A note on circuit graphs

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Submitted: Oct 12, 2009; Accepted: Jan 22, 2010; Published: Jan 31, 2010
Mathematics Subject Classifications: 05C38, 05C40

Abstract

We give a short proof of Gao and Richter's theorem that every circuit graph contains a closed walk visiting each vertex once or twice.

1 Introduction

We only consider finite graphs without loops or multiple edges. For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. A k-*walk* in G is a walk passing through every vertex of G at least once and at most k times. A *circuit graph* (G, C) is a 2-connected plane graph G with outer cycle C such that for each 2-cut S in G, every component of $G - S$ contains a vertex of C. It is immediate that every 3-connected planar graph G is a circuit graph (we may choose C to be any facial cycle of G).

In 1994, Gao and Richter [3] proved that every circuit graph contains a closed 2-walk. The existence of such a walk in every 3-connected planar graph was conjectured by Jackson and Wormald [5]. Gao, Richter, and Yu [4] extended this result by showing that every 3-connected planar graph has a closed 2-walk such that any vertex visited twice is in a vertex cut of size 3. (It is easy to see that this also implies Tutte's theorem [7] that every 4-connected planar graph is Hamiltonian.) The main objective of this note is to present a short proof of Gao and Richter's result.

Theorem 1 *Let (G, C) be a circuit graph and let $u, v \in V(C)$. Then there is a closed 2-walk W in G visiting u and v exactly once and traversing every edge of C exactly once.*

We conclude this section with some notation and terminology. A *plane chain of blocks* is a graph, embedded in the plane, with blocks $B_1, B_2, \ldots, B_k$ such that, for each $i = 1, \ldots, k-1$, B_i and B_{i+1} have a vertex in common, no two of which are the same, and, for each $j = 1, 2, \ldots, k$, $\bigcup_{i \neq j} B_i$ is in the outer face of B_j. We say that B_1 and B_k are *end blocks* of the plane chain of blocks $B_1, B_2, \ldots, B_k$.

Let G be a graph. For any $S \subseteq V(G) \cup E(G)$, define $G - S$ to be the subgraph of G with vertex set $V(G) - (S \cap V(G))$ and edge set $\{e \in E(G) : e \notin S$ or e is not incident with any vertex in $S\}$. Let H be a subgraph of G. We define $H + S$ as the graph with vertex set $V(H) \cup (S \cap V(G))$ and edge set $E(H) \cup \{e \in E(G) : e \in S$ and e is incident with two vertices in $V(H) \cup (S \cap V(G))\}$. When $S = \{s\}$, we simply write $G - s$ and $H + s$ instead of $G - \{s\}$ and $H + \{s\}$.

We write $A := B$ to rename B as A. For any graph G and any $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S.

2 Proof of Theorem 1

The set of circuit graphs has some nice inductive properties. The following ones were proved in [3] and will be used in our later proof.

Lemma 2 *Let (G, C) be a circuit graph.*

(i) *Let C' be any cycle of G and let G' be the subgraph of G contained in the closed disc bounded by C'. Then (G', C') is a circuit graph.*

(ii) *Let $v \in V(C)$, then $G - v$ is a plane chain of blocks $B_1, B_2, \ldots, B_k$. Moreover, one of the neighbors of v in C is in B_1 and the other is in B_k, and none of them is a cut vertex of $G - v$.*

We can now prove our main result.

Proof of Theorem 1. If $V(G) = V(C)$, then let $W := C$ and the assertion of the theorem holds. So we may assume that $V(G) - V(C) \neq \emptyset$. Let w be a neighbor of v in C such that $w \neq u$.

We may also assume that G is 3-connected. For otherwise, suppose that $S := \{x, y\}$ is a 2-cut in G. Since (G, C) is a circuit graph, we conclude that $S \subseteq V(C)$ and $G - S$ has exactly two components, say G_1 and G_2. For $i = 1, 2$, let $G_i^* := G[V(G_i) \cup S] + xy$ and let $C_i^* := (G_i^* \cap C) + xy$. Then it is easy to check that both (G_1^*, C_1^*) and (G_2^*, C_2^*) are circuit graphs. We may assume that x and y are chosen so that $u \neq y$ and $v \neq x$. Let $u_i := u$ if $u \in V(G_i^*)$ and $u_i := x$ if $u \notin V(G_i^*)$, and let $v_i := v$ if $v \in V(G_i^*)$ and $v_i := y$ if $v \notin V(G_i^*)$, for $i = 1, 2$. Since $|V(G_1^*)| < |V(G)|$ and $|V(G_2^*)| < |V(G)|$, we apply the theorem inductively to each (G_i^*, C_i^*) with u_i, v_i playing the roles of u, v, respectively, and obtain a closed 2-walk W_i in G_i^* visiting u_i and v_i exactly once and traversing every edge of C_i^* exactly once. Then $W := (W_1 - xy) \cup (W_2 - xy)$ gives the desired closed 2-walk in G.

Suppose that C is a triangle. Hence $V(C) = \{u, v, w\}$. Since G is 3-connected, we have $G - u$ is 2-connected and so its outer face is bounded by a cycle, say C'. Then it follows from Lemma 2(i) that $(G - u, C')$ is a circuit graph. Let $v' \neq w$ be the other neighbor of v in C'. Hence by Lemma 2(ii), $G-\{u,v\}$ is a plane chain of blocks $B_1, B_2, \ldots, B_k$ with $w \in V(B_1)$, $v' \in V(B_k)$, and neither w nor v' is a cut vertex of $G-\{u,v\}$. Let $v_i := V(B_i) \cap V(B_{i+1})$ for $i = 1, \ldots, k-1$, and let $v_0 := w$ and $v_k := v'$. Clearly, $\{v_0, v_k\} \cap \{v_i | 1 \leqslant i \leqslant k-1\} = \emptyset$. For each $1 \leqslant i \leqslant k$, if $V(B_i) = \{v_{i-1}, v_i\}$, then let $W_i := (v_{i-1}, v_{i-1}v_i, v_i, v_iv_{i-1}, v_{i-1})$; otherwise let C_i be the outer cycle of B_i, and hence by Lemma 2(i), (B_i, C_i) is a circuit graph, then by the induction hypothesis, there exists a closed 2-walk W_i in B_i such that W_i visits v_{i-1} and v_i exactly once and traverses every edge of C_i exactly once. Now let $W := (\bigcup_{i=1}^{k} W_i) + \{u, v, uv, vw, wu\}$. It is easy to see that W is the required closed 2-walk in G.

So we may further assume that C is not a triangle. Let v' (respectively, w') be the other neighbor of v (respectively, w) in C such that $v' \neq w$ (respectively, $w' \neq v$). We now consider $G^* := G/\{vw\}$. Let v^* denote the vertex of G^* resulting from the contraction of vw and let $C^* := (C - \{v,w\}) + \{v^*, v'v^*, v^*w'\}$. Suppose that (G^*, C^*) is a circuit graph. Then since $|V(G^*)| < |V(G)|$, inductively, there is a closed 2-walk W^* in G^* visiting u, v^* exactly once and traversing each edge of C^* exactly once. Now $W := (W^* - v^*) + \{v, w, v'v, vw, ww'\}$ gives the desired closed 2-walk in G.

Therefore, we may assume that (G^*, C^*) is not a circuit graph. Then $\{v, w\}$ is contained in a vertex cut of size 3 in G. Note that it is possible that $\{v, w\}$ is contained in many 3-cuts of G. Without loss of generality, suppose that $\{v, w, z\}$ is a 3-cut in G. Let $C' := \{v, w, z, vw, wz, zv\}$ and let G' be the graph contained in the closed disc bounded by C' such that $G' - \{wz, zv\} \subseteq G$. Then it is easy to check that (G', C') is a circuit graph. We may assume that z is chosen so that $|V(G')|$ is maximum. Then by planarity, for any vertex $z' \in V(G)$ such that $\{v, w, z'\}$ forms a 3-cut in G, we always have $z' \in V(G')$. Let X be the set of vertices in G' not in C' and let $G'' := (G^* - X) + v^*z$. In other words, $G'' = (G - X)/\{vw\} + v^*z$. Then by the choice of z, we have (G'', C^*) is also a circuit graph. By the induction hypothesis, there exists a closed 2-walk W^* in G'' visiting u, v^* exactly once and traversing each edge of C^* exactly once; and there is a closed 2-walk W' in G' visiting v, z exactly once and traversing each edge of C' exactly once. Now $W := ((W^* - v^*) \cup (W' - z)) + \{v'v, ww'\}$ gives the desired closed 2-walk in G. This completes the proof of Theorem 1. ∎

3 Concluding remarks

A *k-tree* is a spanning tree of maximum degree at most k. Barnette [1] showed that every 3-connected planar graph has a 3-tree. It is easy to see that if a graph G has a closed k-walk, then G has a $(k+1)$-tree. Moreover, a vertex visited twice in a closed 2-walk W corresponds to a vertex of degree 3 in the 3-tree corresponding to W. Gao and Richter [3] strengthened the result of Barnette by using Theorem 1. It was also proved in [3] that every 3-connected projective planar graph contains a closed 2-walk, and hence a 3-tree. Brunet et al. [2] showed that every 3-connected graph that embeds in the torus or the Klein bottle has a closed 2-walk, and hence a 3-tree. Recently, Nakamoto, Oda, and Ota [6] proved the following result which bounds the number of vertices of degree 3 of 3-trees in circuit graphs. (They also proved similar results for 3-connected graphs that embed in the projective plane, the torus, and the Klein bottle.)

Theorem 3 *Let (G, C) be a circuit graph. Then G contains a 3-tree with at most* $\max\left\{\frac{|V(G)|-7}{3}, 0\right\}$ *vertices of degree* 3*. Moreover, the estimation for the number of vertices of degree* 3 *is best possible.*

However, our proof as well as the proofs in [3,4] does not bound the number of vertices visited twice in closed 2-walks. In [6], the authors asked for a result for the number of vertices visited twice of closed 2-walks in circuit graphs or in 3-connected planar graphs, similarly to Theorem 3 for 3-trees.

Acknowledgements. The author is indebted to Professors Zhicheng Gao and Xingxing Yu for valuable guidance. He would also like to thank the anonymous referees for their helpful comments.

References

[1] D. W. Barnette, Trees in polyhedral graphs, *Canad. J. Math.* **18** (1966) 731–736.

[2] R. Brunet, M. N. Ellingham, Z. Gao, A. Metzlar, and R. B. Richter, Spanning planar subgraphs of graphs in the torus and Klein bottle, *J. Combin. Theory Ser. B* **65** (1995) 7–22.

[3] Z. Gao and R. B. Richter, 2-walks in circuit graphs, *J. Combin. Theory Ser. B* **62** (1994) 259–267.

[4] Z. Gao, R. B. Richter, and X. Yu, 2-walks in 3-connected planar graphs, *Australas. J. Combin.* **11** (1995) 117–122.

[5] B. Jackson and N. C. Wormald, k-walks of graphs, *Australas. J. Combin.* **2** (1990) 135–146.

[6] A. Nakamoto, Y. Oda, and K. Ota, 3-trees with few vertices of degree 3 in circuit graphs, *Discrete Math.* **309** (2009) 666–672.

[7] W. T. Tutte, A theorem on planar graphs, *Trans. Amer. Math. Soc.* **82** (1956) 99–116.