On the Modes of Polynomials Derived from Nondecreasing Sequences

Donna Q. J. Dou  
School of Mathematics  
Jilin University, Changchun 130012, P. R. China  
qujdou@jlu.edu.cn

Arthur L. B. Yang  
Center for Combinatorics, LPMC-TJKLC  
Nankai University, Tianjin 300071, P. R. China  
yang@nankai.edu.cn

Submitted: Oct 13, 2010; Accepted: Dec 15, 2010; Published: Jan 5, 2011  
Mathematics Subject Classification: 05A20, 33F10

Abstract

Wang and Yeh proved that if $P(x)$ is a polynomial with nonnegative and nondecreasing coefficients, then $P(x + d)$ is unimodal for any $d > 0$. A mode of a unimodal polynomial $f(x) = a_0 + a_1x + \cdots + a_mx^m$ is an index $k$ such that $a_k$ is the maximum coefficient. Suppose that $M_s(P,d)$ is the smallest mode of $P(x + d)$, and $M^*(P,d)$ the greatest mode. Wang and Yeh conjectured that if $d_2 > d_1 > 0$, then $M_s(P,d_1) \geq M_s(P,d_2)$ and $M^*(P,d_1) \geq M^*(P,d_2)$. We give a proof of this conjecture.

Keywords: unimodal polynomials, the smallest mode, the greatest mode.

1 Introduction

This paper is concerned with the modes of unimodal polynomials constructed from nonnegative and nondecreasing sequences. Recall that a sequence $\{a_i\}_{0 \leq i \leq m}$ is unimodal if there exists an index $0 \leq k \leq m$ such that

$$a_0 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_m.$$  

Such an index $k$ is called a mode of the sequence. Note that a mode of a sequence may not be unique. The sequence $\{a_i\}_{0 \leq i \leq m}$ is said to be spiral if

$$a_m \leq a_0 \leq a_{m-1} \leq a_1 \leq \cdots \leq a_{\lfloor m/2 \rfloor},$$  \hspace{1cm} (1.1)
where \([\lfloor \frac{m}{2} \rfloor]\) stands for the largest integer not exceeding \(\frac{m}{2}\). Clearly, the spiral property implies unimodality. We say that a sequence \(\{a_i\}_{0 \leq i \leq m}\) is log-concave if for \(1 \leq k \leq m-1\),

\[
a_k^2 \geq a_{k+1}a_{k-1},
\]

and it is ratio monotone if

\[
\frac{a_m}{a_0} \leq \frac{a_{m-1}}{a_1} \leq \cdots \leq \frac{a_{m-i}}{a_i} \leq \cdots \leq \frac{a_{m-[\frac{m-1}{2}]+1}}{a_{[\frac{m-1}{2}]}} \leq 1 \tag{1.2}
\]

and

\[
\frac{a_0}{a_{m-1}} \leq \frac{a_1}{a_{m-2}} \leq \cdots \leq \frac{a_{i-1}}{a_{m-i}} \leq \cdots \leq \frac{a_{i-[\frac{m-1}{2}]-1}}{a_{m-[\frac{m}{2}]}} \leq 1. \tag{1.3}
\]

It is easily checked that ratio monotonicity implies both log-concavity and the spiral property.

Let \(P(x) = a_0 + a_1 x + \cdots + a_m x^m\) be a polynomial with nonnegative coefficients. We say that \(P(x)\) is unimodal if the sequence \(\{a_i\}_{0 \leq i \leq m}\) is unimodal. A mode of \(\{a_i\}_{0 \leq i \leq m}\) is also called a mode of \(P(x)\). Similarly, we say that \(P(x)\) is log-concave or ratio monotone if the sequence \(\{a_i\}_{0 \leq i \leq m}\) is log-concave or ratio monotone.

Throughout this paper \(P(x)\) is assumed to be a polynomial with nonnegative and nondecreasing coefficients. Boros and Moll [2] proved that \(P(x+1)\), as a polynomial of \(x\), is unimodal. Alvarez et al. [1] showed that \(P(x+n)\) is also unimodal for any positive integer \(n\), and conjectured that \(P(x+d)\) is unimodal for any \(d > 0\). Wang and Yeh [6] confirmed this conjecture and studied the modes of \(P(x+d)\). Llamas and Martínez-Bernal [5] obtained the log-concavity of \(P(x+c)\) for \(c \geq 1\). Chen, Yang and Zhou [4] showed that \(P(x+1)\) is ratio monotone, which leads to an alternative proof of the ratio monotonicity of the Boros-Moll polynomials [3].

Let \(M_*(P,d)\) and \(M^*(P,d)\) denote the smallest and the greatest mode of \(P(x+d)\) respectively. Our main result is the following theorem, which was conjectured by Wang and Yeh [6].

**Theorem 1.1** Suppose that \(P(x)\) is a monic polynomial of degree \(m \geq 1\) with nonnegative and nondecreasing coefficients. Then for \(0 < d_1 < d_2\), we have \(M_*(P,d_1) \geq M_*(P,d_2)\) and \(M^*(P,d_1) \geq M^*(P,d_2)\).

From now on, we further assume that \(P(x)\) is monic, that is \(a_m = 1\). For \(0 \leq k \leq m\), let

\[
b_k(x) = \sum_{j=k}^{m} \binom{j}{k} a_j x^{j-k}. \tag{1.4}
\]

Therefore, \(b_k(x)\) is of degree \(m-k\) and \(b_k(0) = a_k\). For \(1 \leq k \leq m\), let

\[
f_k(x) = b_{k-1}(x) - b_k(x), \tag{1.5}
\]

which is of degree \(m-k+1\). Let \(f_k^{(n)}(x)\) denote the \(n\)-th derivative of \(f_k(x)\).

Our proof of Theorem 1.1 relies on the fact that \(f_k(x)\) has at most one real zero on \((0, +\infty)\). In fact, the derivative \(f_k^{(n)}(x)\) of order \(n \leq m-k\) has the same property. We establish this property by induction on \(n\).
2 Proof of Theorem 1.1

To prove Theorem 1.1, we need the following three lemmas.

Lemma 2.1 For any $0 \leq k \leq m$, we have $b'_k(x) = (k + 1)b_{k+1}(x)$.

Proof. Let $B_{j,k}(x)$ denote the summand of $b_k(x)$. It is readily checked that

$$B'_{j,k}(x) = (k + 1)B_{j,k+1}(x).$$

The result immediately follows.

Lemma 2.2 For $n \geq 1$ and $1 \leq k \leq m$, we have

$$f^{(n)}_k(x) = (k + n - 1)b_{k+n-1}(x) - (k + n)b_{k+n}(x),$$

where $(m)_j = m(m - 1) \cdots (m - j + 1)$.

Proof. Use induction on $n$. For $n = 1$, we have

$$f^{(1)}_k(x) = f'_k(x) = kb_k - (k + 1)b_{k+1}.$$

Assume that the lemma holds for $n = j$, namely,

$$f^{(j)}_k(x) = (k + j - 1)b_{k+j-1}(x) - (k + j)b_{k+j}(x).$$

Therefore,

$$f^{(j+1)}_k(x) = (k + j - 1)b'_{k+j-1}(x) - (k + j)b'_{k+j}(x) = (k + j)(k + j - 1)b_{k+j}(x) - (k + j + 1)(k + j)b_{k+j+1}(x) = (k + j)b_{k+j+1}(x) - (k + j + 1)b_{k+j+1}(x).$$

This completes the proof.

Lemma 2.3 For $1 \leq k \leq m$ and $0 \leq n \leq m - k$, the polynomial $f^{(n)}_k(x)$ has at most one real zero on the interval $(0, +\infty)$. In particular, $f_k(x)$ has at most one real zero on the interval $(0, +\infty)$.

Proof. Use induction on $n$ from $m - k$ to 0. First, we consider the case $n = m - k$. Recall that

$$f_k(x) = \sum_{j=k-1}^{m} \binom{j}{k-1}a_jx^{j-k} + \sum_{j=k}^{m} \binom{j}{k}a_jx^{j-k}.$$

Thus $f_k(x)$ is a polynomial of degree $m - k + 1$. Note that

$$f^{(m-k)}_k(x) = (m - k + 1)!(\binom{m}{k-1}a_m + \binom{m-1}{k-1}a_{m-1} - \binom{m}{k}a_m)(m - k)!.$$
Clearly, $f_k^{(m-k)}(x)$ has at most one real zero $x_0$ on $(0, +\infty)$. So the lemma is true for $n = m - k$.

Suppose that the lemma holds for $n = j$, where $m - k \geq j \geq 1$. We proceed to show that $f_k^{(j-1)}(x)$ has at most one real zero on $(0, +\infty)$. From the inductive hypothesis it follows that $f_k^{(j)}(x)$ has at most one real zero on $(0, +\infty)$. In light of (2.1), it is easy to verify that $f_k^{(j)}(+\infty) > 0$ and

$$f_k^{(j)}(0) = (k + j - 1)a_{k+j-1} - (k + j)a_{k+j} \leq 0.$$ 

It follows that either the polynomial $f_k^{(j-1)}(x)$ is increasing on the entire interval $(0, +\infty)$, or there exists a positive real number $r$ such that $f_k^{(j-1)}(r)$ is decreasing on $(0, r)$ and increasing on $(r, +\infty)$. Again by (2.1) we find $f_k^{(j-1)}(+\infty) > 0$ and

$$f_k^{(j-1)}(0) = (k + j - 2)a_{k+j-2} - (k + j - 1)a_{k+j-1} \leq 0.$$ 

So we conclude that $f_k^{(j-1)}(x)$ has at most one real zero on $(0, +\infty)$. This completes the proof.

Proof of Theorem 1.1. In view of (1.4), we have

$$P(x + d) = \sum_{k=0}^{m} a_k(x + d)^k = \sum_{k=0}^{m} b_k(d)x^k.$$ 

Let us first prove that $M^*(P, d_1) \geq M^*(P, d_2)$. Suppose that $M^*(P, d_1) = k$. If $k = m$, then the inequality $M^*(P, d_1) \geq M^*(P, d_2)$ holds. For the case $0 \leq k < m$, it suffices to verify that $b_k(d_2) > b_{k+1}(d_2)$. By Lemma 2.2, $f_{k+1}(x)$ has at most one real zero on $(0, +\infty)$. Note that

$$f_{k+1}(0) \leq 0 \quad \text{and} \quad f_{k+1}(+\infty) > 0.$$ 

From $M^*(P, d_1) = k$ it follows that $b_k(d_1) > b_{k+1}(d_1)$, that is $f_{k+1}(d_1) > 0$. Therefore, $f_{k+1}(d_2) > 0$, that is, $b_k(d_2) > b_{k+1}(d_2)$.

Similarly, it can be seen that $M^*(P, d_1) \geq M^*(P, d_2)$. Suppose that $M^*(P, d_2) = k$. If $k = 0$, then we have $M^*(P, d_1) \geq M^*(P, d_2)$. If $0 < k \leq m$, it is necessary to show that $b_{k-1}(d_1) < b_k(d_1)$. Again, by Lemma 2.2, we know that $f_k(x)$ has at most one real zero on $(0, +\infty)$. From $M^*(P, d_2) = k$, it follows that $b_{k-1}(d_2) < b_k(d_2)$, that is $f_k(d_2) < 0$. By the boundary conditions

$$f_k(0) \leq 0 \quad \text{and} \quad f_k(+\infty) > 0,$$ 

we obtain $f_k(d_1) < 0$, that is $b_{k-1}(d_1) < b_k(d_1)$. This completes the proof.

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.
References


