On the area discrepancy of triangulations of squares and trapezoids

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Abstract

In 1970 P. Monsky showed that a square cannot be triangulated into an odd number of triangles of equal areas; further, in 1990 E. A. Kasimatis and S. K. Stein proved that the trapezoid $T(\alpha)$ whose vertices have the coordinates $(0,0)$, $(0,1)$, $(1,0)$, and $(\alpha,1)$ cannot be triangulated into any number of triangles of equal areas if $\alpha > 0$ is transcendental.

In this paper we first establish a new asymptotic upper bound for the minimal difference between the smallest and the largest area in triangulations of a square into an odd number of triangles. More precisely, using some techniques from the theory of continued fractions, we construct a sequence of triangulations T_{n_i} of the unit square into n_i triangles, n_i odd, so that the difference between the smallest and the largest area in T_{n_i} is $O\big(\frac{1}{n_i^3}\big)$.

We then prove that for an arbitrarily fast-growing function $f : \mathbb{N} \to \mathbb{N}$, there exists a transcendental number $\alpha > 0$ and a sequence of triangulations T_{n_i} of the trapezoid $T(\alpha)$ into n_i triangles, so that the difference between the smallest and the largest area in T_{n_i} is $O\big(\frac{1}{f(n_i)}\big)$.

Keywords: triangulation, equidissection, area discrepancy, square, trapezoid, continued fraction

1 Introduction

In this paper we consider simplicial triangulations of squares and trapezoids. By 'simplicial' we mean that the intersection of any two triangles in the triangulation, if non-empty,

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is either a common vertex or two vertices and the entire edge that joins them. In other words, a vertex is not allowed to lie in the interior of an edge of another triangle. However, we do allow vertices to lie on the edges of the square (or the trapezoid, respectively). Throughout the paper, by a triangulation we will always mean a simplicial triangulation.

It is a celebrated result of Paul Monsky that a square cannot be triangulated into an odd number of triangles of equal areas [11] (see also [1, 16]). Following Monsky's result, a number of authors have investigated the existence of 'equal-area triangulations' for various other types of polygons, such as trapezoids, regular n-gons, polyominos, etc. (see [7, 5, 12, 4, 6, 14], for example). See also [16] for a nice survey of some basic results in the theory. In recent years, research activities related to 'equal-area triangulations' of polygons have further increased due to some questions and conjectures posed by Richard Kenyon, Sherman Stein, and Günter M. Ziegler [10, 18, 15, 17, 2].

In Section 2 of this paper, we first address the following question asked by Günter M. Ziegler in 2003: *given an odd number $n \in \mathbb{N}$, how small can the difference between the smallest and the largest area in a triangulation of a square into n triangles become?*

Formally, this problem may be described as follows. If for a triangulation T_n of the unit square into n triangles with areas $A_1, \ldots, A_n$, we define

$$\mathrm{Max}(T_n) := \max_{1 \leq i < j \leq n} |A_i - A_j|,$$

then we are interested in

$$\mathrm{M}(n) := \min_{T_n \in S_n} \mathrm{Max}(T_n),$$

where S_n is the set of all triangulations of the unit square into n triangles. It is easy to see that the minimum $\mathrm{M}(n)$ is in fact attained (see [10]). Obviously, we have $\mathrm{M}(n) = 0$ if n is even. So we suppose that n is odd.

The following trivial - though currently best known - asymptotic upper bound for $\mathrm{M}(n)$ was established in [10]:

$$\mathrm{M}(n) = O(\frac{1}{n^2}).$$

In Section 2.2 (Theorem 2.5), we derive

$$\mathrm{M}(n) = O(\frac{1}{n^3}) \tag{1}$$

by constructing a sequence $\{T_{n_i}\}$ of triangulations of the unit square that satisfies $\mathrm{Max}(T_{n_i}) = O(\frac{1}{n_i^3})$.

Some of the difficulties that arise in further improving this upper bound for $\mathrm{M}(n)$ are discussed in Section 2.3.

In Section 3, we study the area discrepancy of triangulations of trapezoids. For any real number $\alpha > 0$, we let $T(\alpha)$ denote the trapezoid whose vertices have the coordinates $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(\alpha, 1)$. Note that we may restrict our attention to such trapezoids, since any trapezoid is affinely equivalent to a trapezoid $T(\alpha)$. Analogously to the definitions above, we let

$$\mathrm{M}(\alpha, n) := \min_{T_n \in S_n^{(\alpha)}} \mathrm{Max}(T_n),$$

where $S_n^{(\alpha)}$ is the set of all triangulations of $T(\alpha)$ into n triangles, and for any triangulation T_n of $T(\alpha)$ into n triangles with areas $A_1, \ldots, A_n$, $\mathrm{Max}(T_n)$ is defined as

$$\mathrm{Max}(T_n) := \max_{1 \le i < j \le n} |A_i - A_j|.$$

It is well known that if α is transcendental, then $T(\alpha)$ cannot be triangulated into triangles of equal areas (see [7] as well as [5, 16, 12, 4], for example), so that for every $n \in \mathbb{N}$ we have

$$\mathrm{M}(\alpha, n) > 0.$$

One might suspect that - due to the large number of degrees of freedom for the vertex coordinates of a triangulation of a trapezoid (or, in particular, of a square) - there exists an exponential asymptotic upper bound for $\mathrm{M}(\alpha, n)$ (see also [10]). We prove in Section 3 (Theorem 3.2) that for suitable transcendental numbers α, the following even stronger statement holds:

Given an (arbitrarily fast-growing) function $f : \mathbb{N} \to \mathbb{N}$, there exists a transcendental number $\alpha > 0$ and a strictly monotone increasing sequence of natural numbers n_i with

$$M(\alpha, n_i) = O\Big(\frac{1}{f(n_i)}\Big). \tag{2}$$

2 Odd triangulations of a square

2.1 Preliminaries

The starting point for our construction of sequences of triangulations which prove (1) are certain triangulations of a trapezoid, as they are described by Stein and Szabó in [16].

Theorem 2.1 *[16] Let t_1, t_2, and t_3 be positive integers such that $t_2^2 - 4t_1t_3$ is positive and is not the square of an integer (i.e., $f(x) = t_3x^2 - t_2x + t_1$ has two positive nonrational roots). Let c be a root of $f(x)$ and let $b = \frac{ct_3}{1+ct_3}$. Then*

(i) $0 < b < 1$;

(ii) the triangulation of the trapezoid $ABCD$ into the triangles Δ_1, Δ_2, and Δ_3 with respective areas A_1, A_2, and A_3 depicted in Figure 1 satisfies

$$\frac{A_2}{A_1} = \frac{t_2}{t_1} \qquad \text{and} \qquad \frac{A_3}{A_1} = \frac{t_3}{t_1}.$$

Corollary 2.2 *[16] A triangulation of Δ_1 into t_1, Δ_2 into t_2, and Δ_3 into t_3 triangles of equal areas gives rise to a triangulation of the trapezoid $ABCD$ into $t_1 + t_2 + t_3$ triangles of equal areas.*

To prove (1) we need the following stronger version of Corollary 2.2:

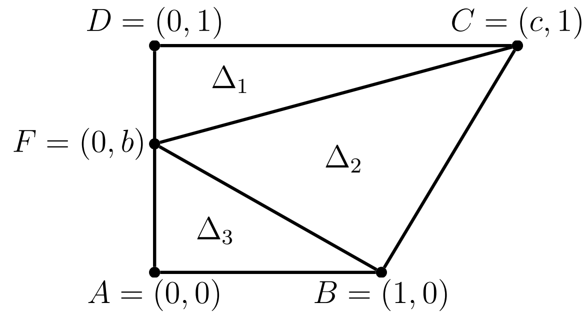

Figure 1: *A triangulation of the trapezoid $ABCD$, where b and c are defined as in Theorem 2.1.*

Corollary 2.3 *Let $a \in \mathbb{N}$, and let t_1, t_2, and t_3 be as in Theorem 2.1. Then the following statements hold:*

(i) If t_2 is odd, then the trapezoid $ABCD$ in Figure 1 can be triangulated into $a(t_1+t_2+t_3)$ triangles of equal areas, so that no vertex lies in the interior of the line segment $\overline{BC}$;

(ii) if t_2 is even, then the trapezoid $ABCD$ in Figure 1 can be triangulated into $a(t_1+t_2+t_3)$ triangles of equal areas, so that one of the vertices of the triangles is the midpoint of the line segment $\overline{BC}$ and no other vertices lie in the interior of $\overline{BC}$.

Proof. (i) Let $a = 2^\alpha a'$, where a' is odd and $\alpha \geq 0$. It is easy to triangulate Δ_3 into at_3 triangles of equal areas by placing $at_3 - 1$ vertices equidistantly on the line segment $\overline{AB}$. Then we triangulate each of the triangles Δ_1 and Δ_2 into 2^α triangles by placing $2^\alpha - 1$ vertices equidistantly on the line segment $\overline{FC}$. Since t_2a' is odd, we can triangulate each of the triangles in the resulting triangulation of Δ_2 into t_2a' triangles of equal areas without placing vertices on edges. If t_1 is odd, the same can be done with the triangulation of Δ_1, yielding a desired triangulation of the trapezoid $ABCD$. If t_1 is even, then we denote the vertices that were added on the line segment $\overline{FC}$ by $V_1, \ldots, V_{2^\alpha-1}$, and triangulate each of the 2^α triangles in the triangulation of Δ_1 into t_1a' triangles of equal areas by placing $t_1a' - 1$ vertices equidistantly on each of the line segments $\overline{DV_{2i-1}}$, $i = 1, \ldots, 2^{\alpha-1}$. This proves (i).

(ii) Let $t_2 = 2^\tau t'$, where t' is odd and $\tau \geq 1$. Then we triangulate the triangle Δ_2 as follows. First, we split Δ_2 into two triangles of equal areas by connecting the vertex F with the midpoint M of the line segment $\overline{BC}$. Then we triangulate each of these two triangles into $2^{\tau-1}t'a$ triangles of equal areas by placing $2^{\tau-1}t'a - 1$ vertices equidistantly on the line segment $\overline{FM}$. The triangles Δ_1 and Δ_3 we triangulate into at_1 and at_3 triangles of equal areas by placing $at_1 - 1$ and $at_3 - 1$ vertices equidistantly on the line segments $\overline{AB}$ and $\overline{DC}$, respectively. This yields a desired triangulation of the trapezoid $ABCD$. □

Throughout this paper, we will need good rational approximations of a real number α; so we will frequently use some basic results from the theory of continued fractions which we summarize in Theorem 2.4. Good sources for these results are [8, 9], for example.

Let the *continued fraction representation* of a real number $\alpha > 0$ be given by

$$\alpha = [a_1, a_2, a_3, \ldots] := a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \ldots}}},$$

where $a_1 \in \mathbb{N} \cup \{0\}$ and $a_i \in \mathbb{N}$ for all $i \geq 1$. Then the rational number

$$[a_1, a_2, \ldots, a_n] := a_1 + \cfrac{1}{a_2 + \ldots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}$$

is called the *nth convergent* of α.

Theorem 2.4 *Let $\alpha \in \mathbb{R}$, $\alpha > 0$, and let $\frac{p_n}{q_n}$ be the nth convergent of α with $gcd(p_n, q_n) = 1$. Then*

(i) the process of representing α as a continued fraction terminates if and only if α is rational;

(ii) $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$;

(iii) $|\alpha - \frac{p_n}{q_n}| \leq \frac{1}{q_n^2}$.

2.2 The main result for the square

Theorem 2.5 *Let $T_{n_0}^{(1)}$, $T_{n_0}^{(2)}$, and $T_{n_0}^{(3)}$ be the triangulations of the rectangle $AECD$ depicted in Figure 2 with $E = (c, 0)$, $G = (1 + \frac{2}{3}(c-1), 0)$, and $M = (1 + \frac{1}{2}(c-1), \frac{1}{2})$; these triangulations extend the triangulation of the trapezoid $ABCD$ in Figure 1. Then for some $k \in \{1, 2, 3\}$, there exists a sequence of triangulations $T_{n_i}^{(k)}$, $i \geq 0$, of $AECD$ into n_i triangles so that*

(i) $n_0 < n_1 < n_2 < \ldots$ $\quad$ (n_i odd for $i \geq 1$);

(ii) $T_{n_i}^{(k)}$ is a refinement of the triangulation $T_{n_0}^{(k)}$ (i.e., each triangle of $T_{n_i}^{(k)}$ is fully contained in a triangle of $T_{n_0}^{(k)}$);

(iii) $Max(T_{n_i}^{(k)}) = O(\frac{1}{n_i^3})$.

Remark 2.1 *By appropriately scaling the x-axis, Theorem 2.5 can immediately be transferred from the rectangle $AECD$ to the unit square.*

Proof of Theorem 2.5. Wlog we assume that $c > 1$ (as it is the case in Figures 1 - 2). For $c < 1$, the proof proceeds analogously. Let A_{trap} denote the area of the trapezoid $ABCD$ and A_{tria} denote the area of the triangle BEC. Then we have

$$\frac{A_{trap}}{A_{tria}} = \frac{c+1}{c-1},$$

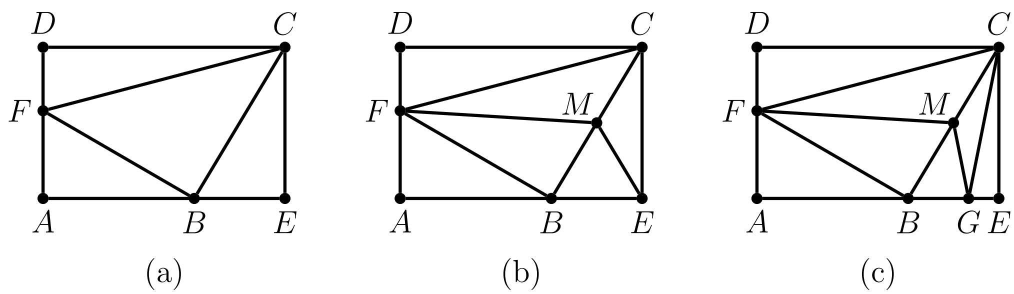

Figure 2: *Triangulations of the rectangle $AECD$:* (a) *the triangulation $T^{(1)}_{n_0}$;* (b) *the triangulation $T^{(2)}_{n_0}$;* (c) *the triangulation $T^{(3)}_{n_0}$.*

and since $c \notin \mathbb{Q}$, $\frac{A_{trap}}{A_{tria}}$ is not rational. We now consider four cases.

Case 1 (see Figure 2 (a)): Suppose that both t_2 and $t_1 + t_2 + t_3$ are odd. By Theorem 2.4 (*iii*), for the nth convergent $\frac{p_n}{q_n}$ of $\frac{A_{trap}}{A_{tria}}$, we have

$$\Big|\frac{A_{trap}}{A_{tria}} - \frac{p_n}{q_n}\Big| \leq \frac{1}{q_n^2},$$

and hence

$$\Big|\frac{A_{trap}}{p_n} - \frac{A_{tria}}{q_n}\Big| \leq \frac{A_{tria}}{p_n} \cdot \frac{1}{q_n^2}.$$

By Theorem 2.4 (*iii*), there exist positive constants c_1 and c_2 such that for all $n \in \mathbb{N}$ we have

$$c_1 q_n \leq p_n \leq c_2 q_n. \tag{3}$$

Therefore,

$$\Big|\frac{A_{trap}}{(t_1+t_2+t_3)p_n} - \frac{A_{tria}}{(t_1+t_2+t_3)q_n}\Big| \leq \frac{A_{tria}}{c_1'} \cdot \frac{1}{q_n^3},$$

where $c_1' = (t_1 + t_2 + t_3)c_1$.

By Corollary 2.3 (*i*), the trapezoid $ABCD$ can be triangulated into $(t_1 + t_2 + t_3)p_n$ triangles of equal areas, and the triangle BEC can be triangulated into $(t_1 + t_2 + t_3)q_n$ triangles of equal areas, so that we obtain a triangulation $T^{(1)}_{n_i}$ of the rectangle $AECD$ into $n_i = (t_1 + t_2 + t_3)(p_n + q_n)$ triangles with

$$\mathrm{Max}(T^{(1)}_{n_i}) \leq \frac{A_{tria}}{c_1'} \cdot \frac{1}{q_n^3}.$$

It follows from (3) that the number n_i of triangles in $T^{(1)}_{n_i}$ is at most $(t_1+t_2+t_3)(c_2+1)q_n$. Moreover, n_i is odd for infinitely many n_i, because if $p_n + q_n$ is even, then it follows from $gcd(p_n, q_n) = 1$ that both p_n and q_n are odd, so that, by Theorem 2.4 (*ii*), $p_{n-1} + q_{n-1}$ is odd. Thus, there exists a sequence $\{T^{(1)}_{n_i}\}_{i\geq 0}$ of triangulations of $AECD$ which satisfies the desired properties.

Case 2 (see again Figure 2 (a)): Suppose that t_2 is odd and that $t_1 + t_2 + t_3$ is even. By Theorem 2.4 (*iii*), for the nth convergent $\frac{p_n}{q_n}$ of

$$\frac{\frac{A_{trap}}{t_1+t_2+t_3}}{A_{tria}},$$

we have

$$\left|\frac{A_{trap}}{(t_1+t_2+t_3)p_n} - \frac{A_{tria}}{q_n}\right| \leq \frac{A_{tria}}{p_n} \cdot \frac{1}{q_n^2}.$$

Thus, analogously to Case 1, Corollary 2.3 (*i*) guarantees the existence of a triangulation $T_{n_i}^{(1)}$ of the rectangle $AECD$ into $n_i = (t_1 + t_2 + t_3)p_n + q_n$ triangles with

$$\mathrm{Max}(T_{n_i}^{(1)}) \leq c \cdot \frac{1}{q_n^3}$$

for some constant c. Since, by Theorem 2.4 (*ii*), q_n and q_{n-1} cannot both be even, n_i is odd for infinitely many n_i. Thus, there exists a sequence $\{T_{n_i}^{(1)}\}_{i \geq 0}$ of triangulations of $AECD$ which satisfies the desired properties.

Case 3 (see Figure 2 (b)): Suppose that t_2 is even and that $t_1 + t_2 + t_3$ is odd. Note that in the triangulation $T_{n_0}^{(2)}$ of $AECD$ depicted in Figure 2 (b), the triangle BEC is triangulated into two triangles of equal areas. By Theorem 2.4 (*iii*), for the nth convergent $\frac{p_n}{q_n}$ of

$$\frac{\frac{A_{trap}}{t_1+t_2+t_3}}{\frac{A_{tria}}{2}},$$

we have

$$\left|\frac{A_{trap}}{(t_1+t_2+t_3)p_n} - \frac{A_{tria}}{2q_n}\right| \leq \frac{A_{tria}}{2p_n} \cdot \frac{1}{q_n^2}.$$

Thus, it follows from Corollary 2.3 (*ii*) that there exists a triangulation $T_{n_i}^{(2)}$ of the rectangle $AECD$ into $n_i = (t_1 + t_2 + t_3)p_n + 2q_n$ triangles with

$$\mathrm{Max}(T_{n_i}^{(2)}) \leq c \cdot \frac{1}{q_n^3}$$

for some constant c. Since, by Theorem 2.4 (*ii*), p_n and p_{n-1} cannot both be even, n_i is odd for infinitely many n_i. Thus, there exists a sequence $\{T_{n_i}^{(2)}\}_{i \geq 0}$ of triangulations of $AECD$ which satisfies the desired properties.

Case 4 (see Figure 2 (c)): Finally, suppose that both t_2 and $t_1 + t_2 + t_3$ are even. Note that in the triangulation $T_{n_0}^{(3)}$ of $AECD$ depicted in Figure 2 (c), the triangle BEC is triangulated into three triangles of equal areas. By Theorem 2.4 (*iii*), for the nth convergent $\frac{p_n}{q_n}$ of

$$\frac{\frac{A_{trap}}{t_1+t_2+t_3}}{\frac{A_{tria}}{3}},$$

we have

$$\left|\frac{A_{trap}}{(t_1+t_2+t_3)p_n} - \frac{A_{tria}}{3q_n}\right| \leq \frac{A_{tria}}{3p_n} \cdot \frac{1}{q_n^2}.$$

Thus, by Corollary 2.3 (*ii*), there exists a triangulation $T_{n_i}^{(3)}$ of the rectangle $AECD$ into $n_i = (t_1+t_2+t_3)p_n + 3q_n$ triangles with

$$\mathrm{Max}(T_{n_i}^{(3)}) \leq c \cdot \frac{1}{q_n^3}$$

for some constant c. Further, we again have that n_i is odd for infinitely many n_i, since, by Theorem 2.4 (*ii*), q_n and q_{n-1} cannot both be even. Thus, there exists a sequence $\{T_{n_i}^{(3)}\}_{i\geq 0}$ of triangulations of $AECD$ which satisfies the desired properties. This completes the proof. □

2.3 Further remarks

In the previous section (Theorem 2.5) we showed that $\mathrm{M}(n) = O(\frac{1}{n^3})$ by constructing a sequence $\{T_{n_i}\}$ of triangulations of the unit square, starting from a suitable triangulation T_{n_0}, with the property that each triangulation T_{n_i} is a refinement of the triangulation T_{n_0}. Can the asymptotic upper bound $O(\frac{1}{n^3})$ for $\mathrm{M}(n)$ be further improved with this method?

Clearly, if the triangles $\Delta_1, \ldots, \Delta_{n_0}$ of T_{n_0} with respective areas $A_1, \ldots, A_{n_0}$ satisfy the property that all quotients $\frac{A_i}{A_1}$, $i = 2, \ldots, n_0$, are rational, then one cannot obtain an analogous result to Theorem 2.5 by refining T_{n_0}, because rational numbers have finite continued fraction representations (recall Theorem 2.4 (*i*)) and $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$ has only a finite number of solutions if α is rational (see [3], for example).

Our analyses in the previous sections suggest to consider triangulations of the following type:

Definition 2.1 We say that a triangulation T_{n_0} of the unit square (or, more generally, of a trapezoid) into triangles $\Delta_1, \ldots, \Delta_{n_0}$ is an *r-triangulation* if for any natural numbers $B_1, \ldots, B_{n_0}$, there exists a natural number B and a refinement of T_{n_0} in which each Δ_i is triangulated into $B \cdot B_i$ triangles of equal areas. (See also Remark 3.1 for further comments on r-triangulations.)

Remark 2.2 *Let T_{n_0} be a triangulation of the unit square whose triangles $\Delta_1, \ldots, \Delta_{n_0}$ have respective areas $A_1, \ldots, A_{n_0}$. If T_{n_0} is an r-triangulation and all quotients $\frac{A_i}{A_1}$, $i = 2, \ldots, n_0$, are rational, then T_{n_0} can of course be refined to a triangulation of the unit square whose triangles all have equal areas. However, it then follows from Monsky's theorem (see [11]) that the number of triangles in this triangulation must be even.*

Remark 2.3 *To improve the asymptotic upper bound for $M(n)$ in Theorem 2.5 it is natural to try the following approach.*

Let $A_1, \ldots, A_{n_0}$ be the areas of the triangles $\Delta_1, \ldots, \Delta_{n_0}$ of an r-triangulation T_{n_0} of the unit square, and let $A'_1, \ldots, A'_{n_0}$ be the areas of the triangles $\Delta'_1, \ldots, \Delta'_{n_0}$ of a *triangulation T'_{n_0} of the unit square, where the coordinates of the vertices of the Δ'_i are rational numbers that approximate the coordinates of the vertices of the Δ_i very well. Moreover, the combinatorial type of the triangulations T_{n_0} and T'_{n_0} shall be the same. Then the quotients $\frac{A'_i}{A'_1}$, $i = 2, \ldots, n_0$, are of course rational, say*

$$\frac{A'_i}{A'_1} = \frac{a_i}{a_1} \qquad \text{with } a_i \in \mathbb{N} \text{ for all } i. \tag{4}$$

Due to the continuity of the area function, the approximation

$$\Big|\frac{A_i}{A_1} - \frac{a_i}{a_1}\Big|$$

is then also very good. It is therefore natural to refine the triangulation T_{n_0} by triangulating each Δ_i into Ba_i triangles of equal areas. This yields a triangulation with $B(a_1 + \ldots + a_{n_0})$ triangles.

Unfortunately, $B(a_1 + \ldots + a_{n_0})$ will always be even, because it follows from (4) that if each triangle Δ'_i is triangulated into Ba_i triangles of equal areas, then one obtains a triangulation of the unit square whose triangles have all equal areas.

The next theorem (Theorem 2.7) shows that if there exist two triangles in T_{n_0} whose ratio of areas is not rational but algebraic over $\mathbb{Q}$, then Theorem 2.5 can also not be improved by refining T_{n_0}. This result is based on the following well-known fact:

Lemma 2.6 (Thue, Siegel, Roth) *[13] Let $\epsilon > 0$, $A > 0$, and $\alpha \in \mathbb{R}$ be nonrational, but algebraic over $\mathbb{Q}$. Then there only exist finitely many fractions $\frac{p}{q}$, $gcd(p, q) = 1$, with*

$$\Big|\alpha - \frac{p}{q}\Big| < \frac{A}{q^{2+\epsilon}}.$$

Theorem 2.7 *Let T_{n_0} be a triangulation of the unit square which contains two triangles Δ_1 and Δ_2 with respective areas A_1 and A_2 so that*

$$\alpha = \frac{A_1}{A_2}$$

is not rational, but algebraic over $\mathbb{Q}$. Let $\epsilon > 0$. Then there exists no sequence of triangulations T_{n_i}, $i \geq 0$, of the unit square into n_i triangles with

(i) $n_0 < n_1 < n_2 < \ldots$;

(ii) T_{n_i} is a refinement of the triangulation T_{n_0} (i.e., each triangle of T_{n_i} is fully contained in a triangle of T_{n_0});

(iii) $Max(T_{n_i}) = O(\frac{1}{n_i^{3+\epsilon}})$.

Proof. Let $\{T_{n_i}\}_{i\geq 0}$ be a sequence of triangulations satisfying the conditions (i), (ii), and (iii). Then $\{T_{n_i}\}_{i\geq 0}$ gives rise to sequences $\{T_{n'_i}\}_{i\geq 0}$ and $\{T_{n''_i}\}_{i\geq 0}$ of triangulations of the triangles Δ_1 and Δ_2 into n'_i and n''_i triangles, respectively. Condition (iii) implies that $\lim_{i\to\infty} n'_i = \infty$ and $\lim_{i\to\infty} n''_i = \infty$. So, wlog, the sequence $\{T_{n_i}\}_{i\geq 0}$ can be chosen so that

$$n'_0 \leq n'_1 \leq n'_2 \leq \dots$$
$$n''_0 \leq n''_1 \leq n''_2 \leq \dots$$

Note that there exist triangles D_1 and D_2 in the triangulations $T_{n'_i}$ and $T_{n''_i}$, respectively, so that the difference between the area of D_1 and the area of D_2 is at least

$$\Big|\frac{A_1}{n'_i} - \frac{A_2}{n''_i}\Big|,$$

because the maximum over all differences between the area of a triangle in $T_{n'_i}$ and the area of a triangle in $T_{n''_i}$ is minimal if both Δ_1 and Δ_2 are triangulated into triangles of equal areas. Thus, we have

$$\text{Max}(T_{n_i}) \geq \Big|\frac{A_1}{n'_i} - \frac{A_2}{n''_i}\Big|. \tag{5}$$

By the definition of α, we have

$$\Big|\frac{A_1}{n'_i} - \frac{A_2}{n''_i}\Big| = \Big|\alpha - \frac{n'_i}{n''_i}\Big| \cdot \frac{A_2}{n'_i}. \tag{6}$$

Now, if condition (iii) holds, then it follows from (5) that there exists a constant $c > 0$ with

$$\Big|\frac{A_1}{n'_i} - \frac{A_2}{n''_i}\Big| \leq \frac{c}{n_i^{3+\epsilon}} \quad \text{for all } i \in \mathbb{N}. \tag{7}$$

Therefore, by (6) and (7), we have

$$\Big|\alpha - \frac{n'_i}{n''_i}\Big| \leq \frac{n'_i}{A_2} \cdot \frac{c}{n_i^{3+\epsilon}} \leq \frac{c}{A_2} \cdot \frac{1}{n_i^{2+\epsilon}} \leq \frac{c}{A_2} \cdot \frac{1}{n_i''^{2+\epsilon}} \quad \text{for all } i \in \mathbb{N}. \tag{8}$$

If $\frac{n'_i}{n''_i}$ takes on infinitely many different values, then (8) contradicts Lemma 2.6.

So, suppose there exists an $i \in \mathbb{N}$, so that

$$\frac{n'_j}{n''_j} = \frac{k_j n'_i}{k_j n''_i}$$

for infinitely many j with $j \geq i$, where $k_j \in \mathbb{N}$, $k_j \geq 0$. Then it follows from (5) that for infinitely many j with $j \geq i$, we have

$$\text{Max}(T_{n_j}) \geq \frac{1}{k_j} \cdot \Big|\frac{A_1}{n'_i} - \frac{A_2}{n''_i}\Big|.$$

Further, we have $n_j \geq k_j(n'_i + n''_i)$, and hence

$$\mathrm{Max}(T_{n_j}) \cdot n_j \geq \left| \frac{A_1}{n'_i} - \frac{A_2}{n''_i} \right| \cdot (n'_i + n''_i)$$

for infinitely many j with $j \geq i$. This contradicts (iii), since the right-hand side of the above inequality is constant. □

Theorem 2.8 *The condition in Theorem 2.7 that the triangulation T_{n_0} contains two triangles Δ_1 and Δ_2 with respective areas A_1 and A_2 so that $\alpha = \frac{A_1}{A_2}$ is not rational but algebraic over $\mathbb{Q}$ may be replaced by the condition that T_{n_0} contains a triangle Δ_1 whose area A_1 is not rational but algebraic over $\mathbb{Q}$.*

Proof. If we assume that there exists a sequence of triangulations of the unit square starting with T_{n_0} and satisfying the conditions $(i)-(iii)$ of Theorem 2.7, then, analogously to the proof of Theorem 2.7, it follows that for each of the triangles Δ_j, $j = 2, \ldots, n_0$, of the triangulation T_{n_0}, there exists a sequence $(n_{j_i})_{i \in \mathbb{N}}$ of natural numbers, so that for appropriate positive constants $c_2, \ldots, c_{n_0}$, we have

$$\left| \frac{A_j}{A_1} - \frac{n_{j_i}}{n_{1_i}} \right| \leq c_j \cdot \frac{1}{n_{1_i}^{2+\epsilon}} \quad \text{for all } i \in \mathbb{N}, \tag{9}$$

where A_j is the area of the triangle Δ_j for each j. Further, it follows from the proof of Theorem 2.7 that for each $j = 1, \ldots, n_0$, $\frac{n_{j_i}}{n_{1_i}}$ takes on infinitely many different values. It follows that the same is also true for n_{1_i}, for otherwise (9) would not hold for all $i \in \mathbb{N}$.

Since we have $A_2 + \ldots + A_{n_0} = 1 - A_1$, summation of the inequalities in (9) yields a constant $c > 0$ and a sequence $(n_i)_{i \in \mathbb{N}}$ with

$$\left| \frac{1 - A_1}{A_1} - \frac{n_i}{n_{1_i}} \right| \leq c \cdot \frac{1}{n_{1_i}^{2+\epsilon}} \quad \text{for all } i \in \mathbb{N}. \tag{10}$$

Since, by assumption, A_1 is not rational but algebraic over $\mathbb{Q}$, the same is also true for $\frac{1-A_1}{A_1}$. Moreover, since n_{1_i} takes on infinitely many different values, the right-hand side of the inequality (10) becomes arbitrarily small as i goes to infinity, and hence $\frac{n_i}{n_{1_i}}$ also takes on infinitely many different values. Thus, the inequality (10) contradicts Lemma 2.6. □

It follows from Theorem 2.7 and the comments in the beginning of this section that an improvement of Theorem 2.5 using refinements of a given triangulation T_{n_0} is only possible if the quotients $\frac{A_i}{A_1}$, $i = 2, \ldots, n_0$, are all either rational or transcendental, and $\frac{A_i}{A_1}$ is transcendental for at least one i.

Note that a triangulation of the unit square into n triangles is determined by the $(n-2)$ 'free' coordinates of the corresponding vertices. If $(n-2)$ of the $(n-1)$ quotients $\frac{A_i}{A_1}$ are rational, then the free coordinates satisfy $(n-2)$ polynomial equations with integer coefficients, so that one may suspect that all the free coordinates - and hence also all the quotients $\frac{A_i}{A_1}$ - are algebraic over $\mathbb{Q}$. In other words, we anticipate that an improvement of Theorem 2.5 can only be obtained if at least two of the quotients $\frac{A_i}{A_1}$ are transcendental.

So, let $\frac{A_2}{A_1}, \ldots, \frac{A_{r+1}}{A_1}$, $r \geq 2$, be transcendental, and let $\frac{A_{r+2}}{A_1}, \ldots, \frac{A_n}{A_1}$ be rational. To improve Theorem 2.7 we then need simultaneous approximations of the form

$$\left|\frac{A_i}{A_1} - \frac{a_i}{a_1}\right| \leq \frac{c}{a_1^{2+\epsilon}}, \quad i = 2, \ldots, r+1, \tag{11}$$

where $\epsilon > 0$ and c is a constant, so that $a_1 + \ldots + a_n$ is odd. It remains open whether there exists such an example.

It is a well known fact (see [3], for example) that there always exists an approximation of the form

$$\left|\frac{A_i}{A_1} - \frac{a_i}{a_1}\right| \leq \frac{c}{a_1^{1+\frac{1}{r}}}, \quad i = 2, \ldots, r+1,$$

where c is a constant, but such an approximation is of course not good enough.

If, on the other hand, the approximation in (11) is 'too good', then $a_1 + \ldots + a_n$ is surely even, since, by Monsky's theorem, the area discrepancy is strictly positive for all odd triangulations of the unit square.

3 Triangulations of trapezoids

While it seems to be difficult to further improve the asymptotic upper bound for $M(n)$ given in Theorem 2.5 using the refinement methods of the previous section, we can use the basic idea of Remark 2.3 to show a surprisingly strong result concerning the area discrepancy of triangulations of certain trapezoids (see Theorem 3.2).

Recall from Section 1 that for any real number $\alpha > 0$, $T(\alpha)$ is the trapezoid whose vertices have the coordinates $(0,0)$, $(0,1)$, $(1,0)$, and $(\alpha, 1)$, and that, as shown in [7], we have

$$\mathrm{M}(\alpha, n) > 0$$

for every $n \in \mathbb{N}$, whenever $\alpha > 0$ is transcendental. To prove (2), we need the following well-known result (see [9]):

Lemma 3.1 *Let $a_1, a_2, \ldots$ be natural numbers and $\alpha = [a_1, a_2, \ldots]$; further, let $\frac{p_n}{q_n} = [a_1, \ldots, a_n]$ be the nth convergent of α, where $gcd(p_n, q_n) = 1$. Then we have*

(i) $\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} \qquad (n > 1)$;

(ii) α is transcendental if there exists an $n_0 \in \mathbb{N}$, so that for all $n \geq n_0$, we have $a_{n+1} > q_n^{n-1}$.

We also need the following remark concerning r-triangulations (recall Definition 2.1):

Remark 3.1 *Let $T(\alpha)$, $\alpha > 0$, be a trapezoid. If T_{n_0} is a triangulation of $T(\alpha)$ with the property that edges of triangles can be removed in such a way that one obtains a dissection of $T(\alpha)$ into quadrilaterals and triangles, where each triangle has at least one edge that lies on an edge of $T(\alpha)$, then T_{n_0} is an r-triangulation.*

Proof. For triangles that have at least one of their edges, say e, on an edge of $T(\alpha)$, the desired dissection can trivially be obtained by adding points appropriately on e. By assumption, we can identify pairs of edge-sharing triangles in T_{n_0}, so that each of the remaining triangles has the property that at least one of its edges lies on an edge of $T(\alpha)$. Let $\Delta_i = ABC$ and $\Delta_j = BDC$ be such a pair of edge-sharing triangles (see Figure 3). We add the midpoint P of the shared edge BC as a new vertex. The edges AP and DP

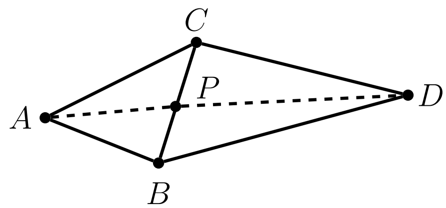

Figure 3: *The edge-sharing triangles ABC and BDC in T_{n_0}.*

triangulate each of the triangles ABC and BDC into two triangles of equal areas. By appropriately choosing $B_i - 1$ points on AP, the triangle Δ_i can be triangulated into $2B_i$ triangles of equal areas. Analogously, one obtains a triangulation of Δ_j into $2B_j$ triangles of equal areas. This yields the desired refinement of T_{n_0} for $B = 2$. □

It remains open, whether there exist triangulations which are not r-triangulations, or whether there exist triangulations which do not satisfy the conditions in Remark 3.1.

We are now ready to prove (2).

Theorem 3.2 *Let $f : \mathbb{N} \to \mathbb{N}$ be an (arbitrarily fast-growing) function. Then there exists a transcendental number $\alpha > 0$ and a strictly monotone increasing sequence of natural numbers n_i with*

$$M(\alpha, n_i) = O\Big(\frac{1}{f(n_i)}\Big).$$

Proof. For $\beta \in \mathbb{R}$, we denote $p(\beta)$ to be the point in $\mathbb{R}^2$ with the coordinates $(\beta, 1)$. Let α' be a positive real number and T'_{n_0} be an r-triangulation of the trapezoid $T(\alpha')$ into n_0 triangles, so that none of the vertices of the triangles in T'_{n_0} lies on the edge between the points $p(\alpha')$ and $(1, 0)$. By Remark 3.1, such a triangulation of $T(\alpha')$ clearly exists. In the following, we denote $p'_2 = (1, 0)$ and we let $p(\alpha'), p'_2, \ldots, p'_m$ be the points in $\mathbb{R}^2$ that correspond to the vertices of the triangles in T'_{n_0}.

We show that there exists a transcendental number $\alpha > 0$ and a triangulation T_{n_0} of $T(\alpha)$, so that

(i) T_{n_0} has the same combinatorial type as T'_{n_0};

(ii) there exists a sequence of triangulations T_{n_j}, $j \geq 0$, of $T(\alpha)$ into n_j triangles with

  (a) $n_0 < n_1 < n_2 < \ldots$;

  (b) all triangulations T_{n_j} are refinements of T_{n_0};

(c) $\mathrm{Max}(T_{n_j}) = O(\frac{1}{f(n_j)})$.

We can perturb the vertices of the triangles in the triangulation T'_{n_0}, so that we obtain a triangulation T''_{n_0} of $T(\alpha'')$ which has the same combinatorial type as T'_{n_0} and whose vertices have all rational coordinates. We denote $p(\alpha''), p_2, \ldots, p_m$ to be the points in $\mathbb{R}^2$ that correspond to the vertices of the triangles in T''_{n_0}. In particular, α'' is then rational, say $\alpha'' = \frac{t_n}{u_n}$ with $gcd(t_n, u_n) = 1$. If A''_i is the area of the triangle Δ''_i in the triangulation T''_{n_0}, then $\frac{A''_i}{A''_1}$ is also rational. Let

$$\frac{A''_i}{A''_1} = \frac{A_{1i}}{A_{11}}, \quad A_{1i}, A_{11} \in \mathbb{N}.$$

Since we do not require that $gcd(A_{1i}, A_{11}) = 1$, we may assume wlog that the denominator A_{11} is the same for all i.

The continued fraction representation of $\alpha'' \in \mathbb{Q}$ is finite, say $\alpha'' = [a_1, \ldots, a_n]$.

We now define α and the desired triangulation T_{n_0} of $T(\alpha)$. To this end, we first recursively construct a suitable sequence of natural numbers $a_{n+1}, a_{n+2}, \ldots$, and define α as

$$\alpha = [a_1, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots].$$

We then define the desired triangulation T_{n_0} of $T(\alpha)$ as the triangulation which is of the same combinatorial type as T''_{n_0} and whose vertices have the positions $p(\alpha), p_2, \ldots, p_m$. This choice for the positions of the vertices will be possible, because we will construct α in such a way that it is 'close enough' to α'', and since, by assumption, none of the points $p_2, \ldots, p_m$ lies on the edge between the points $p(\alpha'')$ and $(1, 0)$. We denote the triangle in T_{n_0} that corresponds to the triangle Δ''_i in T''_{n_0} by Δ_i. The area of the triangle Δ_i is denoted by A_i.

First, we define a_{n+1} appropriately. By definition, $\frac{t_n}{u_n}$ is the nth convergent of α. Let $\frac{t_{n-1}}{u_{n-1}}$ be the $(n-1)$st convergent of α. Both $\frac{t_n}{u_n}$ and $\frac{t_{n-1}}{u_{n-1}}$ are independent of the choice of $a_{n+1}, a_{n+2}, \ldots$. By Lemma 3.1 (i), we have

$$\left|\alpha - \frac{t_n}{u_n}\right| \leq \frac{1}{u_n(a_{n+1}u_n + u_{n-1})}. \tag{12}$$

Thus, if we choose a_{n+1} sufficiently large, say

$$a_{n+1} > N, \tag{13}$$

then T_{n_0} is indeed a triangulation of $T(\alpha)$ which has the same combinatorial type as T'_{n_0} (independent of the choice of the $a_{n+2}, a_{n+3}, \ldots$).

It follows from (12) that for $a_{n+1} \to \infty$, the number α converges to $\frac{t_n}{u_n}$, and hence $\frac{A_i}{A_1}$ converges to $\frac{A_{1i}}{A_{11}}$ (of course, if neither Δ_1 nor Δ_i has $p(\alpha)$ as a vertex, then we have $\frac{A_i}{A_1} = \frac{A_{1i}}{A_{11}}$). Since A_1 is bounded from above and since A_{1i} is a constant, it follows that for a sufficiently large a_{n+1}, we have

$$\left|\frac{A_i}{A_{1i}} - \frac{A_1}{A_{11}}\right| < \frac{1}{f\big(B_1(A_{11} + \ldots + A_{1n_0})\big)}, \tag{14}$$

for all $i = 1, \ldots, n_0$, where B_1 is chosen so that for all i, the triangle Δ_i can be triangulated into $B_1 A_{1i}$ triangles of equal areas. Now we choose a_{n+1} so that it satisfies (13) and (14) and also

$$a_{n+1} > u_n^{n-1}. \tag{15}$$

Condition (15) will be used later to show that the number α is transcendental.

Suppose now that $a_{n+1}, \ldots, a_{n+j-1}$ have already been chosen. Then we define a_{n+j} using the same basic idea that we have used to define a_{n+1} - we merely replace the approximation $\frac{t_n}{u_n}$ of α by the $(n+j-1)$st convergent $[a_1, \ldots, a_{n+j-1}] = \frac{t_{n+j-1}}{u_{n+j-1}}$, and we replace the triangulation T''_{n_0} of $T(\alpha'')$ by the triangulation T'''_{n_0} (of the same combinatorial type) of the trapezoid $T(\frac{t_{n+j-1}}{u_{n+j-1}})$ whose triangles have their vertices at the points $p(\frac{t_{n+j-1}}{u_{n+j-1}}), p_2, \ldots, p_m$.

For sufficiently large a_{n+j}, $\frac{t_{n+j-1}}{u_{n+j-1}}$ approximates the number α again arbitrarily well, and if $\frac{A_{ji}}{A_{j1}}$ are the quotients of the areas of the respective triangles in T'''_{n_0}, then, analogously to (14), we have

$$\Big|\frac{A_i}{A_{ji}} - \frac{A_1}{A_{j1}}\Big| < \frac{1}{f\big(B_j(A_{j1} + \ldots + A_{jn_0})\big)}, \tag{16}$$

for all $i = 1, \ldots, n_0$, where B_j is chosen so that for all i, the triangle Δ_i can be triangulated into $B_j A_{ji}$ triangles of equal areas.

Further, analogously to (15), a_{n+j} is chosen so that

$$a_{n+j} > u_{n+j-1}^{n+j-2}. \tag{17}$$

Since $\frac{A_{ji}}{A_{j1}}$ does not have to be a reduced fraction, we may assume wlog that

$$A_{j1} + \ldots + A_{jn_0} > B_{j-1}(A_{j-1,1} + \ldots + A_{j-1,n_0}). \tag{18}$$

By Lemma 3.1 (ii), it follows from (15) and (17) that α is transcendental. The desired refinements T_{n_j} of the triangulation T_{n_0} of $T(\alpha)$ are now obtained by triangulating the triangle Δ_i into $B_j A_{ji}$ triangles of equal areas, for each $i = 1, \ldots, n_0$. Condition (ii) (c) concerning the area discrepancy of T_{n_j} is then satisfied because of (14) and (16). Moreover, (18) guarantees that the inequalities in (ii) (a) hold. This completes the proof. □

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