Note on highly connected monochromatic subgraphs in 2-colored complete graphs

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Abstract

In this note, we improve upon some recent results concerning the existence of large monochromatic, highly connected subgraphs in a 2-coloring of a complete graph. In particular, we show that if \( n \geq 6.5(k - 1) \), then in any 2-coloring of the edges of \( K_n \), there exists a monochromatic \( k \)-connected subgraph of order at least \( n - 2(k - 1) \). Our result improves upon several recent results by a variety of authors.

1 Introduction

It is easy to see that for any graph \( G \), either \( G \) or its complement is connected. This is equivalent to saying there exists a connected color in any 2-coloring of \( K_n \). However, when we try to find a subgraph with higher connectivity, we cannot hope to find such a spanning subgraph. In order to see this, consider the following example. All standard notation comes from [3].

Consider the following example from [1]. Let \( G_n = H_1 \cup \cdots \cup H_5 \) where \( H_i \) is a red complete graph \( K_{k - 1} \) for \( i \leq 4 \) and \( H_5 \) is a red \( K_{n-2(k-1)} \) where \( n > 4(k - 1) \). To this structure, we add all possible red edges between \( H_5, H_1 \) and \( H_2 \) and from \( H_1 \) to \( H_3 \) and from \( H_2 \) to \( H_4 \). All edges not already colored in red are colored in blue. In either color, there is no \( k \)-connected subgraph of order larger than \( n - 2(k - 1) \).

Since a spanning monochromatic subgraph is more than we could hope for, we consider finding a highly connected subgraph that is as large as possible. Along this line, Bollobás and Gyárfás proposed the following conjecture.

**Conjecture 1 ([1])** For \( n > 4(k - 1) \), every 2-coloring of \( K_n \) contains a \( k \)-connected monochromatic subgraph with at least \( n - 2(k - 1) \) vertices.

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In order to see that the bound on $n$ is the best possible, consider the example $G_n$ above with $n = 4(k-1)$ (so $H_5 = \emptyset$). In [1], the authors showed that this conjecture is true for $k \leq 2$. Also, in [4], Liu, Morris and Prince showed the conjecture holds for $k = 3$, but for other cases, it remains open. As a weaker result, in [6] the authors proved the following.

**Theorem 1 ([6])** If $n \geq 13k - 15$ then every 2-coloring of $K_n$ contains a monochromatic $k$-connected subgraph of order at least $n - 2(k - 1)$.

In a related result, Bollobás and Gyárfás also proved the following.

**Theorem 2 ([1])** If Conjecture 1 holds for $4(k-1) < n < 7(k-1)$ then Conjecture 1 is true.

In this note, we improve both of these results as follows:

**Theorem 3** If $n > 6.5(k - 1)$ then any 2-coloring of $K_n$ contains a monochromatic $k$-connected subgraph of order at least $n - 2(k - 1)$.

By improving the constant from 13 to 6.5, we also slightly improve other results from [5] in some cases. As these improvements are very minor, we omit details. Since any $k$-connected graph has the minimum degree at least $k$, we immediately obtain the following corollary.

**Corollary 4** If $n > 6.5(k - 1)$, then any 2-coloring of $K_n$ contains a monochromatic subgraph of order at least $n - 2(k - 1)$ with the minimum degree at least $k$.

This corollary slightly improves a result in [2], which deals with the monochromatic large subgraph with a specified minimum degree in general graphs. When we focus on complete graphs, their work shows that the conclusion holds if $n \geq 7k + 4$.

## 2 Proof of Theorem 3

Consider a 2-coloring $G$ of $K_n$ with the colors red and blue. The proof proceeds by induction on $k$. The cases for $k \leq 2$ follow from [1] and the case $k = 3$ follows from [4] but we will not need this assumption so we simply suppose $k \geq 3$. By induction, there exists a $(k - 1)$-connected subgraph in one color (suppose red) of order at least $n - 2(k - 2)$. If this subgraph is $k$-connected, this is a desired subgraph so we may assume the connectivity is exactly $k - 1$.

Let $G_r$ be the largest $(k - 1)$-connected red subgraph and consider a minimum cutset $C$ (of order $k - 1$) of $G_r$. Let $A^C$ and $B^C$ be a bipartition of the vertices of $G_r \setminus C$ such that $A^C$ (and likewise $B^C$) is the union of vertices in components of $G_r \setminus C$ and we choose such unions with $|A^C| \geq |B^C|$ and $|B^C|$ maximum. Choose such a cutset $C$ so that $|B^C|$
is maximized and define \(A' = A^C\) and \(B' = B^C\). By definition, all edges between \(A'\) and \(B'\) are blue. This forms a complete bipartite graph in blue. Define \(D' = G \setminus G_r\).

First suppose \(|B'| \geq k\), which implies that this blue complete bipartite graph is \(k\)-connected. Note that \(|D'| \leq 2(k - 2)\) and, since \(|G_r|\) is maximum, every vertex in \(D'\) has at most \(k - 2\) red edges to \(G_r\). This means that each vertex of \(D'\) must have at least \(|G_r| - (k - 2)\) blue edges to \(G_r\). More specifically, each vertex must have at least \(|A' \cup B'| - (k - 2) > k\) blue edges to \(A' \cup B'\) (since \(n \geq 5k - 7\)). This means that \(A' \cup B' \cup D'\) induces a blue \(k\)-connected graph of order exactly \(n - (k - 1)\), thus proving the theorem in this case.

Hence, we assume \(|B'| < k\). Let \(B\) be the set of vertices satisfying the following conditions:

1. \(|B|\) is maximum subject to \(|B| < 3(k - 1)\).
2. Each vertex of \(B\) has at most \(k - 1\) red edges to \(G \setminus B\).

Certainly such a set \(B\) exists since both \(D'\) and \(D' \cup B'\) satisfy Property 2 and we know that \(|D'| \leq 2(k - 1)\) and \(|B'| \geq 1\).

**Claim 1** \(|B| \geq 2(k - 1)\).

**Proof of Claim 1:** Suppose \(|B| < 2(k - 1)\) and consider the graph \(G_r\) induced on the red edges in \(G \setminus B\). If this graph is \(k\)-connected, it would be a desired subgraph so we know \(\kappa(G_r) \leq k - 1\). As above, if there exists a cutset \(C'\) of \(G_r\) and a partition of the components of \(G_r \setminus C'\) so that each part has order at least \(k\), then we could find a \(k\)-connected blue subgraph of order at least \(n - |C'| \geq n - (k - 1)\) which would again be a desired subgraph (note that each vertex of \(B\) has at least \(k\) blue edges to \(G_r\)). Hence, there exists a cutset \(C'\) of order \(|C'| \leq k - 1\) and a set of vertices \(B^*\) (think of a component of \(G_b \setminus C^*\)) of order \(|B^*| \leq k - 1\) which have red edges only to \(C^*\) in \(G_r\). The set \(B \cup B^*\) forms a set larger than \(B\) satisfying Properties 1 and 2, a contradiction. \(\square_{\text{Claim 1}}\)

Let \(A = G \setminus B\) and consider the blue bipartite graph \(G_b\) induced on \(A \cup B\). Since we have assumed \(n > 6.5(k - 1)\), we see that \(|A| \geq 3.5(k - 1) + 1\). At this point, it is worth to note that, by Lemma 10 in [5], Theorem 3 holds for \(n > 8(k - 1)\). Part of what remains of our proof is a strengthening of the ideas presented in [5].

We now claim that there exists a large \(k\)-connected subgraph of \(G_b\) which serves as a desired structure. Hence, we restrict our attention to \(G_b\). Assume \(G_b\) is not \(k\)-connected. Consider a minimum cutset \(C\) with \(|C| \leq k - 1\).

**Claim 2** \(C \subseteq B\).

**Proof of Claim 2:** In order to prove this claim, it suffices to show that a cutset of order at most \(k - 1\) cannot separate two vertices of \(B\). This would imply that any cutset including vertices of \(A\) is not minimal and hence, complete the proof.
Each vertex of $B$ has at least $|A| - (k - 1)$ edges to $A$ which means that each pair of vertices in $B$ shares at least $|A| - 2(k - 1) \geq k$ common neighbors (note that this requires only $n > 6(k-1)$). Hence, no pair of vertices in $B$ can be separated by a cutset of order at most $k - 1$, thereby proving the claim. □

Claim 2

Since $G_b$ is bipartite, every component of $G_b \setminus C$ which does not contain a vertex of $B$ is a single vertex (in $A$). Hence, each of these vertices has degree at most $|C| \leq k - 1$. Let $A^*$ be the vertices $v \in A$ with $d_b(v) \leq k - 1$. Our first goal is to show that $|A^*| = t \leq 2(k - 1)$. From the definitions, there are at most $t(k - 1) + |B|(|A| - t)$ blue edges between $A$ and $B$. Conversely, recall that there are at least $|B|(|A| - (k - 1))$ edges between $A$ and $B$ since each vertex of $B$ has many blue edges to $A$. This means

$t(k - 1) + |B|(|A| - t) \geq |B|(|A| - (k - 1))$

which, using the fact that $|B| \geq 2(k - 1)$, implies

\begin{equation}
    t \leq \frac{|B|(k - 1)}{|B| - (k - 1)} \leq 2(k - 1),
\end{equation}

as required.

Let $A'' = A \setminus A^*$ and let $G''_b = B \cup A'' = G_b \setminus A^*$ (the graph remaining after the removal of the above singleton vertices). We would now like to show that $G''_b$, which has order $n - t \geq n - 2(k - 1)$, is $k$-connected. Let $C''$ be a minimum cutset of $G''_b$ and suppose $|C''| \leq k - 1$. Let $t'$ be the maximum red degree from vertices in $B \setminus C''$ into $A^*$. From this we get the following inequalities

$t(|B| - (k - 1)) \leq e_r(B, A^*) \leq t'|B\setminus C''| + t|B \cap C''|

which implies

\begin{equation}
    t' \geq t - \frac{t(k - 1)}{|B \setminus C''|}.
\end{equation}

We would now like to show that $|A'' \setminus C''| \leq 2(k - 1) - t'$. In order to accomplish this task, let $X$ and $Y$ be the two components (or collections of components) of $G''_b \setminus C''$ and choose a vertex $v \in B \setminus C''$ such that $e_r(v, A^*) = t'$. Notice that, by the definition of $A^*$ and $G''_b = G_b \setminus A^*$, we know that $B \cap X$ and $B \cap Y$ are both nonempty. Without loss of generality, suppose $v \in B \cap X$. Since all edges from $v$ to $A'' \cap Y$ are red, we know that $|A'' \cap Y| \leq k - 1 - t'$. Now let $v'$ be a vertex in $B \cap Y$. Since all edges from $v'$ to $A'' \cap X$ are red, we get $|A'' \cap X| \leq k - 1$. These two bounds show that $|A'' \setminus C''| \leq 2(k - 1) - t'$.

Using (1) and (2), this implies

\begin{align*}
    n &= |C''| + |A'' \setminus C''| + |B \setminus C''| + t \\
    &\leq |C''| + 2(k - 1) - t' + |B \setminus C''| + t \\
    &\leq (k - 1) + 2(k - 1) - \left(t - \frac{t(k - 1)}{|B \setminus C''|}\right) + |B \setminus C''| + t \\
    &\leq 3(k - 1) + |B \setminus C''| + \frac{|B|(k - 1)^2}{|B| - (k - 1)||B \setminus C''|}.
\end{align*}
Hence, we need only show that

**Fact 1**

\[|B \setminus C''| + \frac{|B|(k-1)^2}{|B| - (k-1)|B \setminus C''|} \leq 3.5(k-1).\]

**Proof:** In order to prove this fact, we maximize the left hand side (LHS) over the values \(2(k-1) \leq |B| \leq 3(k-1)\) and \((k-1) \leq |B \setminus C''| \leq 3(k-1)\) (also certainly \(|B| \geq |B \setminus C''|\)). It is easy to see this maximum occurs at one of the boundary points of our allowed values so we need only check these points. The largest value occurs when \(|B| = |B \setminus C''| = 3(k-1)\) which yields \(LHS \leq 3.5(k-1)\).

Hence \(n \leq 3(k-1) + 3.5(k-1) = 6.5(k-1)\) which is a contradiction, completing the proof of Theorem 3. \(\square_{\text{Fact 1}}\)

Since we actually know \(|B| < 3(k-1)\), the result in Fact 1 (and hence Theorem 3) may be improved slightly. For the sake of simplicity, this computation is omitted.

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**References**


