Game colouring directed graphs

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Abstract

In this paper, a colouring game and two versions of marking games (the weak and the strong) on digraphs are studied. We introduce the weak game chromatic number $\chi_{\rm wg}(D)$ and the weak game colouring number $\operatorname{wgcol}(D)$ of digraphs D. It is proved that if D is an oriented planar graph, then $\chi_{\rm wg}(D) \leqslant \operatorname{wgcol}(D) \leqslant 9$, and if D is an oriented outerplanar graph, then $\chi_{\rm wg}(D) \leqslant \operatorname{wgcol}(D) \leqslant 4$. Then we study the strong game colouring number $\operatorname{sgcol}(D)$ (which was first introduced by Andres as game colouring number) of digraphs D. It is proved that if D is an oriented planar graph, then $\operatorname{sgcol}(D) \leqslant 16$. The asymmetric versions of the colouring and marking games of digraphs are also studied. Upper and lower bounds of related parameters for various classes of digraphs are obtained.

1 Introduction

The game chromatic number of graphs was first introduced by Brams for planar graphs (published by Gardner [9]), and then reinvented for arbitrary graphs by Bodlaender in [4].

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Given an (undirected) graph G and a set X of colours. Two players, Alice and Bob, take turns (with Alice having the first move) to colour the vertices of G with colours from X. At the start of the game all vertices are uncoloured. A play by either player colours an uncoloured vertex with a colour from X so that no two adjacent vertices receive the same colour. Alice wins if eventually the whole graph is properly coloured. Bob wins if there comes a time when all the colours have been used on the neighbourhood of some uncoloured vertex u. The *game chromatic number* of G, denoted by $\chi_g(G)$, is the least k such that Alice has a winning strategy in the colouring game on G using a set of k colours.

The game chromatic number of graphs has been studied in many papers. Upper and lower bounds for the maximum game chromatic number of classes of graphs have been obtained in the literature [3-5,7-8,10,12-15,18-19,22-28]. One of the benchmark problems is the maximum game chromatic number of planar graphs. It was conjectured by Bodlaender [4] that the game chromatic number of planar graphs is bounded by a constant. Kierstead and Trotter [15] proved that the conjecture is true and the maximum game chromatic number of planar graphs is at most 33 and at least 8. The upper bound is improved in a sequence of papers [7, 25, 12], and the currently known upper bound for the maximum game chromatic number of planar graphs is 17 [27].

To extend game colouring of graphs to digraphs, we need to define what is a legal partial colouring. Nešetřil and Sopena [20] considered an extension of game colouring to oriented graphs, i.e., digraphs without opposite directed edges. In the non-game version, colouring of oriented graphs is defined as follows: A colouring of an oriented graph D is a homomorphism from D to a tournament T. The *oriented chromatic number* of the oriented graph D is the minimum order of a tournament T such that D admits a homomorphism to T. In other words, the oriented chromatic number of an oriented graph D is the minimum number of colours needed to colour the vertices of D so that no two adjacent vertices receive the same colour, and moreover if (u, v) and (u', v') are directed edges and $c(u) = c(v')$, then $c(v) \neq c(u')$.

Analogue to this definition, Nešetřil and Sopena [20] defined the colouring game of oriented graphs, which is the same as the colouring game of undirected graphs, except that a partial colouring c of an oriented graph D is legal if no two adjacent vertices receive the same colour, and moreover the following hold: (1) if (u, v) and (u', v') are directed edges of D with all the four (not necessarily distinct) vertices u, v, u', v' coloured, and $c(u) = c(v')$, then $c(v) \neq c(u')$. (2) If (u, v, w) is a directed path of length 2 in D, then $c(u) \neq c(w)$. The *oriented game chromatic number* of an oriented graph D is the least number of colours needed so that Alice has a winning strategy in the colouring game. Nešetřil and Sopena [20] showed that the oriented game chromatic number of a graph G is at most $\Delta^2(G)$. It is now known that there exist constant upper bounds on the oriented game chromatic number of oriented outerplanar graphs [20], oriented planar graphs [16], and oriented partial k-trees [17].

The definition above does not apply to digraphs that contain opposite directed edges. In particular, we view an undirected graph G as a symmetric digraph D in which each undirected edge xy of G is replaced by two opposite directed edges (x, y) and (y, x). In this sense, the oriented game chromatic number of oriented graphs is quite different from the game chromatic number of undirected graphs.

This paper introduces another game chromatic number of digraphs. We view an undirected graph as a symmetric digraph. If restricted to symmetric digraphs, the game chromatic number of digraphs introduced here coincides with the original game chromatic number of graphs.

A natural generalization of chromatic number to digraphs was introduced by Neumann -Lara in [21]. A (proper) colouring of a digraph D is a colouring of the vertices of D so that each colour class induces an acyclic digraph. If this definition is applied to symmetric digraphs (i.e., undirected graphs) G, then this is the same as a (proper) colouring of the undirected graph G, because when G is a symmetric digraph, then a colour class is acyclic if and only if it is an independent set.

Suppose D is a digraph and X is a set of colours. Alice and Bob take turns colour the vertices of D, with Alice having the first move (the case that Bob has the first move is similar, and the results in this paper apply to that case as well). A play by either player colours an uncoloured vertex with a colour from X so that no directed cycle is monochromatic. Alice wins if eventually the whole graph is properly coloured. Bob wins if for some uncoloured vertex u, the use of any colour on u will produce a monochromatic directed cycle. The *weak game chromatic number* of D, denoted $\chi_{\rm wg}(D)$, is the least k such that Alice has a winning strategy in this *weak colouring game* on D using a set of k colours.

In the definition above, the digraph D is allowed to have opposite edges. If G is a symmetric digraph, i.e., each directed edge has an opposite directed edge, then the weak colouring game on G and the weak game chromatic number of G defined coincide with the definition of the colouring game and the game chromatic number of undirected graph G (by viewing each pair of opposite directed edges as an undirected edge).

For a digraph D, the underlying graph of D is an undirected graph $\underline{D}$ with the same vertex set and in which xy is an edge of $\underline{D}$ if and only if at least one of (x, y) and (y, x) is a directed edge of D.

By viewing an undirected graph as a symmetric digraph, we can view a digraph D as a sub-digraph of its underlying graph $\underline{D}$. One might expect the weak game chromatic number of D to be bounded from above by the game chromatic number of $\underline{D}$. However, as has been already observed in the case of symmetric digraphs, the weak game chromatic number of digraphs is not monotone, i.e., a sub-digraph may have larger weak game chromatic number. For example, consider the complete bipartite graph $K_{n,n}$. Let M be a perfect matching of $K_{n,n}$. Let D be the digraph obtained from $K_{n,n}$ by assigning a direction to each edge of M, and replace each other edge by two opposite directed edges. So $\underline{D} = K_{n,n}$. It is known and easy to see that $\chi_{\rm wg}(K_{n,n}) = 3$ for $n \geqslant 2$. However, we can show that $\chi_{\rm wg}(D) = n$. It is easy to verify that $\chi_{\rm wg}(D) \leqslant n$. To see that $\chi_{\rm wg}(D) > n-1$, we observe that the following strategy of Bob is a winning strategy when there are at most $n-1$ colours: Whenever Alice colours a vertex v, Bob colours the 'partner' v' of v with the same colour, where two vertices v, v' are partners if vv' is an edge in M.

Nevertheless, for many natural classes of graphs, the best upper bound for their game chromatic number is obtained by considering the game colouring number (see definition in the next section) of these graphs. In Section 2, we shall see that for any digraph D, the weak game chromatic number of D is also bounded above by the game colouring number $\mathrm{gcol}(\underline{D})$ of its underlying graph $\underline{D}$. This implies that if D is a planar digraph, then $\chi_{\mathrm{wg}}(D) \leqslant 17$ [27]; if D is an outerplanar digraph, then $\chi_{\mathrm{wg}}(D) \leqslant 7$ [10]; if D is a digraph whose underlying graph is a partial k-tree, then $\chi_{\mathrm{wg}}(D) \leqslant 3k+2$ [26], etc. However, by simply considering the underlying graphs of digraphs D, the information on the orientation of edges are not used at all. In Section 2, analogue to the game colouring number of undirected graphs, we shall study a weak marking game on digraphs, and define a parameter, called the weak game colouring number for digraphs. We prove that the weak game chromatic number of a digraph is bounded above by its weak game colouring number. Then we prove that if D is an oriented graph, then its weak game colouring number is at most $\lceil \mathrm{gcol}(\underline{D})/2 \rceil$. As a consequence, we know that if D is an oriented planar graph, then its weak game chromatic number is most 9; if D is an oriented outerplanar graph, then its weak game chromatic number is at most 4; if D is an oriented partial k-tree, then its weak game chromatic number is at most $\lceil \frac{3k+2}{2} \rceil$.

In Section 3, we shall prove that the maximum weak game colouring number of oriented partial k-trees is equal to $\lceil \frac{3k+2}{2} \rceil$; the maximum weak game colouring number of oriented interval graphs of clique size $k+1$ is equal to $\lceil \frac{3k+1}{2} \rceil$; the maximum weak game colouring number of oriented outerplanar graphs is equal to 4.

Indeed, in Section 2, we shall also define the weak (a,b)-game colouring number (a,b)-$\mathrm{wgcol}(D)$ of digraphs D, and we shall prove that for an oriented graph D, (a,b)-$\mathrm{wgcol}(D) \leqslant \left\lceil \frac{(a,b)\text{-}\mathrm{gcol}(\underline{D})}{2} \right\rceil$. We shall show that this bound is sharp for many natural classes of graphs in Section 3.

In Section 4 and Section 5, we consider another type of game colouring number of digraphs, which was introduced earlier by Andres [1, 2]. For distinction, we call it the strong game colouring number of digraphs and denote the strong game colouring number of a digraph D by $\mathrm{sgcol}(D)$. This concept and its *asymmetric* variant were introduced by Andres in [1, 2]. Let $\overrightarrow{\mathcal{F}}$ be the class of oriented forests, it is shown in [2] that for $a \geqslant b$, (a,b)- $\mathrm{sgcol}(\overrightarrow{\mathcal{F}}) = b+2$; for $a < b$, (a,b)- $\mathrm{sgcol}(\overrightarrow{\mathcal{F}}) = \infty$. As a consequence, for the class $\overrightarrow{\mathcal{Q}}$ of oriented outerplanar graphs, if $a \geqslant b$, then (a,b)- $\mathrm{sgcol}(\overrightarrow{\mathcal{Q}}) \leqslant b+5$.

For a graph G, the *maximum average degree* of G is defined as $\mathrm{Mad}(G) = \max\{\frac{2|E(H)|}{|V(H)|} : H$ is a non-empty subgraph of $G\}$. The following fact is well-known (cf. [11], Theorem 4):

Fact 1.1 *Let G be a graph. Then G has an orientation such that the maximum outdegree of G is at most k if and only if* $\mathrm{Mad}(G) \leqslant 2k$.

In Section 4, we shall prove that for any undirected graph G, there is an orientation D of G such that $\mathrm{sgcol}(D) \geqslant \mathrm{gcol}(G) - \lceil \mathrm{Mad}(G)/2 \rceil$. In particular, for any planar graph G, there is an orientation D of G such that $\mathrm{sgcol}(D) \geqslant \mathrm{gcol}(G) - 3$. The best known upper bound for the game colouring number of planar graphs is 17 [27]. For oriented planar graphs D, we shall prove that the strong game colouring number of D is at most 16.

In Section 5, we shall study the strong (a, b)-game colouring number (a, b)-sgcol(D) of digraphs D, which was first introduced by Andres in [2]. By extending the Harmonious Strategy to the (a, b)-strong marking game of digraphs, we show that if D is an oriented graph with $\mathrm{Mad}(D) \leqslant 2k$ and $a \geqslant k$, then $(a, 1)$-sgcol$(D) \leqslant k + 2$.

2 Marking games on graphs and weak marking games on digraphs

The *marking game* on graphs was first formally introduced in [25] as a tool in the study of game chromatic number of graphs. The game is also played by two players: Alice and Bob, with Alice playing first. At the start of the game all vertices are unmarked. A play by either player marks an unmarked vertex. The game ends when all the vertices have been marked. Together the players create a linear order L on the vertices of G defined by $u <_L v$ if u is marked before v. For $v \in V(G)$, the neighbourhood of a vertex v is denoted by $N_G(v)$. Let $V_L^+(v) = \{u : u <_L v\}$ and $V_L^-(v) = \{u : v <_L u\}$. Let $N_{G,L}^+(v) = N_G(v) \cap V_L^+(v)$ and $N_{G,L}^+[v] = N_{G,L}^+(v) \cup \{v\}$. The score of the game is s, where $s = \max_{v \in V(G)} |N_{G,L}^+[v]|$. Alice's goal is to minimize the score, while Bob's goal is to maximize the score. The *game colouring number* of G, denoted by $\mathrm{gcol}(G)$, is the least s such that Alice has a strategy that results in a score of at most s.

In the marking game above and the colouring game discussed in Section 1, each move by any player marks or colours exactly one vertex. Given positive integers a, b, the (a, b)-marking game is the same as the marking game, except that in each of Alice's moves, she marks a unmarked vertices, and in each of Bob's moves, he marks b unmarked vertices (in the last move, if there are not enough unmarked vertices, then the player just marks all the remaining unmarked vertices). The (a, b)-colouring game is defined similarly.

The (a, b)*-game colouring number* of a graph G, denoted by (a, b)-$\mathrm{gcol}(G)$, is the least s such that Alice has a strategy that results in a score of at most s, in the (a, b)-marking game of G. The (a, b)*-game chromatic number* of a graph G, denoted by (a, b)-$\chi_{\mathrm{g}}(G)$, is defined similarly through the (a, b)-colouring game of G.

So the original marking game and colouring game is just a $(1, 1)$-marking game and a $(1, 1)$-colouring game. The (a, b)-marking games and the (a, b)-colouring games are called *asymmetric marking games* and *asymmetric colouring games*. Asymmetric marking games and colouring games of undirected graphs were studied in [13, 14, 19, 23, 24]. This concept naturally extends to asymmetric weak colouring games of digraphs. Given a digraph D, the *weak* (a, b)*-game chromatic number* (a, b)-$\chi_{\mathrm{wg}}(D)$ of D is the least number of colours needed so that Alice has a winning strategy in the weak (a, b)-colouring game of D.

It is easy to see that for any graph G, (a, b)-$\chi_{\mathrm{g}}(G) \leqslant (a, b)$-$\mathrm{gcol}(G)$. This upper bound applies to any digraph D.

Lemma 2.1 *If D is a digraph, then (a, b)-$\chi_{\mathrm{wg}}(D) \leqslant (a, b)$-$\mathrm{gcol}(\underline{D})$.*

Proof Assume Alice and Bob play the (a, b)-colouring game on D with (a, b)-$\mathrm{gcol}(\underline{D})$ colours. Alice uses her strategy in the marking game of $\underline{D}$ to choose the next vertex to be coloured, and colour the chosen colour with any legal colour. To prove that this is a winning strategy, it suffices to show that at any moment, any uncoloured vertex has a legal colour. By the definition of (a, b)-game colouring number, any uncoloured vertex v has at most (a, b)-gcol$(\underline{D}) - 1$ coloured neighbours. It is obvious that a colour not used by any neighbour of v is a legal colour for v. So v has a legal colour, and hence this is a winning strategy for Alice. ■

This strategy does not take the orientation of the edges of D into consideration. For a colour to be legal to an uncoloured vertex v, it is not necessary that the colour be not used by any of its neighbours, because two adjacent vertices are allowed to be coloured the same colour. We just need to avoid producing a monochromatic directed cycle. So if a colour α is not used by any in-neighbour of v, or not used by any out-neighbour of v, then α is a legal colour for v. This motivates the definition of the following game colouring number of digraphs.

The weak (a, b)-marking game on a digraph D is defined in the same way the (a, b)-marking game on its underlying graph $\underline{D}$. Except that the score is defined differently. Suppose a linear ordering L of the vertices of D is determined. For a vertex v, let $N_D^+(v)$ denote the set of all out-neighbours of v in D, i.e., $N_D^+(v) = \{u \in V : u \leftarrow v\}$; let $N_D^-(v)$ denote the set of all in-neighbours of v in D, i.e., $N_D^-(v) = \{u \in V : u \rightarrow v\}$. Let $N_{D,L}^{+,+}(v) = N_D^+(v) \cap V_L^+(v)$ and $N_{D,L}^{-,+}(v) = N_D^-(v) \cap V_L^+(v)$. Let $N_{D,L}^{+,+}[v] = N_{D,L}^{+,+}(v) \cup \{v\}$ and $N_{D,L}^{-,+}[v] = N_{D,L}^{-,+}(v) \cup \{v\}$. The score $s(v)$ of a vertex v is defined as

$$s(v) = \min\{\left|N_{D,L}^{+,+}[v]\right|, \left|N_{D,L}^{-,+}[v]\right|\}.$$

The score of the game is

$$s = \max_{v \in V(G)} s(v).$$

The *weak (a, b)-game colouring number* $\mathrm{wgcol}(D)$ of D is the least s such that Alice has a strategy that results in a score of at most s. Suppose v is an uncoloured vertex of D. Then any colour α not used by its out-neighbours or not used by its in-neighbours is a legal colour for v. So the proof of Lemma 2.1 proves the following lemma.

Lemma 2.2 *If D is a digraph, then (a, b)-$\chi_{\mathrm{wg}}(D) \leqslant (a, b)$-wgcol$(D)$.*

If D is a symmetric digraph, then the definition of wgcol(D) coincides with the definition of gcol$(\underline{D})$. However, if D is an oriented graph, then we have the following upper bound for (a, b)-wgcol(D) in terms of (a, b)-gcol$(\underline{D})$.

Lemma 2.3 *If D is an oriented graph, then*

$$(a, b)\text{-wgcol}(D) \leqslant \left\lceil \frac{(a, b)\text{-gcol}(\underline{D})}{2} \right\rceil.$$

Proof Assume (a, b)-gcol$(\underline{D}) = s$. Then Alice has a strategy for the (a, b)-marking game on $\underline{D}$ so that at any moment of the game, any unmarked vertex v has at most $s-1$ marked neighbours. Alice uses the same strategy for playing the weak marking game on D. Since D is an oriented graph, $D^+(v) \cap D^-(v) = \emptyset$. So at any moment of the game, at least one of the sets $D^+(v), D^-(v)$ contains at most $\lfloor (s-1)/2 \rfloor$ marked vertices. Therefore the weak (a,b)-game colouring number of D is at most $\lfloor (s-1)/2 \rfloor + 1 = \lceil s/2 \rceil$. ■

Let $\mathcal{I}_k$ be the class of interval graphs with clique number $k+1$, $\mathcal{Q}$ be the class of outerplanar graphs, $\mathcal{PK}_k$ be the family of partial k-trees, $\mathcal{P}$ be the class of planar graphs. For a class $\mathcal{K}$ of graphs, let $\overrightarrow{\mathcal{K}}$ be the set of all orientations of graphs in $\mathcal{K}$. Denote by $\mathrm{gcol}(\mathcal{K})$ the maximum game colouring number of graphs in $\mathcal{K}$; by $\mathrm{wgcol}(\overrightarrow{\mathcal{K}})$ the maximum weak game colouring number of digraphs in $\overrightarrow{\mathcal{K}}$.

Since planar graphs have game colouring number at most 17 [27], outerplanar graphs have game colouring number at most 7 [10], partial k-trees have game colouring number at most $3k+2$ [26], interval graphs with clique number $k+1$ have game colouring number at most $3k+1$ [8], we have the following corollary.

Corollary 2.4 *The following upper bounds on weak game colouring numbers hold:*

$$\begin{aligned} \mathrm{wgcol}(\overrightarrow{\mathcal{I}_k}) &\leqslant \lceil (3k+1)/2 \rceil, \\ \mathrm{wgcol}(\overrightarrow{\mathcal{Q}}) &\leqslant 4, \\ \mathrm{wgcol}(\overrightarrow{\mathcal{PK}_k}) &\leqslant \lceil (3k+2)/2 \rceil, \\ \mathrm{wgcol}(\overrightarrow{\mathcal{P}}) &\leqslant 9. \end{aligned}$$

Corollary 2.5 *If D is an orientation of G and $\mathrm{Mad}(G) \leqslant 2k$ and $a \geqslant k$, then $(a,1)$-$\mathrm{wgcol}\,(D) \leqslant k+1$.*

Proof By Fact 1.1, if $\mathrm{Mad}(G) \leqslant 2k$, then G has an orientation $\vec{G}$ with maximum outdegree at most k. It was proved in [19] that for a graph G, if $a \geqslant k$, then $(a,1)$-$\mathrm{gcol}\,(G) \leqslant 2k+2$. Thus $(a,1)$-$\mathrm{wgcol}\,(D) \leqslant k+1$. ■

3 Lower bounds for the weak game colouring number

Intuitively, if D is an oriented graph, then D has only half of the directed edges of its underlying graph $\underline{D}$ (by viewing $\underline{D}$ as a symmetric digraph). So it seems reasonable that $\mathrm{wgcol}(D)$ is about half of $\mathrm{gcol}(\underline{D})$. However, for a particular digraph D, it is possible that $\mathrm{wgcol}(D)$ is much less than half of $\mathrm{gcol}(\underline{D})$. For example, if D is an orientation of $K_{n,n}$ with all vertices of $K_{n,n}$ being either a source or a sink, then $\mathrm{wgcol}(D) = 1$ and $\mathrm{gcol}(\underline{D}) = n+1$. Nevertheless, we have the following conjecture:

Conjecture 3.1 *For any undirected graph G, there is an orientation D of G such that*

$$\mathrm{wgcol}(D) = \left\lceil \frac{\mathrm{gcol}(G)}{2} \right\rceil .$$

In particular, for a class $\mathcal{C}$ of undirected graphs,

$$\mathrm{wgcol}(\overrightarrow{\mathcal{C}}) = \left\lceil \frac{\mathrm{gcol}(\mathcal{C})}{2} \right\rceil.$$

The following result shows that this conjecture is true for partial k-trees, interval graphs and outerplanar graphs.

Lemma 3.2 *The weak game colouring numbers of oriented interval graphs, outerplanar graphs and partial k-trees (with $k \geqslant 2$) are as follows:*

$$\begin{aligned}
\mathrm{wgcol}(\overrightarrow{\mathcal{I}_k}) &= \lceil \mathrm{gcol}(\mathcal{I}_k)/2 \rceil = \lceil (3k+1)/2 \rceil, \\
\mathrm{wgcol}(\overrightarrow{\mathcal{Q}}) &= \lceil \mathrm{gcol}(\mathcal{Q})/2 \rceil = 4, \\
\mathrm{wgcol}(\overrightarrow{\mathcal{PK}_k}) &= \lceil \mathrm{gcol}(\mathcal{PK}_k)/2 \rceil = \lceil (3k+2)/2 \rceil.
\end{aligned}$$

Proof By using Corollary 2.4, it suffices to show that $\mathrm{wgcol}(\overrightarrow{\mathcal{I}_k}) \geqslant \lceil (3k+1)/2 \rceil$, $\mathrm{wgcol}(\overrightarrow{\mathcal{Q}}) \geqslant 4$ and $\mathrm{wgcol}(\overrightarrow{\mathcal{PK}_k}) \geqslant \lceil (3k+2)/2 \rceil$. The proof of $\mathrm{wgcol}(\overrightarrow{\mathcal{I}_k}) \geqslant \lceil (3k+1)/2 \rceil$ and $\mathrm{wgcol}(\overrightarrow{\mathcal{Q}}) \geqslant 4$ is provided next in Example 3.5 and Example 3.7.

Here we shall only consider the case of partial k-trees with $k \geqslant 2$. For any $k \geqslant 2$, in [22], a partial k-tree G with $\mathrm{gcol}(G) = 3k+2$ is constructed. The partial k-tree constructed in [22] is as follows: Let P_n^k be the kth power of the path P_n, i.e., P_n^k has vertex set $a_1, a_2, \ldots, a_n$, in which $a_i \sim a_j$ if and only if $|i-j| \leqslant k$. For $k+1 \leqslant i \leqslant n$ which is not a multiple of k, add a vertex b_i and connect b_i to each of $a_i, a_{i-1}, \ldots, a_{i-k+1}$. For $1 \leqslant i < j \leqslant i+k \leqslant n$ and $m = 1, 2$, add a vertex $c_{i,j,m}$ and connect $c_{i,j,m}$ to a_i, a_j. The resulting graph G is a partial k-tree and it is shown in [22] that $\mathrm{gcol}(G) = 3k+2$.

The vertices a_i are called A-vertices, b_i are called B-vertices and $c_{i,j,m}$ are called C-vertices. Let $A' = \{a_{k+1}, a_{k+2}, \ldots, a_{n-k}\}$. Each vertex $a_i \in A'$ has $2k$ A-neighbours (i.e., neighbours that are A-vertices) and $k-1$ B-neighbours and $4k$ C-neighbours. Now we orient the edges of G (the resulting oriented graph is D) so that for $a_j \in A'$, we have $d_A^+(a_j) = d_A^-(a_j) = k$, $d_B^+(a_j) \geqslant \lfloor \frac{k-1}{2} \rfloor$, $d_B^-(a_j) \geqslant \lfloor \frac{k-1}{2} \rfloor$ (this can be easily done). For edges $c_{i,j,m}a_i$ and $c_{i,j,m}a_j$ in $E(G)$, orient the edges $c_{i,j,1}a_i$ and $c_{i,j,1}a_j$ from $c_{i,j,1}$ to a_i, a_j in D, orient the edges $c_{i,j,2}a_i$ and $c_{i,j,2}a_j$ from a_i, a_j to $c_{i,j,2}$ in D. This will make $d_C^+(a_j) = d_C^-(a_j) = 2k$ for $a_j \in A'$.

If n is large enough, by using the same strategy as in [22], Bob can make sure that at a certain step, a vertex a_j in A' is not marked yet, but all its A-neighbours and B-neighbours are marked; moreover, at least for some i, two of its neighbours in C ($c_{i,j,1}$ and $c_{i,j,2}$) are marked. Then the unmarked vertex a_j will have at least $k + \lfloor \frac{k-1}{2} \rfloor + 1$ marked in-neighbours and out-neighbours. Therefore the score of a_j is $s(a_j) \geqslant 1 + \lfloor (3k+1)/2 \rfloor = \lceil (3k+2)/2 \rceil$. ■

The following technical lemma extends Lemma 16 in [19] to the weak marking games of digraphs, we use it to prove our examples for interval graphs and outerplanar graphs. For a digraph $D = (V, E)$, a vertex $v \in V$ and a set $X \subseteq V$, let $d^+(v) = |N^+(v)|$, $d^-(v) = |N^-(v)|$, $d_X(v) = |N(v) \cap X|$, $d_X^+(v) = |N^+(v) \cap X|$, $d_X^-(v) = |N^-(v) \cap X|$. Let $\mathrm{dist}_{\underline{D}}(x, y)$ denote the distance between x and y in the underlying graph $\underline{D}$.

Lemma 3.3 *Let a and d be positive integers and let $D = (V, E)$ be a digraph whose vertices are partitioned into sets L and S. Let $B \subseteq L$ and $T \subseteq S$. If*

1. $d^+(v) \geqslant d$ *and* $d^-(v) \geqslant d$ *for all* $v \in L - B$,
2. $\mathrm{dist}_{\underline{D}}(x, y) > a + 1$ *for all distinct* $x, y \in T$ *and*
3. $a(|B| + |S - T| + 1) < |L - B|$

then $(a, 1)$-wgcol$(D) \geqslant d + 1$.

Proof The proof is analogous to Lemma 16 in [19]. We shall provide Bob with a strategy by which he can obtain a score of at least $d + 1$ in the weak $(a, 1)$-marking game. Bob will begin by making sure that all the vertices in $B \cup (S - T)$ are marked by the end of his first $|B \cup (S - T)|$ plays. Alice can mark at most $a(|B| + |S - T|)$ vertices in $|L - B|$ before Bob accomplishes this task. So by (3) there are still more than a unmarked vertices in $L - B$. Bob's next task is to mark as many of the vertices in T as possible. If all the vertices in S are eventually marked before some vertex in $L - B$ then the last unmarked vertex in $L - B$ will have at least d marked in-neighbours and d marked out-neighbours by (1) and so the score will be at least $d + 1$. So we may assume that for the rest of the game Bob marks vertices in T. Since Alice can only mark a vertices at a time, Bob will eventually have a turn on which $P = (L - B) \cap U$ satisfies $0 < |P| \leqslant a$, where U denotes the set of unmarked vertices. Let Q be a connected component of P. Then by (2) there is at most one neighbour of Q in T, since otherwise T would have distinct vertices whose distance was at most $a + 1$ in $\underline{D}$. Let x be an unmarked element of T and if possible let x be a neighbour of Q. Bob will mark x. Then when the last element of Q is marked it will have at least d marked in-neighbours and d marked out-neighbours. So the score will be at least $d + 1$. ■

We shall apply Lemma 3.3 repeatedly to obtain some sharp results for the classes of orientations of chordal, interval, and outerplanar graphs.

First we consider orientations of chordal graphs and in particular interval graphs. Example 3.4 is analogous to Example 17 in [19]. For a positive integer t, $[t]$ denotes the set $\{1, 2, \ldots, t\}$. Let $I_{k,t}$ be the interval graph determined by the set of intervals $L_{k,t} = \{[i, i + k] : i \in [t]\}$. We identify $V(I_{k,t})$ with $L_{k,t}$ in the natural way and set $v_i = [i, i + k]$. Then, for example, $\{v_i, ..., v_{i+k}\}$ is a $(k + 1)$-clique in $I_{k,t}$. Clearly $I_{k,t}$ has t vertices and $\omega(I_{k,t}) = k + 1$.

In $\overrightarrow{I_{k,t}}$, we orient the edges in $I_{k,t}$ in the following way: suppose $v_i v_j \in E(I_{k,t})$, where $v_i = [i, i + k]$, $v_j = [j, j + k]$. If $i < j$, then orient the edge $v_i v_j$ from v_j to v_i in $\overrightarrow{I_{k,t}}$, i.e., $(v_j, v_i) \in E\left(\overrightarrow{I_{k,t}}\right)$. Then all the vertices of $\overrightarrow{I_{k,t}}$ have outdegree and indegree k except the $2k$ vertices in the *border* set $B_{k,t} = \{v_i : i \in [k] \cup ([t] - [t - k])\}$.

Example 3.4 *For all positive integers k and a there exists an oriented interval graph D with $\omega(D) = k + 1$ and $(a, 1)$-wgcol$(D) \geqslant k + 1$.*

Proof Let $t = (a+1)(2k+1)$, $D = \overrightarrow{I_{k,t}}$, $L = L_{k,t}$, $B = B_{k,t}$, $S = T = \emptyset$, $d = k$. Then

$$\begin{aligned} a(|B| + |S - T| + 1) &= 2ak + a \\ &= t - 2k - 1 \\ &< |L - B|. \end{aligned}$$

So we are done by Lemma 3.3. ■

Example 3.5 is analogous to Example 4.3 in [24]. Let $I_{k,t}^{+}$ be the interval graph determined by the set of intervals $W_{k,t} = L_{k,t} \cup S_{k,t}$, where $S_{k,t} = \left\{\left[i + \frac{1}{2}, i + \frac{1}{2}\right] : k < i < t\right\}$. We identify $V\left(I_{k,t}^{+}\right)$ with $W_{k,t}$ in the natural way and set $x_i = \left[i + \frac{1}{2}, i + \frac{1}{2}\right]$. Notice that $\operatorname{dist}(x_i, x_j) = \left\lfloor \frac{|i-j|}{k} \right\rfloor + 2$.

In $\overrightarrow{I_{k,t}^{+}}$, we oriented the edges in $I_{k,t}^{+}$ in the following way: for edges $v_i v_j \in E(I_{k,t})$, orient them in the same way as $\overrightarrow{I_{k,t}}$. For $v_i x_j \in E\left(I_{k,t}^{+}\right)$, where $v_i = [i, i+k]$, $x_j = \left[j + \frac{1}{2}, j + \frac{1}{2}\right]$, if j is odd, then orient the edge $v_i x_j$ from x_j to v_i in $\overrightarrow{I_{k,t}^{+}}$, i.e., $(x_j, v_i) \in E\left(\overrightarrow{I_{k,t}^{+}}\right)$; otherwise ($j$ is even), orient the edge $v_i x_j$ from v_i to x_j in $\overrightarrow{I_{k,t}^{+}}$, i.e., $(v_i, x_j) \in E\left(\overrightarrow{I_{k,t}^{+}}\right)$.

Example 3.5 *For every positive integer $1 \leqslant a < k$ there exists an oriented interval graph D such that $\omega(D) = k + 1$ and $(a,1)$-wgcol$(D) \geqslant k + \lfloor \frac{k}{2a} \rfloor + 1$.*

Proof Let $r = 3k + 1$, $s = \lfloor \frac{rk^2 + k}{a} \rfloor - \lfloor \frac{k}{a} \rfloor$, $t = rk^2 + 2k$, $L = L_{k,t}$, $B = B_{k,t}$, $S = \left\{x_{ia+1} : i \in \left[\lfloor \frac{rk^2+k}{a} \rfloor\right] - \left[\lfloor \frac{k}{a} \rfloor\right]\right\}$, $T = \{x_{jak+1} : j \in [r]\}$. Let $D = \overrightarrow{I_{k,t}^{+}}$ be the oriented interval graph defined above. Then all the vertices in $L_{k,t}$ have outdegree and indegree at least $k + \lfloor \frac{k}{2a} \rfloor$ except the $2k$ vertices in the *border* set $B_{k,t} = \{v_i : i \in [k] \cup ([t] - [t-k])\}$. Let $d = k + \lfloor \frac{k}{2a} \rfloor$. Note that the distance between any two vertices in T is at least $a + 2$, and

$$\begin{aligned} a(|B| + |S - T| + 1) &= a(2k + s - r + 1) \\ &= a\left(2k + \lfloor \frac{rk^2 + k}{a} \rfloor - \lfloor \frac{k}{a} \rfloor - 3k - 1 + 1\right) \\ &= a\left(\lfloor \frac{rk^2 + k}{a} \rfloor - \lfloor \frac{k}{a} \rfloor - k\right) \\ &< rk^2 \\ &= |L - B|. \end{aligned}$$

So $(a,1)$-wgcol$(D) \geqslant k + \lfloor \frac{k}{2a} \rfloor + 1$ by Lemma 3.3. ■

Next we consider outerplanar graphs. Examples 3.6 and 3.7 are analogous to Example 25 and Example 27 in [19]. Let H_t be the outerplanar graph on the vertex set $W_t = \{v_i : i \in [2t-1] \cup \{0\}\}$ obtained from the union of the cycle $C = v_0 v_1 \ldots v_{2t-1} v_0$ and the path $P = v_{2t-1} v_1 v_{2t-2} v_2 \ldots v_{t+1} v_{t-1}$. In $\overrightarrow{H_t}$, we orient the cycle C by making it a directed cycle in $\overrightarrow{H_t}$, orient the path P by making it a directed path in $\overrightarrow{H_t}$. Let $B'_t = \{v_0, v_{t-1}, v_t, v_{2t-1}\}$. Note that every vertex in $W_t - B'_t$ has outdegree and indegree 2 in $\overrightarrow{H_t}$.

Example 3.6 *For every positive integer a there exists an oriented outerplanar graph D with $(a,1)$-wgcol$(D) \geqslant 3$.*

Proof Let, $D = \overrightarrow{H_t}$, $L = W_t$, $B = B'_t$, $S = T = \emptyset$, $d = 2$, where $5a < 2t - 4$. Then

$$a(|B| + |S - T| + 1) = 5a < |L - B|$$

and we are done by Lemma 3.3. ■

For the next example let $\overrightarrow{H^+}$ be the oriented outerplanar graph obtained from $\overrightarrow{H_{24}}$ by adding the vertices in the set $X_{24} = \{x_0, x_1, ..., x_{22}\} \cup \{x_{24}, ..., x_{46}\}$ so that $(v_i, x_i), (x_i, v_{i+1}) \in E(\overrightarrow{H^+})$. Note that dist $(x_i, x_j) = |i - j| + 1$ if $i, j \leqslant 22$.

Example 3.7 *There exists an oriented outerplanar graph D with* wgcol$(D) = 4$.

Proof Let $D = \overrightarrow{H^+}$, $L = W_{24}$, $B = B'_{24}$, $S = \{x_i : i \in \{0\} \cup [46] - \{23\}\}$, $T = \{x_{2i-1} : i \in [11]\}$, $a = 1$ and $d = 3$. Then $d^+(v) = d^-(v) = 3$ for all $v \in L - B$, the distance between any two vertices in T is at least 3, and

$$a(|B| + |S - T| + 1) = 4 + 46 - 11 + 1 < 44 = |L - B|.$$

So we are done by Lemma 3.3. ■

Conjecture 3.1 can also be extended to asymmetric weak game colouring numbers:

Conjecture 3.8 *For any positive integers a, b, for any undirected graph G, there is an orientation D of G such that*

$$(a,b)\text{-wgcol}(D) = \left\lceil \frac{(a,b)\text{-gcol}(\underline{D})}{2} \right\rceil.$$

Similar to Lemma 3.2, we can show that Conjecture 3.8 is true for those classes of undirected graphs whose game colouring number is known. Note that the upper bounds of Lemma 3.9 can be obtained from Corollary 2.5, the lower bounds are proved in Example 3.4 and Example 3.6.

Lemma 3.9 *The following equalities hold:*

- *If $a \geqslant k$, then* $(a,1)\text{-wgcol}(\overrightarrow{\mathcal{I}_k}) = \left\lceil \frac{(a,1)\text{-gcol}(\mathcal{I}_k)}{2} \right\rceil = k + 1$.
- *If $a \geqslant 2$, then* $(a,1)\text{-wgcol}(\overrightarrow{\mathcal{Q}}) = \left\lceil \frac{(a,1)\text{-gcol}(\mathcal{Q})}{2} \right\rceil = 3$.
- *If $a \geqslant k$, then then* $(a,1)\text{-wgcol}(\overrightarrow{\mathcal{PK}_k}) = \left\lceil \frac{(a,1)\text{-gcol}(\mathcal{PK}_k)}{2} \right\rceil = k + 1$.

4 The strong game colouring number

In the definition of game colouring number of a digraph D, the score $s(v)$ of a vertex v is chosen to be the minimum of two numbers: $s(v) = \min\{|N^{+,+}_{D,L}[v]|, |N^{-,+}_{D,L}[v]|\}$. A natural variation is to fix one of the two numbers to be the score of a vertex. This gives another type of marking game on digraphs. This marking game was first studied by Andres [1, 2].

The *strong marking game* on a digraph D is the same as the weak marking game on D, except the score is calculated differently. If L is the linear ordering produced by a play of the marking game on D, then the score of v is $s(v) = |N^{-,+}_{D,L}[v]|$. The score of the game is $s = \max_{v \in V(D)} s(v)$. Alice's goal is to minimize the score, while Bob's goal is to maximize the score. The *strong game colouring number* of D, denoted by $\mathrm{sgcol}\,(D)$, is the least s such that Alice has a strategy that results in a score of at most s. If $\overrightarrow{\mathcal{C}}$ is a class of digraphs then $\mathrm{sgcol}(\overrightarrow{\mathcal{C}}) = \max_{D \in \overrightarrow{\mathcal{C}}} \mathrm{sgcol}\,(D)$.

If G is a symmetric digraph, then $\mathrm{sgcol}(G) = \mathrm{gcol}(G)$. So the strong game colouring number can also be viewed as a generalization of the game colouring number of undirected graphs to digraphs. By Theorem 2.3, if D is an oriented graph, then $\mathrm{wgcol}(D)$ is bounded from above by half of $\mathrm{gcol}(\underline{D})$. The following lemma shows that the behavior of the strong game colouring number is quite different.

Lemma 4.1 *For any undirected graph G, there is a digraph D which is an orientation of G such that* $\mathrm{sgcol}(D) \geqslant \mathrm{gcol}(G) - \lceil \mathrm{Mad}(G)/2 \rceil$.

Proof By Fact 1.1, there is an orientation D of G such that D has maximum outdegree at most $\lceil \mathrm{Mad}(G)/2 \rceil$. Now for any linear ordering L of the vertices of D, for any vertex v of D,

$$|N^{-,+}_{D,L}[v]| \geqslant |N^{+}_{G,L}[v]| - \lceil \mathrm{Mad}(G)/2 \rceil.$$

Thus if Bob has a strategy to ensure that when playing the marking game on G, there is a vertex of score at least k, then the same strategy ensures that when playing the marking game on D, there is a vertex of score at least $k - \lceil \mathrm{Mad}(G)/2 \rceil$. This completes the proof of the lemma. ■

Corollary 4.2 *For the classes of oriented planar graphs, outerplanar graphs, partial k-trees, interval graphs of clique size $k+1$, we have*

$$\begin{aligned} \mathrm{sgcol}(\overrightarrow{\mathcal{P}}) &\geqslant 8, \\ \mathrm{sgcol}(\overrightarrow{\mathcal{Q}}) &\geqslant 5, \\ \mathrm{sgcol}(\overrightarrow{\mathcal{PK}}_k) &\geqslant 2k+2, \\ \mathrm{sgcol}(\overrightarrow{\mathcal{I}}_k) &\geqslant 2k+1. \end{aligned}$$

Proof This follows from the known lower bounds on the game colouring number of these classes of graphs and the upper bound on $\mathrm{Mad}(G)$ for these graphs.

For planar graphs, we have $\mathrm{gcol}(\mathcal{P}) \geqslant 11$ (by Theorem 4 of [22]) and $\mathrm{Mad}(\mathcal{P}) \leqslant 6$, so $\mathrm{sgcol}(\overrightarrow{\mathcal{P}}) \geqslant 8$. For outerplanar graphs, we have $\mathrm{gcol}(\mathcal{Q}) \geqslant 7$ (by Example 27 of [19]) and $\mathrm{Mad}(\mathcal{Q}) \leqslant 4$, so $\mathrm{sgcol}\left(\overrightarrow{\mathcal{Q}}\right) \geqslant 5$. For partial k-trees, if $k \geqslant 2$, we have $\mathrm{gcol}(\mathcal{PK}_k) \geqslant 3k+2$ (by Theorem 3 of [22]), so $\mathrm{sgcol}\left(\overrightarrow{\mathcal{PK}}\right) \geqslant 3k+2-k = 2k+2$. For interval graphs of clique size $k+1$, we have $\mathrm{gcol}(\mathcal{I}_k) \geqslant 3k+1$ (by Example 4.3 of [24]), so $\mathrm{sgcol}\left(\overrightarrow{\mathcal{I}}_k\right) \geqslant 3k+1-k = 2k+1$. ■

It is shown in [1] that the maximum game colouring number of oriented forests is 3, where the maximum game colouring number of (undirected) forests is 4. However, it is unknown whether or not the trivial upper bound $\mathrm{sgcol}(D) \leqslant \mathrm{gcol}(\underline{D})$ can be improved in general (excluding some trivial cases such as an empty graph or a star). The currently known best upper bound for the game colouring number of planar graphs is 17. So for any oriented planar graph D, $\mathrm{sgcol}(D) \leqslant 17$. Using a similar technique, we can improve this upper bound by 1, i.e., for any oriented planar graph D, $\mathrm{sgcol}\,(D) \leqslant 16$. For the proof of this result, we use the activation strategy.

Suppose D is a digraph and L is a linear ordering of the vertices of D. For each vertex v of D, choose a subset $W(v)$ of $N^{-,-}_{D,L}(v)$ (where $N^{-,-}_{D,L}(v) = N^-_D(v) \cap V^-_L(v)$). We say a vertex u is *two-reachable from v with respect to L and $W(v)$* if either $u \in N^{+,+}_{D,L}(v)$, or $u <_L v$ and there is a vertex $z \in (N^{-,-}_{D,L}(v) - W(v)) \cap N^{-,-}_{D,L}(u)$. The latter case means that $u <_L v$ and there is a path $P = (v, z, u)$ of length 2 from v to u (so u can be reached from v in two steps), where z is not in $W(v)$ and is larger than v (in the ordering L) and the two edges are oriented as (z, v) and (z, u). Let $R^2_{D,L}(v)$ denote the set of all vertices u that are two-reachable from v in D with respect to L and $W(v)$, let $a(v) = |R^2_{D,L}(v)|$, let

$$s_{L,W}(v) = 2a(v) + |N^{-,+}_{D,L}(v)| + |W(v)| + 2.$$

Lemma 4.3 *Suppose D is a digraph, L is a linear ordering of $V(D)$, $W(v)$ and $s_{L,W}(v)$ are defined as above. Then Alice has a strategy for playing the strong marking game so that for any vertex v, the score of v is at most $s_{L,W}(v)$.*

Proof The strategy is the activation strategy, which is widely used in the marking game and colouring game of undirected graphs. Alice will activate and mark the least vertex in her first move (the order in the proof always refers to the linear ordering L). Here by activating a vertex, it means that vertex is put into a set A of active vertices. Alice will use the set A in determining her later moves. Suppose Bob has marked a vertex b. Alice activates b if it is inactive. If all vertices in $N^{+,+}_{D,L}(b)$ are marked, then Alice activates and marks the least unmarked vertex. Otherwise, Alice *jumps* from b to the least unmarked vertex in $N^{+,+}_{D,L}(b)$. Then she repeats the following process until she marks a vertex: Suppose Alice has jumped to a vertex v. If v is inactive, then she activates v and then jumps to the least unmarked vertex in $N^{+,+}_{D,L}[v]$. If v is active, then Alice marks v.

We say v *made a contribution* to u (or u *received a contribution* from v) if Alice made a jump from v to u. Observe that when a vertex v is activated, it makes a contribution to its least unmarked out-neighbour, if v has an unmarked out-neighbour preceding it in L. Otherwise, v makes a contribution to itself. When a vertex receives the first contribution, it is activated. When it receives the second contribution, it is marked. So a vertex receives at most two contributions.

Assume Alice has just finished a move and v is an unmarked vertex. The set $M(v)$ of marked in-neighbours of v is partitioned into three subsets: $M(v) \cap N^{-,+}_{D,L}(v)$, $M(v) \cap W(v)$ and $M(v) \cap N^{-,-}_{D,L}(v) \backslash W(v)$. Note that each vertex u in $M(v) \cap N^{-,-}_{D,L}(v) \backslash W(v)$ is active, and hence made a contribution to some vertices in $V^+_L(u)$. Since v is unmarked, and hence v can receive contributions from u, so the vertex which received a contribution from u precedes v in L and hence is two-reachable from v with respect to L and $W(v)$. (Note that if the contribution goes to v, since v is unmarked, this contribution is passed on to some vertex in $N^{+,+}_{D,L}(v)$ immediately.) As each vertex can receive at most two contributions, we conclude that $|M(v) \cap N^{-,-}_{D,L}(v) \backslash W(v)| \leqslant 2a(v)$. Therefore $|M(v)| \leqslant 2a(v) + |N^{-,+}_{D,L}(v)| + |W(v)|$. If Bob makes another move, the size of $M(v)$ may increase by 1. Thus the score of v is at most $s_{L,W}(v)$. ■

Theorem 4.4 *If $D = (V, E)$ is an oriented planar graph, then* $\operatorname{sgcol}(D) \leqslant 16$.

Proof By Lemma 4.3, it suffices to construct a linear ordering L of the vertices of D, and choose $W(v)$ for each vertex v so that for each vertex v, $s_{L,W}(v) \leqslant 16$.

Fix a planar drawing of D. Initially we have a set of *chosen* vertices $C = \emptyset$ and a set of *unchosen* vertices $U = V$. At any stage of the construction we choose a vertex $u \in U$, and then add the vertex to L, such that for any $x \in U \backslash \{u\}$, $y \in C$, we have $x <_L u <_L y$. Then let $U := U - \{u\}$; $C := C \cup \{u\}$.

Let H be the planar digraph obtained from D by deleting all arcs between vertices in C; deleting all arcs with tails in U and heads in C; deleting each vertex $y \in C$ such that $|N^+_D(y) \cap U| \leqslant 3$, and adding one directed edge (in either direction) between any two nonadjacent vertices of $N^+_D(y) \cap U$. Clearly H is still an oriented planar graph.

For each vertex v of H, let $c(v) = \frac{2}{3} d^+_H(v) + d^-_H(v)$ be the initial charge of v. Then

$$\sum_{v \in V(H)} c(v) = \frac{5}{3}|E(H)| < 5|V(H)|.$$

If xy is an edge of H with $x \in U$ and $y \in C$, then move a charge of $\frac{7}{12}$ from x to y. Denote by $c'(v)$ the new charge of a vertex $v \in V(H)$. If $v \in C$, then since v has outdegree at least 4, we conclude that $c'(v) \geqslant 4(\frac{2}{3} + \frac{7}{12}) = 5$. As $\sum_{v \in V(H)} c'(v) < 5|V(H)|$, there is a vertex $u \in U$ with $c'(u) < 5$. We choose a vertex $u \in U$ with $c'(u) < 5$ and add u to C, and the ordering is that for any $x \in U - \{u\}$, $y \in C - \{u\}$, we have $x <_L u <_L y$. Let $W(u) = C \cap N_H(u)$. Although the linear ordering L is not completely determined yet, the value of $s_{L,W}(u)$ is determined.

Indeed, $R^2_{D,L}(u) \subseteq N_H(u) \cap U$. To see this, it suffices to note that if a vertex $z \in (N^{-,-}_{D,L}(u) - W(u))$, then $|N^+_D(z) \cap U| \leqslant 3$. Therefore, by the construction of H, if w is two-reachable from u with respect to L and $W(u)$, either $w \in N^{+,+}_{H,L}(u)$ or $w \in N^{-,+}_{H,L}(u)$.

Also note that $N^{-,+}_{D,L}(u) \subseteq N^-_H(u) \cap U$ and $W(u) = N^-_H(u) \cap C$. Define $\sigma = \left|N^{+,+}_{H,L}(u)\right|$, $\alpha = \left|N^{-,+}_{H,L}(u)\right|$, $\beta = \left|N^{-,-}_{H,L}(u)\right|$. Then

$$\begin{aligned} c'(u) &= \frac{2}{3}\sigma + \alpha + \frac{5}{12}\beta. \\ s_{L,W}(u) &= 2|R^2_{D,L}(u)| + |N^{-,+}_{D,L}(u)| + |W(u)| + 2 \\ &\leqslant 2(\sigma + \alpha) + \alpha + \beta + 2 \\ &= 2\sigma + 3\alpha + \beta + 2. \end{aligned}$$

Since σ, α and β are nonnegative integers satisfying $c'(u) < 5$, we have

$$s_{L,W}(u) \leqslant 16.$$

This finishes the proof. ■

5 Asymmetric strong marking games on digraphs

For the strong marking game on directed graphs, we have seen that for an oriented graph D, sgcol(D) can be much larger than half of gcol($\underline{D}$). However, for certain asymmetric strong marking games, the situation can be different.

The *strong* (a,b)*-game colouring number* (a,b)-sgcol(D) of D is the least s such that Alice has a strategy that results in a score of at most s in the strong (a,b)-marking game of D. Let $\mathcal{G}_k$ be the class of graphs G with $\mathrm{Mad}(G) \leqslant 2k$. It follows from results in [19] that for $a \geqslant k$, $2k+1 \leqslant (a,1)$-gcol$(\mathcal{G}_k) \leqslant 2k+2$. We shall show that $(a,1)$-sgcol$(\overrightarrow{\mathcal{G}}_k)$ is about half of this number.

Theorem 5.1 *If $a \geqslant k$, then $k+1 \leqslant (a,1)$ -sgcol$(\overrightarrow{\mathcal{G}_k}) \leqslant k+2$.*

Proof Since $(a,1)$ -gcol$(\mathcal{G}_k) \geqslant 2k+1$, there is a graph $G \in \mathcal{G}_k$ such that $(a,1)$ -gcol$(G) \geqslant 2k+1$. As $\mathrm{Mad}(G) \leqslant 2k$, by Fact 1.1, there is an orientation D of G such that $\Delta^+(D) \leqslant k$. When playing the strong $(a,1)$-marking game on D, Bob uses his strategy for the $(a,1)$-marking game on G. Thus there is a vertex $v \in V(G)$ which has at least $2k$ neighbours marked before v. Since $d^+(v) \leqslant k$, so v has at least k in-neighbours in D marked before v. I.e., for the strong marking game on D, $s(v) \geqslant k+1$. Hence $(a,1)$ -sgcol$(D) \geqslant k+1$.

It remains to show that for any $G \in \mathcal{G}_k$, for any orientation D of G, $(a,1)$-sgcol$(D) \leqslant k+2$. Since $G \in \mathcal{G}_k$, there is an orientation $\vec{G}$ of G with $\Delta^-(\vec{G}) \leqslant k$. Define two auxiliary digraphs $\vec{G}_r$ and $\vec{G}_l$ as follows:

1. Both $\vec{G}_r$ and $\vec{G}_l$ have the same vertex set as D.
2. For any arc $(u,v) \in E(D)$, if also $(u,v) \in E(\vec{G})$, then add (u,v) to $E(\vec{G}_r)$; else (that is the case in which $(u,v) \in E(D)$ and $(v,u) \in E(\vec{G})$), then add (v,u) to $E(\vec{G}_l)$.

Thus $\vec{G}_r$ and $\vec{G}_l$ are subdigraphs of $\vec{G}$ and $E(\vec{G}) = E(\vec{G}_r) \cup E(\vec{G}_l)$, $E(\vec{G}_r) \cap E(\vec{G}_l) = \emptyset$.

Let G_l be the underlying (undirected) graph of $\vec{G}_l$. Instead of playing the strong $(a, 1)$-marking game on D, Alice views it as a $(a, 1)$-marking games on G_l, and she uses the so-called *Harmonious Strategy* for this game.

Here is a formal description of the strategy. To unify the description we consider an equivalent version of the marking game in which Bob plays first by marking a new vertex x_0 with no neighbours in $V(G_l)$. Let $\Pi(G_l)$ be the set of linear orderings on the vertex set of G_l. Fix any $L \in \Pi(G_l)$. For a subset X of vertices of D, L-min$X$ means the smallest vertex of X with respect to the order of L. In the description of the strategy, U denotes the set of unmarked vertices. For a vertex v, S_v denotes the set of in-neighbours u to whom v has not yet contributed.

Initialization: $U := V(G_l)$; for $v \in V(G)$ do $S_v := N^-_{\vec{G}_l}(v)$ end do;

Now suppose that Bob has just marked a vertex x. Alice plays by performing the following steps.
Alice's play: for i from 1 to a while $U \neq \emptyset$ do

1. if $S_x \cap U \neq \emptyset$ then $y := L\text{-min}\, S_x \cap U$ else $y := L\text{-min}\, U$ end if;
   $S_x := S_x - \{y\}$;
2. while $S_y \cap U \neq \emptyset$ do $z := L\text{-min}\, S_y \cap U$; $S_y := S_y - \{z\}$; $y := z$ end do;
3. $U := U - \{y\}$ end do;

For an unmarked vertex u, we say u *receives a contribution* from v and v *made a contribution* to u, if: in Line 1, we have $u = y := L\text{-min}\, S_x \cap U$ and $v = x$; or in Line 2, we have $u = z := L\text{-min}\, S_y \cap U$ and $v = y$.

So in Step 1 Alice selects a vertex y and then x contributes to y. In Step 2 this contribution is passed along until finally it arrives at a vertex y that has already contributed to all its in-neighbours. This strategy is used in [19] for $(a, 1)$-marking games on undirected graphs. The following lemmas were proved in [19].

Lemma 5.2 *The Harmonious Strategy always terminates with Alice completing her turn.*

Lemma 5.3 *Suppose that $k \leqslant a$ and Alice follows the Harmonious Strategy. Consider a time when Alice has just marked a vertex v. Then*

1. *Any unmarked vertex has received the same number of contributions as it has made.*
2. *The vertex v has contributed to all its unmarked in-neighbours, i.e., $S_v \cap U = \emptyset$.*
3. *If Alice has completed her turn then every marked vertex x satisfies $S_x \cap U = \emptyset$.*

Suppose that Alice uses the Harmonious Strategy on the graph $\vec{G}_l$. Consider any time when a vertex v has just been marked by Alice. If Alice has not yet completed her turn, let x be the last vertex marked by Bob. Otherwise x is undefined. It suffices to show that any unmarked vertex u has at most k marked in-neighbours in D other than x. This allows for the fact that if x is defined, then it may be an in-neighbour of u in D; and otherwise it is Bob's turn, and he may be about to mark a vertex that is an in-neighbour of u. In the former case we treat x separately because it may have not yet contributed to all of its unmarked in-neighbours in $\vec{G}_l$.

Suppose $s = |N^-_{\vec{G}_r}(u)|$ and $t = |N^-_{\vec{G}_l}(u)|$. By Lemma 5.3 (2,3) every marked out-neighbour of u in $\vec{G}_l$ other than x has contributed to u and by Lemma 5.3 (1) each contribution to u is matched by a unique contribution to an in-neighbour of u in $\vec{G}_l$. Therefore, the number of marked out-neighbours of u in $\vec{G}_l$ other than x is at most t.

Every marked in-neighbour of u in D is either an in-neighbour of u in $\vec{G}_r$ or an out-neighbour of u in $\vec{G}_l$. As $s + t = |N^-_{\vec{G}}(u)| \leqslant k$, we conclude that u has at most k marked in-neighbours other than x in D. ■

Corollary 5.4

- *If $a \geqslant k$, then* $(k,1)$-sgcol$\left(\overrightarrow{\mathcal{PK}_k}\right) \leqslant k+2$.
- *If $a \geqslant 3$, then* $(a,1)$-sgcol$\left(\overrightarrow{\mathcal{P}}\right) \leqslant 5$.
- *If $a \geqslant 2$, then* $(a,1)$-sgcol$\left(\overrightarrow{\mathcal{Q}}\right) \leqslant 4$.

The Harmonious Strategy only applies to $(a,1)$-marking games on graphs G with $a \geqslant \mathrm{Mad}(G)/2$. It is known that if $a < k$ then $(a,1)$-gcol$(\mathcal{G}_k) = \infty$ (refer Example 14, [19]). The first paragraph of the proof of Theorem 5.1 shows that this implies $(a,1)$-gcol$(\overrightarrow{\mathcal{G}}_k) = \infty$. One technique can be used in the game if G can be decomposed into two graphs: one with small maximum average degree, and the other with small maximum indegree (refer Observation 1, [2]).

Corollary 5.5 *If G is the edge disjoint union of G_1 and G_2, where $\lceil \mathrm{Mad}(G_1)/2 \rceil = k \leqslant a$ and $\Delta^-(G_2) = k'$, then for any orientation D of G,* $(a,1)$-sgcol$(D) \leqslant k + k' + 2$.

Proof Alice plays the marking game on the orientation of G_1 so that each unmarked vertex v has at most $k+1$ marked in-neighbours in the orientation of G_1. Plus those marked in-neighbours of v in the orientation of G_2, we conclude that v has at most $k + k' + 1$ marked in-neighbours in D. So $(a,1)$-sgcol$(D) \leqslant k + k' + 2$. ■

It was proved in [25] that every planar graph G can be edge-partitioned into two subgraphs G_1 and G_2 such that $\mathrm{Mad}(G_1) \leqslant 4$ and $\Delta(G_2) \leqslant 8$. Thus we have the following corollary:

Corollary 5.6 *If D is an orientation of a planar graph, then* $(2,1)$-sgcol$(D) \leqslant 12$.

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