must not be the same as the one associated to $a < b < c < d$ as their reduced critical 3412 embeddings are different.

Now we first consider the case where $\mathcal{D}$ has a part of the critical embedding but $\mathcal{A}$ does not. If $i \in \mathcal{D}$, then $i > a$, so the critical region as well as the regions $A$ and $B$ associated to $i < j < k < m$ do not intersect $\{(p,q) \mid p \leq a, \delta' \leq q \leq \alpha'\}$, and $i < j < k < m$ is also a reduced critical 3412-embedding producing a type II component of the singular locus of $X_w$. If $j \in \mathcal{D}$, then $v(i) < k$, and, since $i \notin \mathcal{A}$ by assumption, $v(i) < \delta'$. Since $j > a$, we therefore also have that $\{(p,q) \mid p \leq a, \delta' \leq q \leq \alpha'\}$ fails to intersect the critical regions or the regions $A$ and $B$, and $i < j < k < m$ is a critical 3412 embedding producing a type II component of the singular locus of $X_w$. Otherwise, $i < d_1$ and $v(i) > \alpha'$, so by Lemma 5.1 (i) and (ii), $i > a$, implying that $i < j < k < m$ produces a type II component of the singular locus of $X_w$. Since $i < j < k < m$ is not $a < b < c < d$, we must have produced a second component of the singular locus of $X_w$ in all of these cases.

Now suppose $\mathcal{A}$ has part of the critical embedding. We cannot have $k \in \mathcal{A}$ or $m \in \mathcal{A}$, since otherwise we would have $i < j < a$ with $v(a) < v(i) < v(j)$, which forces $j \notin \mathcal{A}$. Then $j < a$ and $w(j) = v(j) > \alpha'$, violating Lemma 5.1 (i) or (ii). Therefore, $j \in \mathcal{A}$ or $i \in \mathcal{A}$.

If $j \in \mathcal{A}$, then since $i < j$ and $v(i) < v(j)$, $v(i) < \delta'$, and so $v(k) < v(m) < \delta'$. Now if $k < b$, $i < j < k < m$ is a reduced critical 3412 embedding in $w$. It may have an element in its $A_2$ region in $w$ that when there is none in its $A_2$ region in $v$, but in that case either the $B$ region is empty or $w$ fails to avoid 463152. When $w$ avoids 463152, $i < j < k < m$ produces a second type II component of the singular locus of $X_w$. If $j \in \mathcal{A}$ and $k > b$, then by Lemma 5.1 (x) and (xi), $k \geq c$. Moreover, we cannot have $k = c$ as, in that case, $c < m$ and $\gamma < v(m) = w(m) < \delta'$, violating Lemma 5.1 (viii) or (ix). Therefore, $c < k < m$, and, as $m < \delta'$, $m < d$ by Lemma 5.1 (vii) and (viii). Since $j < c < k$, we now must have that $v(i) = w(i) > \gamma$ in order for $i < j < k < m$ to be a critical 3412 embedding in $v$. If $h = 1$ and hence $\mathcal{D}_1$ is empty, then $i < j < k < m$ is a critical 3412 embedding in $w$ with $A$ or $B$ empty as they are in $v$. If $h > 1$, let $i' = \max\{p \mid p < b, \gamma < w(p) \leq w(i)\}$; this set is nonempty because $i$ is an element. Then $i' < d_{h-1} < k < m$ is a reduced critical embedding of 3412 in $w$, and the component of the singular locus of $X_w$ it produces, whether it is type I or type II, must be different from the one associated to $a < b < c < d$. This last case is illustrated in Figure 7.

Finally we tackle the case where $i \in \mathcal{A}$. If $m < b$, then $i < j < k < m$ is a reduced critical 3412 embedding in $w$. Otherwise, $m \geq c$ by Lemma 5.1 (x) and (xi), and hence

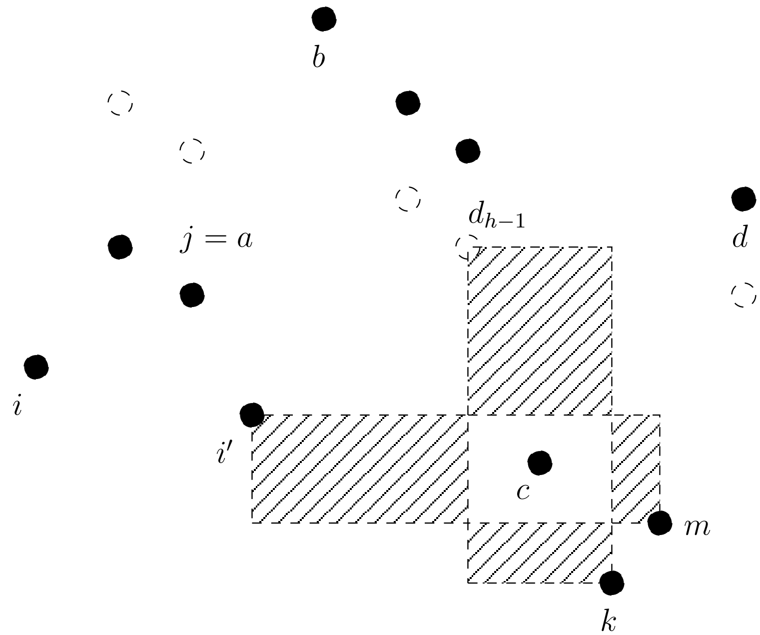


Figure 7: The case of a type II configuration in $v$, using points in $\mathcal{A}$, with $h > 1$ and $k > b$. The hollow points are in $w$, and the shaded regions are the critical regions of the associated 3412 embedding in $w$.

$v(m) < \delta'$. If $k < c$, then $v(k) < \delta'$, so $k \leq M$ by definition. We then have $j < M$ with $v(j) > \alpha'$, which is forbidden by Lemma 5.1 (i), (ii), and (iii) and Lemma 5.2 (ii). We cannot have $k = c$ since in that case $\gamma < w(m) = v(m) < \delta'$ and $m > c$, violating Lemma 5.1 (viii) or (ix). If $k > c$ then we have $c < k < m < d$. In this case $a < b < k < m$ is a critical 3412 embedding in $w$. In particular, $\{p \mid b < p < k, \alpha < w(p) < \beta\}$ is empty by Lemma 5.1 (iv) and Lemma 5.2 (iv). Since $k \neq c$ and $m \neq d$, the associated component of the singular locus of $X_w$ must be different from the component associated to $a < b < c < d$.

We have now shown that, unless 463152 embeds in $w$, no matter what singularity $X_v$ may have, it must produce a second component of the singular locus of $X_w$, either directly or through the use of Lemma 5.1 or Lemma 5.2. Therefore, if the singular locus of $X_w$ has only one component and $w$ avoids 463152, $X_v$, and hence $Z$, is nonsingular. □

Now we continue on to proving the lemmas of Section 4.2.

**Lemma 4.2.** *The fiber of* $\pi_2$ *over a flag* $F_\bullet$ *is*

$$\{G \in Gr_\kappa(\mathbb{C}^n) \mid E_{\delta'-1} + F_M \subseteq G \subseteq E_{\alpha'} \cap F_N\}.$$

*Proof.* By definition of $Z$, $E_{\delta'-1} \subseteq G \subseteq E_{\alpha'}$. We need to show that $F_M \subseteq G$, that $G \subseteq F_N$, and that any such subspace $G$ is in $\pi_2^{-1}(F_\bullet)$.

To show that $F_M \subseteq G$, we show that $r_v(M, \kappa) = M$. This is equivalent to showing that $\{p \mid p \leq M, v(p) > \kappa\}$ is empty, which is in turn equivalent to showing that $\{p \mid p \leq$

$M, \delta'-1 < w(p) < \alpha\}$ and $\{p \mid p \leq M, \alpha' < w(p)\}$ are both empty. The first follows from Lemma 5.1 (vi) since $M < b$ by Lemma 5.2 (i). The second follows from Lemma 5.1 (i) and (ii) and Lemma 5.2 (ii).

Now we show $G \subseteq F_N$. This means showing that $r_v(N,\kappa) = \kappa$, or that $\{p \mid p > N, v(p) \leq \kappa\}$ is empty. This is equivalent to showing that $\{p \mid p > N, w(p) < \delta'\}$ and $\{p \mid p > N, \alpha \leq w(p) \leq \alpha'\}$ are both empty. The first is empty by the definition of $N$, and the second is empty by the definition of $\alpha'$.

To show that any $G$ satisfying $E_{\delta'-1} + F_M \subseteq G \subseteq E_{\alpha'} \cap F_N$ is in $\pi_2^{-1}(F_\bullet)$, we need to show that $\dim(G \cap F_j) \geq r_v(j,\kappa)$ for any $j$ with $M < j < N$. It suffices to show that $r_v(j,\kappa) = r_v(M,\kappa) = M$ when $M < j < c$, and that $r_v(j,\kappa) = r_v(j-1,\kappa)+1$ when $c \leq j \leq N$. Equivalently, this means that $v(j) > \kappa$ when $M < j < c$ and $v(j) \leq \kappa$ when $c \leq j \leq N$.

Since $v(j) \leq \kappa$ if and only if $w(j) < \delta'$ or $\alpha \leq w(j) \leq \alpha'$, the first condition is clear from the definition of $M$. We also have that $N < d$ by Lemma 5.2 (iii), so we need that $\{p \mid c < p < N, w(p) > \alpha'\}$ is empty, which follows from Lemma 5.2 (iv). Therefore, $r_v(j,\kappa) = r_v(j-1,\kappa)+1$ when $c \leq j \leq N$, and any $G$ satisfying $E_{\delta'-1}+F_M \subseteq G \subseteq E_{\alpha'} \cap F_N$ is in $\pi_2^{-1}(F_\bullet)$. □

**Lemma 4.3.** *Suppose that the singular locus of $X_w$ has only one component and $w$ avoids 546213. Then* $\dim(E_{\delta'-1} + E_M) = \kappa - 1$.

*Proof.* Since $r_v(M,\kappa) = M$, $c > M$, and $v(c) < \kappa$, $M = r_v(M,\kappa) + 1 \leq r_v(c,\kappa) \leq \kappa$, so $M \leq \kappa - 1$. If $\alpha = \alpha'$, then $\delta' = \alpha' - \alpha + \delta' = \kappa$, so $\delta' - 1 = \kappa - 1$. Otherwise, we need to show that $M = \kappa - 1$. Since $M \geq a$, so that $\{p \mid p > M, \alpha \leq p \leq \alpha'\}$ is empty, this is equivalent to showing that $\{p \mid p > M, w(p) < \delta'\}$ has only one element, namely $c$.

By the definition of $M$, $\{p \mid M < p < c, w(p) < \delta'\}$ is empty. Furthermore, by Lemma 5.1 (vii), (viii), and (ix), $\{p \mid p > d, w(p) < \gamma\}$, $\{p \mid p < d, \gamma < w(p) < \delta'\}$, and $\{p \mid c < p < d, \gamma < w(p) < \delta\}$ are empty. This leaves $\{p \mid c < p < d, w(p) < \gamma\}$, which is empty since $\alpha' \neq \alpha$ and $w$ avoids 546213. □

**Lemma 4.4.** *Suppose that the singular locus of $X_w$ has only one component and $w$ avoids 465132. Then* $\dim(E_{\alpha'} \cap E_N) = \kappa + h$.

*Proof.* First, note that $N \geq \kappa + h$, since $N = \#\{p \mid p < N, v(p) \leq \kappa\} + \#\{p \mid p < N, v(p) > \kappa\}$, and the first summand is $r_v(N,\kappa) = \kappa$, while the second summand is at least $h$ since the $h$ elements $b, w^{-1}(\alpha-1), \ldots, w^{-1}(\delta+1)$ are in the set. If $\delta' = \delta$, then $\alpha' = \kappa + \alpha - \delta = \kappa + h$. Otherwise, we need to show that $N = \kappa + h$. This means showing that $\{p \mid p < N, v(p) > \kappa\}$ has exactly $h$ elements, or, equivalently, that $\{p \mid p < N, w(p) > \alpha'\}$ contains only $b$.

We know that $\{p \mid c < p < N, w(p) > \alpha'\}$ is empty by Lemma 5.2 (iv), and, by Lemma 5.1 (i), (ii), and (iii), $\{p \mid p < a, w(p) > \beta\}$, $\{p \mid p < a, \alpha' < w(p) < \beta\}$, and $\{p \mid a < p < b, \alpha < w(p) < \beta\}$ are empty. This leaves $\{p \mid a < p < b, w(p) > \beta\}$, which is empty since $\delta' \neq \delta$ and $w$ avoids 465132. □

**Lemma 4.5.** *Suppose the singular locus of $X_w$ has only one component, and $h > 1$. Then the image of the exceptional locus of $\pi_2$ is*

$$\{F_\bullet \mid \dim(E_{\delta'-1} \cap F_M) > r_w(M, \delta' - 1)\}.$$

*Proof.* First we show that $r_w(N, \alpha') = \kappa + h - 1$. By the definition of $N$, $r_v(N, \kappa) = \kappa$, so the two sets $\{p \mid p < N, w(p) < \delta'\}$ and $\{p \mid p < N, \alpha \leq w(p) \leq \alpha'\}$ have $\kappa$ elements combined. Since $c < N$ by definition and $N < d$ by Lemma 5.1 (vii) and (viii), $\{p \mid p < N, \delta' \leq w(p) < \alpha\}$ has precisely the $h-1$ elements $w^{-1}(\alpha - 1), \ldots, w^{-1}(\delta + 1)$. Therefore, $\dim(E_{\alpha'} \cap F_N) = r_w(N, \alpha') = \kappa + h - 1$ generically.

Now we calculate $\dim(E_{\delta'-1} + F_M)$. Note that

$$\dim(E_{\delta'-1} + F_M) = \delta' - 1 + M - \dim(E_{\delta'-1} \cap F_M).$$

Generically,

$$\dim(E_{\delta'-1} \cap F_M) = r_w(M, \delta' - 1),$$

and

$$\begin{aligned} r_w(M, \delta' - 1) &= r_v(M, \kappa) - \#\{p \mid p < M, \alpha \leq w(p) \leq \alpha'\} \\ &= M - (\alpha' - \alpha + 1). \end{aligned}$$

Therefore, generically,

$$\begin{aligned} \dim(E_{\delta'-1} + F_M) &= \delta' - 1 + M - M + \alpha' - \alpha + 1 \\ &= \delta' + \alpha' - \alpha \\ &= \kappa. \end{aligned}$$

Recall that, by Lemma 4.2, the fiber over a flag $F_\bullet$ is

$$\{G \in Gr_\kappa(\mathbb{C}^n) \mid E_{\delta'-1} + F_M \subseteq G \subseteq E_{\alpha'} \cap F_N\}.$$

Therefore, since $\dim G = \kappa$, the fiber over $F_\bullet$ consists of the single point corresponding to the subspace $E_{\delta'-1} + F_M$ generically; here, the generic situation occurs whenever $\dim(E_{\delta'-1} \cap F_M) = r_w(M, \delta' - 1)$. When $h = 1$, we also have that $\dim(E_{\alpha'} \cap F_N) = \kappa$ in the generic situation, and we also need $\dim(E_{\alpha'} \cap F_N) > \kappa$ in order for the fiber over $F_\bullet$ to consist of more than a point. However, when $h > 1$, $\pi^{-1}(F_\bullet)$ has more than one point whenever $\dim(E_{\delta'-1} \cap F_M) > r_w(M, \delta' - 1)$, so the image of the exceptional locus is

$$\{F_\bullet \mid \dim(E_{\delta'-1} \cap F_M) > r_w(M, \delta' - 1)\},$$

as desired. □

Recall that $u$ is defined by $u = \sigma w$, where $\sigma \in S_n$ is the cycle $(\gamma, \delta + 1, \delta + 2, \ldots, \alpha)$.

**Lemma 4.6.** *Assume that $h > 1$ and $w$ avoids 526413. Then the image of the exceptional locus of $\pi_2$ is $X_u$, $\ell(w) - \ell(u) = h$, and the generic fiber over $X_u$ is isomorphic to $\mathbb{P}^{h-1}$.*

*Proof.* Suppose $F_\bullet$ is in the image of the exceptional locus, and let $X_x^\circ$ be the Schubert cell containing $F_\bullet$. Our strategy is to show using rank matrices that $x \leq u$. As part of this proof, we show that a certain region of the graph of $w$ is empty, which will imply that $\ell(w) - \ell(u) = h$.

First we compare the rank matrices $r_u$ and $r_w$. Let $R_1$ denote the region $\{(p,q) \mid w^{-1}(\delta+1) \leq p < c, \gamma \leq q < \delta+1\}$ and $R_i = \{(p,q) \mid w^{-1}(\delta+i-1) \leq p < w^{-1}(\delta+i), \gamma \leq q < \delta+i\}$ when $1 < i \leq h$. Since $u = t_h \cdots t_1 w$ where $t_1 = (\gamma, \delta+1)$ and $t_i = (\delta+i-1, \delta+i)$ when $1 < i \leq h$, we get that $r_u(p,q) = r_w(p,q) + 1$ if $(p,q)$ is in $R_i$ for some $i$, and $r_u(p,q) = r_w(p,q)$ otherwise. Let $R$ denote the union $R = \bigcup_{i=1}^h R_i$. The region $R$ is drawn in Figure 8.

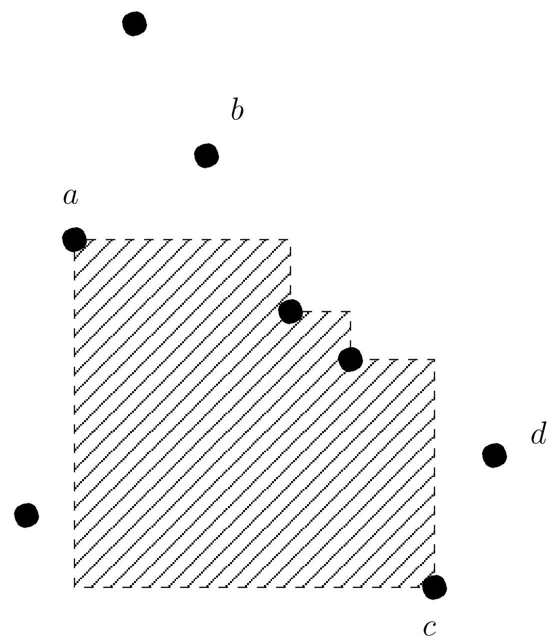


Figure 8: The region $R$ "between" $u$ and $w$.

Now we show that, when $(p,q) \in R$, then $r_w(p,q)$ is as small as possible given that $r_w(M, \delta'-1) = M$ and given that, as for any permutation and any $p$ and $q$, $0 \leq r_w(p+1,q) - r_w(p,q) \leq 1$ and $0 \leq r_w(p,q+1) - r_w(p,q) \leq 1$. To be precise, this means that, assuming $(p,q)$ and $(p,q+1)$ are both in $R$, $r_w(p,q) = r_w(p,q+1) - 1$ if $\gamma \leq q < \delta'$ and $r_w(p,q) = r_w(p,q+1)$ otherwise, and, assuming $(p,q)$ and $(p+1,q)$ are both in $R$, $r_w(p,q) = r_w(p+1,q) - 1$ if $a \leq p < M$ with $r_w(p,q) = r_w(p+1,q)$ otherwise.

To prove the above claim, we need to show that $R$ contains no point in the graph of $w$, that $w(p) < \gamma$ when $a < p \leq M$, and that $w^{-1}(q) < a$ when $\gamma < q \leq \delta' - 1$. Since $h > 1$ and $w$ avoids 526413, $\{p \mid a < p < b, \gamma < w(p) < \delta'\}$. Also, $\{p \mid a < p < b, \delta' \leq w(p) < \alpha\}$ and $\{p \mid b \leq p < c, \gamma < w(p) < \delta'\}$ are empty by Lemma 5.1 (vi) and (x). The remaining portion of $R$ contains no point in the graph of $w$ since, as $a < b < c < d$ is a 3412 embedding of minimal height, $b < w^{-1}(\alpha - 1) < \cdots < w^{-1}(\delta+1) < c$ by $a < b < c < d$.

Furthermore, using that $R$ contains no point of the graph of $w$, $w(p) < \gamma$ when $a < p \leq M$ by Lemma 5.2 (ii), and $w^{-1}(q) < a$ when $\gamma < q \leq \delta' - 1$ by Lemma 5.1 (viii) and (ix).

Suppose $X_x^\circ$ is in the image of the exceptional locus, so that $x \leq w$ and $r_x(M, \delta'-1) \geq r_w(M, \delta'-1) + 1$. We show that $r_x(p,q) \geq r_u(p,q)$ for all $p$ and $q$. If $(p,q)$ is not in $R$, this follows since $x \leq w$. For $(p,q) \in R$, $r_w(p,q)$ is the minimum possible given that $r_w(M, \delta'-1) = M$. Since $r_x(M, \delta'-1) > r_w(M, \delta'-1)$, it follows that $r_x(p,q) > r_w(p,q)$ for $(p,q) \in R$. Therefore, $r_x(p,q) \geq r_u(p,q)$ when $(p,q)$ is in $R$, and $x \leq u$.

Since the regions $R_i$ are empty, multiplication by each transposition $t_1, \ldots, t_h$ decreases the length of $w$ by 1, and $\ell(u) = \ell(w) - h$.

Since $u(p) \leq \alpha'$ if and only if $w(p) \leq \alpha'$, $r_u(N, \alpha') = r_w(N, \alpha') = \kappa + h - 1$. Therefore, $\dim(E'_\alpha \cap F_N) = \kappa + h - 1$ for $F_\bullet \in X_u^\circ$. Moreover, for $F_\bullet \in X_u^\circ$,

$$\begin{aligned}\dim(E_{\delta'-1} + F_M) &= \delta' - 1 + M - r_u(M, \delta'-1)\\ &= \delta' - 1 + M - M + \alpha' - \alpha\\ &= \kappa - 1.\end{aligned}$$

Therefore, the generic fiber over $X_u$ is isomorphic to $\mathbb{P}^{h-1}$. □

# A A Purely Pattern Avoidance Characterization (by Sara Billey and Jonathan Weed)

Let $S_\infty$ be the union of $S_n$ for all $n \geq 1$. There exists a partial order on $S_\infty$ determined by pattern embeddings; we say $v \prec w$ if there is a pattern embedding of $v$ into $w$. If the embedding of $v$ into $w$ is given by the set of indices $Z = \{i_1, \ldots, i_m\}$, then we write $fl_Z(w) = v$, i.e., that the "flattened" version of $w$ consisting only of the indices in $Z$ is the permutation $v$.

Consider the set $KL_m = \{w \in S_\infty \,|\, P_{id,w}(1) \leq m\}$ for any positive integer $m$. By [6, Thm. 1], we know $KL_m$ is the complement of a lower order ideal in the poset of pattern embeddings. Therefore, $KL_m$ can be characterized by pattern avoidance for every $m \geq 1$. For example, $KL_1$ is the set of permutations avoiding 4231 and 3412. The following theorem gives a minimal set of patterns characerizing $KL_2$.

**Theorem A.1.** *$KL_2$ is equal to the set of permutations avoiding the 66 patterns*

| | | | | |
|---|---|---|---|---|
| 45123 | 34512 | 53412 | 52341 | |
| 45231 | 351624 | 523614 | 526314 | |
| 624153 | 524613 | 462513 | 526413 | |
| 546213 | 361452 | 461352 | 364152 | |
| 463152 | 536142 | 465132 | 426351 | |
| 632541 | 635241 | 642531 | 653421 | |
| 3612745 | 6231745 | 6241735 | 3416725 | |
| 4236715 | 4263715 | 4267315 | 3712564 | |
| 7231564 | 3715264 | 3751264 | 7523164 | (A.1) |
| 6251734 | 7261453 | 3417562 | 3517462 | |
| 4517362 | 4237561 | 5347261 | 4275631 | |
| 34127856 | 42317856 | 34172856 | 42371856 | |
| 42731856 | 35127846 | 52317846 | 52417836 | |
| 34128675 | 42318675 | 34182675 | 42381675 | |
| 42831675 | 34186275 | 42386175 | 42863175 | |
| 35128674 | 52318674 | 36128574 | 62318574 | |
| 52418673 | 62518473. | | | |

Given $w \in S_n$, the irreducible components of the singular locus of the Schubert variety $X_w$ are themselves Schubert varieties. The set of permutations indexing these irreducible components is called the *maximal singular locus*, and is denoted by $\mathrm{maxsing}(w)$. The proof of Theorem A.1 follows from the next two lemmas relating the maximal singular locus of a Schubert variety with patterns.

**Lemma A.2.** *[8, Sec. 13] Consider a set $Z$ such that $fl_Z(w) = 4231$ or $3412$. Then $Z$ corresponds to a unique element of $\mathrm{maxsing}(w)$ if and only if the pattern does not occur as the dotted part of one of the following patterns:*

$$\begin{array}{lll} \dot{3}\dot{5}4\dot{1}\dot{2} & \dot{4}3\dot{5}\dot{1}\dot{2} & \dot{4}\dot{5}\dot{1}3\dot{2} \\ \dot{4}\dot{5}2\dot{1}\dot{3} & \dot{5}\dot{2}3\dot{4}\dot{1} & \dot{5}\dot{2}\dot{4}3\dot{1} \\ \dot{5}3\dot{2}\dot{4}\dot{1} & \dot{5}\dot{3}\dot{4}2\dot{1} & \dot{5}4\dot{2}\dot{3}\dot{1} \\ \dot{6}\dot{3}52\dot{4}\dot{1} & \dot{5}\dot{6}34\dot{1}\dot{2} & \dot{5}2\dot{6}4\dot{1}\dot{3} \\ \dot{4}\dot{6}3\dot{1}5\dot{2}. & & \end{array} \qquad (A.2)$$

**Remark A.3.** In contrast to Theorem A.1, it is interesting to note that the set $MS_2 = \{w \in S_\infty : |\mathrm{maxsing}(w)| \geq 2\}$ is not characterized by pattern avoidance. For instance, the permutation $x = 4631725$ has a maximal singular locus of size 2, so $x \in MS_2$. However $x \prec w = 47318625$, but $w$ has maximal singular locus of size 1.

**Lemma A.4.** *If* $|\operatorname{maxsing}(w)| \geq k$*, then there exists a pattern* $v \prec w$ *with at most* $4k$ *entries such that* $|\operatorname{maxsing}(v)| \geq k$*.*

*Proof.* For each element $x_i$ of $\operatorname{maxsing}(w)$, let $Z_i$ be the indices of $w$ such that $fl_{Z_i}(w)$ is the 4231 or 3412 pattern corresponding to $x_i$, and let $Z$ be the union of $Z_1, Z_2, \ldots, Z_k$. Then $|Z| \leq 4k$, since each element of $\operatorname{maxsing}(w)$ adds at most 4 indices to $Z$, so $\mathrm{fl}_Z(w)$ has at most $4k$ entries. Let $v = \mathrm{fl}_Z(w)$.

Given $x_i \in \operatorname{maxsing}(w)$, $\mathrm{fl}_{Z_i}(w)$ is a 4231 or 3412 pattern which is not a subpattern of one of the dotted patterns in (A.2). Since $Z_i \subset Z$, $\mathrm{fl}_{Z_i}(v) = \mathrm{fl}_{Z_i}(\mathrm{fl}_Z(w)) = \mathrm{fl}_{Z_i}(w)$, so $Z_i$ is a 4231 or 3412 pattern in $v$ as well. Furthermore, it cannot be a subpattern of one of the patterns in (A.2), since then it would be a subpattern of that pattern in $w$ as well. Hence $Z_i$ corresponds to a unique element of $\operatorname{maxsing}(v)$. So $|\operatorname{maxsing}(v)| \geq |\operatorname{maxsing}(w)| \geq k$, as desired. □

*Proof of Theorem A.1.* By Theorem 1.1, $KL_2$ is the set of permutations which have at most 1 elements in the maximal singular locus and avoid

$$\{653421, 632541, 463152, 526413, 546213, 465132\}. \tag{A.3}$$

If $w \notin KL_2$, then either it contains a pattern in (A.3) or it has at least two elements in its maximal singular locus. The patterns of (A.3) are in (A.1). We claim that any $w \in S_\infty$ with $|\operatorname{maxsing}(w)| \geq 2$ contains a pattern in (A.1). Therefore, $w \notin KL_2$ contains a pattern from (A.1).

To prove the claim, note by Lemma A.4 that there exists $v \in S_{\leq 8}$ such that $v \prec w$ and $|\operatorname{maxsing}(v)| \geq 2$. A computer check establishes that (A.1) contains all the minimal patterns in $S_{\leq 8}$ not in $KL_2$, hence $w$ contains one such pattern.

Conversely, if $w \in S_\infty$ contains a pattern $v$ in (A.1), then a computer verification shows that $P_{id,v}(1) > 2$ so by [6], $P_{id,w}(1) > 2$. Hence, $w$ is not in $KL_2$. Therefore, the patterns of (A.1) characterize $KL_2$, as desired. □

The structure of $KL_m$ for $m \geq 1$ gives a pattern avoidance "filtration" on $S_\infty$. This suggests the following questions.

1. Can $KL_m$ always be characterized by a finite number of patterns?
2. If so, can the minimal elements of the complement of $KL_m$ be determined efficiently?
3. We know the maximal singular locus is efficient to calculate. Can we use information about $\operatorname{maxsing}(w)$ to give bounds for $P_{id,w}(1)$?

The following conjecture has been tested through $S_8$.

**Conjecture A.5.** Let $w \in S_n$.

1. If $P_{id,w}(1) \leq 3$ then $|\operatorname{maxsing}(w)| \leq 3$.
2. If $P_{id,w}(1) = 3$ and $|\operatorname{maxsing}(w)| = 1$ then $P_{id,w} = 1 + q^a + q^b$.
3. If $P_{id,w}(1) = 3$ and $|\operatorname{maxsing}(w)| = 2$ then $P_{id,w} = 1 + q^a + q^b$.
4. If $P_{id,w}(1) = 3$ and $|\operatorname{maxsing}(w)| = 3$ then $P_{id,w} = 1 + 2q^a$.

The following conjecture has been tested for $B_5$, $C_5$, and $D_5$.

**Conjecture A.6.** For other Weyl group types, $P_{id,w}(1) = 2$ implies $|\,\mathrm{maxsing}(w)| = 1$.

# References


[1] A. Beilinson and J. Bernstein, *Localisation de g-modules*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), 15–18.

[2] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, in *Analyse et topologie sur les espaces singuliers (I)*, Astérisque **100** (1982), 3–171.

[3] A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2), **98** (1973), 480–497.

[4] A. Białynicki-Birula, *On fixed points of torus actions on projective varieties*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **22** (1974), 1097–1101.

[5] L. Billera and F. Brenti, *Quasisymmetric functions and Kazhdan-Lusztig polynomials*, arXiV:0710.3965

[6] S. Billey and T. Braden, *Lower bounds for Kazhdan-Lusztig polynomials from patterns*, Transform. Groups **8** (2003), 321–332.

[7] S. Billey and G. Warrington, *Kazhdan-Lusztig polynomials for 321-hexagon-avoiding permutations*, J. Algebraic Combin. **13** (2001), 111–136.

[8] S. Billey and G. Warrington, *Maximal singular loci of Schubert varieties on $SL(n)/B$*, Trans. Amer. Math. Soc. **355** (2003), 3915–3945.

[9] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics **231**, Springer-Verlag, New York-Heidelberg, 2005.

[10] R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958), 964–1029.

[11] T. Braden and R. Macpherson, *From moment graphs to intersection cohomology*, Math. Ann. **321** (2001), 533–551.

[12] F. Brenti, *Lattice paths and Kazhdan-Lusztig polynomials*, J. Amer. Math. Soc. **11** (1998), 229–259.

[13] J.-L. Brylinski and M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math. **64** (1981), 387–410.

[14] J. Carrell, *Torus actions and cohomology*, in *Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action*, Encyclopaedia Math. Sci. **131**, Springer-Verlag, Berlin (2002), 83–158.

[15] A. Cortez, *Singularités génériques des varieétés de Schubert covexillaires*, Ann. Inst. Fourier (Grenoble) **51** (2001), 375–393.

[16] A. Cortez, *Singularités génériques et quasi-résolutions des variétés de Schubert pour le groupe linéaire*, Adv. Math. **178** (2003), 396–445.

[17] V. Deodhar, *Local Poincaré duality and nonsingularity of Schubert varieties*, Comm. Algebra **13** (1985), 1379–1388.

[18] V. Deodhar, *A combinatorial setting for questions in Kazhdan-Lusztig theory*, Geom. Dedicata **36** (1990), 95–119.

[19] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. (4), **7** (1974), 53–88.

[20] W. Fulton, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. **65** (1992), 381–420.

[21] J. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, Cambridge, 1990.

[22] R. Irving, *The socle filtration of a Verma module*, Ann. Sci. Ecole Norm. Sup. ser. 4 **21** (1988), 47–65.

[23] C. Kassel, A. Lascoux and C. Reutenauer, *The singular locus of a Schubert variety*, J. Algebra **269** (2003), 74–108.

[24] D. Kazhdan and G. Lusztig, *Representations of Coxeter Groups and Hecke Algebras*, Invent. Math. **53** (1979), 165–184.

[25] D. Kazhdan and G. Lusztig, *Schubert varieties and Poincaré duality*, in *Geometry of the Laplace operator*, Proc. Sympos. Pure Math., **36**, Amer. Math. Soc., Providence, RI (1980), 185–203.

[26] V. Lakshmibai and B. Sandhya, *Criterion for smootheness of Schubert varieties in* $SL(n)/B$, Proc. Indian Acad. Sci. Math. Sci. **100** (1990), 45–52.

[27] A. Lascoux, *Polynômes de Kazhdan-Lusztig pour les variétés de Schubert vexillaires*, C. R. Acad. Sci. Paris, **321** (1995), 667–670.

[28] A. Lascoux and M.P. Schützenberger, *Polynômes de Kazhdan-Lusztig pour les grassmanniennes*, Astérisque, **87–88** (1981), 249–266.

[29] P. Magyar, *Schubert polynomials and Bott-Samelson varieties*, Commentarii Mathematici Helvetici, **73** (1998), 603–636.

[30] L. Manivel, *Le lieu singulier des variétés de Schubert*, Internat. Math. Res. Notices **16** (2001), 849–871.

[31] L. Manivel, *Generic singularities of Schubert varieties*, math.AG/0105239.

[32] P. Polo, *Construction of arbitrary Kazhdan-Lusztig polynomials in symmetric groups*, Represent. Theory **3** (1999), 90–104 (electronic).

[33] G. Warrington, *KLPOL* (2002).

[34] G. Warrington, *A formula for certain inverse Kazhdan-Lusztig polynomials in* $S_n$, J. Combin. Theory Ser. A **104** (2003), 301–316.

[35] A. Woo and A. Yong, *Governing singularities of Schubert varieties*, J. Algebra **320** (2008), 495–520.

[36] A. Zelevinsky, *Small resolutions of singularities of Schubert varieties*, Funct. Anal. Applic. **17** (1982), 142–144.