Aztec Diamonds and Baxter Permutations

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Abstract

We present a proof of a conjecture about the relationship between Baxter permutations and pairs of alternating sign matrices that are produced from domino tilings of Aztec diamonds. It is shown that a tiling corresponds to a pair of ASMs that are both permutation matrices if and only if the larger permutation matrix corresponds to a Baxter permutation.

There has been a thriving literature on both pattern-avoiding permutations of various kinds [Baxter 1964, Dulucq and Guibert 1988] and tilings of regions using dominos or rhombuses as tiles [Elkies et al. 1992, Kuo 2004]. However, there have not as of yet been many links between these two areas of enumerative combinatorics. This paper gives one such link.

1   Introduction

Figure 1 shows two of the 64 different ways of tiling an order three Aztec diamond with dominoes. Aztec Diamonds are defined in "Alternating-Sign Matrices and Domino Tilings," by Elkies, Kuperberg, Larsen, and Propp [3]. Each tiling will be referred to as a TOAD, a Tiling Of an Aztec Diamond. In all of the illustrations, we have rotated the Aztec Diamonds and drawn spurs on each corner to simplify later calculations.

An Alternating Sign Matrix (ASM) is a square matrix whose entries are 0, 1, or -1, such that the entries of each row or column must add up to 1, and all nonzero entries must alternate in sign. Examples:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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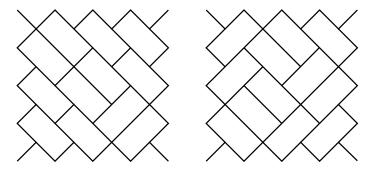
Figure 1: Two examples of tilings of Aztec diamonds.

Elkies, Kuperberg, Larsen, and Propp showed that there are $2^{n(n+1)/2}$ tilings of an order n Aztec diamond with dominoes [3]. As part of their proof, they used a relationship between TOADs and pairs of compatible ASMs. *Compatibility* of ASMs was defined by Robbins and Rumsey in terms of the ASMs themselves, without reference to domino tilings [5].

The interior vertices of an order n TOAD are arranged as a n-by-n square matrix inside an $(n+1)$-by-$(n+1)$ matrix. Each interior vertex is connected to two or more of its four nearest neighbors. For the smaller square assign a 0 to each vertex incident to exactly three edges, a 1 to each vertex incident to exactly two edges, and a -1 to each vertex incident to exactly four edges. For the larger matrix, do the same thing, but reverse the roles of 1's and -1's. The two above examples of ASMs come from the Aztec Diamond in Figure 2.
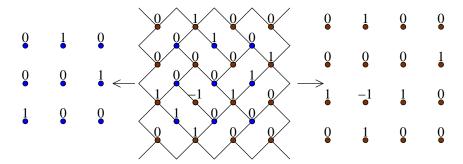

Figure 2: How to produce two ASMs from a TOAD.

A pair of ASMs of order n and order $n+1$ is said to be compatible if there is a TOAD that is related to the pair by the above algorithm. A pair of compatible ASMs uniquely determines a TOAD. It has been shown that an order $n+1$ ASM with k -1's is compatible with 2^k order n ASMs. Consequently an ASM with no -1's is only compatible with one smaller ASM. An ASM with no -1's is a permutation matrix.

The study of Baxter permutations began outside of the field of combinatorics [1], and even though they show up many places, it was surprising to find a connection between

domino tilings and Baxter permutations. We will give two definitions of Baxter permutations. The latter follows from the former and is described in terms of permutation matrices.

A permutation π of n objects is a Baxter permutation if for each $i \in \{1, 2, 3, \ldots, n-1\}$, there is a k_i such that $\pi(k_i)$ is between $\pi(i)$ and $\pi(i+1)$ (allowing k_i to be i but not $i+1$) and $m \leqslant i$ for every $\pi(m)$ between $\pi(i)$ and $\pi(k_i)$ (including $\pi(k_i)$), and $m > i+1$ for every $\pi(m)$ between $\pi(k_i)$ and $\pi(i+1)$ (not including $\pi(k_i)$) [2].

The easiest way to understand this definition is by looking at at a permutation matrix. Let B be a permutation matrix. By definition, each row and each column of B contains exactly one entry that is a 1: for each i, $B_{i,\pi(i)} = 1$ and the rest are zeros.

Take any two adjacent rows in B, the ith and $(i+1)$th. Then $B_{i,\pi(i)} = B_{i+1,\pi(i+1)} = 1$. Then, we look at the columns between the $\pi(i)$th column and the $\pi(i+1)$th column.

If the permutation is Baxter then there will be a vertical dividing line between two of these columns, so that every 1 on the $\pi(i+1)$ side of the line is in a row below the $i+1$th row, and every 1 on the $\pi(i)$ side of the line is in a row above the ith row. For example, let us test permutation 45123 between the second and third rows:

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Because there is a place to draw a vertical line between columns 1 and 5, this matrix passes the test for $i = 2$. Since it passes the test for each of the other rows as well, it is Baxter.

Now we are ready to present our theorem, which we will prove in Section 3:

Theorem 1 *An order $(n+1)$ ASM B without -1's is compatible with an order n ASM A without -1's if and only if B is a Baxter permutation matrix.*

As a consequence of this theorem, we know the number of order n TOADs with the property that vertices corresponding to the smaller ASM never have four edges incident to them, and vertices corresponding to the larger ASM never have two edges incident to them. That number is the same as the number of Baxter permutations of order $n + 1$, which is given by Chung, Graham, Hoggatt, and Kleiman [6]. The number of Baxter permutations on order n is

$$\sum_{r=0}^{n} \frac{\binom{n+2}{r} \binom{n+2}{r+1} \binom{n+2}{r+2}}{\binom{n+2}{1} \binom{n+2}{2}}.$$

2 Construction of the smaller ASM

To prove Theorem 1, we will present an algorithm that will produce the smaller ASM that is compatible with a given permutation matrix. The following lemma is a formal

description of the algorithm. We will use the relationship between ASMs and Aztec diamonds to prove the lemma by the end of section 2.

Lemma 1 *If B is an order $n+1$ permutation matrix, and A is the order n ASM that is compatible with B, then $A_{i,j} = 0$ if and only if*

$$(\forall\ k \leqslant i)\ B_{k,j} = B_{k,j+1} = 0\ or\ (\forall\ k \geqslant i+1)\ B_{k,j} = B_{k,j+1} = 0\ or$$
$$(\forall\ k \leqslant j)\ B_{i,k} = B_{i+1,k} = 0\ or\ (\forall\ k \geqslant j+1)\ B_{i,k} = B_{i+1,k} = 0.$$

The nonzero elements of A alternate between 1 and -1.

2.1 Using the Algorithm

We will use an example to show how to apply the algorithm. Suppose we are given the permutation 31425. First, let us define the matrix B which corresponds to our permutation.

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Now draw the smaller matrix A interspersed within B.

$$\begin{pmatrix} 0 & & 0 & & 1 & & 0 & & 0 \\ & A_{11} & & A_{12} & & A_{13} & & A_{14} & \\ 1 & & 0 & & 0 & & 0 & & 0 \\ & A_{21} & & A_{22} & & A_{23} & & A_{24} & \\ 0 & & 0 & & 0 & & 1 & & 0 \\ & A_{31} & & A_{32} & & A_{33} & & A_{34} & \\ 0 & & 1 & & 0 & & 0 & & 0 \\ & A_{41} & & A_{42} & & A_{43} & & A_{44} & \\ 0 & & 0 & & 0 & & 0 & & 1 \end{pmatrix}$$

From the description of the algorithm in Lemma 1, $A_{11} = 0$ because $B_{11} = B_{12} = 0$ and $A_{14} = 0$ because $B_{14} = B_{15} = 0$ and $A_{24} = 0$ because $B_{14} = B_{15} = B_{24} = B_{25} = 0$. Repeated applications of this method will give : $A_{11} = A_{14} = A_{13} = A_{24} = A_{31} = A_{33} = A_{34} = A_{41} = A_{42} = A_{43} = 0$. Graphically:

$$\begin{pmatrix} 0 & & 0 & & 1 & & 0 & & 0 \\ & 0 & & A_{12} & & 0 & & 0 & \\ 1 & & 0 & & 0 & & 0 & & 0 \\ & A_{21} & & A_{22} & & A_{23} & & 0 & \\ 0 & & 0 & & 0 & & 1 & & 0 \\ & 0 & & A_{32} & & 0 & & 0 & \\ 0 & & 1 & & 0 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & A_{44} & \\ 0 & & 0 & & 0 & & 0 & & 1 \end{pmatrix}$$

Lemma 1 tells us that the other elements—A_{12}, A_{21}, A_{22}, A_{23}, A_{32}, and A_{44}—are nonzero and alternate between 1 and -1. Thus,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The proof of Lemma 1 comes in two parts. First we show that entries in A that have a certain property are equal to 0. Second, we show that entries that lack this property are nonzero. The values of the nonzero entries are given by the definition of an ASM.

2.2 Zeros

First let us show by induction on j that for all $k \leqslant j$, $B_{i,k} = B_{i+1,k} = 0$ implies $A_{i,j} = 0$.

Let $j = 1$. Figure 3 shows a situation where $B_{i,1} = B_{i+1,1} = 0$. We need to show that $A_{i,1}$ must be zero. To do this, we look at the positioning of the dominoes in the TOAD that corresponds to A and B. The square located between the vertices of $B_{i,1}$ and $B_{i+1,1}$ must be part of a domino. Since it is on the left edge of the Aztec diamond, the domino must be pointing northeast or southeast. Figure 3 shows both cases. The figure also
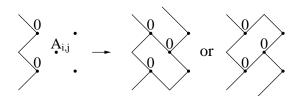

Figure 3: Determining $A_{i,1}$

shows the $B_{i,1}$ and $B_{i+1,1}$ vertices with three edges incident to them. Either configuration leads to $A_{i,1} = 0$.

For the induction step choose any $j \geqslant 1$ such that all of the $B_{i,k}$'s and $B_{i+1,k}$'s to the left of $A_{i,j}$ are zero. We assume not only that all of the $A_{k,j}$'s to the left of $A_{i,j}$ are zero, but also that the zig-zag shape from the far left edge of the Aztec diamond has propagated inward towards the jth column. Since $B_{i,j} = B_{i+1,j} = 0$, both of these vertices
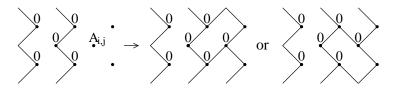

Figure 4: The induction step.

meets exactly three edges. As Figure 4 shows, once we know the shape of the tiling to the left of $A_{i,j}$, and that $B_{i,j} = B_{i+1,j} = 0$, we can determine the value of $A_{i,j}$ just as we determined the value of $A_{i,1}$.

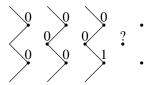Figure 5 shows that this induction can continue until we encounter a 1.

Figure 5: Zeros in A propagate until they hit a 1 in B.

In other words, 0's propagate from the edge of A until they hit a nonzero entry in the surrounding rows and columns of B. This proof can also be applied to the top, bottom, and right edges of the ASMs. Then we have proved half of the lemma.

2.3   Non-zero matrix entries

To finish proving Lemma 1, we will need to look at how the tiling patterns must propagate inside the matrix A. We will concentrate on those elements of A whose value is still undetermined. To represent those entries, we will use the symbol □.

We will now look at one individual row of A, and look at the two rows of B that are above and below it. Here is an example: the second and third rows from the permutation 34215.

$$
\begin{array}{ccccc}
0 & 0 & 0 & 1 & 0 \\
& 0 & \square & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{array}
$$

Notice first that all of the unknown entries on any row in A will be between the 1's in the two closest rows of B. As Figure 6 shows, a 0 will show up between the 1's only if there is a vertical path of zeros from either the top edge or the bottom edge. In this example, the zero is there because of a path from the bottom edge. We can say that a □ arises when the columns of B to the immediate left and right of the box have their 1's in opposite directions, and that if these two columns of B have their 1's in the same direction, there is a 0.

Our next example is the (non-Baxter) permutation 3142. We choose this because the second row of A will have three □s.

$$
\begin{array}{cccc}
0 & 0 & 1 & 0 \\
& 0 & \square & 0 \\
1 & 0 & 0 & 0 \\
& \square & \square & \square \\
0 & 0 & 0 & 1 \\
& 0 & \square & 0 \\
0 & 1 & 0 & 0
\end{array}
$$

```
                        1
                        ⋮
        0      0      0      1      0
            0      □      0      0
        0      1      0      0      0
```
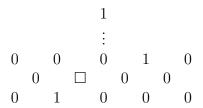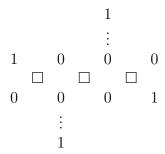
Figure 6: From the position of the □s in a row of A, we can reconstruct where 1's are located in B.

Looking only at the center row, we can reconstruct where the 1's are in B:

```
                        1
                        ⋮
        1      0      0      0
           □      □      □
        0      0      0      1
                        ⋮
                        1
```

It can now be seen that there is an odd number of boxes in each row, because each box always corresponds to a switching in the direction of the 1's, and the 1 on the left of the leftmost box must be in the direction opposite of the 1 in the column to the right of the rightmost box.

Consequently, □s can be in either of two configurations. The important thing is the direction of the 1 in the two adjacent rows and two adjacent columns of B.

There are only two types of configurations, up to various symmetries, as Figure 7 illustrates. Call the configuration represented by the one on the right *windmilled*, and the configuration represented by the one on the left *non-windmilled*.
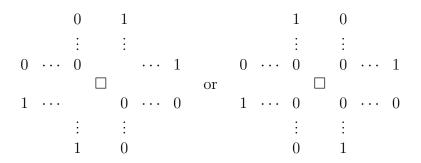
```
        0      1                    1      0
        ⋮      ⋮                    ⋮      ⋮
  0 ⋯ 0          ⋯ 1          0 ⋯ 0      0 ⋯ 1
        □              or           □
  1 ⋯         0 ⋯ 0          1 ⋯ 0      0 ⋯ 0
        ⋮      ⋮                    ⋮      ⋮
        1      0                    0      1
```

Figure 7: The non-windmilled (left) and windmilled (right) □ configurations.
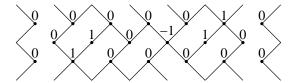
It can be seen that the □s along a row alternate between non-windmilled and windmilled, starting with a non-windmilled □. It is claimed that the left configuration will

lead to $\square = 1$ and the windmilled always lead to $\square = -1$. To prove this, we will have to look at the tiling patterns between the 1's in two consecutive rows of B.

2.4   Tilings and $\square$s

In this section, we will assume for a moment that we know the values of each entry in the ith row of A and the ith and $i+1$th rows of B. The reader might find it useful to generate example rows and try to draw the corresponding tilings. For example, these rows:

$$
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 & \\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

will produce this partial tiling:

Notice that in the interesting area—between the 1's in B—the tiling is uniquely determined by these three matrix rows. We will prove this, and use the machinery of the proof to show that $\square$s are always non-zero.
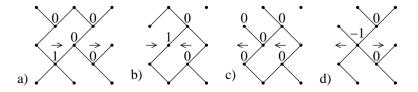

Figure 8: Here are examples of how the titling patterns propagate from left to right. The arrows serve to highlight the change that 1's and -1's make to the pattern. **(a)** and **(c)** show configurations that go unchanged when they encounter a 0 in A. **(b)** and **(d)** show how encountering a 1 or a -1 in A changes the configuration.

Figure 8 shows how the tiling patterns that propagate in from the left side of the TOAD change first when they hit a 1 in B and later when they hit 1's and -1's in A.

Notice that in between the two 1's in B, the tiling can take on one of two patterns: one will happen if the next nonzero element to the right in A is 1, and the other will happen if if the next nonzero element to the right is -1. Figure 9 shows the two patterns.

To help us keep this straight, the components in Figures 8 and 9 have an arrow that points to the right if the next nonzero element on the right in A is 1, and to the left if the next nonzero element on the left in A is a 1.
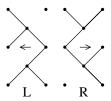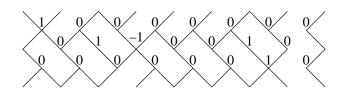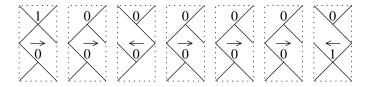
Figure 9: Once we are to the right of a 1 in B, the tiling can have two patterns, L and R. L corresponds to having the closest non-zero A entry on the left being 1, and R corresponds to being on the right of a 1 in A. If R and L fit together one way, there is a -1 in between. If they are fit together the other way, there is a 1 in between.

For example, look at this row from an ASM, with the arrows filled in.

$$\rightarrow \quad 0 \quad \rightarrow \quad 1 \quad \leftarrow \quad -1 \quad \rightarrow \quad 0 \quad \rightarrow \quad 0 \quad \rightarrow \quad 1 \quad \leftarrow \quad 0 \quad \leftarrow$$

Suppose that this row had been determined by a larger compatible permutation matrix. Here is what it might look like:

$$
\begin{array}{ccccccccc}
1 & & 0 & & 0 & & 0 & 0 & 0 & 0 & 0 \\
\rightarrow & 0 & \rightarrow & 1 & \leftarrow & -1 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 1 & \leftarrow & 0 & \leftarrow \\
0 & & 0 & & 0 & & 0 & 0 & 0 & 1 & 0 \\
\end{array}
$$

The last picture shows the tiling broken into components. Notice that the components between the 1's in B are all the same, up to a rotation. Each rotation corresponds to a nonzero element in the ASM A that fits between the two components.

This constitutes an algorithm to determine the shape of the local tiling from the two ASMs.
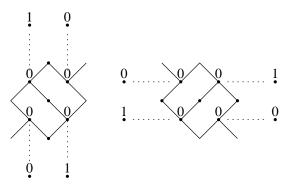
Now we have assembled all of the tools necessary to finish proving Lemma 1. We know that the □s alternate between two configurations: windmilled and non-windmilled; if we were to show that the windmilled boxes can not contain 1's or 0's, then the alternating-sign property of A would force the non-windmilled boxes to contain 1's, and the lemma would be proved.

2.5 Windmilled □s

Here is what a windmilled configuration looks like, up to reflection.

```
              1       0
              ⋮       ⋮
     0  ⋯  0     0  ⋯  1
              □
     1  ⋯  0     0  ⋯  0
              ⋮       ⋮
              0       1
```
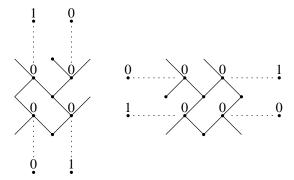
Let us assume that □ = 1, and find a contradiction. The contradiction arises when we try to draw the tiling components that correspond to the arrows that point towards the 1.

```
              1       0
              ⋮       ⋮
     0  ⋯  0 ↓ 0  ⋯  1
            → 1 ←
     1  ⋯  0 ↑ 0  ⋯  0
              ⋮       ⋮
              0       1
```

But the next figure shows that the tilings that result from looking at the column and the row don't match, so we have a contradiction.

Let us now assume that □ = 0, and find a contradiction. If we have a 0 in A, then the arrows on either side point in the same direction. This is true for the arrows in the row and in the column. In the next figure, we have chosen to make the arrows point up and to the left. Because of symmetry, we will get the same result for any equivalent configuration.

$$
\begin{array}{ccccccccc}
 & & & 1 & & 0 & & & \\
 & & & \vdots & & \vdots & & & \\
0 & \cdots & 0 & \uparrow & 0 & \cdots & 1 \\
 & & & \leftarrow & 0 & \leftarrow & & & \\
1 & \cdots & 0 & \uparrow & 0 & \cdots & 0 \\
 & & & \vdots & & \vdots & & & \\
 & & & 0 & & 1 & & &
\end{array}
$$

But the next figure shows that the tilings that result from looking at the column and the row don't match, so we have a contradiction.

This concludes our proof of Lemma 1.

3   Baxter Permutations and -1's

With an algorithm in hand that will produce the unique order n ASM that is compatible with an order $n + 1$ permutation matrix, we are prepared to show that the conditions in the permutation that produce -1's in the smaller ASM are identical to the conditions that would prevent that permutation from being Baxter.

First, assume that a permutation B is compatible with a smaller ASM A that contains no -1's. If A contains no -1's then each row of A must contain exactly one $\square$. Remember that the $\square$ represents the place where the direction that the 1's of B are located switches from up to down or down to up. Then for each row, the location of the $\square$ is the natural place to draw a dividing line between the columns to pass the Baxter test. Consequently, B is Baxter.

Assume that a permutation B is compatible with a smaller ASM A that contains at least one -1. The row of A that contains the -1 has at least three $\square$s, and consequently, has no natural place to draw the dividing line between columns. Let us show that this will keep B from being Baxter. From our earlier discussion, it is apparent that B contains a windmilled configuration of 1's.

$$B = \begin{pmatrix} & & & \vdots & \vdots & & & \\ & & & 1 & 0 & & & \\ & & & \vdots & \vdots & & & \\ \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 & \cdots \\ \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots \\ & & & \vdots & \vdots & & & \\ & & & 0 & 1 & & & \\ & & & \vdots & \vdots & & & \end{pmatrix}$$

It is easily checked that a permutation that looks like this is not Baxter, because there can be no vertical dividing line between columns that properly segregates the 1's into two groups.

This concludes the proof of Theorem 1.

Thanks

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