# On the linearity of higher-dimensional blocking sets

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**Abstract**

A small minimal $k$-blocking set $B$ in $\mathrm{PG}(n,q)$, $q=p^t$, $p$ prime, is a set of less than $3(q^k+1)/2$ points in $\mathrm{PG}(n,q)$, such that every $(n-k)$-dimensional space contains at least one point of $B$ and such that no proper subset of $B$ satisfies this property. The *linearity conjecture* states that all small minimal $k$-blocking sets in $\mathrm{PG}(n,q)$ are linear over a subfield $\mathbb{F}_{p^e}$ of $\mathbb{F}_q$. Apart from a few cases, this conjecture is still open. In this paper, we show that to prove the linearity conjecture for $k$-blocking sets in $\mathrm{PG}(n,p^t)$, with exponent $e$ and $p^e \geq 7$, it is sufficient to prove it for one value of $n$ that is at least $2k$. Furthermore, we show that the linearity of small minimal blocking sets in $\mathrm{PG}(2,q)$ implies the linearity of small minimal $k$-blocking sets in $\mathrm{PG}(n,p^t)$, with exponent $e$, with $p^e \geq t/e+11$.

**Keywords:** blocking set, linear set, linearity conjecture

## 1 Introduction and preliminaries

If V is a vectorspace, then we denote the corresponding projective space by $\mathrm{PG}(V)$. If V has dimension $n$ over the finite field $\mathbb{F}_q$, with $q$ elements, $q=p^t$, $p$ prime, then we also write V as $\mathrm{V}(n,q)$ and $\mathrm{PG}(V)$ as $\mathrm{PG}(n-1,q)$. A $k$-dimensional space will be called a $k$-space.

A *$k$-blocking set* in $\mathrm{PG}(n,q)$ is a set $B$ of points such that every $(n-k)$-space of $\mathrm{PG}(n,q)$ contains at least one point of $B$. A $k$-blocking set $B$ is called *small* if $|B|<3(q^k+1)/2$ and *minimal* if no proper subset of $B$ is a $k$-blocking set. The points of a $k$-space of $\mathrm{PG}(n,q)$ form a $k$-blocking set, and every $k$-blocking set containing a $k$-space is called *trivial*. Every small minimal $k$-blocking set $B$ in $\mathrm{PG}(n,p^t)$, $p$ prime, has an *exponent* $e$, defined to be the largest integer for which every $(n-k)$-space intersects $B$ in 1 mod $p^e$ points. The fact that every small minimal $k$-blocking set has an exponent $e \geq 1$ follows from a result of Szőnyi and Weiner and will be explained in Section 2. A minimal $k$-blocking set $B$ in $\mathrm{PG}(n,q)$ is of *Rédei-type* if there exists a hyperplane containing $|B|-q^k$ points of $B$; this

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is the maximum number possible if $B$ is small and spans $\mathrm{PG}(n,q)$. For a long time, all constructed small minimal $k$-blocking sets were of Rédei-type, and it was conjectured that all small minimal $k$-blocking sets must be of Rédei-type. In 1998, Polito and Polverino [9] used a construction of Lunardon [8] to construct small minimal *linear* blocking sets that were not of Rédei-type, disproving this conjecture. Soon people conjectured that all small minimal $k$-blocking sets in $\mathrm{PG}(n,q)$ must be linear. In 2008, the 'Linearity conjecture' was for the first time formally stated in the literature, by Sziklai [15].

A point set $S$ in $\mathrm{PG}(V)$, where V is an $(n+1)$-dimensional vector space over $\mathbb{F}_{p^t}$, is called *linear* if there exists a subset $U$ of V that forms an $\mathbb{F}_{p_0}$-vector space for some $\mathbb{F}_{p_0} \subset \mathbb{F}_{p^t}$, such that $S = \mathcal{B}(U)$, where

$$\mathcal{B}(U) := \{\langle u \rangle_{\mathbb{F}_{p^t}} \ : \ u \in U \setminus \{0\}\}.$$

If we want to specify the subfield we call $S$ an $\mathbb{F}_{p_0}$*-linear set* (of $\mathrm{PG}(n,p^t)$).

We have a one-to-one correspondence between the points of $\mathrm{PG}(n,p_0^h)$ and the elements of a Desarguesian $(h-1)$-spread $\mathcal{D}$ of $\mathrm{PG}(h(n+1)-1,p_0)$. This gives us a different view on linear sets; namely, an $\mathbb{F}_{p_0}$-linear set is a set $S$ of points of $\mathrm{PG}(n,p_0^h)$ for which there exists a subspace $\pi$ in $\mathrm{PG}(h(n+1)-1,p_0)$ such that the points of $S$ correspond to the elements of $\mathcal{D}$ that have a non-empty intersection with $\pi$. We identify the elements of $\mathcal{D}$ with the points of $\mathrm{PG}(n,p_0^h)$, so we can view $\mathcal{B}(\pi)$ as a subset of $\mathcal{D}$, i.e.

$$\mathcal{B}(\pi) = \{S \in \mathcal{D} | S \cap \pi \neq \emptyset\}.$$

If we want to denote the element of $\mathcal{D}$ corresponding to the point $P$ of $\mathrm{PG}(n,p_0^h)$, we write $\mathcal{S}(P)$, analogously, we denote the set of elements of $\mathcal{D}$ corresponding to a subspace $H$ of $\mathrm{PG}(n,p_0^h)$, by $\mathcal{S}(H)$. For more information on this approach to linear sets, we refer to [7].

To avoid confusion, subspaces of $\mathrm{PG}(n,p_0^h)$ will be denoted by capital letters, while subspaces of $\mathrm{PG}(h(n+1)-1,p_0)$ will be denoted by lower-case letters.

**Remark 1.** The following well-known property will be used throughout this paper: if $\mathcal{B}(\pi)$ is an $\mathbb{F}_{p_0}$-linear set in $\mathrm{PG}(n,p_0^h)$, where $\pi$ is a $d$-dimensional subspace of $\mathrm{PG}(h(n+1)-1,p_0)$, then for every point $x$ in $\mathrm{PG}(h(n+1)-1,p_0)$, contained in an element of $\mathcal{B}(\pi)$, there is a $d$-dimensional space $\pi'$, through $x$, such that $\mathcal{B}(\pi) = \mathcal{B}(\pi')$. This is a direct consequence of the fact that the elementwise stabilisor of $\mathcal{D}$ in $\mathrm{P\Gamma L}(h(n+1),p_0)$ acts transitively on the points of one element of $\mathcal{D}$.

To our knowledge, the Linearity conjecture for $k$-blocking sets $B$ in $\mathrm{PG}(n,p^t)$, $p$ prime, is still open, except in the following cases:

- $t=1$ (for $n=2$, see [1]; for $n>2$, this is a corollary of Theorem 1 (i));
- $t=2$ (for $n=2$, see [13]; for $k=1$, see [12]; for $k \geq 1$, see [3] and [16]);
- $t=3$ (for $n=2$, see [10]; for $k=1$, see [12]; for $k \geq 1$, see [6] and independently [4],[5]);

- $B$ is of Rédei-type (for $n = 2$, see [2]; for $n > 2$, see [11]);
- $B$ spans an $tk$-dimenional space (see [14, Theorem 3.14]).

It should be noted that in $\mathrm{PG}(2, p^t)$, for $t = 1, 2, 3$, all small minimal blocking sets are of Rédei-type. Storme and Weiner show in [12] that small minimal 1-blocking sets in $\mathrm{PG}(n, p^t)$, $t = 2, 3$, are of Rédei-type too. The proofs rely on the fact that for $t = 2, 3$, small minimal blocking sets in $\mathrm{PG}(2, p^t)$ are listed. The special case $k = 1$ in Main Theorem 1 of this paper shows that using the (assumed) linearity of planar small minimal blocking sets, it is possible to prove the linearity of small minimal 1-blocking sets in $\mathrm{PG}(n, p^t)$, which reproofs the mentioned statements of Storme and Weiner in the cases $t = 2, 3$.

The techniques developed in [6] to show the linearity of $k$-blocking sets in $\mathrm{PG}(n, p^3)$, using the linearity of 1-blocking sets in $\mathrm{PG}(n, p^3)$, can be modified to apply for general $t$. This will be Main Theorem 2 of this paper. In particular, this theorem reproofs the results of [16], [6], [4], [5].

In this paper, we prove the following main theorems. Recall that the exponent $e$ of a small minimal $k$-blocking set is the largest integer such that every $(n-k)$-space meets in 1 mod $p^e$ points. Theorem 1 (i) will assure that the exponent of a small minimal blocking set is at least 1.

**Main Theorem 1.** *If for a certain pair $(k, n^*)$ with $n^* \geq 2k$, all small minimal $k$-blocking sets in $\mathrm{PG}(n^*, p^t)$ are linear, then for all $n > k$, all small minimal $k$-blocking sets with exponent $e$ in $\mathrm{PG}(n, p^t)$, $p$ prime, $p^e \geq 7$, are linear.*

In particular, this shows that if the linearity conjecture holds in the plane, it holds for all small minimal 1-blocking sets with exponent $e$ in $\mathrm{PG}(n, p^t)$, $p^e \geq 7$.

**Main Theorem 2.** *If all small minimal 1-blocking sets in $\mathrm{PG}(n, p^t)$ are linear, then all small minimal $k$-blocking sets with exponent $e$ in $\mathrm{PG}(n, p^t)$, $n > k$, $p^e \geq t/e + 11$, are linear.*

Combining the two main theorems yields the following corollary.

**Corollary 1.** *If the linearity conjecture holds in the plane, it holds for all small minimal $k$-blocking sets with exponent $e$ in $\mathrm{PG}(n, p^t)$, $n > k$, $p$ prime, $p^e \geq t/e + 11$.*

## 2 Previous results

In this section, we list a few results on the linearity of small minimal $k$-blocking sets and on the size of small $k$-blocking sets that will be used throughout this paper. The first of the following theorems of Szőnyi and Weiner has the linearity of small minimal $k$-blocking sets in projective spaces over prime fields as a corollary.

**Theorem 1.** *Let $B$ be a $k$-blocking set in $\mathrm{PG}(n, q)$, $q = p^t$, $p$ prime.*

*(i)* [14, Theorem 2.7] *If $B$ is small and minimal, then $B$ intersects every subspace of* $\mathrm{PG}(n,q)$ *in* 1 *mod $p$ or zero points.*

*(ii)* [14, Lemma 3.1] *If $|B| \leq 2q^k$ and every $(n-k)$-space intersects $B$ in* 1 *mod $p$ points, then $B$ is minimal.*

*(iii)* [14, Corollary 3.2] *If $B$ is small and minimal, then the projection of $B$ from a point $Q \notin B$ onto a hyperplane $H$ skew to $Q$ is a small minimal $k$-blocking set in $H$.*

*(iv)* [14, Corollary 3.7] *The size of a non-trivial $k$-blocking set in* $\mathrm{PG}(n,p^t)$*, $p$ prime, with exponent $e$, is at least $p^{tk}+1+p^e\lceil\frac{p^{tk}/p^e+1}{p^e+1}\rceil$.*

Part (iv) of the previous theorem gives a lower bound on the size of a $k$-blocking set. In this paper, we will work with the following, weaker, lower bound.

**Corollary 2.** *The size of a non-trivial $k$-blocking set in* $\mathrm{PG}(n,p^t)$*, $p$ prime, with exponent $e$, is at least $p^{tk}+p^{tk-e}-p^{tk-2e}$.*

If a blocking set $B$ in $\mathrm{PG}(2,q)$ is $\mathbb{F}_{p_0}$-linear, then every line intersects $B$ in an $\mathbb{F}_{p_0}$-linear set. If $B$ is small, many of these $\mathbb{F}_{p_0}$-linear sets are $\mathbb{F}_{p_0}$-sublines (i.e. $\mathbb{F}_{p_0}$-linear sets of rank 2). The following theorem of Sziklai shows that for *all* small minimal blocking sets, this property holds.

**Theorem 2.** *(i)* [15, Proposition 4.17 (2)] *If $B$ is a small minimal blocking set in* $\mathrm{PG}(2,q)$*, with $|B|=q+\kappa$, then the number of $(p_0+1)$-secants to $B$ through a point $P$ of $B$ lying on a $(p_0+1)$-secant to $B$, is at least*

$$q/p_0 - 3(\kappa-1)/p_0 + 2.$$

*(ii)* [15, Theorem 4.16] *Let $B$ be a small minimal blocking set with exponent $e$ in* $\mathrm{PG}(2,q)$*. If for a certain line $L$, $|L\cap B| = p^e+1$, then $\mathbb{F}_{p^e}$ is a subfield of $\mathbb{F}_q$ and $L\cap B$ is $\mathbb{F}_{p^e}$-linear.*

The next theorem, by Lavrauw and Van de Voorde, determines the intersection of an $\mathbb{F}_p$-subline with an $\mathbb{F}_p$-linear set; all possibilities for the size of the intersection that are obtained in this statement, can occur (see [7]). The bound on the characteristic of the field appearing in Main Theorem 2 arises from this theorem.

**Theorem 3.** [7, Theorem 8] *An $\mathbb{F}_{p_0}$-linear set of rank $k$ in* $\mathrm{PG}(n,p^t)$ *and an $\mathbb{F}_{p_0}$-subline (i.e. an $\mathbb{F}_{p_0}$-linear set of rank* 2*), intersect in $0,1,2,\ldots,k$ or $p_0+1$ points.*

The following lemma is a straightforward extension of [6, Lemma 7], where the authors proved it for $h=3$.

**Lemma 1.** *If $B$ is a subset of* $\mathrm{PG}(n,p_0^h)$*, $p_0\geq 7$, intersecting every $(n-k)$-space, $k\geq 1$, in $1 \mod p_0$ points, and $\Pi$ is an $(n-k+s)$-space, $s<k$, then either*

$$|B\cap\Pi| < p_0^{hs}+p_0^{hs-1}+p_0^{hs-2}+3p_0^{hs-3}$$

*or*

$$|B \cap \Pi| > p_0^{hs+1} - p_0^{hs-1} - p_0^{hs-2} - 3p_0^{hs-3}.$$

*Furthermore,* $|B| < p_0^{hk} + p_0^{hk-1} + p_0^{hk-2} + 3p_0^{hk-3}$.

*Proof.* Let $\Pi$ be an $(n-k+s)$-space of $\mathrm{PG}(n, p_0^h)$, $s \leq k$, and put $B_\Pi := B \cap \Pi$. Let $x_i$ denote the number of $(n-k)$-spaces of $\Pi$ intersecting $B_\Pi$ in $i$ points. Counting the number of $(n-k)$-spaces, the number of incident pairs $(P, \Sigma)$ with $P \in B_\Pi, P \in \Sigma, \Sigma$ an $(n-k)$-space, and the number of triples $(P_1, P_2, \Sigma)$, with $P_1, P_2 \in B_\Pi$, $P_1 \neq P_2$, $P_1, P_2 \in \Sigma$, $\Sigma$ an $(n-k)$-space yields:

$$\sum_i x_i = \begin{bmatrix} n-k+s+1 \\ n-k+1 \end{bmatrix}_{p_0^h}, \quad (1)$$

$$\sum_i i x_i = |B_\Pi| \begin{bmatrix} n-k+s \\ n-k \end{bmatrix}_{p_0^h}, \quad (2)$$

$$\sum i(i-1)x_i = |B_\Pi|(|B_\Pi|-1) \begin{bmatrix} n-k+s-1 \\ n-k-1 \end{bmatrix}_{p_0^h}. \quad (3)$$

Since we assume that every $(n-k)$-space intersects $B$ in $1 \mod p_0$ points, it follows that every $(n-k)$-space of $\Pi$ intersect $B_\Pi$ in $1 \mod p_0$ points, and hence $\sum_i (i-1)(i-1-p_0)x_i \geq 0$. Using Equations (1), (2), and (3), this yields that

$$|B_\Pi|(|B_\Pi|-1)(p_0^{hn-hk+h}-1)(p_0^{hn-hk}-1) - (p_0+1)|B_\Pi|(p_0^{hn-hk+hs}-1)(p_0^{hn-hk+h}-1)$$
$$+(p_0+1)(p_0^{hn-hk+hs+h}-1)(p_0^{hn-hk+hs}-1) \geq 0.$$

Putting $|B_\Pi| = p_0^{hs} + p_0^{hs-1} + p_0^{hs-2} + 3p_0^{hs-3}$ in this inequality, with $p_0 \geq 7$, gives a contradiction; putting $|B_\Pi| = p_0^{hs+1} - p_0^{hs-1} - p_0^{hs-2} - 3p_0^{hs-3}$ in this inequality, with $p_0 \geq 7$, gives a contradiction if $s < k$. For $s = k$, it is sufficient to note that when $|B|$ is the size of a $k$-space, the inequality holds, to deduce that $|B| < p_0^{hk} + p_0^{hk-1} + p_0^{hk-2} + 3p_0^{hk-3}$. The statement follows. □

Let $B$ be a subset of $\mathrm{PG}(n, p_0^h)$, $p_0 \geq 7$, intersecting every $(n-k)$-space, $k \geq 1$, in $1 \mod p_0$ points. From now on, we call an $(n-k+s)$-space *small* if it meets $B$ in less than $p_0^{hs} + p_0^{hs-1} + p_0^{hs-2} + 3p_0^{hs-3}$ points, and *large* if it meets $B$ in more than $p_0^{hs+1} - p_0^{hs-1} - p_0^{hs-2} - 3p_0^{hs-3}$ points, and it follows from the previous lemma that each $(n-k+s)$-space is either small or large.

The following Lemma and its corollaries show that if all $(n-k)$-spaces meet a $k$-blocking set $B$ in 1 mod $p_0$ points, then every subspace that intersects $B$, intersects it in 1 mod $p_0$ points.

**Lemma 2.** *Let $B$ be a small minimal $k$-blocking set in $\mathrm{PG}(n, p_0^h)$ and let $L$ be a line such that $1 < |B \cap L| < p_0^h + 1$. For all $i \in \{1, \ldots, n-k\}$ there exists an $i$-space $\pi_i$ through $L$ such that $B \cap \pi_i = B \cap L$.*

*Proof.* It follows from Theorem 1 that every subspace through $L$ intersects $B \setminus L$ in zero or at least $p$ points, where $p_0 = p^e$, $p$ prime. We proceed by induction on the dimension $i$. The statement obviously holds for $i = 1$. Suppose there exists an $i$-space $\Pi_i$ through $L$ such that $\Pi_i \cap B{=}L \cap B$, with $i \leq n - k - 1$. If there is no $(i+1)$-space intersecting $B$ only in points of $L$, then the number of points of $B$ is at least

$$|B \cap L| + p(p_0^{h(n-i-1)} + p_0^{h(n-i-2)} + \ldots + p_0^h + 1),$$

but by Lemma 1 $|B| \leq p_0^{hk} + p_0^{hk-1} + p_0^{hk-2} + p_0^{hk-3}$. If $i < n - k$ this is a contradiction. We may conclude that there exists an $i$-space $\Pi_i$ through $L$ such that $B \cap L = B \cap \Pi_i$, $\forall i \in \{1, \ldots, n-k\}$. □

Using Lemma 2, the following corollaries follow easily.

**Corollary 3.** (see also [14, Corollary 3.11]) *Every line meets a small minimal $k$-blocking set in $\mathrm{PG}(n, p^t)$, $p$ prime, with exponent $e$ in $1$ mod $p^e$ or zero points.*

*Proof.* Suppose the line $L$ meets the small minimal $k$-blocking set in $x$ points, where $1 \leq x \leq p^t$. By Lemma 2, the line $L$ is contained in an $(n-k)$-space $\pi$ such that $B \cap \pi = B \cap L$. Since every $(n-k)$-space meets the $k$-blocking set $B$ with exponent $e$ in $1$ mod $p^e$ points, the corollary follows. □

By considering all lines through a certain point of $B$ in some subspace, we get the following corollary.

**Corollary 4.** (see also [14, Corollary 3.11]) *Every subspace meets a small minimal $k$-blocking set in $\mathrm{PG}(n, p^t)$, $p$ prime, with exponent $e$ in $1$ mod $p^e$ or zero points.*

## 3 On the $(p_0{+}1)$-secants to a small minimal $k$-blocking set

In this section, we show that Theorem 2 on planar blocking sets can be extended to a similar result on $k$-blocking sets in $\mathrm{PG}(n, q)$.

**Lemma 3.** *Let $B$ be a small minimal $k$-blocking set with exponent $e$ in $\mathrm{PG}(n, p_0^h)$, $p_0 := p^e \geq 7$, $p$ prime, $\mathbf{n \geq 2k+1}$. The number of points, not in $B$, that do not lie on a secant line to $B$ is at least*

$$(p_0^{h(n+1)} - 1)/(p_0^h + 1) - (p_0^{2hk-2} + 2p_0^{2hk-3})(p_0^h + 1) - p_0^{hk} - p_0^{hk-1} - p_0^{hk-2} - 3p_0^{hk-3},$$

*and this number is larger than the number of points in $\mathrm{PG}(n-1, p_0^h)$.*

*Proof.* By Corollary 3, the number of secant lines to $B$ is at most $\frac{|B|(|B|-1)}{(p_0+1)p_0}$. By Lemma 1, the number of points in $B$ is at most $p_0^{hk} + p_0^{hk-1} + p_0^{hk-2} + 3p_0^{hk-3}$, hence the number of secant lines is at most $p_0^{2hk-2} + 2p_0^{2hk-3}$. This means that the number of points on at least one secant line is at most $(p_0^{2hk-2}+2p_0^{2hk-3})(p_0^h+1)$. It follows that the number of points in $\mathrm{PG}(n,p_0^h)$, not in $B$, not on a secant to $B$ is at least $(p_0^{h(n+1)}-1)/(p_0^h+1)-(p_0^{2hk-2}+2p_0^{2hk-3})(p_0^h+1)-p_0^{hk}-p_0^{hk-1}-p_0^{hk-2}-3p_0^{hk-3}$. Since we assume that $n\geq 2k+1$ and $p_0\geq 7$, the last part of the statement follows. □

We first extend Theorem 2 (i) to 1-blocking sets in $\mathrm{PG}(n,q)$.

**Lemma 4.** *A point of a small minimal 1-blocking set $B$ with exponent $e$ in $\mathrm{PG}(n,p_0^h)$, $p_0:=p^e\geq 7$, $p$ prime, lying on a $(p_0+1)$-secant, lies on at least $p_0^{h-1}-4p_0^{h-2}+1$ $(p_0+1)$-secants.*

*Proof.* We proceed by induction on the dimension $n$. If $n=2$, by Theorem 2, the number of $(p_0+1)$-secants through $P$ is at least $q/p_0-3(\kappa-1)/p_0+2$, where $|B|=q+\kappa$. By Lemma 1, $\kappa$ is at most $p_0^{h-1}+p_0^{h-2}+3p_0^{h-3}$, which means that the number of $(p_0+1)$-secants is at least $p_0^{h-1}-4p_0^{h-2}+1$. This proves the statement for $n=2$.

Now assume $n\geq 3$. From Lemma 3 (observe that, since $n\geq 3$ and $k=1$, $n\geq 2k+1$), we know that there is a point $Q$, not lying on a secant line to $B$. Project $B$ from the point $Q$ onto a hyperplane through $P$ and not through $Q$. It is clear that the number of $(p_0+1)$-secants through $P$ to the projection of $B$ is the number of $(p_0+1)$-secants through $P$ to $B$. By the induction hypothesis, this number is at least $p_0^{h-1}-4p_0^{h-2}+1$. □

**Lemma 5.** *Let $\Pi$ be an $(n-k)$-space of $\mathrm{PG}(n,p_0^h)$, $k>1$, $p_0\geq 7$. If $\Pi$ intersects a small minimal $k$-blocking set $B$ with exponent $e$ in $\mathrm{PG}(n,p_0^h)$, $p_0:=p^e\geq 7$, $p$ prime in $p_0+1$ points, then there are at most $3p_0^{hk-h-3}$ large $(n-k+1)$-spaces through $\Pi$.*

*Proof.* Suppose there are $y$ large $(n-k+1)$-spaces through $\Pi$. A small $(n-k+1)$-space through $\Pi$ meets $B$ clearly in a small 1-blocking set, which is in this case, non-trivial and hence, by Theorem 2, has at least $p_0^h+p_0^{h-1}-p_0^{h-2}$ points.

Then the number of points in $B$ is at least

$$y(p_0^{h+1}-p_0^{h-1}-p_0^{h-2}-3p_0^{h-3}-p_0-1)+$$
$$((p_0^{hk}-1)/(p_0^h-1)-y)(p_0^h+p_0^{h-1}-p_0^{h-2}-p_0-1)+p_0+1 \;(*)$$

which is at most $p_0^{hk}+p_0^{hk-1}+p_0^{hk-2}+3p_0^{hk-3}$. This yields $y\leq 3p_0^{hk-h-3}$. □

**Theorem 4.** *A point of a small minimal $k$-blocking set $B$ with exponent $e$ in $\mathrm{PG}(n,p_0^h)$, $p_0:=p^e\geq 7$, $p$ prime, $k>1$, lying on a $(p_0+1)$-secant, lies on at least $((p_0^{hk}-1)/(p_0^h-1)-3p_0^{hk-h-3})(p_0^{h-1}-4p_0^{h-2})+1$ $(p_0+1)$-secants.*

*Proof.* Let $P$ be a point on a $(p_0+1)$-secant $L$. By Lemma 2, there is an $(n-k)$-space $\Pi$ through $L$ such that $B\cap\Pi=B\cap L$. Let $\Sigma$ be a small $(n-k+1)$-space. It is clear that the space $\Sigma$ meets $B$ in a small 1-blocking set $B'$. Every $(n-k)$-space contained in $\Sigma$ meets $B'$ in 1 mod $p_0$ points. By Theorem 1 (ii), $B'$ is a small minimal 1-blocking set in $\Sigma$. For every small $(n-k+1)$-space $\Sigma_i$ through $\pi$, $P$ is a point in $\Sigma_i$, lying on a $(p_0+1)$-secant in $\Sigma_i$, and hence, by Lemma 4, $P$ lies on at least $p_0^{h-1}-4p_0^{h-2}+1$ $(p_0+1)$-secants to $B$ in $\Sigma_i$. From Lemma 5, we get that the number of small $(n-k+1)$-spaces $\Sigma_i$ through $\Pi$ is at least $(p_0^{hk}-1)/(p_0^h-1)-3p_0^{hk-h-3}$, hence, the number of $(p_0+1)$-secants to $B$ through $P$ is at least $((p_0^{hk}-1)/(p_0^h-1)-3p_0^{hk-h-3})(p_0^{h-1}-4p_0^{h-2})+1$. □

We will now show that Theorem 2 (ii) can be extended to $k$-blocking sets in $\mathrm{PG}(n, q)$. We start with the case $k = 1$.

**Lemma 6.** *Let $B$ be a small minimal 1-blocking set with exponent $e$ in $\mathrm{PG}(n, q)$, $q = p^t$. If for a certain line $L$, $|L \cap B| = p^e + 1$, then $\mathbb{F}_{p^e}$ is a subfield of $\mathbb{F}_q$ and $L \cap B$ is $\mathbb{F}_{p^e}$-linear.*

*Proof.* We proceed by induction on $n$. For $n = 2$, the statement follows from Theorem 2 (ii), hence, let $n > 2$. Let $L$ be a line, meeting $B$ in $p^e + 1$ points and let $H$ be a hyperplane through $L$. A plane through $L$ containing a point of $B$, not on $L$, contains at least $p^{2e}$ points of $B$, not on $L$ by Theorem 1 (i). If all $q^{n-2}$ planes through $L$, not in $H$, contain an extra point of $B$, then $|B| \geq p^{2e} q^{n-2}$, which is larger than $p^h + p^{h-1} + p^{h-2} + 3p^{h-3}$, a contradiction by Lemma 1. Let $Q$ be a point on a plane $\pi$ through $L$, not in $H$ such that $\pi$ meets $B$ only in points of $L$. The projection of $B$ onto $H$ is a small minimal 1-blocking set $B'$ in $H$ (see Theorem 1 (iii)), for which $L$ is a $(p^e + 1)$-secant. The intersection $B' \cap L$ is by the induction hypothesis an $\mathbb{F}_{p^e}$-linear set. Since $B \cap L = B' \cap L$, the statement follows. □

Finally, we extend Theorem 2 (ii) to a theorem on $k$-blocking sets in $\mathrm{PG}(n, q)$.

**Theorem 5.** *Let $B$ be a small minimal $k$-blocking set with exponent $e$ in $\mathrm{PG}(n, q)$, $q = p^t$. If for a certain line $L$, $|L \cap B| = p^e + 1$, $p^e \geq 7$, then $\mathbb{F}_{p^e}$ is a subfield of $\mathbb{F}_q$ and $L \cap B$ is $\mathbb{F}_{p^e}$-linear.*

*Proof.* Let $L$ be a $p^e + 1$-secant to $B$. By Lemma 5, there is at least one small $(n - k + 1)$-space $\Pi$ through $L$. Since $\Pi \cap B$ is a small 1-blocking set to $B$, and every $(n - k)$-space, contained in $\Pi$ meets $B$ in 1 mod $p^e$ points, by Theorem 1 (ii), $B$ is minimal. By Lemma 6, $L \cap B$ is an $\mathbb{F}_{p^e}$-linear set. □

# 4 The proof of Main Theorem 1

In this section, we will prove Main Theorem 1, that, roughly speaking, states that if we can prove the linearity for $k$-blocking sets in $\mathrm{PG}(n, q)$ for a certain value of $n$, then it is true for all $n$. It is clear from the definition of a $k$-blocking set that we can only consider $k$-blocking sets in $\mathrm{PG}(n, q)$ where $1 \leq k \leq n - 1$, and whenever we use the notation $k$-blocking set in $\mathrm{PG}(n, q)$, we assume that the above condition is satisfied.

> From now on, if we want to state that for the pair $(k, n^*)$, all small minimal $k$-blocking sets in $\mathrm{PG}(n^*, q)$ are linear, we say that the condition $(H_{k,n^*})$ holds.

To prove Main Theorem 1, we need to show that if $(H_{k,n^*})$ holds, then $(H_{k,n})$ holds for all $n \geq k + 1$. The following observation shows that we only have to deal with the case $n \geq n^*$.

**Lemma 7.** *If $(H_{k,n^*})$ holds, then $(H_{k,n})$ holds for all $n$ with $k + 1 \leq n \leq n^*$.*

*Proof.* A small minimal $k$-blocking set $B$ in $\mathrm{PG}(n,q)$, with $k+1 \le n \le n^*$, can be embedded in $\mathrm{PG}(n^*,q)$, in which it clearly is a small minimal $k$-blocking set. Since $(H_{k,n^*})$ holds, $B$ is linear, hence, $(H_{k,n})$ holds. □

The main idea for the proof of Main Theorem 1 is to prove that all the $(p_0+1)$-secants through a particular point $P$ of a $k$-blocking set $B$ span a $hk$-dimensional space $\mu$ over $\mathbb{F}_{p_0}$, and to prove that the linear blocking set defined by $\mu$ is exactly the $k$-blocking set $B$.

**Lemma 8.** *Assume $(H_{k,n-1})$ and $n-1 \ge 2k$, and let $B$ denote a small minimal $k$-blocking set with exponent $e$ in $\mathrm{PG}(n,p^t)$, $p$ prime, $p^e \ge 7$, $t \ge 2$. Let $\Pi$ be a plane in $\mathrm{PG}(n,p^t)$.*

- *(i) There is a 3-space $\Sigma$ through $\Pi$ meeting $B$ only in points of $\Pi$ and containing a point $Q$ not lying on a secant line to $B$* **if $\mathbf{k>2}$**.
- *(ii) The intersection $\Pi \cap B$, is a linear set* **if $\mathbf{k>2}$**.

*Proof.* Let $\Pi$ be a plane of $\mathrm{PG}(n,p^t)$, $p_0 := p^e \ge 7$. By Lemma 3, there are at least

$$s := (p_0^{h(n+1)}-1)/(p_0^h+1) - (p_0^{2hk-2}+2p_0^{2hk-3})(p_0^h+1) - p_0^{hk} - p_0^{hk-1} - p_0^{hk-2} - 3p_0^{hk-3},$$

points $Q \notin \{B\}$ not lying on a secant line to $B$. This means that there are at least $r := (s-(p_0^{2h}+p_0^h+1))/p_0^{3h}$ 3-spaces through $\Pi$ that contain a point that does not lie on a secant line to $B$ and is not contained in $B$ nor in $\Pi$. If all $r$ 3-spaces contain a point $Q$ of $B$ that is not contained in $\Pi$, then the number of points in $B$ is at least $r$. It is easy to check that this is a contradiction if $n-1 \ge 2k$, $p^e \ge 7$, and $k>2$.

Hence, there is a 3-space $\Sigma$ through $\Pi$ meeting $B$ only in points of $\Pi$ and containing a point $Q$ not lying only on a secant line to $B$. The projection of $B$ from $Q$ onto a hyperplane containing $\Pi$ is a small minimal $k$-blocking set $\bar{B}$ in $\mathrm{PG}(n-1,q)$ (see Theorem 1(iii)), which is, by $(H_{k,n-1})$, a linear set. Now $\Pi \cap \bar{B} = \Pi \cap B$, since the space $\langle Q, \pi \rangle$ meets $B$ only in points of $\Pi$, and hence, the set $\Pi \cap B$ is linear. □

**Corollary 5.** *Assume $(H_{k,n-1})$, $k>2$, $(n-1) \ge 2k$ and let $B$ denote a small minimal $k$-blocking set with exponent $e$ in $\mathrm{PG}(n,p^t)$, $p$ prime, $p^e \ge 7$, $t \ge 2$. The intersection of a line with $B$ is an $\mathbb{F}_{p^e}$-linear set.*

**Remark 2.** The linear set $\mathcal{B}(\mu)$ does not determine the subspace $\mu$ in a unique way; by Remark 1, we can choose $\mu$ through a fixed point $S(P)$, with $P \in \mathcal{B}(\mu)$. Note that there may exist different spaces $\mu$ and $\mu'$, *through the same point of* $\mathrm{PG}(h(n+1)-1,p)$, such that $\mathcal{B}(\mu) = \mathcal{B}(\mu')$. If $\mu$ is a line, however, if we fix a point $x$ of an element of $\mathcal{B}(\mu)$, then there is a unique line $\mu'$ through $x$ such that $\mathcal{B}(\mu) = \mathcal{B}(\mu')$ since, in this case, $\mu'$ is the unique transversal line through $x$ to the regulus $\mathcal{B}(\mu)$. This observation is crucial for the proof of the following lemma.

**Lemma 9.** *Assume $(H_{k,n-1})$, $n-1 \ge 2k$, and let $B$ be a small minimal $k$-blocking set with exponent $e$ in $\mathrm{PG}(n,p^t)$, $p$ prime, $p_0 := p^e \ge 7$. Denote the $(p_0+1)$-secants through a point $P$ of $B$ that lies on at least one $(p_0+1)$-secant, by $L_1,\ldots,L_s$. Let $x$ be a point of $\mathcal{S}(P)$ and let $\ell_i$ be the line through $x$ such that $\mathcal{B}(\ell_i) = L_i \cap B$. The following statements hold:*

*(i) The space $\langle \ell_1, \ldots, \ell_s \rangle$ has dimension $hk$.*

*(ii) $\mathcal{B}(\langle \ell_i, \ell_j \rangle) \subseteq B$ for $1 \leq i \neq j \leq s$.*

*Proof.* (i) Let $P$ be a point of $B$ lying on a $(p_0+1)$-secant, and let $H$ be a hyperplane through $P$. By Lemma 6, there is a point $Q$, not in $B$ and not in $H$, not lying on a secant line to $B$. The projection of $B$ from $Q$ onto $H$ is a small minimal $k$-blocking set $\bar{B}$ in $H \cong \mathrm{PG}(n-1,q)$ (Theorem 1 (iii)). By $(H_{k,n-1})$, $\bar{B}$ is a linear set. Every line meets $B$ in 1 mod $p_0$ or 0 points, which implies that every line in $H$ meets $\bar{B}$ in 1 mod $p_0$ or 0 points, hence, $\bar{B}$ is $\mathbb{F}_{p_0}$-linear. Take a fixed point $x$ in $\mathcal{S}(P)$. Since $\bar{B}$ is an $\mathbb{F}_{p_0}$-linear set, there is an $hk$-dimensional space $\mu$ in $\mathrm{PG}(h(n+1)-1,p_0)$, through $x$, such that $\mathcal{B}(\mu)=\bar{B}$.

From Lemma 4, we get that the number of $(p_0+1)$-secants through $P$ to $B$ is at least $z := ((p_0^{hk}-1)/(p_0^h-1)-3p_0^{hk-h-3})(p_0^{h-1}-4p_0^{h-2})+1$, denote them by $L_1,\ldots,L_s$ and let $\ell_1,\ldots,\ell_s$ be the lines through $x$ such that $\mathcal{B}(\ell_i)=B\cap L_i$. These lines exist by Theorem 5. Note that, by Remark 2, $\mathcal{B}(\ell_i)$ determines the line $\ell_i$ through $x$ in a unique way, and that $\ell_i \neq \ell_j$ for all $i \neq j$.

We will prove that the projection of $\ell_i$ from $\mathcal{S}(Q)$ onto $\langle \mathcal{S}(H)\rangle$ in $\mathrm{PG}(h(n+1)-1,p_0)$ is contained in $\mu$. Since $L_1$ is projected onto a $(p_0+1)$-secant $M$ to $\bar{B}$ through $P$, there is a line $m$ through $x$ in $\mathrm{PG}(h(n+1)-1,p_0)$ such that $\mathcal{B}(m)=M\cap\bar{B}$. Now $\bar{B}=\mathcal{B}(\mu)$, and $|\bar{B}\cap M|=p_0+1$, hence, there is a line $m'$ through $x$ in $\mu$ such that $\mathcal{B}(m')=\bar{B}\cap M$. Since $m$ is the unique transversal line through $x$ to $M\cap\bar{B}$ (see Remark 2), $m=m'$, and $m$ is contained in $\mu$.

This implies that the space $W := \langle \ell_1,\ldots,\ell_s\rangle$ is contained in $\langle \mathcal{S}(Q),\mu\rangle$, hence, $W$ has dimension at most $hk+h$. Suppose that $W$ has dimension at least $hk+1$, then it intersects the $(h-1)$-dimensional space $\mathcal{S}(Q)$ in at least a point. But this holds for all $\mathcal{S}(Q)$ corresponding to points, not in $B$, such that $Q$ does not lie on a secant line to $B$. This number is at least

$$(p_0^{h(n+1)}-1)/(p_0^h+1)-(p_0^{2hk-2}+2p_0^{2hk-3})(p_0^h+1)-p_0^{hk}-p_0^{hk-1}-p_0^{hk-2}-3p_0^{hk-3}$$

by Lemma 3, which is larger than the number of points in $W$, since $W$ is at most $(hk+h)$-dimensional, a contradiction.

From Theorem 4, we get that $W$ contains at least

$$(((p_0^{hk}-1)/(p_0^h-1)-3p_0^{hk-h-3})(p_0^{h-1}-4p_0^{h-2})+1)p_0+1$$

points, which is larger than $(p_0^{hk}-1)/(p_0-1)$ if $p_0 \geq 7$, hence, $W$ is at least $hk$-dimensional. Since we have already shown that $W$ is at most $hk$-dimensional, the statement follows.

(ii) W.l.o.g. we choose $i=1, j=2$. Let $m$ be a line in $\langle \ell_1,\ell_2\rangle$, not through $\ell_1\cap\ell_2$. Let $M$ be the line of $\mathrm{PG}(n,q^t)$ containing $\mathcal{B}(m)$ and let $H$ be a hyperplane of $\mathrm{PG}(n,q^t)$ containing the plane $\langle L_1,L_2\rangle$. We claim that there exists a point $Q$, not in $H$, such that the planes $\langle Q,L_1\rangle$, $\langle Q,L_2\rangle$ and $\langle Q,M\rangle$ only contain points of $B$ that are in $H$.

If $k>2$, this follows from Lemma 8(i). Now assume that $1\leq k\leq 2$. There are $q^{n-2}$ planes through $M$, not in in $H$. Since $M$ is at least a $(p_0+1)$-secant (Theorem 1 (i)), it holds that if a plane $\Pi$ through $M$ contains a point of $B$, that is not contained in $M$, then, $\Pi$ contains at least $p_0^2$ points of $B$, not in $M$ (again by Theorem 1(i)). Since $|B| \leq q^k + q^{k-1} + q^{k-2} + 3q^{k-3}$ (Lemma 1), and $n-1 \geq 2k$, there is at least one plane $\Pi$ through $M$, not contained in $H$ that contains only points of $B$ that are contained in $M$. Now, there is one of the $q^2$ points in $\Pi$, say $Q$, that is not contained in $M$ for which the planes $\langle Q, L_i\rangle$, $i=1,2$ only contain points of $B$ on the line $L_i$, $i=1,2$, since otherwise, the number of points in $B$ would be at least $p_0^2q^2$, a contradiction since $k \leq 2$ and $|B| \leq q^k + q^{k-1} + q^{k-2} + 3q^{k-3}$ by Lemma 1. This proves our claim.

The projection of $B$ from $Q$ onto $H$ is a small minimal $k$-blocking set $\bar{B}$ in $\mathrm{PG}(n,q)$ (Theorem 1 (iii)). By $(H_{k,n-1})$, $\bar{B}$ is a linear set, hence, it meets $\langle L_1, L_2\rangle$ in a linear set. This means that there is a space $\pi$ through $x$ such that $\langle L_1, L_2\rangle \cap B = \mathcal{B}(\pi)$. Note that, since $\langle Q, L_1\rangle$ and $\langle Q, L_2\rangle$ only contain points of $B$ that are contained in $H$, the lines $L_1$ and $L_2$ are $(p_0+1)$-secants to $\bar{B}$.

Hence, the space $\pi$ contains $\ell_i$ since $\mathcal{B}(\pi) \cap L_i = \mathcal{B}(\ell_i)$ and $\ell_i$ is the unique transversal line to the regulus $B \cap L_i$, $i=1,2$. Hence, $\mathcal{B}(\langle \ell_1, \ell_2\rangle) \subset \bar{B}$, so $\mathcal{B}(m) \subset \bar{B}$. The plane $\langle Q, M\rangle$ only contains points of $B$ that are on $M$, so $M \cap B = M \cap \bar{B}$, hence, $\mathcal{B}(m) \subset B$. Since every point of $\langle \ell_1, \ell_2\rangle$, not on $\ell_1, \ell_2$, lies on a line $m$ meeting $\ell_1$ and $\ell_2$ in different points, $\mathcal{B}(\langle \ell_1, \ell_2\rangle) \subseteq B$. □

**Proof of Main Theorem 1.**

Let $B$ be a small minimal $k$-blocking set with exponent $e$ in $\mathrm{PG}(n,p^t)$, $p$ prime, $p_0 = p^e \geq 7$ and assume that $(H_{k,n-1})$ holds with $n-1 \geq 2k$. Let $P$ be a point of $B$, lying on a $(p_0+1)$-secant. By Theorem 4, there are at least $((p_0^{hk}-1)/(p_0^h-1)-3p_0^{hk-h-3})(p_0^{h-1}-4p_0^{h-2})+1$ $(p_0+1)$-secants $L_1 \ldots, L_s$ through $P$, and by Lemma 9, the corresponding lines $\ell_1, \ldots, \ell_s$ in $\mathrm{PG}(h(n+1)-1, p_0)$, with $\mathcal{B}(\ell_i) = B \cap L_i$, $\ell_i$ through a fixed point $x$ of $\mathcal{S}(P)$, span an $hk$-dimensional space $W$. Suppose that $\mathcal{B}(W) \not\subseteq B$, and let $w$ be a point of $W$ for which $\mathcal{B}(w) \notin B$. Since the number of points lying on one of the lines of the set $\{\ell_1, \ldots, \ell_s\}$, is at least $(((p_0^{hk}-1)/(p_0^h-1)-3p_0^{hk-h-3})(p_0^{h-1}-4p_0^{h-2})+1)p_0+1$, at least one of the $(p_0^{hk}-1)/(p_0-1)$ lines through $w$, say $m$, contains two points lying on one of the lines of the set $\{\ell_1, \ldots, \ell_s\}$. By Lemma 9 (b), $\mathcal{B}(m)$ is contained in $B$, a contradiction since $\mathcal{B}(w) \in \mathcal{B}(m)$, and $\mathcal{B}(w) \notin B$.

Hence, $\mathcal{B}(W) \subseteq B$, and since $\mathcal{B}(W)$ is a small minimal linear $k$-blocking set $\mathrm{PG}(n,p^t)$, contained in the minimal $k$-blocking set $B$, $B$ equals the linear set $\mathcal{B}(W)$. Hence, we have shown that if $(H_{k,n-1})$ holds, with $n-1 \geq 2k$, then $(H_{k,n})$ holds, and repeating this argument shows that if $(H_{k,n^*})$ holds for some $n^*$, then $(H_{k,n})$ holds for all $n \geq n^*$. Since Lemma 7 shows the desired property for all $n$ with $k+1 \leq n \leq n^*$, the statement follows. □

## 5 The proof of Main Theorem 2

In this section, we will prove Main Theorem 2, stating that, if all small minimal 1-blocking sets in $\mathrm{PG}(n,p_0^h)$ are linear, then all small minimal $k$-blocking sets in $\mathrm{PG}(n,p_0^h)$, are linear, provided a condition on $p_0$ and $h$ holds.

We proved in Lemma 1 that a subspace meets the small minimal $k$-blocking set $B$ in either in a 'small' number, or in a 'large' number of points. To simplify the terminology, we call a $(n-k+s)$-space $\Pi$, $s \leq k$, for which $|B \cap \Pi| < p_0^{hs} + p_0^{hs-1} + p_0^{hs-2} + 3p_0^{hs-3}$ points, a *small* $(n-k+s)$-space. An $(n-k+s)$-space which is not small is called *large*.

**Lemma 10.** *Let $\Pi$ be an $(n-k)$-space of $\mathrm{PG}(n, p_0^h)$ and let $B$ be a small minimal $k$-blocking set with exponent $e$ in $\mathrm{PG}(n, p^t)$, $p$ prime, $p_0 := p^e \geq 7$, $k > 1$.*

*(i) If $B \cap \Pi$ is a point, then there are at most $p_0^{hk-h-2} + 4p_0^{hk-h-3} - 1$ large $(n-k+1)$-spaces through $\Pi$.*

*(ii) If $\Pi$ intersects $B$ in $p_0 + 1$ points, then there are at most $3p_0^{hk-h-3}$ large $(n-k+1)$-spaces through $\Pi$.*

*Proof.* (i) A small $(n-k+1)$-space through $\Pi$ meets $B$ in at least $p_0^h + 1$ points. Suppose there are $y$ large $(n-k+1)$-spaces through $\Pi$. Then the number of points in $B$ is at least

$$y(p_0^{h+1} - p_0^{h-1} - p_0^{h-2} - 3p_0^{h-3} - 1) + ((p_0^{hk} - 1)/(p_0^h - 1) - y)p_0^h + 1$$

which is at most $p_0^{hk} + p_0^{hk-1} + p_0^{hk-2} + 3p_0^{hk-3}$. This yields $y \leq p_0^{hk-h-2} + 4p_0^{hk-h-3} - 1$.

(ii) Suppose there are $y$ large $(n-k+1)$-spaces through $\Pi$. A small $(n-k+1)$-space through $\Pi$ meets $B$ in a linear 1-blocking set, which is in this case, non-trivial and hence, by Theorem 2, has at least $p_0^h + p_0^{h-1} - p_0^{h-2}$ points.

Then the number of points in $B$ is at least

$$y(p_0^{h+1} - p_0^{h-1} - p_0^{h-2} - 3p_0^{h-3} - p_0 - 1) +$$

$$((p_0^{hk} - 1)/(p_0^h - 1) - y)(p_0^h + p_0^{h-1} - p_0^{h-2} - p_0 - 1) + p_0 + 1 \ (*)$$

which is at most $p_0^{hk} + p_0^{hk-1} + p_0^{hk-2} + 3p_0^{hk-3}$. This yields $y \leq 3p_0^{hk-h-3}$. □

**Lemma 11.** *If $B$ is a non-trivial small minimal $k$-blocking set with exponent $e$ in $\mathrm{PG}(n, p^t)$, $p$ prime, $p_0 := p^e \geq 7$, $k > 1$, then there exist a point $P \in B$, a tangent $(n-k)$-space $\Pi$ at the point $P$ and small $(n-k+1)$-spaces $H_i$, through $\Pi$, such that there is a $(p_0 + 1)$-secant through $P$ in $H_i$, $i = 1, \ldots, p_0^{hk-h} - 5p_0^{hk-h-1}$.*

*Proof.* Let $L$ be a $(p_0 + 1)$-secant to $B$ and let $P$ be a point of $B \cap L$. Lemma 2 shows that there is an $(n-k)$-space $\Pi_L$ such that $B \cap \Pi_L = B \cap L$. By Theorem 4, $P$ lies on $((p_0^{hk} - 1)/(p_0^h - 1) - 3p_0^{hk-h-3})(p_0^{h-1} - 4p_0^{h-2}) + 1$ other $(p_0 + 1)$-secants. By Lemma 10 (ii), there are at least $(p_0^{hk} - 1)/(p_0^h - 1) - 3p_0^{hk-h-3}$ small hyperplanes through $\Pi_L$, which each contain at least $p_0^h + p_0^{h-1} - p_0^{h-2} - p_0 - 1$ points of $B$ not on $L$. Since $|B| < p_0^{hk} + p_0^{hk-1} + p_0^{hk-2} + 3p_0^{hk-3}$ (see Lemma 2), there are less than $2p_0^{hk-1}$ points of $B$ left in large $(n-k+1)$-spaces through $\Pi_L$. Hence, $P$ lies on less than $2p_0^{hk-h-1}$ lines that are completely contained in $B$.

Since $B$ is minimal, $P$ lies on a tangent $(n-k)$-space $\Pi$ to $B$. There are at most $p_0^{hk-h-2} + 4p_0^{hk-h-3} - 1$ large $(n-k+1)$-spaces through $\Pi$ (Lemma 10 (i)). Moreover, since at least $\frac{p_0^{hk}-1}{p_0^h-1} - (p_0^{hk-h-2} + 4p_0^{hk-h-3} - 1) - (2p_0^{hk-h-1})$ $(n-k+1)$-spaces through $\Pi$

contain at least $p_0^h+p_0^{h-1}-p_0^{h-2}$ points of $B$, and at most $2p_0^{hk-h-1}$ of the small $(n-k+1)$-spaces through $\Pi$ contain exactly $p_0^h+1$ points of $B$, there are at most $p_0^{hk-2}$ points of $B$ contained in large $(n-k+1)$-spaces through $\Pi$. Hence, $P$ lies on at most $p_0^{hk-3}$ $(p_0+1)$-secants of the large $(n-k+1)$-spaces through $\Pi$. This implies that there are at least $(((p_0^{hk}-1)/(p_0^h-1)-3p_0^{hk-h-3})(p_0^{h-1}-4p_0^{h-2})+1)-p_0^{hk-3}$ $(p_0+1)$-secants through $P$ left in small $(n-k+1)$-spaces through $\Pi$. Since in a small $(n-k+1)$-space through $\Pi$, there can lie at most $(p_0^h-1)/(p_0-1)$ $(p_0+1)$-secants through $P$, this implies that there are at least $p_0^{hk-h}-5p_0^{hk-h-1}$ $(n-k+1)$-spaces $H_i$ through $\Pi$ such that $P$ lies on a $(p_0+1)$-secant in $H_i$. □

We continue with the following hypothesis:

(H) A small minimal $j$-blocking set in $\mathrm{PG}(n,q)$, $1 \le j < k$ is linear.

**Lemma 12.** *Let $B$ be a non-trivial small minimal $k$-blocking set with exponent $e$ in $\mathrm{PG}(n,p^t)$, $p$ prime, $p_0 := p^e \ge 7$, $k>1$. If we assume (H), then the following statements hold.*

*(i) A small $(n-k+s)$-dimensional space $\Pi$ of $\mathrm{PG}(n,p^t)$, $s<k$, intersects $B$ in a linear set and $|\Pi \cap B| \le (p_0^{hs+1}-1)/(p_0-1)$.*

*(ii) Let $L$ be a $(p_0+1)$-secant to $B$ and let $S$ be a point of $B$, not on $L$. There exists a small $(n-2)$-space through $L$, skew to $S$.*

*(iii) A line intersects $B$ in a linear set.*

*(iv) Let $\Pi$ be a small $(n-2)$-space containing a $(p_0+1)$-secant to $B$. Then the number of large $(n-1)$-spaces through $\Pi$ is at most $4p_0^{h-3}$.*

*Proof.* (i) It is clear that an $(n-k+s)$-space $\Pi$ meets $B$ in a small $s$-blocking set $B'$. Every $(n-k)$-space contained in $\Pi$ meets $B'$ in 1 mod $p_0$ points, hence, by Theorem 1 (ii), $B'$ is a small minimal $s$-blocking set in $\mathrm{PG}(n-k+s,p_0^h)$, which is, by the hypothesis (H), $\mathbb{F}_{p_0}$-linear. It follows that $|B'| \le (p_0^{hs+1}-1)/(p_0-1)$.

(ii) Lemma 2 shows that there is an $(n-k)$-space $\Pi_{n-k}$ through $L$, such that $B \cap L = B \cap \Pi_{n-k}$. By Lemma 1, an $(n-k+1)$-space through $\Pi_{n-k}$ contains at most $(p_0^{h+1}-1)/(p_0-1)$ or at least $p_0^{h+1}-p_0^{h-1}-p_0^{h-2}-3p_0^{h-3}$ points of $B$. If all $(n-k+1)$-spaces through $\Pi_{n-k}$ (except possibly $\langle \Pi_{n-k}, S\rangle$) would be large, the number of points in $B$ would be at least $((p_0^{hk}-1)/(p_0^h-1)-1)(p_0^{h+1}-p_0^{h-1}-p_0^{h-2}-3p_0^{h-3}-p_0^h)$, which is larger than $p_0^{hk}+p_0^{hk-1}+p_0^{hk-2}+3p_0^{hk-3}$, a contradiction. Hence, there is a small $(n-k+1)$-space through $\Pi_{n-k}$.

Suppose, by induction, that there exists a small $(n-k+s)$-space $\Pi_{n-k+s}$ through $L$, skew to $S$ and suppose all $(p_0^{h(k-s)}-1)/(p_0^h-1)-1$ $(n-k+s)$-spaces through $\Pi_{n-k+s-1}$, different from $\langle \Pi_{n-k+s}, S\rangle$ are large. Then the number of points in $B$ is larger than $p_0^{hk}+p_0^{hk-1}+p_0^{hk-2}+3p_0^{hk-3}$ if $s \le k-2$, a contradiction. We conclude that there exists a small $(n-2)$-space through $L$, skew to $S$.

(iii) Let $L$ be a line, with $0 < |L \cap B| < p^t + 1$, otherwise the statement trivially holds. The previous part of this lemma shows that $L$ is contained in a small $(n-k+1)$-space, which has, by the first part of this lemma, a linear intersection with $B$. Hence, $B \cap L$ is a linear set.

(iv) A small $(n-1)$-space through $\Pi$ meets $B$ in at least $p_0^{hk-h} + p^{hk-h-1} - p^{hk-h-2}$ points (see Corollary 2) and a small $(n-2)$-space contains at most $(p_0^{hk-2h+1} - 1)/(p_0 - 1)$ points by the first part of this lemma. By Lemma 1, a large $(n-1)$-space through $\Pi$ contains at least $p^{hk-h+1} - p^{hk-h-1} - p^{hk-h-2} - 3p^{hk-h-3}$ points of $B$. Suppose there are $y$ large $(n-1)$-spaces through $\Pi$. Then the number of points in $B$ is at least

$$y(p_0^{hk-h+1} - p_0^{hk-h-1} - p_0^{hk-h-2} - 3p_0^{hk-h-3} - (p_0^{hk-2h+1} - 1)/(p_0 - 1))+$$

$$(p_0^h + 1 - y)(p_0^{hk-h} + p^{hk-h-1} - p^{hk-h-2} - (p_0^{hk-h+1} - 1)/(p_0 - 1)) + (p_0^{hk-2h+1} - 1)/(p_0 - 1)$$

which is at most $p_0^{hk} + p_0^{hk-1} + p_0^{hk-2} + 3p_0^{hk-3}$. This yields $y \leq 4p_0^{h-3}$. □

**Lemma 13.** *Assume (H). Let $B$ be a non-trivial small minimal $k$-blocking set with exponent $e$ in $\mathrm{PG}(n, p^t)$, $p$ prime, $p_0 := p^e \geq 7$ and let $P$ be a point of $B$, and let $\Pi$ be a tangent $(n-k)$-space to $B$ through $P$. Let $H_1$ and $H_2$ be two $(n-k+1)$-spaces through $\Pi$ for which $B \cap H_i = \mathcal{B}(\pi_i)$, for some $h$-space $\pi_i$ through a point $x \in \mathcal{S}(P)$, such that $P$ lies on a $(p_0+1)$-secant in $H_i$, $i = 1, 2$. Then $\mathcal{B}(\langle \pi_1, \pi_2 \rangle) \subset B$.*

*Proof.* Let $L$ be a $(p_0+1)$-secant through $P$ in $H_1$ and let $\ell$ be the line in $\pi$ through $x$ such that $\langle \mathcal{B}(\ell) \rangle = L$. Let $s$ be a point of $\pi_2$. By Lemma 12 (ii), there is a small $(n-2)$-space $\Pi_{n-2}$ through $L$, skew to $\mathcal{B}(s)$. There are at least $p_0^{h-1} - 4p_0^{h-2}$ $(p_0+1)$-secants through $P$, of which at least $p_0^{h-1} - 4p_0^{h-2} - (p_0^{h-1} - 1)/(p_0 - 1)$ span an $(n-1)$-space together with $\Pi_{n-2}$. By Lemma 12 (iv), there are at most $4p_0^{h-3}$ large spaces through $\Pi_{n-2}$, so at least $p_0^{h-1} - 4p_0^{h-2} - (p_0^{h-1} - 1)/(p_0 - 1) - 4p_0^{h-3}$ of the $(p_0+1)$-secants through $P$ have a transversal line $\ell_k$, for which $\mathcal{B}(\langle \ell, \ell_k \rangle) \subset B$. This gives in total at least $p_0^{h+1} - 6p_0^h$ points $Q$ in $\langle \ell, \pi_2 \rangle$ for which $\mathcal{B}(Q) \subset B$, denote this pointset by $G$. This means that every point $t$ of $\langle \ell, \pi_2 \rangle$ lies on a line $m$ with at least $p_0 - 5$ points of $G$. Since $\langle \mathcal{B}(m) \rangle$ either is contained in $B$, or it meets $B$ in a linear set of rank at most $h$ (see Lemma 12 (iii)), and $p_0 - 5 > h$, again by Theorem 3, $\mathcal{B}(m) \subset B$ by Theorem 3, and hence, $\mathcal{B}(t) \subset B$.

Hence, for all $(p_0+1)$-secants $\mathcal{B}(\ell)$, with $\ell$ through $x$, in $H_1$, $\mathcal{B}(\langle \ell, \pi_2 \rangle) \subset B$. This shows that there are at least $(p_0^{h-1} - 4p_0^{h-2})p_0^{h+1} + (p_0^{h+1} - 1)/(p_0 - 1)$ points $Q$ in the $2h$-space $\langle \pi_1, \pi_2 \rangle$ such that $\mathcal{B}(Q) \subset B$. Every point $t$ of $\langle \pi_1, \pi_2 \rangle$ lies on a line $m$ with at least $p_0 - 5$ points of $G$. Again, since $p_0 - 5 > h$, by Theorem 3, $\mathcal{B}(m) \subset B$ and hence, $\mathcal{B}(t) \subset B$. It follows that $\mathcal{B}(\langle \pi_1, \pi_2 \rangle) \subseteq B$.

□

**Proof of Main Theorem 2.** Let $B$ be a non-trivial small minimal $k$-blocking set with exponent $e$ in $\mathrm{PG}(n, p^t)$, $p$ prime, $p_0 := p^e \geq 7$. We will show that, assuming that all small minimal 1-blocking sets with exponent $e$ in $\mathrm{PG}(n, p^t)$, $p$ prime, $p_0 := p^e \geq 7$, are $\mathbb{F}_{p_0}$-linear, $B$ is $\mathbb{F}_{p_0}$-linear. By induction, we may assume (H) holds. If $B$ is a $k$-space, then $B$ is $\mathbb{F}_{p_0}$-linear. If $B$ is a non-trivial small minimal $k$-blocking set, Lemma 11 shows that there exists a point $P$ of $B$, a tangent $(n-k)$-space $\Pi$ at the point $P$ and at least $p_0^{hk-h} - 5p_0^{hk-h-1}$ $(n-k+1)$-spaces $H_i$ through $\Pi$ for which $B \cap H_i$ is small and linear, where $P$ lies on at least one $(p_0+1)$-secant of $B \cap H_i$, $i = 1, \ldots, s$, $s \geq p_0^{hk-h} - 5p_0^{hk-h-1}$. Let $B \cap H_i = \mathcal{B}(\pi_i), i = 1, \ldots, s$, with $\pi_i$ an $h$-dimensional space in $\mathrm{PG}(h(n+1)-1, p_0)$, where $x \in \pi_i$, with $x \in \mathcal{S}(P)$.

Lemma 13 shows that $\mathcal{B}(\langle \pi_i, \pi_j \rangle) \subseteq B$, $0 \leq i \neq j \leq s$.

If $k = 2$, the set $\mathcal{B}(\langle \pi_1, \pi_2 \rangle)$ corresponds to a linear 2-blocking set $B'$ in $\mathrm{PG}(n, p_0^h)$. Since $B$ is minimal, $B = B'$, and the Theorem is proven.

Let $k > 2$. Denote the $(n-k+1)$-spaces through $\Pi$, different from $H_i$, by $K_j, j = 1, \ldots, z$. It follows from Lemma 11 that $z \leq 5p_0^{hk-h-1} + (p_0^{hk-h} - 1)/(p_0 - 1) \leq 6p_0^{hk-h-1}$. There are at least $(p_0^{hk-h} - 5p_0^{hk-h-1} - 1)/p_0^h$ different $(n-k+2)$-spaces $\langle H_1, H_j \rangle$, $1 < j \leq s$. If all $(n-k+2)$-spaces $\langle H_1, H_j \rangle$, contain at least $10p_0^{h-1}$ of the spaces $K_i$, then $z \geq 10p_0^{h-1}(p_0^{hk-h} - 5p_0^{hk-h-1} - 1)/p_0^h > 6p_0^{hk-h-1}$, a contradiction if $p_0 > h + 10$. Let $\langle H_1, H_2 \rangle$ be an $(n-k+2)$-spaces containing less than $10p_0^{h-1}$ spaces $K_i$.

Suppose by induction that for any $1 < i < k$, there is an $(n-k+i)$-space $\langle H_1, H_2, \ldots, H_i \rangle$ containing at most $10p_0^{hi-h-1}$ of the spaces $K_i$ such that $\mathcal{B}(\langle \pi_1, \ldots, \pi_i \rangle) \subseteq B$.

There are at least

$$\frac{p_0^{hk-h} - 6p_0^{hk-h-1} - (p_0^{hi} - 1)/(p_0^h - 1)}{p_0^h}$$

different $(n-k+i+1)$-spaces $\langle H_1, H_2, \ldots, H_i, H_r \rangle$, $H_r \not\subseteq \langle H_1, H_2, \ldots, H_i \rangle$. If all of these contain at least $10p_0^{hi-1}$ of the spaces $K_i$, then $z \geq 6p_0^{hk-h-1}$, a contradiction. Let $\langle H_1, \ldots, H_{i+1} \rangle$ be an $(n-k+i+1)$-space containing less than $10p_0^{hi-1}$ spaces $K_i$. We still need to prove that $\mathcal{B}(\langle \pi_1, \ldots, \pi_{i+1} \rangle) \subseteq B$. Since $\mathcal{B}(\langle \pi_{i+1}, \pi \rangle) \subseteq B$, with $\pi$ an $h$-space in $\langle \pi_1, \ldots, \pi_i \rangle$ for which $\mathcal{B}(\pi)$ is not contained in one of the spaces $K_i$, there are at most $10p_0^{hi-h-1}$ $2h$-dimensional spaces $\langle \pi_{i+1}, \mu \rangle$ for which $\mathcal{B}(\langle \pi_{i+1}, \mu \rangle)$ is not necessarily contained in $B$, giving rise to at most $v := 10p_0^{hi-h-1}(p_0^{2h+1} - 1)/(p_0 - 1)$ points $t$ for which $\mathcal{B}(t)$ is not necessarily contained in $B$. Let $u$ be a point of such a space $\langle \pi_{i+1}, \mu \rangle$, and suppose that $\mathcal{B}(u) \notin B$. If each of the $(p_0^{hi+h} - 1)/(p_0 - 1)$ lines through $u$ in $\langle \pi_1, \ldots, \pi_{i+1} \rangle$ contains at least 10 of the points $t$ for which $\mathcal{B}(t)$ is not in $B$, then there are more than $v$ such points $t$, a contradiction. Hence, there is a line $n$ through $u$ for which for at least $p_0 - 10$ points $v \in n$, $\mathcal{B}(v) \in B$. Every line $L$ meets $B$ in a linear set (see Lemma 12 (iii)), and if this linear set has rank at least $h + 1$, then $L$ is completely contained in $B$. This implies that $\langle \mathcal{B}(n) \rangle \cap B$ has rank at most $h$, and that the subline $\mathcal{B}(n)$ contains at least $p_0 - 10$ points of the linear set $\langle \mathcal{B}(n) \rangle \cap B$. Since $p_0 - 10 > h$, by Theorem 3, $\mathcal{B}(n)$ is contained in $\langle \mathcal{B}(n) \rangle \cap B$, so $\mathcal{B}(u) \subset B$, a contradiction.

This implies that $\mathcal{B}(\langle \pi_1, \ldots, \pi_{i+1} \rangle) \subseteq B$.

Since $\mathcal{B}(\langle \pi_1, \ldots, \pi_k \rangle) \subseteq B$, and $\mathcal{B}(\langle \pi_1, \ldots, \pi_k \rangle)$ corresponds to a linear $k$-blocking set $B'$ in $\mathrm{PG}(n, p_0^h)$ contained in the minimal $k$-blocking set $B$, $B = B'$ and hence, $B$ is $\mathbb{F}_{p_0}$-linear. □

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