Long path lemma concerning connectivity and independence number

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Abstract

We show that, in a k-connected graph G of order n with $\alpha(G) = \alpha$, between any pair of vertices, there exists a path P joining them with $|P| \geq \min\left\{n, \frac{(k-1)(n-k)}{\alpha} + k\right\}$. This implies that, for any edge $e \in E(G)$, there is a cycle containing e of length at least $\min\left\{n, \frac{(k-1)(n-k)}{\alpha} + k\right\}$. Moreover, we generalize our result as follows: for any choice S of $s \leq k$ vertices in G, there exists a tree T whose set of leaves is S with $|T| \geq \min\left\{n, \frac{(k-s+1)(n-k)}{\alpha} + k\right\}$.

1 Introduction

In this work, we present a tool which we believe will be useful in many applications. Much work has been devoted to finding long paths and cycles in graphs. In particular, in [4], O, West and Wu recently proved a conjecture by Fouquet and Jolivet [3] stated as follows.

Theorem 1 ([4]) *Let $k \geq 2$ and let G be a k-connected graph of order n with $\alpha(G) = \alpha$. Then there is a cycle in G of length at least* $\min\{n, \frac{k(n+\alpha-k)}{\alpha}\}$.

In various situations including this work, it often becomes necessary to find a long path between a chosen pair of vertices. For this reason, O, West and Wu proved the following theorem which they used in their proof of the conjecture.

Theorem 2 ([4]) *Let G be a k-connected graph for $k \geq 1$. If $H \subseteq G$ and u and v are distinct vertices in G, then G contains a u,v-path P such that $V(H) \subseteq V(P)$ or $\alpha(H - P) \leq \alpha(H) - (k-1)$.*

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We also use this theorem and, following the proofs presented in [4], we prove the following lemma which is our main result.

Lemma 1 *Let $k \geq 1$ be an integer and let G be a graph of order n with $\kappa(G) = k$ and $\alpha(G) = \alpha$. Then for any pair of vertices u, v in G, there exists a u, v-path of order at least* $\min\{n, \frac{(k-1)(n-k)}{\alpha} + k\}$.

Our hope is that this lemma may be applied to produce other results like Theorem 3, which follows immediately from Lemma 1 by choosing u and v to be the ends of e.

Theorem 3 *Let $k \geq 2$ be an integer and let G be a k-connected graph of order n with $\alpha(G) = \alpha$. Then for any edge $e \in E(G)$, there exists a cycle of length at least* $\min\left\{n, \frac{(k-1)(n-k)}{\alpha} + k\right\}$ *in G containing the edge e.*

Lemma 1 can be generalized to the following result concerning large trees with specified sets of leaves. Let $\ell(T)$ denote the set of leaves in a tree T.

Theorem 4 *Let k and s be integers with $2 \leq s \leq k$ and let G be a k-connected graph of order n with $\alpha(G) = \alpha$. Then for any set of s vertices $V_s = \{v_1, \ldots, v_s\} \subseteq G$, there exists a tree $T \subseteq G$ with $V_s = \ell(T)$ and* $|T| \geq \min\left\{n, \frac{(k-s+1)(n-k)}{\alpha} + k\right\}$.

The proofs of Lemma 1 and Theorem 4 are presented in Section 3. As we will observe in Section 4, our results are all best possible.

2 Preliminaries

In our proof, we use the following corollary to break the problem into cases. We also state and prove a path version of Theorem 6. Both of these results come from [4].

Corollary 5 ([4]) *If a graph G admits no vertex partition (V_1, V_2) such that $\alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2])$, then G is 2-connected or $G \in \{K_1, K_2\}$. Also, for distinct vertices $u, v \in G$, there is a u, v-path P such that $\alpha(G - P) < \alpha(G)$.*

Theorem 6 ([4]) *Let k be an integer greater than 1. If C is a cycle with size at least k in a k-connected graph G, then for any non-empty subgraph $H \subseteq G - C$, there exists a cycle C' in G such that $|C - C'| \leq \frac{|C|}{k} - 1$ and $\alpha(H - C') \leq \alpha(H) - 1$.*

We will also make use of the following classical result of Chvátal and Erdős [2]. A graph is said to be *hamiltonian connected* if, between any pair of vertices, there exists a path covering the entire graph.

Theorem 7 ([2]) *For any graph G, if $\kappa(G) > \alpha(G)$, then G is hamiltonian connected.*

Following the notation of [4], let P be a path and u and v be vertices in P. Define $P(u, v)$ to be the subpath of P strictly between (not including) u and v. Also, for a vertex v and a set of vertices or subgraph A, define a (v, A) *k-fan* to be a set of k paths from v to A which are all pairwise vertex disjoint except at v. All other standard notation comes from [1].

3 Proofs of our Main Results

We begin by proving a key lemma used to obtain our main result. The main idea of the proof is based on that of Theorem 6.

Lemma 2 *Let $k \geq 2$ be an integer, and suppose G is a k-connected graph containing vertices u, v. If P is a u, v-path of order at least k in G, then for any non-empty subgraph $H \subseteq G \setminus P$, there is a u, v-path P' in G such that $|P \setminus P'| \leq \frac{|P|-k}{k-1}$ and $\alpha(H \setminus P') \leq \alpha(H) - 1$.*

Proof: Suppose there exists a subgraph H for which there is no desired path P' and choose H to be the smallest such subgraph. By Corollary 5, either

(1) H can be bipartitioned into non-empty subgraphs H_1 and H_2 so that $\alpha(H) = \alpha(H_1) + \alpha(H_2)$, or

(2) H is 2-connected or $H \in \{K_1, K_2\}$. Also, for any distinct vertices $x, y \in H$, there exists an x, y-path P_{xy} in H such that $\alpha(H \setminus P_{xy}) < \alpha(H)$.

If (1) holds, we simply apply Lemma 2 on H_1 (since H was the smallest counterexample) and obtain a path P' satisfying the desired conditions. Hence we may assume (2) holds.

Let B be the block of $G \setminus P$ containing H. First we assume $|B| \geq k$. By Menger's Theorem, there exist k vertex-disjoint paths from P to B. Choose the shortest such set of paths, meaning that each path contains exactly one vertex of B and one vertex of P. This means that there must exist a pair of these paths, say $P_1 = p_1 \ldots b_1$ and $P_2 = p_2 \ldots b_2$ for $p_i \in V(P)$ and $b_i \in V(B)$ such that there are at most $\frac{|P|-k}{k-1}$ vertices between p_1 and p_2 on P. Since B is 2-connected, there exist vertex-disjoint paths P_{b_i} in B from b_i to $h_i \in V(H)$ for $i = 1, 2$. Note that $h_1 = h_2$ is only possible if $|H| = 1$. (Suppose $P_{b_i} \cap H = h_i$.) By (2), there is a path P_H in H from h_1 to h_2 for which $\alpha(H \setminus P_H) < \alpha(H)$. Then $P' = (P \setminus P(p_1, p_2)) \cup (P_1 \cup P_{b_1} \cup P_H \cup P_{b_2} \cup P_2)$ is the desired path. Hence, we may assume $|B| < k$.

Let $V(B) = \{b_1, \ldots, b_\ell\}$, where we have assumed $\ell < k$. Note that we may possibly have $\ell = 1$. Let C be the component of $G \setminus P$ containing B. Let $S = \{p_1, \ldots, p_m\}$ be the set of vertices of P (in order along P) with at least one neighbor in C. Note that, by Menger's Theorem, $m \geq k$.

For each edge e from p_i to C, there exists a unique vertex $b_j \in B$ such that there is a unique path $Q_{i,j}$ from b_j to p_i containing e with all interior vertices in $C \setminus B$. Let X_j be the set of vertices p_i for which such a path $Q_{i,j}$ exists. Note that the sets $\{X_j\}$ are not necessarily disjoint. Also note that, since B is a block, $Q_{i,j}$ and $Q_{i',j'}$ are internally disjoint when $j \neq j'$. Call a segment $P(p_i, p_j)$ for $i < j$ *large* if $p_i \in X_{i'}$ and $p_j \in X_{j'}$ for some $i' \neq j'$. Otherwise, as long as the segment $P(p_i, p_j)$ is not contained in a large segment, it will be called *small*.

Using the same argument as above, the following fact is immediate.

Fact 1 *For any large segment $P(p_i, p_j)$, we have*

$$|P(p_i, p_j)| > \frac{|P| - k}{k-1}.$$

Let t be the number of segments $P(p_i, p_{i+1})$ for $1 \le i \le m$ which are large. Since large segments contain at least $\frac{|P|-k+1}{k-1}$ vertices, we see that

$$|P| \ge t\left(\frac{|P| - k + 1}{k-1}\right) + k,$$

which implies that $t < k - 1$. For each $b_i \in B$, there exists a $(b_i - P)$ k-fan. Choose such a fan so that each path intersects P in exactly one vertex. Let $v_1, \dots, v_k$ (in this order on P) be the vertices of P at the ends of this fan. For each pair v_j, v_{j+1}, we already know that $v_j, v_{j+1} \in X_i$, but if one of these is also in $X_{i'}$ for some $i' \neq i$, then $P(v_j, v_{j+1})$ must be a large segment of P. This means that, for each vertex in B, there are at least $k - 1 - t$ corresponding small segments of P. Since the ends of these small segments corresponding to b_i are all in X_i, these segments must then be disjoint from all small segments corresponding to b_j for $j \neq i$ since the ends of those segments would be in X_j. Therefore there are $(k - 1 - t)\ell$ small segments *all* pairwise disjoint. This implies that the average order of small segments is at most

$$\frac{|P| - t\left(\frac{|P|-k+1}{k-1}\right) - k}{(k-1-t)\ell}.$$

By the pigeonhole principle, if we choose the shortest small segment corresponding to each vertex $b_i \in B$, then the sum of the orders of these shortest segments is at most

$$\frac{|P| - t\left(\frac{|P|-k+1}{k-1}\right) - k}{(k-1-t)} \le \frac{|P| - k}{k-1}.$$

We now replace each of these small segments with the corresponding b_i using the paths $Q_{i,j}$ and $Q_{i,j+1}$ for the appropriate choice of j. This creates a new u, v-path P' such that $H \subseteq B \subseteq P'$ and $|P \setminus P'| \le \frac{|P|-k}{k-1}$. $\square$

Before our next lemma, we observe an easy fact without proof.

Fact 2 *Let G be a k-connected graph for $k \ge 2$ and let u and v be two distinct vertices in G. Then for any u, v-path P with $|P| < k$, there is another u, v-path P' with $|P'| \ge k$ such that $P \subseteq P'$.*

Lemma 3 *Let G be a graph with $\kappa(G) = k$ and $\alpha(G) = \alpha$. If u, v are two vertices in G, ℓ is an integer satisfying $0 \le \ell \le \alpha - k + 1$, then there exists a set of u, v-paths $P_0, \dots, P_\ell$ satisfying:*

1. $\alpha\left(G \setminus \bigcup_{i=0}^{\ell} P_i\right) \le \alpha - k + 1 - \ell$

2. $\left| P_i \setminus \bigcup_{j=0}^{j-1} P_j \right| \leq \frac{|P_0|-k}{k-1}$ *for* $1 \leq i \leq \ell$

Proof: Induct on ℓ. If $\ell = 0$, Theorem 2 gives a u, v-path P_0 with $\alpha(G \setminus P_0) \leq \alpha - k + 1$. Now suppose we have u, v-paths $P_0, \ldots, P_{\ell-1}$ satisfying Properties 1 and 2 for $\ell - 1$.

Let $H = G \setminus \cup_{i=0}^{\ell-1} P_i$ be so that $\alpha(H) \leq \alpha - k + 1 - (\ell - 1)$. Assume $\alpha(H) \geq 1$ since otherwise we could simply set $P_\ell = P_0$. By Lemma 2 with $P_0 = P$ (note that Fact 2 implies we may assume $|P_0| \geq k$), there is a u, v-path P' such that $|P_0 \setminus P'| \leq \frac{|P_0|-k}{k-1}$ and $\alpha(H \setminus P') \leq \alpha(H) - 1 \leq \alpha - k + 1 - \ell$.

Case 1 $|P'| \leq |P_0|$

Then $\left| P' \setminus \cup_{i=0}^{\ell-1} P_i \right| \leq |P' \setminus P_0| \leq |P_0 \setminus P'| \leq \frac{|P_0|-k}{k-1}$, so we can set $P' = P_\ell$ to satisfy the desired properties.

Case 2 $|P'| > |P_0|$

Relabel the paths as follows: $P'_0 = P'$ and $P'_i = P_{i-1}$ for $1 \leq i \leq \ell$. This new labelling gives $\alpha\left(G \setminus \cup_{i=0}^{\ell} P'_i\right) \leq \alpha - k + 1 - \ell$ so Property 1 is satisfied. For Property 2, first consider the case $i = 1$. $|P'_i \setminus P'_0| = |P_0 \setminus P'| \leq \frac{|P'|-k}{k-1}$ as desired. For $2 \leq i \leq \ell$, we have

$$\left| P'_i \setminus \bigcup_{j=0}^{i-1} P'_j \right| \leq \left| P_{i-1} \setminus \bigcup_{j=0}^{i-2} P_j \right| \leq \frac{|P_0| - k}{k-1} \leq \frac{|P'_0| - k}{k-1}$$

so this labelling satisfies Properties 1 and 2, and we have our desired result. □

Using these lemmas, the proof of our main result is easy.

Proof of Lemma 1: For $k = 1$, the result is trivial so we will assume $k \geq 2$. When $k > \alpha$, the assertion holds by Theorem 7. Thus, we may also assume $\alpha \geq k$.

Set $\ell = \alpha - k + 1$ and apply Lemma 3. By Property 1, the set of paths $P_0, \ldots, P_\ell$ must cover all of $V(G)$. Using Property 2, this implies

$$n = |P_0| + \sum_{i=1}^{\ell} \left| P_i \setminus \bigcup_{j=0}^{i-1} P_j \right| \leq |P_0| + (\alpha - k + 1)\left(\frac{|P_0| - k}{k-1}\right).$$

Solving for $|P_0|$, we get get the desired result $|P_0| \geq \frac{(k-1)(n-k)}{\alpha} + k$. □

Proof of Theorem 4: This proof is by induction on s. If $s = 2$, the result follows immediately from Lemma 1. Now suppose $s > 3$ and consider $G \setminus v_s$. This graph has $\kappa(G \setminus v_s) \geq k - 1$ and we will assume $\alpha(G \setminus v_s) = \alpha(G)$ (otherwise a stronger result is possible). By induction on s, there exists a tree $T_{s-1} \subseteq G$ with $\ell(T_{s-1}) = \{v_1, \ldots, v_{s-1}\}$ and

$$
\begin{aligned}
|T_{s-1}| &\geq \min\left\{n-1, \frac{(k-s+1)(n-k)}{\alpha}+k-1, \frac{(k-s+2)(n-k-1)}{\alpha}+k\right\}\\
&\geq \min\left\{n-1, \frac{(k-s+1)(n-k)}{\alpha}+k-1\right\}
\end{aligned}
$$

as long as $n \geq 2k+2-s-\alpha$. Otherwise, if we assume $n < 2k+2-s-\alpha$, then since $n \geq k+1$, if we let $H = G \setminus \{v_3, v_4, \ldots, v_s\}$, we have $\kappa(H) \geq \alpha+1$. By Theorem 7, this means that H is hamiltonian connected so we can find a path P from v_1 to v_2 using all of H. Since G is k-connected, each vertex v_i for $3 \leq i \leq s$ has at least k paths to P. Since $k \geq s$, there is an edge from each v_i to $P \setminus \{v_1, v_2\}$, forming the desired tree of order n. Hence, we may suppose the above inequality holds.

In G, there are k disjoint (except at v_s) paths from v_s to T_{s-1} so there is at least one such path Q which avoids the set $\{v_1, \ldots, v_{s-1}\}$. Hence, the tree $T = T_{s-1} \cup Q$ is the desired tree with $|T| \geq |T_{s-1}|+1$. □

4 Conclusion

The results contained in this work are all sharp by the following example. Let $C = K_k$ and let $H_i = K_{\frac{n-k}{\alpha}}$ for $1 \leq i \leq \alpha$ where we assume α divides $n-k$. Let $G = C + (\cup H_i)$ where $+$ is the standard join operation such that $V(A+B) = V(A) \cup V(B)$ and $E(A+B) = E(A) \cup E(B) \cup \{u, v : u \in A, v \in B\}$. Choose $u, v \in C$ and let P be a u, v-path that uses all vertices of C and all of $H_1, \ldots, H_{k-1}$. This is the longest u, v-path in G, which shows that Lemma 1 is sharp. The same example, with the inclusion of the edge uv to complete a cycle, shows that Theorem 3 is sharp.

For Theorem 4, choose $v_1, \ldots, v_s$ from C to obtain the desired bound. In this situation, because these vertices must be leaves of the constructed tree, we may use the vertices of at most $k-s+1$ components H_i in building T. Note also that if $s > k$, a similar result cannot hold because, if we choose all of C and at least one vertex of $G \setminus C$, at least one vertex of C must not be a leaf of a tree including these vertices.

The authors hope that the results contained in this work may be applied in other works. Like Theorems 3 and 4 we believe that many results will follow from this work and perhaps other proofs may be simplified through use of Lemma 1.

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References

[1] G. Chartrand and L. Lesniak. *Graphs & Digraphs.* Chapman & Hall/CRC, Boca Raton, FL, fourth edition, 2005.

[2] V. Chvátal and P. Erdős, *A note on Hamiltonian circuits*, Discrete Math 2, (1972). 111-113.

[3] J.L. Fouquet and J.L. Jolivet. *Problèmes combinatoires et théorie des graphes Orsay, Problèmes*. 1976.

[4] S. O, D. B. West, and H. Wu. *Longest cycles in k-connected graphs with given independence number.* J. Combin. Th. (B), In Press.