Congruences involving alternating multiple harmonic sums

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Abstract

We show that for any prime prime $p \neq 2$,

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} \equiv - \sum_{k=1}^{(p-1)/2} \frac{1}{k} \pmod{p^3}$$

by expressing the left-hand side as a combination of alternating multiple harmonic sums.

1 Introduction

In [8] Van Hamme presented several results and conjectures concerning a curious analogy between the values of certain hypergeometric series and the congruences of some of their partial sums modulo power of prime. In this paper we would like to discuss a new example of this analogy. Let us consider

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} &= \left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1 \cdot 3}{2 \cdot 4}\right) + \frac{1}{3}\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right) + \frac{1}{4}\left(\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}\right) + \cdots \\ &= \int_0^{-1} \frac{1}{x}\left(\frac{1}{\sqrt{1+x}} - 1\right) dx = -2\left[\log\left(\frac{1+\sqrt{1+x}}{2}\right)\right]_0^{-1} = 2\log 2. \end{aligned}$$

Let p be a prime number, what is the p-adic analogue of the above result?
The real case suggests to replace the logarithm with some p-adic function which behaves in a similar way. It turns out that the right choice is the *Fermat quotient*

$$q_p(x) = \frac{x^{p-1} - 1}{p}$$

(which is fine since $q_p(x \cdot y) \equiv q_p(x) + q_p(y) \pmod p$), and, as shown in [7], the following congruence holds for any prime $p \neq 2$,

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} \equiv 2\, q_p(2) \pmod p.$$

Here we improve this result to the following statement.

Theorem 1.1. *For any prime $p > 3$,*

$$\begin{aligned}\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} &\equiv 2q_p(2) - pq_p(2)^2 + \frac{2}{3}p^2 q_p(2)^3 + \frac{7}{12}p^2 B_{p-3} \\ &\equiv -\sum_{k=1}^{(p-1)/2} \frac{1}{k} \pmod{p^3}\end{aligned}$$

where B_n is the n-th Bernoulli number.

In the proof we will employ some new congruences for alternating multiple harmonic sums which are interesting in themselves, such as

$$\begin{aligned}H(-1,-2;p-1) &= \sum_{0<i<j<p} \frac{(-1)^{i+j}}{ij^2} \equiv -\frac{3}{4} B_{p-3} \pmod p, \\ H(-1,-1,1;p-1) &= \sum_{0<i<j<k<p} \frac{(-1)^{i+j}}{ijk} \equiv q_p(2)^3 + \frac{7}{8} B_{p-3} \pmod p.\end{aligned}$$

2 Alternating multiple harmonic sums

Let $r > 0$ and let $(a_1, a_2, \ldots, a_r) \in (\mathbb{Z}^*)^r$. For any $n \geqslant r$, we define the *alternating multiple harmonic sum* as

$$H(a_1, a_2, \ldots, a_r; n) = \sum_{1 \leqslant k_1 < k_2 < \cdots < k_r \leqslant n} \prod_{i=1}^{r} \frac{\mathrm{sign}(a_i)^{k_i}}{k_i^{|a_i|}}.$$

The integers r and $\sum_{i=1}^r |a_i|$ are respectively the *depth* and the *weight* of the harmonic sum. From the definition one derives easily the *shuffle relations*:

$$\begin{aligned}H(a;n) \cdot H(b;n) &= H(a,b;n) + H(b,a;n) + H(a \oplus b;n) \\ H(a,b;n) \cdot H(c;n) &= H(c,a,b;n) + H(a,c,b;n) + H(a,b,c;n) \\ &\quad + H(a \oplus b, c;n) + H(a, b \oplus c;n)\end{aligned}$$

where $a \oplus b = \mathrm{sign}(ab)(|a| + |b|)$. Moreover, if p is a prime, by replacing k_i with $p - k_i$ we get the *reversal relations*:

$$\begin{aligned}H(a,b;p-1) &\equiv H(b,a;p-1)(-1)^{a+b}\mathrm{sign}(ab) \pmod p, \\ H(a,b,c;p-1) &\equiv H(c,b,a;p-1)(-1)^{a+b+c}\mathrm{sign}(abc) \pmod p.\end{aligned}$$

The values of several *non-alternating* (i.e. when all the indices are positive) harmonic sums modulo a power of a prime are well known:

(i). ([4], [11]) for $a, r > 0$ and for any prime $p > ar + 2$

$$H(\{a\}^r; p-1) \equiv \begin{cases} (-1)^r \frac{a(ar+1)}{2(ar+2)} p^2 B_{p-ar-2} & \pmod{p^3} \quad \text{if } ar \text{ is odd,} \\ (-1)^{r-1} \frac{a}{ar+1} p B_{p-ar-1} & \pmod{p^2} \quad \text{if } ar \text{ is even;} \end{cases}$$

(ii). ([6]) for any prime $p > 3$

$$H\left(1; \frac{p-1}{2}\right) \equiv -2q_p(2) + pq_p(2)^2 - \frac{2}{3} p^2 q_p(2)^3 - \frac{7}{12} p^2 B_{p-3} \pmod{p^3}.$$

and for $a > 1$ and for any prime $p > a + 1$

$$H\left(a; \frac{p-1}{2}\right) \equiv \begin{cases} -\frac{2^a-2}{a} B_{p-a} & \pmod{p} \quad \text{if } a \text{ is odd,} \\ \frac{a(2^{a+1}-1)}{2(a+1)} p B_{p-a-1} & \pmod{p^2} \quad \text{if } a \text{ is even;} \end{cases}$$

(iii). ([4], [10]) for $a, b > 0$ and for any prime $p > a + b + 1$

$$H(a, b; p-1) \equiv \frac{(-1)^b}{a+b} \binom{a+b}{a} B_{p-a-b} \pmod{p}$$

(note that $B_{2n+1} = 0$ for $n > 0$).

The following result will allow us to compute the mod p values of multiple harmonic sums of depth $\leqslant 2$ when the indices are all negative.

Theorem 2.1. *Let $a, b > 0$; then for any prime $p \neq 2$,*

$$H(-a; p-1) = -H(a; p-1) + \frac{1}{2^{a-1}} H\left(a; \frac{p-1}{2}\right),$$
$$2H(-a, -a; p-1) = H(-a; p-1)^2 - H(2a; p-1),$$

and

$$\begin{aligned} H(-a, -b; p-1) \equiv & -\left(1 - \frac{1}{2^{a+b-1}}\right) H(a, b; p-1) \\ & - \frac{(-1)^b}{2^{a+b-1}} H\left(a; \frac{p-1}{2}\right) H\left(b; \frac{p-1}{2}\right) \pmod{p}. \end{aligned}$$

Proof. The first shuffling relation applied to $H(-a;p-1)^2$ yields the second equation. As regards the first equation we simply observe that $(-1)^i/i^a$ is positive if and only if i is even. We use a similar argument for the congruence: since $(-1)^{i+j}/(i^a j^b)$ is positive if and only if i and j are both even or if $(p-i)$ and $(p-j)$ are both even then we have mod p,

$$H(-a,-b;p-1) \equiv -H(a,b;p-1) + \frac{2}{2^{a+b}}\left(H\left(a,b;\frac{p-1}{2}\right) + (-1)^{a+b} H\left(b,a;\frac{p-1}{2}\right)\right).$$

Moreover, by decomposing the sum $H(a,b;p-1)$ we obtain

$$\begin{aligned} H(a,b;p-1) \equiv H\left(a,b;\frac{p-1}{2}\right) &+ H\left(a;\frac{p-1}{2}\right)(-1)^b H\left(b;\frac{p-1}{2}\right) \\ &+ (-1)^{a+b} H\left(b,a;\frac{p-1}{2}\right), \end{aligned}$$

that is,

$$\begin{aligned} &H\left(a,b;\frac{p-1}{2}\right) + (-1)^{a+b} H\left(b,a;\frac{p-1}{2}\right) \\ &\qquad \equiv H(a,b;p-1) - H\left(a;\frac{p-1}{2}\right)(-1)^b H\left(b;\frac{p-1}{2}\right), \end{aligned}$$

and the desired congruence follows immediately. □

Corollary 2.2. *For any prime $p > 3$,*

$$\begin{aligned} H(-1;p-1) &\equiv -2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2 q_p(2)^3 - \frac{1}{4}p^2 B_{p-3} \pmod{p^3}, \\ H(-1,-1;p-1) &\equiv 2q_p(2)^2 - 2pq_p(2)^3 - \frac{1}{3}p B_{p-3} \pmod{p^2}. \end{aligned}$$

Moreover, for $a > 1$ and for any prime $p > a+1$,

$$H(-a;p-1) \equiv -\frac{2^a - 2}{a2^{a-1}} B_{p-a} \pmod{p}.$$

Proof. The proof is straightforward: apply Theorem 2.1, (i), (ii), and (iii). □

The following theorem is a variation of a result presented in [9].

Theorem 2.3. *Let $r > 0$; then for any prime $p > r+1$,*

$$H(\{1\}^{r-1},-1;p-1) \equiv (-1)^{r-1} \sum_{k=1}^{p-1} \frac{2^k}{k^r} \pmod{p}.$$

Proof. For $r \geqslant 1$, let

$$F_r(x) = \sum_{0<k_1<\cdots<k_r<p} \frac{x^{k_r}}{k_1 \cdots k_r} \in \mathbb{Z}_p[x] \quad \text{and} \quad f_r(x) = \sum_{0<k<p} \frac{x^k}{k^r} \in \mathbb{Z}_p[x].$$

We show by induction that

$$F_r(x) \equiv (-1)^{r-1} f_r(1-x) \pmod{p}$$

then our congruence follows by taking $x = -1$.
For $r = 1$, since $\binom{p}{k} = (-1)^{k-1}\frac{p}{k} \pmod{p^2}$ for $0 < k < p$ then

$$f_1(x) \equiv \frac{1}{p}\sum_{k=1}^{p-1}(-1)^{k-1}\binom{p}{k}x^k = -\frac{1}{p}\sum_{k=1}^{p-1}\binom{p}{k}(-x)^k = \frac{1-(1-x)^p - x^p}{p} \pmod{p}.$$

Hence $F_1(x) = f_1(x) \equiv f_1(1-x) \pmod{p}$.
Assume that $r > 1$, then the formal derivative yields

$$\begin{aligned}
\frac{d}{dx}F_r(x) &= \sum_{0<k_1<\cdots<k_r<p} \frac{k_r x^{k_r-1}}{k_1\cdots k_r} = \sum_{0<k_1<\cdots<k_{r-1}<p} \frac{1}{k_1\cdots k_{r-1}} \sum_{k_r=k_{r-1}+1}^{p-1} x^{k_r-1} \\
&= \sum_{0<k_1<\cdots<k_{r-1}<p} \frac{1}{k_1\cdots k_{r-1}} \cdot \frac{x^{p-1}-x^{k_{r-1}}}{x-1} \\
&= \frac{x^{p-1}}{x-1}\, H(\{1\}^{r-1}; p-1) - \frac{1}{x-1}\, F_{r-1}(x) \equiv \frac{F_{r-1}(x)}{1-x} \pmod{p}
\end{aligned}$$

where we used (i). Moreover

$$\frac{d}{dx}f_r(1-x) = -\sum_{0<k<p}\frac{(1-x)^{k-1}}{k^{r-1}} = -\frac{f_{r-1}(1-x)}{1-x}.$$

Hence, by the induction hypothesis

$$(1-x)\frac{d}{dx}\left(F_r(x) + (-1)^r f_r(1-x)\right) \equiv F_{r-1}(x) + (-1)^{r-1} f_{r-1}(1-x) \equiv 0 \pmod{p}.$$

Thus $F_r(x) + (-1)^r f_r(1-x) \equiv c_1 \pmod{p}$ for some constant c_1 because this polynomial has degree $< p$. Substituting in $x = 0$ we find that by (i),

$$F_r(x) + (-1)^r f_r(1-x) \equiv c_1 \equiv F_r(0) + (-1)^r f_r(1) = (-1)^r H(r; p-1) \equiv 0 \pmod{p}.$$

□

With the next two corollaries we have a complete the list of the mod p values of the alternating multiple harmonic sums of depth and weight $\leqslant 3$.

Corollary 2.4. *The following congruences mod p hold for any prime $p > 3$,*

$$H(1,-1;p-1) \equiv -H(-1,1;p-1) \equiv q_p(2)^2\,,$$
$$H(-1,2;p-1) \equiv H(1,-2;p-1) \equiv H(2,-1;p-1) \equiv H(-2,1;p-1) \equiv \frac{1}{4}B_{p-3}\,,$$
$$H(-1,-2;p-1) \equiv -H(-2,-1;p-1) \equiv -\frac{3}{4}B_{p-3}\,.$$

Proof. By Theorem 2.3 and by [2] we have that

$$H(1,-1;p-1) \equiv -\sum_{k=1}^{p-1}\frac{2^k}{k^2} \equiv q_p(2)^2 \pmod p.$$

By the first shuffling relation applied to the product $H(-1;p-1)H(2;p-1)$, by (i), and by Corollary 2.2 we get

$$H(-1,2;p-1) = \frac{1}{2}H(-1;p-1)H(2;p-1) - \frac{1}{2}\,H(-3;p-1) \equiv \frac{1}{4}B_{p-3} \pmod p.$$

By (ii) and by Theorem 2.1

$$H(-1,-2;p-1) \equiv -\frac{3}{4}H(1,2;p-1) - \frac{1}{4}H\left(1;\frac{p-1}{2}\right)H\left(2;\frac{p-1}{2}\right) \equiv -\frac{3}{4}B_{p-3} \pmod p.$$

The remaining congruences follow by applying the reversal relation of depth 2. □

Corollary 2.5. *The following congruences mod p hold for any prime $p > 3$,*

$$H(-1,1,-1;p-1) \equiv 0\,,$$
$$H(1,1,-1;p-1) \equiv H(-1,1,1;p-1) \equiv -\frac{1}{3}q_p(2)^3 - \frac{7}{24}B_{p-3}\,,$$
$$H(-1,-1,1;p-1) \equiv -H(1,-1,-1;p-1) \equiv q_p(2)^3 + \frac{7}{8}B_{p-3}\,,$$
$$H(1,-1,1;p-1) \equiv \frac{2}{3}q_p(2)^3 + \frac{1}{12}B_{p-3}\,,$$
$$H(-1,-1,-1;p-1) \equiv -\frac{4}{3}q_p(2)^3 - \frac{1}{6}B_{p-3}\,.$$

Proof. By the reversal relation of depth 3, $H(-1,1,-1;p-1) \equiv -H(-1,1,-1;p-1) \equiv 0$. By Theorem 2.3, by Theorem 1 in [3], and by Corollary 2.2 we have that

$$H(1,1,-1;p-1) \equiv \sum_{k=1}^{p-1}\frac{2^k}{k^3} \equiv -\frac{1}{3}q_p(2)^3 + \frac{7}{12}H(-3,p-1) \equiv -\frac{1}{3}q_p(2)^3 - \frac{7}{24}B_{p-3} \pmod p.$$

By the second shuffling relation applied to the products

$H(1,-1;p-1)H(-1;p-1)$, $H(1,-1;p-1)H(1;p-1)$, and $H(-1,-1;p-1)H(-1;p-1)$

we respectively find that

$$2H(1,-1,-1;p-1) \equiv H(1,-1;p-1)H(-1;p-1) - H(1,2;p-1) - H(-2,-1;p-1)\,,$$
$$H(1,-1,1;p-1) \equiv -2H(1,1,-1,p-1) - 2H(2,-1;p-1)\,,$$
$$3H(-1,-1,-1;p-1) \equiv H(-1,-1;p-1)H(-1;p-1) - 2H(2,-1;p-1).$$

The remaining congruences follow by applying the reversal relation of depth 3. □

3 Proof of Theorem 1.1

The following useful identity appears in [7]. Here we give an alternative proof by using Riordan's array method (see [5] for more examples of this technique), hoping that it will help the interested reader to find some generalization.

Theorem 3.1. *Let $n \geqslant d > 0$,*

$$d\sum_{k=1}^{n}\binom{2k}{k+d}\frac{x^{n-k}}{k} = \sum_{k=0}^{n-d}\binom{2n}{n+d+k}v_k - \binom{2n}{n+d}$$

where $v_0 = 2$, $v_1 = x-2$ and $v_{k+1} = (x-2)v_k - v_{k-1}$ for $k \geqslant 1$.

Proof. We first note that

$$\begin{aligned}\binom{2k}{k+d} &= \binom{2k}{k-d} = (-1)^{k-d}\binom{-k-d-1}{k-d}\\ &= [z^{k-d}]\frac{1}{(1-z)^{k+d+1}} = [z^{-1}]\frac{z^{d-1}}{(1-z)^{d+1}}\cdot\left(\frac{1}{z(1-z)}\right)^k.\end{aligned}$$

Since the residue of a derivative is zero then

$$\begin{aligned}d\sum_{k=1}^{n}\binom{2k}{k+d}\frac{x^{n-k}}{k} &= [z^{-1}]\,x^n\,\frac{dz^{d-1}}{(1-z)^{d+1}}\,G\left(\frac{1}{xz(1-z)}\right)\\ &= -[z^{-1}]\,x^n\,\frac{z^d}{(1-z)^d}\,G'\left(\frac{1}{xz(1-z)}\right)\cdot\left(\frac{1}{xz(1-z)}\right)'\\ &= [z^{-1}]\,\frac{z^{d-n-1}}{(1-z)^{n+d+1}}\,\frac{1-x^nz^n(1-z)^n}{1-xz+xz^2}\cdot(1-2z)\\ &= [z^{-1}]\,\frac{z^{d-n-1}}{(1-z)^{n+d+1}}\,\frac{1-2z}{1-xz+xz^2}\end{aligned}$$

where $G(z) = \sum_{k=1}^{n}\frac{z^k}{k}$ and $G'(z) = \sum_{k=1}^{n} z^{k-1} = \frac{1-z^n}{1-z}$. Moreover

$$\begin{aligned}\binom{2n}{n+d+k} &= \binom{2n}{n-d-k} = (-1)^{n-d-k}\binom{-n-d-k-1}{n-d-k}\\ &= [z^{n-d-k}]\frac{1}{(1-z)^{n+d+k+1}} = [z^{-1}]\frac{z^{d-n-1}}{(1-z)^{n+d+1}}\cdot\left(\frac{z}{1-z}\right)^k.\end{aligned}$$

Letting $F(z) = \sum_{k=0}^{\infty} v_k z^k = \frac{2-(x-2)z}{1-(x-2)z+z^2}$ then

$$\begin{aligned}\sum_{k=0}^{n-d} \binom{2n}{n+d+k} v_k - \binom{2n}{n+d} &= [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \cdot F\left(\frac{z}{1-z}\right) - [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \\ &= [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \left(\frac{(2-xz)(1-z)}{1-xz+xz^2} - 1 \right) \\ &= [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \frac{1-2z}{1-xz+xz^2}.\end{aligned}$$

□

Notice that it is easy to verify by comparing the recurrence relations or the generating functions that

$$v_k = 2T_k\left(\frac{x}{2} - 1\right)$$

where $T_k(x)$ denote the Chebyshev polynomials of the first kind (I thank the referee for this valuable remark). We wonder if Theorem 3.1 can be interpreted as a known identity for Chebyshev polynomials.

Corollary 3.2. *For any $n > 0$,*

$$4^n \sum_{k=1}^{n} \binom{-\frac{1}{2}}{k} \frac{(-1)^k}{k} = -4(-1)^n \sum_{d=0}^{n-1} \frac{(-1)^d}{n-d} \sum_{j=0}^{d-1} \binom{2n}{j} - 2(-1)^n \sum_{d=0}^{n-1} \frac{(-1)^d}{n-d} \binom{2n}{d}.$$

Proof. Since

$$0 = \sum_{d=-k}^{k} (-1)^d \binom{2k}{k+d} = \binom{2k}{k} + 2 \sum_{d=1}^{k} (-1)^d \binom{2k}{k+d}$$

then for any $n \geqslant k$,

$$(-1)^k \binom{-\frac{1}{2}}{k} = 4^{-k} \binom{2k}{k} = -2 \cdot 4^{-k} \sum_{d=1}^{n} (-1)^d \binom{2k}{k+d}.$$

For $x = 4$ we have $v_k = 2$ for all $k \geqslant 0$ and with Theorem 3.1 we get

$$\begin{aligned}4^n \sum_{k=1}^{n} \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} &= -2 \sum_{k=1}^{n} \frac{4^{n-k}}{k} \sum_{d=1}^{n} (-1)^d \binom{2k}{k+d} = -2 \sum_{d=1}^{n} (-1)^d \sum_{k=1}^{n} \frac{4^{n-k}}{k} \binom{2k}{k+d} \\ &= -2 \sum_{d=1}^{n} \frac{(-1)^d}{d} \left(2 \sum_{k=0}^{n-d} \binom{2n}{n+d+k} - \binom{2n}{n+d} \right) \\ &= -4 \sum_{d=1}^{n} \frac{(-1)^d}{d} \sum_{k=1}^{n-d} \binom{2n}{n-d-k} - 2 \sum_{d=1}^{n} \frac{(-1)^d}{d} \binom{2n}{n-d} \\ &= -4(-1)^n \sum_{d=0}^{n-1} \frac{(-1)^d}{n-d} \sum_{j=0}^{d-1} \binom{2n}{j} - 2(-1)^n \sum_{d=0}^{n-1} \frac{(-1)^d}{n-d} \binom{2n}{d}.\end{aligned}$$

□

We will make use of the following lemma.

Lemma 3.3. *For any prime $p \neq 2$ and for $0 < j < p$,*

$$\binom{2p}{j} \equiv -2p\frac{(-1)^j}{j} + 4p^2\frac{(-1)^j}{j}H(1;j-1) \pmod{p^3}$$

and

$$\binom{2p}{p} \equiv 2 - \frac{4}{3}p^3 B_{p-3} \pmod{p^4}.$$

Proof. It suffices to expand the binomial coefficient as follows

$$\binom{2p}{j} = -2p\frac{(-1)^j}{j}\prod_{k=1}^{j-1}\left(1-\frac{2p}{k}\right) = \frac{(-1)^j}{j}\sum_{k=1}^{j-1}(-2p)^k\, H(\{1\}^{k-1};j-1),$$

and apply (i). □

Proof of Theorem 1.1. Letting $n = p$ in the identity given by Corollary 3.2 we obtain

$$4^p\sum_{k=1}^{p}\frac{(-1)^k}{k}\binom{-\frac{1}{2}}{k} = 4\sum_{0\leqslant j<d<p}\frac{(-1)^d}{p-d}\binom{2p}{j} + 2\sum_{0\leqslant d<p}\frac{(-1)^d}{p-d}\binom{2p}{d},$$

that is,

$$4^{p-1}\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\binom{-\frac{1}{2}}{k} = \frac{2-\binom{2p}{p}}{4p} - \sum_{0<d<p}\frac{(-1)^d}{d} + \sum_{0<j<d<p}\frac{(-1)^d}{p-d}\binom{2p}{j} + \frac{1}{2}\sum_{0<d<p}\frac{(-1)^d}{p-d}\binom{2p}{d}.$$

Now we consider each term of the right-hand side separately. By Lemma 3.3,

$$\frac{2-\binom{2p}{p}}{4p} \equiv \frac{1}{3}\,p^2 B_{p-3} \pmod{p^3}.$$

By Corollary 2.2,

$$\sum_{0<d<p}\frac{(-1)^d}{d} = H(-1;p-1) = -2q_p(2) + pq_p(2)^2 - \frac{2}{3}\,p^2 q_p(2)^3 - \frac{1}{4}\,p^2 B_{p-3} \pmod{p^3}.$$

Since for $0 < d < p$,

$$\frac{1}{p-d} = -\frac{1}{d(1-\frac{p}{d})} \equiv -\frac{1}{d} - \frac{p}{d^2} \pmod{p^2}$$

then by Lemma 3.3, (i), and (iii) we have that

$$\begin{aligned}
\sum_{0<d<p}\frac{(-1)^d}{p-d}\binom{2p}{d} &\equiv \sum_{0<d<p}\left(-\frac{(-1)^d}{d} - p\frac{(-1)^d}{d^2}\right)\left(-2p\frac{(-1)^d}{d} + 4p^2\frac{(-1)^d}{d}H(1;d-1)\right)\\
&\equiv 2pH(2;p-1) + 2p^2H(3;p-1) - 4p^2H(1,2;p-1)\\
&\equiv -\frac{8}{3}\,p^2 B_{p-3}. \pmod{p^3}.
\end{aligned}$$

In a similar way, by Lemma 3.3 and Corollaries 2.4 and 2.5 we get

$$\sum_{0<j<d<p} \frac{(-1)^d}{p-d}\binom{2p}{j}$$
$$\equiv \sum_{0<j<d<p}\left(-\frac{(-1)^d}{d}-p\frac{(-1)^d}{d^2}\right)\left(-2p\frac{(-1)^j}{j}+4p^2\frac{(-1)^j}{j}H(1;j-1)\right)$$
$$\equiv 2pH(-1,-1;p-1)+2p^2H(-1,-2;p-1)-4p^2H(1,-1,-1;p-1)$$
$$\equiv 4pq_p(2)^2+\frac{4}{3}p^2B_{p-3} \pmod{p^3}.$$

Thus,

$$4^{p-1}\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\binom{-\frac{1}{2}}{k} = 2q_p(2)+3pq_p(2)^2+\frac{2}{3}p^2q_p(2)^3+\frac{7}{12}p^2B_{p-3} \pmod{p^3}.$$

Since $4^{p-1}=(q_p(2)p+1)^2=1+2q_p(2)p+q_p(2)^2p^2$ then

$$4^{-(p-1)}=(1+2q_p(2)p+q_p(2)^2p^2)^{-1}\equiv 1-2q_p(2)p+3q_p(2)^2p^2 \pmod{p^3}.$$

Finally,

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\binom{-\frac{1}{2}}{k}$$
$$\equiv \left(1-2q_p(2)p+3q_p(2)^2p^2\right)\left(2q_p(2)+3pq_p(2)^2+\frac{2}{3}p^2q_p(2)^3+\frac{7}{12}p^2B_{p-3}\right)$$
$$\equiv 2q_p(2)-pq_p(2)^2+\frac{2}{3}p^2q_p(2)^3+\frac{7}{12}p^2B_{p-3} \pmod{p^3}.$$

Note that by (ii) the right-hand side is just $-H(1,(p-1)/2)=-\sum_{k=1}^{(p-1)/2}\frac{1}{k}$. □

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