A Colorful Involution for the Generating Function for Signed Stirling Numbers of the First Kind

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Abstract

We show how the generating function for signed Stirling numbers of the first kind can be proved using the involution principle and a natural combinatorial interpretation based on cycle-colored permuations.

We seek an involution-based proof of the generating function for signed Stirling numbers of the first kind, written here as

$$\sum_k (-1)^k c(n,k)x^k = (-1)^n (x)(x-1)\cdots(x-n+1)$$

where $c(n,k)$ is the number of permutations of $[n]$ with k cycles. The standard proof uses [2] an algebraic manipulation of the generating function for unsigned Stirling numbers of the first kind.

Fix an unordered x-set A; for example a set of x letters or "colors". For $\pi \in S_n$, let K_π be the set of disjoint cycles of π (including any cycles of length one). Let $S_{n,A} = \{(\pi, f) : \pi \in S_n; f : K_\pi \to A\}$ be the set of *cycle-colored permutations of* $[n]$, where f is interpreted as a "coloring" of the cycles of π using the "colors" of A. (We follow [1] in using colored permutations). Further let $K_\pi(i)$ be the unique cycle of π containing i for any $1 \leqslant i \leqslant n$, and $\kappa(\pi) = |K_\pi|$ be the number of cycles of π. Note that

$$\sum_{(\pi,f)\in S_{n,A}} (-1)^{\kappa(\pi)} = \sum_{\pi \in S_n} (-1)^{\kappa(\pi)} x^{\kappa(\pi)} = \sum_k (-1)^k c(n,k)x^k$$

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For $(\pi, f) \in S_{n,A}$, let $R_{(\pi,f)} = \{(i,j) : 1 \leqslant i < j \leqslant n; f(K_\pi(i)) = f(K_\pi(j))\}$ be the set of pairs of distinct elements of $[n]$ in cycles–not necessarily distinct–colored the same way by f.

Define a map ϕ on $S_{n,A}$ as follows for $(\pi, f) \in S_{n,A}$: If $R_{(\pi,f)} = \emptyset$, let $\phi((\pi, f)) = (\pi, f)$. Otherwise, let $(i, j) \in R_{(\pi,f)}$ be minimal under the lexicographic ordering of $R_{(\pi,f)}$. Let $\tilde{\pi} = (i, j) \circ \pi$, the product of the transposition (i, j) and π in S_n. Note that, if $K_\pi(i) = K_\pi(j)$, left-multiplication by (i, j) splits the cycle $K_\pi(i)$ into two cycles; if $K_\pi(i) \neq K_\pi(j)$, left-multiplication by (i, j) concatenates the distinct cycles $K_\pi(i)$ and $K_\pi(j)$ into a single cycle. Since $f(K_\pi(i)) = f(K_\pi(j))$, define $\tilde{f} : K_{\tilde{\pi}} \to A$ consistently and uniquely by $\tilde{f}(K_{\tilde{\pi}}(p)) = f(K_\pi(p))$ for all $1 \leqslant p \leqslant n$. Let $\phi((\pi, f)) = (\tilde{\pi}, \tilde{f})$.

Note that $R_{(\pi,f)} = R_{\phi((\pi,f))}$ for all $(\pi, f) \in S_{n,A}$, and that therefore ϕ is involutive. Note further that, if $(\pi, f) \neq \phi((\pi, f)) = (\tilde{\pi}, \tilde{f})$, $\kappa(\pi) = \kappa(\tilde{\pi}) \pm 1$. Note finally that $(\pi, f) = \phi((\pi, f))$ if and only if $R_{(\pi,f)} = \emptyset$, or if and only if $\kappa(\pi) = n$ (so $\pi = e_n$, the identity permutation of S_n) and $f : K_\pi \to A$ is injective. Therefore $|Fix(\phi)| = (x)(x-1)\ldots(x-n+1)$. This suffices.

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References

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