B_h Sequences in Higher Dimensions

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Abstract

In this article we look at the well-studied upper bounds for $|A|$, where $A \subset \mathbb{N}$ is a B_h sequence, and generalise these to the case where $A \subset \mathbb{N}^d$. In particular we give d-dimensional analogues to results of Chen, Jia, Graham and Green.

1 Introduction

1.1 Infinite B_h sequences

Let $h, d \in \mathbb{N}$ with $h \geqslant 2$. A d-dimensional set $A \subset \mathbb{N}^d$ is called a d-dimensional B_h sequence if all sums $a_1 + a_2 + \cdots + a_h$, where $a_1, a_2, \ldots, a_h \in A$, are different up to rearrangement of summands.

We denote $A(n)$ as number of elements of A in a box $[1, n]^d$. If A is a d-dimensional B_h sequence, then $\binom{A(n)}{h} \leqslant (hn)^d$ which implies

$$A(n) = \mathcal{O}(n^{d/h}). \tag{1}$$

Erdős improved this inequality for one-dimensional B_2 sequences showing that

$$\liminf_{n\to\infty} A(n)\sqrt{\frac{\log n}{n}} < \infty.$$

This result was generalised for d-dimensional B_2 sequences by J. Cilleruelo:

Theorem 1.1. [1] *If $A \subset \mathbb{N}^d$ is a B_2 sequence, then*

$$\liminf_{n\to\infty} A(n)\sqrt{\frac{\log n}{n^d}} < \infty.$$

and for one dimensional B_{2k} sequences by S. Chen:

Theorem 1.2. [2] *If $A \subset \mathbb{N}$ is a B_{2k} sequence, then*

$$\liminf_{n\to\infty} A(n)\sqrt[2k]{\frac{\log n}{n}} < \infty.$$

As noted in [2], no results of this type are known for h odd.

1.2 Finite B_h sequences

Erdős and Turán gave the first upper bound for finite B_2 sequences, showing that if $A \subseteq [1, N]$ is a B_2 sequence then

$$|A| \leqslant N^{\frac{1}{2}} + \mathcal{O}\big(N^{\frac{1}{4}}\big).$$

Lindström [7] improved the method of this paper to obtain

$$|A| \leqslant N^{\frac{1}{2}} + N^{\frac{1}{4}} + 1.$$

If $A \subseteq [1, N]$ is a B_h sequence a simple counting argument gives

$$|A| \leqslant (hh!N)^{\frac{1}{h}}.$$

Lindström [8] improved this for $A \subseteq [1, N]$ a B_4 sequence, proving

$$|A| \leqslant 8^{\frac{1}{4}} N^{\frac{1}{4}} + \mathcal{O}\big(N^{\frac{1}{8}}\big).$$

Jia generalised this argument for even h to obtain:

Theorem 1.3 ([6], see also [5])**.** *If $A \subseteq [1, N]$ is a B_{2k} sequence, then*

$$|A| \leqslant k^{\frac{1}{2k}} (k!)^{\frac{1}{k}} N^{\frac{1}{2k}} + \mathcal{O}\big(N^{\frac{1}{4k}}\big).$$

For the case h is odd, the best known upper bound was given by Chen and Graham:

Theorem 1.4 ([5],[3])**.** *If $A \subseteq [1, N]$ is a B_{2k-1}, then*

$$|A| \leqslant (k!)^{\frac{2}{2k-1}} N^{\frac{1}{2k-1}} + \mathcal{O}\big(N^{\frac{1}{4k-2}}\big).$$

Finally, Green used the techniques of Fourier analysis to improve above theorems in three special cases:

Theorem 1.5. [4] *If $A \subseteq [1, N]$ is a B_3 sequence, then*

$$|A| \leqslant \left(\tfrac{7}{2}\right)^{\frac{1}{3}} N^{\frac{1}{3}} + o\left(N^{\frac{1}{3}}\right).$$

Theorem 1.6. [4] *If $A \subseteq [1, N]$ is a B_4 sequence, then*

$$|A| \leqslant (7)^{\frac{1}{4}} N^{\frac{1}{4}} + o\left(N^{\frac{1}{4}}\right).$$

Theorem 1.7. [4] *For sufficiently large k:*

(i) If $A \subseteq [1, N]$ is a B_{2k} sequence, then

$$|A| \leqslant \pi^{\frac{1}{4k}} k^{\frac{1}{4k}} (k!)^{\frac{1}{k}} (1 + \epsilon(k)) N^{\frac{1}{2k}} + \mathcal{O}\left(N^{\frac{1}{4k}}\right).$$

(ii) If $A \subseteq [1, N]$ is a B_{2k-1} sequence, then

$$|A| \leqslant \pi^{\frac{1}{2(2k-1)}} k^{\frac{-1}{2(2k-1)}} (k!)^{\frac{2}{2k-1}} (1 + \epsilon(k)) N^{\frac{1}{2k-1}} + \mathcal{O}\left(N^{\frac{1}{2(2k-1)}}\right).$$

2 Preliminaries

We denote

$$rA = \{x = x_1 + ... + x_r : x_s \in A, 1 \leqslant s \leqslant r\},$$
$$r * A = \{x = x_1 + ... + x_r : x_s \in A, x_i \neq x_j, 1 \leqslant i < j \leqslant r\}.$$

For any $x = x_1 + \cdots + x_r \in rA$, we let $\overline{x}$ be the set $\{x_1, \ldots, x_r\}$ (counting multiplicities). For a B_h-sequence $A \subseteq [1, N]^d$ we define

$$D_j(z; r) \quad = \quad \{(x, y) : x - y = z, x, y \in jA, |\overline{x} \cap \overline{y}| = r\},$$

and write $d_j(z; r)$ for its cardinality.

Lemma 2.1.1. *Let $A \subseteq [1, N]^d$.*

(i) If A is a B_{2k} sequence, for $1 \leqslant j \leqslant k$,

$$d_j(z; 0) \leqslant 1;$$

(ii) If A is B_{2k} sequence, for $1 \leqslant r \leqslant k$,

$$\sum_{z \in \mathbb{Z}^d} d_k(z; r) \leqslant |A|^{2k-r}.$$

Proof.

(i) If $(x,y),(x',y')\in D_j(z;0)$ then we have $x+y'=x'+y$. Since A is a B_h sequence, the two representations correspond to different permutations of the same h elements and as $\overline{x}\cap\overline{y}=\overline{x'}\cap\overline{y'}=\emptyset$, then $x=x'$ and $y=y'$.

(ii) There are at most $|A|^r$ possible values for $\overline{x}\cap\overline{y}$ (where the intersection is taken with multiplicities), so
$$d_k(z;r)\leqslant |A|^r d_{k-r}(z;0).$$
Then
$$\begin{aligned}\sum_{z\in\mathbb{Z}^d} d_k(z;r) &\leqslant |A|^r \sum_{z\in\mathbb{Z}^d} d_{k-r}(z;0)\\ &\leqslant |A|^r|(k-r)A|^2 \qquad \text{(using (i))}\\ &\leqslant |A|^{2k-r}.\end{aligned}$$

□

Similarly for a B_h-sequence $A\subseteq[1,N]^d$ we define
$$\begin{aligned} D_j^*(z;r) &= \{(x,y): x-y=z, x,y\in j*A, |\overline{x}\cap\overline{y}|=r\},\\ D_j^*(z;r;a) &= \{(x,y)\in D_j^*(z,r): a\in\overline{x}\}\end{aligned}$$
and write $d_j^*(z;r)$ and $d_j^*(z;r;a)$ for their respective cardinalities.

Lemma 2.1.2. *Let $A\subseteq[1,N]^d$.*

(i) If A is a B_{2k-1} sequence, for $1\leqslant j\leqslant k-1$,
$$d_j^*(z;0)\leqslant 1;$$

(ii) If A is a B_{2k-1} sequence,
$$d_k^*(z;0)\leqslant \frac{|A|}{k}.$$

(iii) If A is a B_{2k-1} sequence, for $1\leqslant r\leqslant k$,
$$\sum_{z\in\mathbb{Z}^d} d_k^*(z;r)\leqslant |A|^{2k-r}.$$

Proof.

(i) We may use the same proof as in (i) previous lemma.

(ii) We show that $d_k^*(z;0;a) \leqslant 1$. Assume not. Then there exists $x = x_1 + \ldots + x_k, x' = x_1' + \ldots + x_k', y = y_1 + \ldots + y_k, y' = y_1' + \ldots + y_k' \in k * A$ such that $x - y = x' - y' = z$. In addition, without loss of generality, we may assume $x_k = x_k' = a$. Hence we have

$$x_1 + \ldots + x_{k-1} + y_1' + \ldots y_k' = x_1' + \ldots + x_{k-1}' + y_1 + \ldots y_k.$$

Once again, since A is a B_{2k-1} sequence, the two representations correspond to different permutations of the same $2k-1$ elements and as $\overline{x} \cap \overline{y} = \overline{x} \cap \overline{y} = \emptyset$ we must have $x = x'$ and $y = y'$, giving a contradiction.

Notice that

$$\sum_{a \in A} d_k^*(z;0;a) = k d_k^*(z;0)$$

and the statement of the lemma follows.

(iii) We may use the same proof as in (ii) in previous lemma.

□

3 Infinite d-dimensional B_{2k} sequences

In this section we prove the following amalgamation of Theorems 1.1 and 1.2:

Theorem 3.1. *If $A \subset \mathbb{N}^d$ is a B_{2k} sequence, then*

$$\liminf_{n \to \infty} A(n) \sqrt[2k]{\frac{\log n}{n^d}} < \infty$$

We fix a large enough positive integer n and set $u = \lfloor n^{1/(2k-1)} \rfloor$. For any d-dimensional vector $\vec{i}$ use the L_∞ norm defined as follows:

$$|\vec{i}|_\infty = |(i_1, i_2, ..., i_d)|_\infty = \max_{1 \leqslant k \leqslant d} \{|i_k|\}.$$

For any d-dimensional set B denote

$$B_{\vec{i}} = B \cap \bigotimes_{j=1}^{d} ((i_j - 1)kn, i_j kn].$$

We set

$$\begin{aligned} A' &= A \cap [1, un]^d, \\ C &= kA', \\ c_{\vec{i}} &= |C_{\vec{i}}|, \\ \Delta_j &= \sum_{|\vec{i}|_\infty = j} c_{\vec{i}}, \\ \tau(n) &= \min_{n \leqslant m \leqslant un} \frac{A(m)}{m^{d/2k}}. \end{aligned}$$

Lemma 3.1.1.

$$\tau(n)^{2k} n^d \log n = \mathcal{O}\left(\sum_{\vec{i}\in[1,u]^d} c_{\vec{i}}^2\right).$$

Proof. Note that

$$\begin{aligned}
\left(\sum_{\vec{i}\in[1,u]^d} \frac{c_{\vec{i}}}{|\vec{i}|_\infty^{d/2}}\right)^2 &\leqslant \left(\sum_{\vec{i}\in[1,u]^d} \frac{1}{|\vec{i}|_\infty^{d}}\right)\left(\sum_{\vec{i}\in[1,u]^d} c_{\vec{i}}^2\right) \\
&\leqslant \left(\sum_{i=1}^{u} \frac{di^{d-1}}{i^d}\right)\left(\sum_{\vec{i}\in[1,u]^d} c_{\vec{i}}^2\right) \\
&\leqslant \mathcal{O}\left(\log n \sum_{\vec{i}\in[1,u]^d} c_{\vec{i}}^2\right). \qquad (2)
\end{aligned}$$

On the other hand, for any positive i ($1 \leqslant i \leqslant u$),

$$C(ikn) \geqslant cA(in)^k,$$

where $c > 0$ is an absolute constant depending only on k, and

$$\begin{aligned}
A(in)^k &= \left(\frac{A(in)}{(in)^{d/2k}}\right)^k (in)^{d/2} \\
&\geqslant \tau(n)^k (in)^{d/2}.
\end{aligned}$$

Hence, for absolute constants c_1, c_2, c_3 depending on d and k,

$$\begin{aligned}
\sum_{\vec{i}\in[1,u]^d} \frac{c_{\vec{i}}}{|\vec{i}|_\infty^{d/2}} &= \sum_{i=1}^{u} \frac{\Delta_i}{i^{d/2}} \\
&= \sum_{i=1}^{u}\left(\frac{1}{i^{d/2}} - \frac{1}{(i+1)^{d/2}}\right)\sum_{j=1}^{i} \Delta_j + \frac{1}{(u+1)^{d/2}}\sum_{j=1}^{u}\Delta_j \\
&\geqslant c_1 \sum_{i=1}^{u} \frac{C(ikn)}{i^{d/2+1}} \\
&\geqslant c_2 \sum_{i=1}^{u} \frac{\tau(n)^k (in)^{d/2}}{i^{d/2+1}} \\
&= c_2 \tau(n)^k n^{d/2} \sum_{i=1}^{u} \frac{1}{i} \\
&\geqslant c_3 \tau(n)^k n^{d/2} \log n. \qquad (3)
\end{aligned}$$

Combining inequalities (2) and (3), Lemma 3.1.1 follows. □

Lemma 3.1.2.

$$\sum_{\vec{i}\in[1,u]^d} c_{\vec{i}}^2 = \mathcal{O}(n^d).$$

Proof. We have

$$\begin{aligned}
\sum_{\vec{i}\in[1,u]^d} c_{\vec{i}}^2 &\leqslant \sum_{r=0}^{k} \sum_{|z|_\infty \leqslant kn} d_k(z;r) \\
&= \sum_{|z|_\infty \leqslant kn} d_k(z;0) + \sum_{r=1}^{k} \sum_{|z|_\infty \leqslant kn} d_k(z;r) \\
&\leqslant \sum_{|z|_\infty \leqslant kn} 1 + \sum_{r=1}^{k} |A'|^{2k-r} \quad \text{(using Lemma 2.1.1 (i) and (iv))} \\
&= (2kn)^d + \mathcal{O}\left((un)^{d(1-1/(2k))}\right) \quad \text{(using equation (1))} \\
&= \mathcal{O}(n^d).
\end{aligned}$$

□

We are now able to prove Theorem 3.1:

Proof of Theorem 3.1. From Lemmas 3.1.1 and 3.1.2 we have $\tau(n)^{2k}\log n = \mathcal{O}(1)$. Hence,

$$\begin{aligned}
\liminf_{n\to\infty} A(n) \sqrt[2k]{\frac{\log n}{n^d}} &= \lim_{n\to\infty} \inf_{n\leqslant m\leqslant un} A(m) \sqrt[2k]{\frac{\log m}{m^d}} \\
&\leqslant \lim_{n\to\infty} \inf_{n\leqslant m\leqslant un} \frac{A(m)}{m^{d/2k}} \sqrt[2k]{\log un} \\
&\leqslant 2 \lim_{n\to\infty} \tau(n) \sqrt[2k]{\log n} < \infty.
\end{aligned}$$

□

4 Finite d-dimensional B_h-sequences

4.1 Preliminaries

The following lemma will be our main tool for the subsequent two sections:

Lemma 4.1.1. *Let G be an additive group and $A_1, A_2, X \subset G$ such that $A_1 + A_2 = X$. Write*

$$\begin{aligned}
d_{A_i}(g) &= \#\{(a,a') : a, a' \in A_i, a - a' = g\}, i = 1, 2, \\
r_{A_1+A_2}(g) &= \#\{(a,a') : a \in A_1, a' \in A_2, a + a' = g\}.
\end{aligned}$$

Then

$$\sum_{g\in G} d_{A_1}(g)d_{A_2}(g) - \frac{|A_1|^2|A_2|^2}{|X|} = \sum_{g\in X}\left(r_{A_1+A_2}(g) - \frac{|A_1||A_2|}{|X|}\right)^2.$$

In particular, we have

$$\sum_{g\in G} d_{A_1}(g)d_{A_2}(g) - \frac{|A_1|^2|A_2|^2}{|X|} \geqslant 0. \tag{4}$$

Proof. Note that

$$\begin{aligned}\sum_{g\in X} r_{A_1+A_2}(g)^2 &= \#\{(a_1,a_2,a_3,a_4) : a_1,a_3\in A_1, a_2,a_4\in A_2, a_1+a_2=a_3+a_4\}\\ &= \#\{(a_1,a_2,a_3,a_4) : a_1,a_3\in A_1, a_2,a_4\in A_2, a_1-a_3=a_2-a_4\}\\ &= \sum_{g\in G} d_{A_1}(g)d_{A_2}(g).\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{g\in X}\left(r_{A_1+A_2}(g) - \frac{|A_1||A_2|}{|X|}\right)^2\\ &=\sum_{g\in X} r_{A_1+A_2}(g)^2 - 2\frac{|A_1||A_2|}{|X|}\sum_{g\in X} r_{A_1+A_2}(g) + \sum_{g\in X}\frac{|A_1|^2|A_2|^2}{|X|^2}\\ &=\sum_{g\in G} d_{A_1}(g)d_{A_2}(g) - 2\frac{|A_1||A_2|}{|X|}|A_1||A_2| + \frac{|A_1|^2|A_2|^2}{|X|^2}|X|\\ &=\sum_{g\in G} d_{A_1}(g)d_{A_2}(g) - \frac{|A_1|^2|A_2|^2}{|X|}.\end{aligned}$$

□

4.2 Finite d-dimensional B_{2k} sequences

In this section we show the multidimensional analogue of Theorem 1.3:

Theorem 4.1. *If $A \subseteq [1,N]^d$ is a B_{2k} sequence, then*

$$|A| \leqslant N^{\frac{d}{2k}} k^{\frac{d}{2k}} (k!)^{\frac{1}{k}} + \mathcal{O}\left(N^{\frac{d^2}{2k(d+1)}}\right).$$

We first prove the following lemma:

Lemma 4.2.1. *For $I = [0, u-1]^d$,*

$$\sum_{z\in\mathbb{Z}^d} d_{kA}(z)d_I(z) \leqslant u^{2d} + \mathcal{O}(u^d|A|^{2k-1}).$$

Proof.

$$\begin{aligned}
\sum_{z\in\mathbb{Z}^d} d_{kA}(z)d_I(z) &= \sum_{z\in\mathbb{Z}^d} d_I(z)\sum_{r=0}^{k} d_k(z;r) \\
&= \sum_{z\in\mathbb{Z}^d} d_I(z)d_k(z;0) + \sum_{r=1}^{k}\sum_{z\in\mathbb{Z}^d} d_I(z)d_k(z;r) \\
&\leqslant u^{2d} + \mathcal{O}(u^d|A|^{2k-1}). \quad \text{(using Lemma 2.1.1 (i) and (ii))}
\end{aligned}$$

□

Proof of Theorem 4.1. We will use Lemma 4.1.1 with $G = \mathbb{Z}^d$, $A_1 = kA$, $A_2 = I = [0, u-1]^d$ (where the positive integer u will be chosen later) and $X = kA + I$.

$$\begin{aligned}
|kA| &\geqslant \frac{1}{k!}|A|^k, \\
|I| &= u^d, \\
|X| &\leqslant (kN+u)^d.
\end{aligned}$$

Thus, using Lemma 4.2.1 and equation (4), we have (after simplification)

$$\frac{|A|^{2k}u^d}{k!^2(kN+u)^d} \leqslant u^d + \mathcal{O}\left(|A|^{2k-1}\right),$$

or

$$\begin{aligned}
|A|^{2k} &\leqslant k!^2(kN+u)^d + \mathcal{O}\left(\left(\frac{kN}{u}+1\right)^d |A|^{2k-1}\right) \\
&\leqslant k!^2(kN+u)^d + \mathcal{O}\left(\left(\frac{kN}{u}+1\right)^d N^{\frac{(2k-1)d}{2k}}\right). \quad \text{(using equation (1))}
\end{aligned}$$

To minimise the error term we need $\left(\frac{N}{u}\right)^d N^{\frac{(2k-1)d}{2k}} = uN^{d-1}$, so we take $u = N^{1-\frac{d}{(d+1)2k}}$ giving

$$\begin{aligned}
|A|^{2k} &\leqslant k!^2k^dN^d + \mathcal{O}\big(N^{d-\frac{d}{(d+1)2k}}\big) \\
&\leqslant k!^2k^dN^d\big(1+\mathcal{O}(N^{-\frac{d}{(d+1)2k}})\big).
\end{aligned}$$

Taking $2k^{\text{th}}$ roots ends the proof. □

4.3 Finite d-dimensional B_{2k-1} sequences

In this section we show the multidimensional analogue of Theorem 1.4.

Theorem 4.2. *If $A \subset [1,N]^d$ is a B_{2k-1} sequence, then*

$$|A| \leqslant (k!)^{\frac{2}{2k-1}} k^{\frac{d-1}{2k-1}} N^{\frac{d}{2k-1}} + \mathcal{O}\big(N^{\frac{d^2}{(d+1)(2k-1)}}\big).$$

Lemma 4.3.1. *For $I = [0, u-1]^d$,*

$$\sum_{z \in \mathbb{Z}^d} d_{k*A}(z) d_I(z) \leqslant \frac{|A|}{k} u^{2d} + \mathcal{O}\big(u^d |A|^{2k-1}\big).$$

Proof. The proof follows the same course as that of Lemma 4.2.1 except using Lemma 2.1.2 (i), (ii) and (iii) in the final step. □

Proof of Theorem 4.2. As before we make use of Lemma 4.1.1, taking $G = \mathbb{Z}^d$, $A_1 = k * A$, $A_2 = I = [0, u-1]^d$ (where the positive integer u will be chosen later) and $X = A_1 + A_2$.

We have

$$|k * A| \geqslant \frac{1}{k!} |A|^k (1 - \frac{c}{|A|}),$$

where constant c depends on k, which with Lemma 4.3.1 and equation (4) gives:

$$\frac{(1 - \frac{c}{|A|})^2 |A|^{2k} u^{2d}}{(k!)^2 (kN + u)^d} \leqslant u^{2d} \frac{|A|}{k} + \mathcal{O}(|A|^{2k-1} u^d),$$

or

$$\frac{|A|^{2k} u^{2d}}{(k!)^2 (kN + u)^d} \leqslant u^{2d} \frac{|A|}{k} + \mathcal{O}(|A|^{2k-1} u^d)$$

thus

$$\begin{aligned} |A|^{2k-1} &\leqslant \frac{(k!)^2 (kN+u)^d}{k} + \mathcal{O}\left(\left(\frac{kN}{u} + 1\right)^d |A|^{2k-2}\right) \\ &\leqslant \frac{(k!)^2 (kN+u)^d}{k} + \mathcal{O}\left(\left(\frac{kN}{u} + 1\right)^d N^{d\frac{2k-2}{2k-1}}\right). \end{aligned}$$

To minimise the error term we need $N^{d-1} u = N^d N^{d(2k-2)/(2k-1)}$ so we take $u = N^{1 - \frac{d}{(d+1)(2k-1)}}$ which gives

$$\begin{aligned} |A|^{2k-1} &\leqslant (k!)^2 N^d k^{d-1} + \mathcal{O}(N^{d - \frac{d}{(d+1)(2k-1)}}) \\ &\leqslant (k!)^2 N^d k^{d-1} \big(1 + \mathcal{O}(N^{-\frac{d}{(d+1)(2k-1)}})\big). \end{aligned}$$

Taking $2k-1^{\text{th}}$ roots gives the result. □

4.4 Finite B_h sequences for large h

4.4.1 Fourier Analysis Prerequisites

We use the notation of Green [4].

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be any function. We define the dot product of two vectors $a = (a_1, a_2, \ldots, a_d)$ and $b = (b_1, b_2, \ldots, b_d)$ from an orthonormal vector space as

$$a \cdot b = \sum_{i=1}^{d} a_i b_i.$$

For $r \in \mathbb{Z}_N^d$, we define the Fourier transform

$$\hat{f}(r) = \sum_{x \in \mathbb{Z}_N^d} f(x) e^{\frac{2\pi i r \cdot x}{N}}.$$

If $f, g : G \to \mathbb{C}$ are two functions on an abelian group G, we define the convolution

$$(f * g)(x) = \sum_{y \in G} f(y)\overline{g(y-x)}.$$

We adopt the convention that

$$f_1 * f_2 * \cdots * f_k = f_1 * (f_2 * \cdots * (f_{k-1} * f_k)).$$

We shall denote $A^{*2k}(x) = (\underbrace{A * A * \cdots * A}_{2k \text{ times}})(x)$. Notice that $A^{*2k}(x)$ is the number of ordered representations of $x = a_1 + \cdots + a_k - a_{k+1} - \cdots - a_{2k}$ for $a_1, a_2, \ldots, a_{2k} \in A$. We shall use the following two well-known identities:

Lemma 4.4.1 (Parseval's Identity)**.** *If $f, g : \mathbb{Z}_N^d \to \mathbb{C}$ are two functions then*

$$N^d \sum_{x \in \mathbb{Z}_N^d} f(x)\overline{g(x)} = \sum_{r \in \mathbb{Z}_N^d} \hat{f}(r)\overline{\hat{g}(r)}.$$

Lemma 4.4.2. *If $f, g : \mathbb{Z}_N^d \to \mathbb{C}$ are two functions then*

$$\widehat{(f * g)}(r) = \hat{f}(r)\overline{\hat{g}(r)}.$$

From now on we will let $A(x)$ be the characteristic function of the set, i.e.

$$A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$$

4.4.2 B_h sequences for large h

In this section we show the multidimensional analogue of Theorem 1.7.

Theorem 4.3. *For k sufficiently large and $A \subseteq [1,N]^d$*

(i) If A is a B_{2k} sequence

$$|A| \leqslant (\pi d)^{\frac{d}{4k}}(1+\epsilon(k))k^{\frac{d}{4k}}(k!)^{\frac{1}{k}}N^{\frac{d}{2k}} + \mathcal{O}\Big(N^{\frac{d^2}{2k(d+1)}}\Big).$$

(ii) If A is a B_{2k-1} sequence

$$|A| \leqslant (\pi d)^{\frac{d}{2(2k-1)}}(1+\epsilon(k))k^{\frac{d-2}{2(2k-1)}}(k!)^{\frac{2}{2k-1}}N^{\frac{d}{2k-1}} + \mathcal{O}\Big(N^{\frac{d^2}{(2k-1)(d+1)}}\Big).$$

Proof.

(i) We regard A as a subset of $\mathbb{Z}^d_{kN+v}$ where $v \ll N$ so that $A^{*2k}(x)$ remains the same for $x \in [-v,v]^d$ as it was when we regarded A as a subset of $\mathbb{Z}^d$.
Let $I = [0, u-1]^d$ where $u \ll v$.
Notice that, for all $x \in [-v,v]^d$, $A^{*2k}(x) \leqslant (k!)^2 d_{kA}(x)$ and $I * I(x) = d_I(x)$.
Hence, arguing as in the proof of Lemma 4.2.1, we obtain

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d_{kN+v}} A^{*2k}(x)(I*I)(x) &= \sum_{x\in[-u+1,u-1]^d} A^{*2k}(x)(I*I)(x) \\ &\leqslant (k!)^2u^{2d} + \mathcal{O}\left(|A|^{2k-1}u^d\right). \qquad (5)\end{aligned}$$

Parseval's identity (Lemma 4.4.1) and Lemma 4.4.2 give

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d_{kN+v}} A^{*2k}(x)(I*I)(x) &= \frac{1}{(kN+v)^d}\sum_{r\in\mathbb{Z}^d_{kN+v}} \widehat{A^{*2k}}(x)\overline{\widehat{I*I}(x)} \\ &= \frac{1}{(kN+v)^d}\sum_{r\in\mathbb{Z}^d_{kN+v}} |\hat{A}(r)|^{2k}|\hat{I}(r)|^2 \\ &\geqslant \frac{1}{(kN+v)^d}\sum_{|r_1|+\cdots+|r_d|\leqslant k/2} |\hat{A}(r)|^{2k}|\hat{I}(r)|^2. \qquad (6)\end{aligned}$$

Claim 1. $|\hat{I}(r)| \geqslant u^d - \frac{2\pi|r_1+r_2+\cdots+r_d|u^{d+1}}{kN}$.

$$\begin{aligned}
|u^d - \hat{I}(r)| &\leqslant \sum_{x\in[0,u-1]^d} \left|1 - e^{\frac{2\pi i r\cdot x}{kN+v}}\right| \\
&= \sum_{x\in[0,u-1]^d} \left|1 - \cos\left(\frac{2\pi r\cdot x}{kN+v}\right) - i\sin\left(\frac{2\pi r\cdot x}{kN+v}\right)\right| \\
&\leqslant u^d \left(\frac{2\pi(|r_1|+|r_2|+\cdots+|r_d|)(u-1)}{kN+v}\right) \\
&\leqslant \frac{2\pi(|r_1|+|r_2|+\cdots+|r_d|)u^{d+1}}{kN},
\end{aligned}$$

proving Claim 1.

Claim 2. $\displaystyle\sum_{|r_1|+\cdots+|r_d|\leqslant k/2} |\hat{A}(r)|^{2k} \geqslant |A|^{2k}\left(\frac{k}{\pi d}\right)^{\frac{d}{2}}(1-\epsilon(k)).$

Note that the set

$$\{x_1r_1+\cdots+x_dr_d : |r_1|+\cdots+|r_d| \leqslant k/2, x\in[1,N]^d\}$$

is contained in an interval of length $\frac{k}{2}N$. Therefore for such r, vectors in the complex plane corresponding to elements of A in Fourier transform will not cancel each other. Furthermore, we can expect elements of A to be more-or-less distributed in the whole of $[1,N]^d$, thus rotating by $N/2$ in each dimension should almost align the sum of the these vectors with the real axis.

$$\begin{aligned}
|\hat{A}(r)|^{2k} &= \left|\sum_{x\in\mathbb{Z}^d_{kN+v}} A(x)e^{2\pi i\frac{x_1r_1+\cdots+x_dr_d}{kN+v}}\right|^{2k} \\
&= \left|\sum_{x\in\mathbb{Z}^d_{kN+v}} A(x)e^{2\pi i\frac{(x_1-N/2)r_1+\cdots+(x_d-N/2)r_d}{kN+v}}\right|^{2k} \\
&\geqslant \left|\sum_{x\in\mathbb{Z}^d_{kN+v}} A(x)\cos\left(\frac{\pi(r_1+\cdots+r_d)}{k}\right)\right|^{2k}.
\end{aligned}$$

Since $|r_1|+\cdots+|r_d|\leqslant k/2$, this is greater or equal than

$$|A|^{2k}\left|1-\frac{\pi^2(r_1+\cdots+r_d)^2}{2k^2}\right|^{2k}.$$

Now we can give a bound for the sum:

$$\begin{aligned}\sum_{|r_1|+\cdots+|r_d|\leqslant k/2} |\hat{A}(r)|^{2k} &\geqslant |A|^{2k} \sum_{|r_1|+\cdots+|r_d|\leqslant k/2} \left|1-\frac{\pi^2(r_1+\cdots+r_d)^2}{2k^2}\right|^{2k} \\ &\geqslant |A|^{2k} \sum_{|r_1|+\cdots+|r_d|\leqslant k^{5/8}} \left|1-\frac{\pi^2(r_1+\cdots+r_d)^2}{2k^2}\right|^{2k}.\end{aligned}$$

Since k is large, this is greater or equal than

$$|A|^{2k} \sum_{|r_1|+\cdots+|r_d|\leqslant k^{5/8}} \left|1-\frac{\pi^4(r_1+\cdots+r_d)^4}{4k^4}\right|^{2k} e^{\frac{-\pi^2(r_1+\cdots+r_d)^2}{k}}.$$

In the last step we used inequality $1-s \geqslant e^{-s}(1-s^2)$, which is true for $s \leqslant 1$. Note that, under restrictions $|r_1|+\cdots+|r_d| \leqslant k^{5/8}$, we have

$$\left|1-\frac{\pi^4(r_1+\cdots+r_d)^4}{4k^4}\right|^{2k} \to 1$$

as $k \to \infty$. The remaining sum can be rearranged using the Cauchy-Schwarz inequality:

$$\begin{aligned}\sum_{|r_1|+\cdots+|r_d|\leqslant k^{5/8}} e^{\frac{-\pi^2(r_1+\cdots+r_d)^2}{k}} &\geqslant \sum_{|r_i|\leqslant \frac{k^{5/8}}{d}} e^{\frac{-d\pi^2(r_1^2+\cdots+r_d^2)}{k}} \\ &= \prod_{i=1}^{d} \sum_{|r_i|\leqslant \frac{k^{5/8}}{d}} e^{\frac{-\pi^2 d r_i^2}{k}}.\end{aligned}$$

Now the claim follows from the fact

$$\sum_{|r_i|\leqslant \frac{k^{5/8}}{d}} e^{\frac{-\pi^2 d r_i^2}{k}} \to \int_{-\infty}^{\infty} e^{\frac{-\pi^2 d t^2}{k}}\, dt = \left(\frac{k}{\pi d}\right)^{1/2}.$$

Combining equations (5) and (6) with Claims 1 and 2, we obtain

$$\begin{aligned}(k!)^2 u^{2d} + \mathcal{O}\left(|A|^{2k-1}u^d\right) &\geqslant \frac{u^{2d}}{(kN+v)^d}\left(1-\frac{\pi u d}{N}\right)^2 \sum_{|r_1|+|r_2|+\cdots+|r_d|\leqslant \frac{k}{2}} |\hat{A}(r)|^{2k} \\ &\geqslant \frac{u^{2d}}{(kN+v)^d}\left(1-\frac{\pi u d}{N}\right)^2 |A|^{2k}\left(\frac{k}{\pi d}\right)^{\frac{d}{2}}(1-\epsilon(k)).\end{aligned}$$

So, using equation (1),

$$|A|^{2k} \leqslant \frac{(k!)^2(kN+v)^d + \mathcal{O}\left(N^{d(2-\frac{1}{2k})}u^{-d}\right)}{\frac{u^d}{(kN+v)^d}\left(1-\frac{\pi u d}{N}\right)\left(\frac{k}{\pi d}\right)^{\frac{d}{2}}(1-\epsilon(k))}.$$

We can minimise the error term by choosing $u = v = N^{1-\frac{d}{2k(d+1)}}$ which, using Taylor's expansions, gives

$$|A|^{2k} \leqslant (\pi d)^{\frac{d}{2}}(1+\epsilon(k))k^{\frac{d}{2}}(k!)^2 N^d \left(1 + \mathcal{O}\left(N^{-\frac{d}{2k(d+1)}}\right)\right).$$

Taking $2k^{\text{th}}$ roots gives the result.

(ii) This uses essentially the same proof except arguing as in Lemma 4.3.1 to obtain the equivalent of equation (5):

$$\sum_{x \in \mathbb{Z}^d_{kN+v}} A^{*2k}(x)(I * I)(x) \leqslant |A|k!(k-1)!\, u^{2d} + \mathcal{O}\left(|A|^{2k-1}u^d\right).$$

□

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