

A generalization of generalized Paley graphs and new lower bounds for $R(3, q)$

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Abstract

Generalized Paley graphs are cyclic graphs constructed from quadratic or higher residues of finite fields. Using this type of cyclic graphs to study the lower bounds for classical Ramsey numbers, has high computing efficiency in both looking for parameter sets and computing clique numbers. We have found a new generalization of generalized Paley graphs, i.e. automorphism cyclic graphs, also having the same advantages. In this paper we study the properties of the parameter sets of automorphism cyclic graphs, and develop an algorithm to compute the order of the maximum independent set, based on which we get new lower bounds for 8 classical Ramsey numbers: $R(3, 22) \geq 131$, $R(3, 23) \geq 137$, $R(3, 25) \geq 154$, $R(3, 28) \geq 173$, $R(3, 29) \geq 184$, $R(3, 30) \geq 190$, $R(3, 31) \geq 199$, $R(3, 32) \geq 214$. Furthermore, we also get $R(5, 23) \geq 521$ based on $R(3, 22) \geq 131$. These nine results above improve their corresponding best known lower bounds.

1 Lower bounds for Ramsey numbers and generalized Paley graphs

Let $q_1, q_2, \dots, q_m \geq 3$ be given integers with $m \geq 2$. The classical Ramsey number $R(q_1, \dots, q_m)$ is the minimum positive integer n satisfying the following condition: For an arbitrary coloring of the complete graph K_n with m colors, there is always a complete subgraph K_{q_i} for some $1 \leq i \leq m$ such that every edge of K_{q_i} has the i -th color. The determination of Ramsey numbers is a very difficult problem in combinatorics [1]. Various methods have been designed to compute their bounds.

When Greenwood and Gleason determined the exact values of a few small Ramsey numbers in 1955 [4], they utilized the quadratic residues of finite fields to construct self complementary graphs and thus obtained the lower bounds $R(3, 3) \geq 6$ and $R(4, 4) \geq 18$. These graphs were Paley graphs, and the same method produced results such like $R(6, 6) \geq 102$ [6] and $R(8, 8) \geq 282$ [2].

In 1979, Clapham generalized the Paley graphs by a more general approach than the construction of self complementary graphs, and obtained the results $R(7, 7) \geq 114$ [3]. Another generalization of Paley graphs is the construction of non-self complementary graphs by using cubic residues of finite fields in [11, 12], and some new lower bounds such like $R(4, 4, 4) \geq 128$ [12] and $R(6, 6, 6) \geq 1070$ [11] were obtained.

In the past few years we have used the cyclic graphs of prime order to study the lower bounds for classical Ramsey numbers to the effect of improvements and generalizations of the method of Paley graphs. For example, in [7] we pointed out the isomorphism properties of the self complementary graphs could be used to enhance the computing efficiency for the computation of clique numbers, from which some new lower bounds such like $R(17, 17) \geq 8917$, $R(18, 18) \geq 11005$ and $R(19, 19) \geq 17885$ were obtained. In [7, 11] we constructed cyclic graphs by using cubic residues of finite fields and obtained new results such like $R(4, 12) \geq 128$, $R(6, 16) \geq 434$, $R(6, 17) \geq 548$. In [8, 13] we constructed cyclic graphs by using higher residues of finite fields and obtained new lower bounds such like $R(3, 3, 12) \geq 182$, $R(3, 3, 13) \geq 212$, $R(3, 28) \geq 164$.

As far as we know, all generalized Paley graphs considered so far have been restricted to finite fields. This is one limit of this method. However it is not easy to generalize the generalized Paley graphs to cyclic graphs of arbitrary order.

We have noticed that the parameter set of a generalized Paley graph of prime order is related to a cyclic group and automorphism is an important tool to deduce the isomorphism properties of generalized Paley graphs. In this paper we use this tool to study cyclic graphs of non-prime order and give a new generalization of generalized Paley graphs, which we will describe as automorphism cyclic graphs. The parameter sets of such graphs are also related to cyclic groups. The search for parameter sets and the computation of clique numbers by using this tool have higher efficiency.

2 The parameter sets of automorphism cyclic graphs

For basic concepts and terms refer to [1].

For a give integer $n \geq 8$, let $m = \lfloor \frac{n}{2} \rfloor$ be the integer part of $n/2$. For integers $s < t$, denote $[s, t] = \{s, s + 1, \dots, t\}$. Let $\mathbb{Z}_n = [-m, m]$ or $[1 - m, m]$ depending on n is odd or even.

The results of all arithmetic operations modulo n are understood to be in the set \mathbb{Z}_n unless mentioned specifically. The equality sign “=” for elements in \mathbb{Z}_n generally means “ $\equiv \pmod{n}$ ”.

Definition 1 For a partition $S = S_1 \cup S_2$ of the set $S = [1, m]$ let $A_i = \{x \mid |x| \in S_i\}$ for $i = 1, 2$. Let $V = \mathbb{Z}_n$ be the vertex set of the complete graph K_n and let $E = \{(x, y) \in$

$\mathbb{Z}_n \times \mathbb{Z}_n \setminus \{x = y\}$ be the edge set of K_n . Let

$$E_i = \{(x, y) \in E \mid x - y \in A_i\}$$

for $i = 1, 2$. An edge in E_i is said to be an A_i -colored edge. Denote by $G_n(A_i)$ the subgraph of K_n derived from the A_i -colored edges. The clique number of $G_n(A_i)$ is denoted by $[G_n(A_i)]$. This gives a 2-coloring of K_n in terms of the parameter set A_1 or A_2 (i.e., S_1 or S_2). We say that the cyclic graph $G_n(A_i)$ of order n is generated by the parameter set S_i .

By Ramsey's theorem, it is obvious that $R([G_n(A_1)] + 1, [G_n(A_2)] + 1) \geq n + 1$.

Lemma 1 Assume that $k \in \mathbb{Z}_n$ and $(k, n) = 1$. Then the transform $f : x \mapsto kx$ of \mathbb{Z}_n gives rise to isomorphisms of the graphs $G_n(A_i)$ for $i = 1, 2$.

In general, the transform f changes the parameter set S_i into S_i^* and $G_n(A_i)$ into another cyclic graph $G_n(A_i^*)$, where $S_i^* = \{|kx| \mid x \in S_i\}$, $A_i^* = \{kx \mid x \in A_i\}$. In particular, when $G_n(A_i) = G_n(A_i^*)$ we have

Definition 2 For a parameter set S_i , if there is some $k \in \mathbb{Z}_n$ such that $kA_i = A_i$, then the transform $f : x \mapsto kx$ is called an automorphism transform of $G_n(A_i)$, and $G_n(A_i)$ is called an automorphism cyclic graph. The set S_i is called an automorphism parameter set. The number k is called a special element for $G_n(A_i)$ which is also called a special element of S_i or simply of n .

Let P denote the set of all special elements for $G_n(A_i)$. Obviously $\pm 1 \in P$. They are called trivial special elements. If $P = \{1, -1\}$ then $G_n(A_i)$ is called a trivial automorphism cyclic graphs.

By convention, in what follows all automorphism cyclic graphs are nontrivial ones.

Lemma 2 The graph $G_n(A_i)$ is an automorphism cyclic graph if and only if there exists $k \in [2, m]$ with $(k, n) = 1$ such that $a \in A_i \Leftrightarrow ka \in A_i$ (i.e. $a \in S_i \Leftrightarrow |ka| \in S_i$).

Lemma 3 The set P under multiplication in \mathbb{Z}_n is a group.

Proof. Assume that $k, h \in P$. It follows from Lemma 2 that $kA_i = A_i$ and $hA_i = A_i$. Thus $khA_i = A_i$. Hence $kh \in P$, which means that P is closed under multiplication.

Obvious $1 \in P$. It remains to show that every $k \in P$ has an inverse. Since $(k, n) = 1$, there is some $j \in \mathbb{Z}_n$ such that $kj = 1$. Hence $jA_i = A_i$, which implies that j is the inverse of k . \square

Now that P is a group, it can be decomposed as a union of cyclic subgroups. For any $k \in P$, $\langle k \rangle$ denotes the cyclic subgroup of P generated by k .

For integer $n \geq 8$ and $k \in [2, m]$ with $(k, n) = 1$, let s be the smallest positive integer such that $|k^s| = 1$. Then k is called a special element of n with order s .

Lemma 4 *Let k be a special element of n with order s . For any $a \in A_i$ with $a \neq \pm 1$, let*

$$a(k) = \{|k^j a| \mid 1 \leq j \leq s\}. \quad (1)$$

Then $G_n(A_i)$ is an automorphism cyclic graph if and only if $a \in S_i \Leftrightarrow a(k) \subseteq S_i$ for $i = 1, 2$.

Proof. Necessity: Let $G_n(A_i)$ be an automorphism cyclic graph. By Lemma 2 there is $k \in [2, m]$ with $(k, n) = 1$ such that $a \in S_i \Leftrightarrow |ka| \in S_i$ for $i = 1, 2$. It follows that $|k^2 a| \in S_i, |k^3 a| \in S_i, \dots$. Thus $a(k) \subseteq S_i$.

Sufficiency: It follows from (1) that $|ka| \in a(k)$. Since $a \in S_i \Leftrightarrow a(k) \subseteq S_i$, we obtain $a \in S_i \Leftrightarrow |ka| \subseteq S_i$. Hence $G_n(A_i)$ is an automorphism cyclic graph. \square

It is easy to see that $1(k) = \{1, |k|, |k^2|, \dots, |k^{s-1}|\}$ is a cyclic group of order s . Since $a \in S_i \Leftrightarrow a(k) \subseteq S_i$, the automorphism parameter set S_i is related to cyclic groups. It is the union of several subsets in the form of (1). More precisely

$$S_i = a_1(k) \cup \dots \cup a_r(k) \quad (2)$$

of which the right hand side is simply denoted by $[a_1, \dots, a_r]$.

Example 1 *Let $n = 145$ and let*

$$S_1 = \{1, 12, 17, 20, 28, 41, 46, 50, 55, 57, 59, 65\}.$$

Then $P = (12) \cup (17) \cup (59) \cup (28) \cup (41)$. The elements 12, 17, 59 are special elements of order 2 and 28, 41, 46, 57 are special elements of order 4. Choose any one from these seven special elements and then the parameter set S_1 can be expressed in the form of (2). For instance, we may choose $k = 12$ of order 2. Then $1(12) = \{1, 12\}$ and

$$S_1 = 1(12) \cup 17(12) \cup 20(12) \cup 28(12) \cup 41(12) \cup 55(12) = [1, 17, 20, 28, 41, 55].$$

If we choose the special element $k = 46$ of order 4, then $1(46) = \{1, 46, 59, 41\}$ and $S_1 = [1, 17, 20, 28, 41, 55]$.

From Example 1 one can obtain the result $R(3, 25) \geq 146$. Although this lower bound is not good enough, but it illustrates an extreme case in which the set P is not a cyclic group and the expression (2) is not unique. We will soon see that the choice of a special element of highest order among the 7 different expressions of S_1 has the advantage of enhancing the computing efficiency of the clique number of $G_n(A_i)$.

3 The computation of the clique number of the automorphism cyclic graph $G_n(A_i)$.

In the following discussion we will mainly restrict ourselves to the case of subgroup (k) in P . For instance, we restrict to the case of one subgroup among (12), (17), (28), (41), (59) in example.

Definition 3 Let k be a special element of order s in P , and $H = \{\pm k^j | 1 \leq j \leq s\}$. Two element a and b in A_i are said to be equivalent if there is $k \in H$ such that $b = ka$. The equivalence class represented by a is denoted by $\langle a \rangle$.

This equivalence relation gives rise to a partition of A_i . In fact, an equivalence class is an orbit of A_i under the action of H .

Lemma 5 Let k be a special element of order s of the automorphism cyclic graph $G_n(A_i)$. For an arbitrary $a \in A_i$, let $r = |a(k)|$. If $r > 1$, then the equivalence class $\langle a \rangle = \{a, -a, ka, -ka, \dots, k^{s-1}a, -k^{s-1}a\}$ contains $2r$ elements. If $r = 1$ then $\langle a \rangle = \{a, -a\}$. In particular,

$$|\langle a \rangle| = \begin{cases} 2, & \text{if } a \neq -a \\ 1, & \text{if } a = -a \end{cases}$$

By the symmetry of the graph $G_n(A_i)$ the clique number of $G_n(A_i)$ is the maximal order of cliques containing the vertex 0. Thus it suffices to consider the cliques containing 0. By Definition 1 all nonzero vertices of such cliques are also elements of A_i . Thus we have

Lemma 6 Denote the subgraph of $G_n(A_i)$ derived from the vertex set A_i by $G_n[A_i]$ and its clique number by $[A_i]$. Then

$$[G_n(A_i)] = [A_i] + 1.$$

This amounts to saying that we only need to compute the clique number of $G_n[A_i]$ in order to find that of $G_n(A_i)$. To find $[A_i]$ we introduce a total order in A_i .

Definition 4 For $x \in S_i$, denote $d_i(x) = |\{y \in A_i | x - y \in A_i\}|$. The total order \prec in A_i is defined as follows:

(1) Every subset $\langle a \rangle = \{a, -a, ka, -ka, \dots, k^{s-1}a, -k^{s-1}a\}$ of A_i forms an interval under \prec with $a \prec -a \prec ka \prec -ka \prec \dots \prec k^{s-1}a \prec -k^{s-1}a$.

(2) Assume that $x \in \langle a \rangle, y \in \langle b \rangle$ and a is not equivalent to b . If $d_i(a) < d_i(b)$, then $x \prec y$; if $d_i(a) = d_i(b)$ and $a < b$ then $x \prec y$.

Remark 1 (1) In the subset $\langle a \rangle = \{a, -a, ka, -ka, \dots, k^{s-1}a, -k^{s-1}a\}$ of A_i there is at least one element belonging to S_i .

(2) For arbitrary $a, y \in A_i$, it follows from Lemma 2 that $a - y \in A_i \Leftrightarrow \pm k^j(a - y) \in A_i$, where $0 \leq j \leq s - 1$. Hence $d_i(a) = d_i(-a)$ and $d_i(a) = d_i(k^j a)$, so

$$d_i(a) = d_i(-a) = d_i(ka) = d_i(-ka) = \dots = d_i(k^{s-1}a) = d_i(-k^{s-1}a).$$

Remark 1 shows that the total order \prec is well-defined and (A_i, \prec) is an ordered set. When $x \prec y$ we say that x precedes y .

Lemma 7 Let M be a set of representatives of all equivalence classes in (A_i, \prec) . If $d_i(x) = 0$ for every $x \in M$, then $[A_i] = 1$.

Proof. Otherwise suppose $[A_i] \geq 2$. Then $[G_n(A_i)] \geq 3$. There would be a 3-clique $\{0, x, y\}$ in $G_n(A_i)$ in which $x, y \in A_i$ and $x - y \in A_i$. There are following two cases:

Case I) $x \in M$ or $y \in M$. Then $d_i(x) \geq 1$ or $d_i(y) \geq 1$, contradicting the hypothesis.

Case II) $-x \in M$ or $-y \in M$. Lemma 1 implies that $\{0, -x, -y\}$ is also a 3-clique of $G_n(A_i)$, thus $d_i(-x) \geq 1$, which leads to contradiction again. \square

Definition 5 A chain $x_0 \prec x_1 \prec \cdots \prec x_t$ of length $t \geq 1$ in (A_i, \prec) is called an A_i -colored chain of length t starting at x_0 . The length of a maximal chain starting at x_0 is denoted by $\ell_i(x_0)$. If there is no chain of positive length starting at x_0 , then denote $\ell_i(x_0) = 0$.

Lemma 8

$$[A_i] = 1 + \max\{\ell_i(a) | a \in M\}.$$

Proof. First assume that $[A_i] = 1$. Then for arbitrary $a \in M$ and $y \in A_i$, the element $y - a$ is not in A_i . By Definition 5 $\ell_i(a) = 0$. Hence $\max\{\ell_i(a) | a \in M\} = 0$ and the equality holds.

Then assume that $[A_i] \geq 2$. If $t = \max\{\ell_i(a) | a \in M\} \geq 1$, then there is an A_i -colored chain $x_0 \prec x_1 \prec \cdots \prec x_t$ of length t . According to Definition 5 the $t + 1$ elements of this chain form a clique of $G_n[A_i]$. Hence $[A_i] \geq t + 1 = 1 + \max\{\ell_i(a) | a \in M\}$. It remains to show that $[A_i] \leq 1 + \max\{\ell_i(a) | a \in M\}$.

Assume that $[A_i] = 1 + t \geq 2$. Then there exist some $t + 1$ -cliques in $G_n[A_i]$. These cliques form chains of length t in (A_i, \prec) . Among all these chains of length t choose $x_0 \prec \cdots \prec x_t$ whose starting vertex x_0 precedes the starting vertices of all other chains. We assert that $x_0 \in M$. Otherwise in the equivalent class represented by x_0 there is an element, say kx_0 , belonging to M . Lemma 1 implies that the transform $f : x \mapsto kx$ in \mathbb{Z}_n is an automorphism of $G_n(S_i)$, which is an automorphism of $G_n[A_i]$ as well. It carries the $t + 1$ -clique $\{x_0, x_1, \dots, x_t\}$ into another one $\{kx_0, kx_1, \dots, kx_t\}$. From Definition 5 we know that the elements kx_0, kx_1, \dots, kx_t form a chain of length t in (A_i, \prec) . Recall the total order in (A_i, \prec) as defined in Definition 4 and we know that this chain is in fact $kx_0 \prec kx_1 \prec \cdots \prec kx_t$, whose starting vertex is exactly kx_0 . This contradicts the hypothesis that x_0 precedes all other starting vertices of chains of length t . \square

Lemma 8 tells us that in order to find the clique number of $G_n[A_i]$ it suffices to find the longest chain starting from $a \in M$. Since most equivalence classes contain $2s$ elements, the amount of computation is reduced by a factor of $1/2s$.

Moreover, since Definition 4 follows the principle of “the vertices with small degrees have priority”, the efficiency of computation can be raised several more times. In total, the computation of clique numbers can be enhanced for several dozen times by using this technique.

Example 2 Choose $n = 35$ and a special element $k = 11$ of order 3. Let $S_1 = [1, 7] = \{1, 7, 11, 16\}$. By Lemma 4 S_i is an automorphism parameter set and $G_{35}(A_i)$ is an automorphism cyclic graph. It is easy to verify that the clique number of $G_{35}(A_1)$ is $[G_{35}(A_1)] = 2$.

In order to compute the clique number of $G_{35}(A_2)$, follow Lemma 5 to subdivide A_2 into 5 equivalence classes:

$$\langle 2 \rangle = \{2, -2, -13, 13, -3, 3\},$$

$$\langle 14 \rangle = \{14, -14\},$$

$$\langle 4 \rangle = \{4, -4, 9, -9, -6, 6\},$$

$$\langle 5 \rangle = \{5, -5, -15, 15, 10, -10\},$$

$$\langle 8 \rangle = \{8, -8, -17, 17, -12, 12\}.$$

Endow A_2 with a total order in terms of Definition 5 and the set of representatives of equivalence classes is $M = \{2, 14, 4, 5, 8\}$. Compute all A_2 -colored chains starting at $a \in M$ and we obtain an A_2 -colored chain $2 \prec -2 \prec 4 \prec -4 \prec -6 \prec 6 \prec 8$ of length 6, which is the longest. It follows from Lemma 8 that $[A_2] = 1 + \max\{\ell_2(a) | a \in M\} = 7$, and thus $[G_{35}(A_2)] = [A_2] + 1 = 8$. By Ramsey's theorem we have $R(3, 9) \geq 36$.

For brevity, in the following examples we only display n, k, S_1 and the new lower bounds $R(3, q)$.

Example 3 $n = 45$, special element $k = 19$ of order 2, $S_1 = [1, 3, 5] = \{1, 3, 5, 12, 19\}$. It is easy to verify that $G_{45}(A_i)$ is an automorphism cyclic graph and $R(3, 11) \geq 46$.

Example 4 $n = 72$, special element $k = 23$ of order 3, $S_1 = [1, 3, 12, 18, 33] = \{1, 3, 12, 18, 23, 25, 33\}$. It is easy to verify that $G_{45}(A_i)$ is an automorphism cyclic graph and $R(3, 15) \geq 73$.

Example 5 $n = 121$, special element $k = 3$ of order 5, $S_1 = [1, 17] = \{1, 3, 9, 17, 25, 27, 32, 40, 46, 51\}$. It is easy to verify that $G_{121}(A_i)$ is an automorphism cyclic graph and $R(3, 21) \geq 122$.

These four examples give the best know lower bounds for their corresponding Ramsey numbers (compare [10]), in which $R(3, 9) = 36$ is even the exact value. The computation of all these examples took less than one minute on a PC with CPU model AMD6400+, which shows the high efficiency of our method.

4 The main results

Theorem 1 $R(3, 22) \geq 131, R(3, 23) \geq 137, R(3, 25) \geq 154, R(3, 28) \geq 173, R(3, 29) \geq 184, R(3, 30) \geq 190, R(3, 31) \geq 199, R(3, 32) \geq 214$.

Proof. To save space, except for 1) where the first A_2 -colored chain of length $[A_2] - 1$ in the automorphism cyclic graph $G_n(A_2)$ is explicitly given, we only write down n, k, S_1 and the new lower bounds for $R(3, q)$.

1) Choose $n = 130$ and a special element $k = 57$ of order 2. Let $S_1 = [2, 12, 13, 20, 38, 65]$, i.e.,

$$S_1 = \{2, 12, 13, 16, 20, 30, 34, 38, 39, 44, 65\}.$$

By Lemma 4 S_i is an automorphism parameter set and $G_{130}(A_i)$ is an automorphism cyclic graph. It is easy to verify that the clique number of $G_{130}(A_1)$ is $[G_{130}(A_1)] = 2$. Compute all A_2 -colored chains starting at $a \in M$ and we obtain the first longest A_2 -colored chain of length 19 :

$3 \prec -3 \prec 6 \prec 48 \prec 55 \prec 29 \prec -29 \prec -61 \prec -25 \prec -8 \prec -51 \prec 51 \prec -58 \prec 58 \prec 54 \prec -54 \prec -50 \prec -26 \prec -4 \prec -18$. It follows from Lemma 8 that $[A_2] = 1 + \max\{\ell_2(a) | a \in M\} = 20$, and thus $[G_{130}(A_2)] = [A_2] + 1 = 21$. By Ramsey's theorem we have $R(3, 22) \geq 131$.

2) $n = 136$, special element $k = 67$ of order 2, $S_1 = [1, 5, 8, 20, 26, 32, 42, 44, 56] = \{1, 5, 8, 20, 26, 32, 42, 44, 56, 63, 67\}$. Computation shows that $R(3, 23) \geq 137$.

3) $n = 153$, special element $k = 50$ of order 3, $S_1 = [1, 19, 36, 48, 60, 63, 66, 75] = \{1, 19, 32, 36, 48, 50, 52, 60, 63, 66, 70, 75\}$. Computation shows that $R(3, 25) \geq 154$.

4) $n = 172$, special element $k = 85$ of order 2,

$$\begin{aligned} S_1 &= [1, 23, 34, 44, 54, 60, 70, 72, 76, 80, 82] \\ &= \{1, 23, 34, 44, 54, 60, 63, 70, 72, 76, 80, 82, 85\}. \end{aligned}$$

Computation shows that $R(3, 28) \geq 173$.

5) $n = 183$, special element $k = 62$ of order 2,

$$\begin{aligned} S_1 &= [1, 4, 13, 15, 27, 33, 43, 51, 72, 90] \\ &= \{1, 4, 13, 15, 27, 33, 43, 51, 62, 65, 72, 74, 79, 90\}. \end{aligned}$$

Computation shows that $R(3, 29) \geq 184$.

6) $n = 189$, special element $k = 62$ of order 3,

$$\begin{aligned} S_1 &= [1, 3, 10, 15, 24, 36, 42, 69, 81, 90] \\ &= \{1, 3, 10, 15, 24, 36, 42, 53, 62, 64, 69, 73, 81, 90\}. \end{aligned}$$

Computation shows that $R(3, 30) \geq 190$.

7) $n = 198$, special element $k = 65$ of order 3,

$$\begin{aligned} S_1 &= [2, 7, 21, 24, 27, 39, 69, 72, 81, 84] \\ &= \{2, 7, 21, 24, 27, 39, 59, 64, 68, 69, 72, 73, 81, 84\}. \end{aligned}$$

Computation shows that $R(3, 31) \geq 199$.

8) $n = 213$, special element $k = 70$ of order 2,

$$\begin{aligned} S_1 &= [1, 3, 10, 18, 24, 30, 44, 57, 65, 84, 93] \\ &= \{1, 3, 10, 18, 24, 30, 44, 57, 61, 65, 70, 77, 84, 93, 98\}. \end{aligned}$$

Computation shows that $R(3, 32) \geq 214$. \square

The computing time for these results on a PC with AMD6400+ CPU is about six hours. By comparing the corresponding results in [10] the 8 results in Theorem 1 improve the corresponding best known results $R(3, 22) \geq 125$, $R(3, 23) \geq 136$, $R(3, 25) \geq 153$, $R(3, 28) \geq 172$, $R(3, 29) \geq 182$, $R(3, 30) \geq 187$, $R(3, 31) \geq 198$, $R(3, 32) \geq 212$.

We would also like to mention that our result $R(3, 22) \geq 131$ and the formula $R(5, t) \geq 4R(3, t - 1) - 3$, ($t \geq 5$) in [14] imply $R(5, 23) \geq 521$, which is superior to the result $R(5, 23) \geq 509$ in [10].

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