Linear Recurrence Relations for Sums of Products of Two Terms

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Abstract

For a sum of the form $\sum_k F(\boldsymbol{n},k)G(\boldsymbol{n},k)$, we set up two systems of equations involving shifts of $F(\boldsymbol{n},k)$ and $G(\boldsymbol{n},k)$. Then we solve the systems by utilizing the recursion of $F(\boldsymbol{n},k)$ and the method of undetermined coefficients. From the solutions, we derive linear recurrence relations for the sum. With this method, we prove many identities involving Bernoulli numbers and Stirling numbers.

1. Introduction

Finding recurrence relations is a basic method for proving identities. Fasenmyer [4–6] proposed a systematic method to find linear recurrence relations for hypergeometric sums. Wilf and Zeilberger [10–13] provided an efficient algorithm, called Zeilberger's algorithm, to construct linear recurrence relations for hypergeometric sums. Chyzak [2] extended Zeilberger's algorithm to holonomic systems. Kauers [8] presented algorithms for sums involving Stirling-like sequences. Chyzak, Kauers and Salvy [3] further considered non-holonomic systems.

In this paper, we focus on deriving a linear recurrence relation for the sum

$$f(\boldsymbol{n}) = \sum_{k=-\infty}^{\infty} F(\boldsymbol{n},k)G(\boldsymbol{n},k), \tag{1.1}$$

where $\boldsymbol{n} = (n_1, \ldots, n_r)$ and $F(\boldsymbol{n},k), G(\boldsymbol{n},k)$ are two functions of $\boldsymbol{n}$ and k. Our approach can be described as follows. We first take a finite subset S of $\mathbb{Z}^{r+1}$ as the set of shifts of

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the variables $\boldsymbol{n}$ and k. Then we make an ansatz

$$\sum_{(\boldsymbol{\alpha},\beta)\in S} \lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)F(\boldsymbol{n}-\boldsymbol{\alpha},k-\beta)=0, \tag{1.2}$$

where $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)$ are functions to be determined. Denote

$$A_S=\{\boldsymbol{\alpha}\in\mathbb{Z}^r\colon \text{there exists } \beta\in\mathbb{Z} \text{ such that } (\boldsymbol{\alpha},\beta)\in S\}$$

and

$$S_{\boldsymbol{\alpha}}=\{\beta\in\mathbb{Z}\colon (\boldsymbol{\alpha},\beta)\in S\}.$$

For each $\boldsymbol{\alpha}\in A_S$, we take a finite subset $S'_{\boldsymbol{\alpha}}$ of $\mathbb{Z}^r$ and make an ansatz

$$\sum_{\beta\in S_{\boldsymbol{\alpha}}} \lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k+\beta)F(\boldsymbol{n}-\boldsymbol{\alpha},k)G(\boldsymbol{n},k+\beta) = \sum_{\boldsymbol{\gamma}\in S'_{\boldsymbol{\alpha}}} c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})F(\boldsymbol{n}-\boldsymbol{\gamma},k)G(\boldsymbol{n}-\boldsymbol{\gamma},k), \tag{1.3}$$

where $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$ are functions which are independent of k and need to be determined. The system of equations consisting of (1.2) and (1.3) for all $\boldsymbol{\alpha}\in A_S$ is called a *coupling system*. Suppose that we obtain a solution $(\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k), c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n}))$ to the coupling system such that $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$ are not all zeros. Then (by Lemma 2.1) we are led to a non-trivial linear recurrence relation for $f(\boldsymbol{n})$:

$$\sum_{\boldsymbol{\alpha}\in A_S}\sum_{\boldsymbol{\gamma}\in S'_{\boldsymbol{\alpha}}} c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})f(\boldsymbol{n}-\boldsymbol{\gamma})=0. \tag{1.4}$$

We mainly investigate the case in which $F(\boldsymbol{n},k)$ is independent of $n_1,n_2,\ldots,n_s$ and

$$S'_{\boldsymbol{\alpha}}\subset\{\boldsymbol{\alpha}+\ell_1\boldsymbol{e}_1+\cdots+\ell_s\boldsymbol{e}_s\colon \ell_i\in\mathbb{Z}, i=1,2,\ldots,s\},$$

where each $\boldsymbol{e}_i\in\mathbb{Z}^r$ is the unit vector whose i-th component is 1. In this case, Equation (1.3) reduces to

$$\sum_{\beta\in S_{\boldsymbol{\alpha}}} \lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k+\beta)G(\boldsymbol{n},k+\beta) = \sum_{\boldsymbol{\gamma}\in S'_{\boldsymbol{\alpha}}} c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})G(\boldsymbol{n}-\boldsymbol{\gamma},k). \tag{1.5}$$

We call such a coupling system a *split system*. We will see that both Sister-Celine's method and Zeilberger's algorithm fall into the framework of split systems.

We use the above method to prove identities of sums involving special combinatorial sequences. We first split the summand into a product of two terms $F(\boldsymbol{n},k)$ and $G(\boldsymbol{n},k)$ so that $F(\boldsymbol{n},k)$ depends on as few variables as possible and satisfies a simple recurrence relation. Then by solving the equations (1.2) and (1.5), we get a recurrence relation for the sum. Finally we prove the identity by checking the initial values. Notice that the method given in [3] treats the summand as a whole.

The paper is organized as follows. In Section 2, we prove the recursion (1.4) and the existence of non-trivial solutions to a kind of split systems. In Section 3, we consider the case in which $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)$ are given a priori. In Section 4, we study the split systems in which $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)$ can be expressed in terms of $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$.

2. Split systems

We follow the notation of Section 1. The following lemma is valid for a general coupling system.

Lemma 2.1 *Suppose that* $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)$ *and* $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$ *satisfy* (1.2) *and* (1.3). *Then* (1.4) *holds.*

Proof. By direct calculation, we see that

$$\begin{aligned}
0 &= \sum_{k=-\infty}^{\infty} \sum_{(\boldsymbol{\alpha},\beta)\in S} \lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k) F(\boldsymbol{n}-\boldsymbol{\alpha},k-\beta) G(\boldsymbol{n},k) \\
&= \sum_{\boldsymbol{\alpha}\in A_S} \sum_{k=-\infty}^{\infty} \sum_{\beta\in S_{\boldsymbol{\alpha}}} \lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k+\beta) F(\boldsymbol{n}-\boldsymbol{\alpha},k) G(\boldsymbol{n},k+\beta) \\
&= \sum_{\boldsymbol{\alpha}\in A_S} \sum_{k=-\infty}^{\infty} \sum_{\boldsymbol{\gamma}\in S'_{\boldsymbol{\alpha}}} c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n}) F(\boldsymbol{n}-\boldsymbol{\gamma},k) G(\boldsymbol{n}-\boldsymbol{\gamma},k) \\
&= \sum_{\boldsymbol{\alpha}\in A_S} \sum_{\boldsymbol{\gamma}\in S'_{\boldsymbol{\alpha}}} c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n}) f(\boldsymbol{n}-\boldsymbol{\gamma}).
\end{aligned}$$

∎

From now on until the end of the paper, we always assume that the coupling systems are split.

Consider the split system with $r=1$, $F(n,k)=1$,

$$S = \{(j,j)\colon 0 \le j \le J\} \quad \text{and} \quad S'_j = \{0,1,2,\dots,I\},\ \forall\, 0 \le j \le J,$$

where I, J are two non-negative integers. Then (1.5) reduces to

$$\lambda_{j,j}(n,k+j) G(n,k+j) = \sum_{i=0}^{I} c_{j,i}(n) G(n-i,k),$$

so that

$$\lambda_{j,j}(n,k) = \sum_{i=0}^{I} c_{j,i}(n) G(n-i,k-j)/G(n,k).$$

Now substitute the above equation into (1.2). We finally obtain an equation

$$\sum_{j=0}^{J} \sum_{i=0}^{I} c_{j,i}(n) G(n-i,k-j) = 0,$$

which is the same as the one appearing in Sister-Celine's method. Now consider the split system with $r=1$, $F(n,k)=1$,

$$S = \{(0,0),(0,1)\} \quad \text{and} \quad S'_0 = \{0,1,\dots,I\}.$$

By equation (1.2), we derive that $\lambda_{0,0}(n,k) = -\lambda_{0,1}(n,k)$. Thus, (1.5) reduces to

$$\lambda_{0,1}(n,k+1)G(n,k+1) - \lambda_{0,1}(n,k)G(n,k) = \sum_{i=0}^{I} c_i(n)G(n-i,k),$$

which is the skew recurrence relation appearing in Zeilberger's algorithm. Therefore, both Sister-Celine's method and Zeilberger's algorithm fall into the framework of split systems.

The following theorem ensures the existence of non-trivial solutions to a kind of split systems.

Theorem 2.2 *Suppose that $F(\boldsymbol{n},k)$ is independent of $n_1,\dots,n_s$ and satisfies a non-trivial linear recurrence relation*

$$F(\boldsymbol{n},k) = \sum_{(\boldsymbol{\alpha},\beta)\in R} a_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)F(\boldsymbol{n}-\boldsymbol{\alpha},k-\beta), \tag{2.1}$$

where R is a finite subset of

$$\{(0,\dots,0,n_{s+1},\dots,n_r,k) \in \mathbb{Z}^{r+1}\}$$

and $a_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)$ are rational functions of $\boldsymbol{n}$ and k. Assume that $G(\boldsymbol{n},k)$ is a proper hypergeometric term (see [10] for the definition). Then there exist S and $S'_{\boldsymbol{\alpha}}$ such that the corresponding split system has a non-trivial solution $(\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k), c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n}))$ with $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)$ being polynomials in k.

Proof. We will set up a system of linear equations on $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)$ and $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$ according to (1.2) and (1.5). Then the theorem follows from the fact that the number of unknowns is larger than that of equations.

Assume that

$$\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k) = \sum_{\ell=0}^{D} \lambda_{\boldsymbol{\alpha},\beta,\ell}(\boldsymbol{n})k^{\ell}. \tag{2.2}$$

We take

$$S = \{(0,\dots,0,n_{s+1},\dots,n_r,k)\colon 0 \le k \le I_0,\ 0 \le n_j \le I_j,\ j = s+1,\dots,r\}$$

and

$$S'_{\boldsymbol{\alpha}} = \{\boldsymbol{\alpha} + \ell_1\boldsymbol{e}_1 + \cdots + \ell_s\boldsymbol{e}_s\colon 0 \le \ell_i \le I_i,\ i = 1,2,\dots,s\}.$$

Then the corresponding split system leads to a system of linear equations on $\lambda_{\boldsymbol{\alpha},\beta,\ell}(\boldsymbol{n})$ and $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$. We will derive an upper bound for the number of equations in terms of D and $I_0, I_1, \dots, I_r$.

We first consider (1.2). Without loss of generality, we may assume that each element in R is strictly greater than the zero vector in the lexicographic order. Let

$$S_R = \{(\boldsymbol{\alpha},\beta) \in S\colon \text{there exists } (\boldsymbol{\alpha}',\beta') \in R \text{ such that } (\boldsymbol{\alpha},\beta) + (\boldsymbol{\alpha}',\beta') \notin S\} \tag{2.3}$$

be the boundary set with respect to the recurrence relation (2.1). By iterating use of the recursion (2.1), we can express the terms $F(\boldsymbol{n}-\boldsymbol{\alpha}, k-\beta)$ with $(\boldsymbol{\alpha}, \beta) \in S$ in terms of those with $(\boldsymbol{\alpha}, \beta) \in S_R$. Therefore,

$$\sum_{(\boldsymbol{\alpha},\beta)\in S} \lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n}, k) F(\boldsymbol{n}-\boldsymbol{\alpha}, k-\beta) = \sum_{(\boldsymbol{\alpha},\beta)\in S_R} \mu_{\boldsymbol{\alpha},\beta}(\boldsymbol{n}, k) F(\boldsymbol{n}-\boldsymbol{\alpha}, k-\beta), \tag{2.4}$$

where

$$\mu_{\boldsymbol{\alpha},\beta}(\boldsymbol{n}, k) = \lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n}, k) + \sum_{(\boldsymbol{\alpha}',\beta')\in S\setminus S_R} A_{\boldsymbol{\alpha},\beta,\boldsymbol{\alpha}',\beta'}(\boldsymbol{n}, k) \lambda_{\boldsymbol{\alpha}',\beta'}(\boldsymbol{n}, k), \tag{2.5}$$

and $A_{\boldsymbol{\alpha},\beta,\boldsymbol{\alpha}',\beta'}(\boldsymbol{n}, k)$ are linear combinations of terms of the form

$$\prod_i a_{\boldsymbol{\alpha}_i,\beta_i}(\boldsymbol{n}-\boldsymbol{\delta}_i, k-\delta_i).$$

By setting all $\mu_{\boldsymbol{\alpha},\beta}(\boldsymbol{n}, k)$ to zeros, we reduce (1.2) to a system of linear equations on $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n}, k)$. The number of equations of the system equals the cardinality of S_R, which is bounded by a multi-variable polynomial P_1 in $I_0, I_{s+1}, \ldots, I_r$ of total degree $r-s$. By multiplying the common denominators, we transfer the coefficients of the system into polynomials in k. The degrees of these polynomials are bounded by $P_2 = C(I_0+1)(I_{s+1}+1)\cdots(I_r+1)$, where C is a constant. Now substituting (2.2) into the system and equating the coefficient of each power of k to zero, we finally obtain a system of linear equations on $\lambda_{\boldsymbol{\alpha},\beta,\ell}(\boldsymbol{n})$. The number of equations of the new system is bounded by $P_1(P_2+D+1)$.

We next consider (1.5). Dividing $G(\boldsymbol{n}, k)$ on both sides and multiplying the common denominators, we are led to a system of linear equations on $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n}, k)$ and $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$. The number of equations of the system is equal to the cardinality of A_S, which is $(I_{s+1}+1)\cdots(I_r+1)$. Since $G(\boldsymbol{n}, k)$ is a proper hypergeometric term, the coefficients of the system are polynomials in k whose degrees are bounded by a linear function P_3 of $I_0, I_1, \ldots, I_r$ (see [11]). Once again, we substitute (2.2) into the system and equate the coefficient of each power of k to zero. This leads to a system of linear equations on $\lambda_{\boldsymbol{\alpha},\beta,\ell}(\boldsymbol{n})$ and $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$. The number of equations of the new system is bounded by $(P_3+D+1)(I_{s+1}+1)\cdots(I_r+1)$.

Finally, we combine the equations deduced from (1.2) and (1.5) together. The total number of equations is bounded by

$$E = P_1(P_2+D+1) + (P_3+D+1)(I_{s+1}+1)\cdots(I_r+1).$$

While the total number of unknowns is

$$U = (I_0+1)(I_{s+1}+1)\cdots(I_r+1)(D+1) + (I_1+1)\cdots(I_r+1).$$

The leading coefficient of E in variable D is a polynomial in $I_0, I_{s+1}, \ldots, I_r$ of total degree $r-s$. While the leading coefficient of U in variable D is $(I_0+1)(I_{s+1}+1)\cdots(I_r+1)$. Hence $U > E$ holds for sufficiently large $I_0, I_1, \ldots, I_r$ and D, which implies that the split system has a non-trivial solution. ∎

Remark. For $s > 1$, the product $(I_1+1)\cdots(I_r+1)$ which is the number of unknowns $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$ will be larger than E for sufficiently large $I_1, \ldots, I_s$. Thus we derive that $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$ are not all zeros. But for $s = 1$, we could not ensure that $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$ are not all zeros.

3. Split systems with given λ

In this section, we consider split systems in which $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)$ are given a priori. Assume that $F(\boldsymbol{n},k)$ is independent of $n_1,\ldots,n_s$ and satisfies a linear recurrence relation of the form (2.1). We take

$$S = R \cup \{\mathbf{0}\}$$

and set

$$\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k) = \begin{cases} a_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k), & (\boldsymbol{\alpha},\beta)\in R, \\ -1, & (\boldsymbol{\alpha},\beta)=\mathbf{0}. \end{cases}$$

It is straightforward to see that those $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)$ satisfy (1.2). The remaining task is to solve (1.5). Noting that $G(\boldsymbol{n},k)$ is clearly equal to itself, we need only solve the following equations:

$$\sum_{\beta\in R_{\boldsymbol{\alpha}}} a_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k+\beta)G(\boldsymbol{n},k+\beta) = \sum_{\boldsymbol{\gamma}\in S'_{\boldsymbol{\alpha}}} c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})G(\boldsymbol{n}-\boldsymbol{\gamma},k), \quad \boldsymbol{\alpha}\in A_R, \tag{3.1}$$

where

$$A_R = \{\boldsymbol{\alpha}\colon \text{there exists } \beta\in\mathbb{Z} \text{ such that } (\boldsymbol{\alpha},\beta)\in R\},$$

and $R_{\boldsymbol{\alpha}} = \{\beta\in\mathbb{Z}\colon (\boldsymbol{\alpha},\beta)\in R\}$. Each solution $\{c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})\}$ to (3.1) leads to a recurrence relation

$$f(\boldsymbol{n}) = \sum_{\boldsymbol{\alpha}\in A_R}\sum_{\boldsymbol{\gamma}\in S'_{\boldsymbol{\alpha}}} c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n}) f(\boldsymbol{n}-\boldsymbol{\gamma})$$

for

$$f(\boldsymbol{n}) = \sum_{k=-\infty}^{\infty} F(\boldsymbol{n},k)G(\boldsymbol{n},k).$$

We illustrate the method by an identity involving Bernoulli numbers B_k.

Example 3.1 *We have*

$$\sum_{k=0}^{m}\binom{m}{k}B_{n+k} = (-1)^{m+n}\sum_{k=0}^{n}\binom{n}{k}B_{m+k}. \tag{3.2}$$

Chen and Sun [1] found a recurrence relation satisfied by both sides based on the integral representation of B_k. Here we provide a proof in the framework of split systems.
Proof. Let

$$F(n,m,k) = \binom{m}{k} \quad \text{and} \quad G(n,m,k) = B_{n+k}.$$

We see that $F(n,m,k)$ is independent of n and

$$F(n,m,k) = F(n,m-1,k) + F(n,m-1,k-1).$$

In this case, (3.1) becomes

$$G(n,m,k)+G(n,m,k+1)=\sum_{\ell} c_\ell(n,m)G(n-\ell,m-1,k). \tag{3.3}$$

Observing that

$$G(n,m,k)=G(n,m-1,k) \quad \text{and} \quad G(n,m,k+1)=G(n+1,m-1,k),$$

we obtain a solution $c_0=c_{-1}=1$ to (3.3). Therefore, the sum

$$f(n,m)=\sum_{k=0}^{m}\binom{m}{k}B_{n+k}$$

satisfies the recurrence relation

$$f(n,m)=f(n,m-1)+f(n+1,m-1). \tag{3.4}$$

Similarly, by taking

$$F(n,m,k)=(-1)^n\binom{n}{k} \quad \text{and} \quad G(n,m,k)=(-1)^m B_{m+k},$$

we derive that the sum

$$g(n,m)=(-1)^{m+n}\sum_{k=0}^{n}\binom{n}{k}B_{m+k}$$

satisfies

$$g(n,m)=-g(n-1,m)+g(n-1,m+1),$$

which is equivalent to (3.4).

Finally, from the identity

$$\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}=\frac{x}{e^x-1}=e^{-x}\frac{-x}{e^{-x}-1},$$

we see that

$$B_n=(-1)^n\sum_{k=0}^{n}\binom{n}{k}B_k,$$

that is, (3.2) holds for $m=0$. Hence by the recurrence relation (3.4), (3.2) holds for all $m,n\geq 0$. ∎

By a similar argument as above, we prove most of the identities in [1]. We always take F to be a binomial coefficient which satisfies a triangular recurrence relation. As another example, we derive the recurrence relation satisfied by both sides of a convolution identity for Bernoulli numbers.

Example 3.2 *We have [1, Theorem 4.4]*

$$\begin{aligned}
&\sum_{j=0}^{n}\binom{n}{j}B_{k+j}B_{m+n-j}\\
=&-\frac{k!m!}{(m+k+1)!}(n+\delta(m,k)(m+k+1))B_{m+n+k}\\
&+\sum_{r=0}^{m+k}(-1)^r\frac{B_{m+k+1-r}}{m+k+1-r}(-1)^k\binom{k+1}{r}\left(\frac{k+1-r}{k+1}n-\frac{rm}{k+1}\right)B_{n+r-1}\\
&+\sum_{r=0}^{m+k}(-1)^r\frac{B_{m+k+1-r}}{m+k+1-r}(-1)^m\binom{m+1}{r}\left(\frac{m+1-r}{m+1}n-\frac{rk}{m+1}\right)B_{n+r-1}, \qquad (3.5)
\end{aligned}$$

where

$$\delta(m,k)=\begin{cases}-1, & \text{if } (m,k)=(0,0),\\ 0, & \text{if } mk=0 \text{ but } (m,k)\neq(0,0),\\ 1, & \text{otherwise.}\end{cases}$$

We first consider the sum $L(m,n,k)$ on the left hand side. Let

$$F(m,n,k,j)=\binom{n}{j} \quad \text{and} \quad G(m,n,k,j)=B_{k+j}B_{m+n-j}.$$

We have

$$F(m,n,k,j)=F(m,n-1,k,j)+F(m,n-1,k,j-1).$$

Observing that

$$G(m,n,k,j)=G(m+1,n-1,k,j) \quad \text{and} \quad G(m,n,k,j+1)=G(m,n-1,k+1,j),$$

we derive that

$$L(m,n,k)=L(m+1,n-1,k)+L(m,n-1,k+1),$$

which is equivalent to

$$L(m,n+1,k)=L(m+1,n,k)+L(m,n,k+1).$$

Now let us consider the right hand side of (3.5). We split the first sum into the difference of

$$n\sum_{r=0}^{m+k}(-1)^r\frac{B_{m+k+1-r}}{m+k+1-r}(-1)^k\binom{k}{r}B_{n+r-1}=n\cdot R_1(m,n,k)$$

and

$$m\sum_{r=0}^{m+k}(-1)^r\frac{B_{m+k+1-r}}{m+k+1-r}(-1)^k\binom{k}{r-1}B_{n+r-1}=m\cdot R_2(m,n,k).$$

By taking $F = \binom{k}{r}$ and $F = \binom{k}{r-1}$, respectively, we derive that

$$R_i(m, n, k) = -R_i(m+1, n, k-1) + R_i(m, n+1, k-1), \quad i = 1, 2,$$

which is equivalent to

$$R_i(m, n+1, k) = R_i(m+1, n, k) + R_i(m, n, k+1), \quad i = 1, 2.$$

Similarly, let

$$R_3(m, n, k) = \sum_{r=0}^{m+k} (-1)^r \frac{B_{m+k+1-r}}{m+k+1-r} (-1)^m \binom{m}{r} B_{n+r-1},$$
$$R_4(m, n, k) = \sum_{r=0}^{m+k} (-1)^r \frac{B_{m+k+1-r}}{m+k+1-r} (-1)^m \binom{m}{r-1} B_{n+r-1},$$

we have

$$R_i(m, n+1, k) = R_i(m+1, n, k) + R_i(m, n, k+1), \quad i = 3, 4.$$

Let

$$R_5(m, n, k) = -\frac{k!m!}{(m+k+1)!}(n + \delta(m, k)(m+k+1))B_{m+n+k},$$

by direct calculation, we derive that

$$R_5(m, n+1, k) = R_5(m+1, n, k) + R_5(m, n, k+1).$$

Finally, let $R(m, n, k)$ be the right hand side of (3.5). Replacing the index of summation r in $R_2(m+1, n, k)$ and $R_4(m, n, k+1)$ by $r+1$, we see that

$$\begin{aligned} &R(m, n+1, k) - R(m+1, n, k) - R(m, n, k+1) \\ &= R_1(m, n+1, k) + R_2(m+1, n, k) + R_3(m, n+1, k) + R_4(m, n, k+1) \\ &= 0. \end{aligned}$$

Hence both sides of (3.5) satisfy the same recurrence relation. ∎

We can also apply this method to sums involving Stirling numbers (Eulerian numbers, respectively). Let $S_1(n, k)$ and $S_2(n, k)$ be the Stirling numbers of the first kind and the second kind, respectively. It is well-known that

$$S_1(n, k) = -(n-1)S_1(n-1, k) + S_1(n-1, k-1),$$
$$S_2(n, k) = kS_2(n-1, k) + S_2(n-1, k-1).$$

With these recursions, we prove identities (6.15)–(6.19), (6.28), (6.29), (6.38), and (6.39) of [7]. Here are two examples.

Example 3.3 *Find a recurrence relation for the sum [7, Identity (6.28)]*

$$f(l,m,n) = \sum_{k=l}^{n-m} \binom{n}{k} S_2(k,l) S_2(n-k,m).$$

Let

$$F(l,m,n,k) = \binom{n}{k} \quad \text{and} \quad G(l,m,n,k) = S_2(k,l)S_2(n-k,m).$$

We have

$$F(l,m,n,k) = F(l,m,n-1,k) + F(l,m,n-1,k-1).$$

In this case, (3.1) becomes

$$G(l,m,n,k) + G(l,m,n,k+1) = \sum_{i,j} c_{i,j}(l,m,n) G(l-i, m-j, n-1, k). \tag{3.6}$$

Substituting

$$S_2(n-k,m) = mS_2(n-1-k,m) + S_2(n-1-k,m-1)$$

and

$$S_2(k+1,l) = lS_2(k,l) + S_2(k,l-1)$$

into the left hand side of (3.6), we find a solution

$$c_{0,0} = (m+l), \;\; c_{1,0} = c_{0,1} = 1.$$

Thus,

$$f(l,m,n) = (m+l)f(l,m,n-1) + f(l,m-1,n-1) + f(l-1,m,n-1). \qquad \blacksquare$$

Example 3.4 *We have [7, Identity (6.19)]*

$$\sum_{k=0}^{m} \binom{m}{k} k^n (-1)^{m-k} = m! S_2(n,m). \tag{3.7}$$

Proof. Let

$$F(m,n,k) = k^n \quad \text{and} \quad G(m,n,k) = (-1)^{m-k}\binom{m}{k}.$$

We have

$$F(m,n,k) = kF(m,n-1,k).$$

We take $S'_{0,1} = \{(0,1),(1,1)\}$ so that (3.1) becomes

$$kG(m,n,k) = c_0(m,n)G(m,n-1,k) + c_1(m,n)G(m-1,n-1,k).$$

Dividing both sides by $G(m,n,k)$, we derive that

$$k = c_0(m,n) - c_1(m,n)\frac{m-k}{m},$$

which has a solution

$$c_0(m,n) = c_1(m,n) = m.$$

Therefore, the sum

$$f(m,n) = \sum_{k=0}^{m} \binom{m}{k} k^n (-1)^{m-k}$$

satisfies

$$f(m,n) = mf(m,n-1) + mf(m-1,n-1).$$

It is easy to check that $m!S_2(n,m)$ satisfies the same recursion and that

$$f(m,0) = \delta_{m,0} = m!S_2(0,m).$$

The proof follows by induction on n. ∎

4. Partially λ-free split systems

In this section, we consider a kind of split systems in which $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)$ can be expressed in terms of $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$.

Suppose that $F(\boldsymbol{n},k)$ is independent of $n_1,\ldots,n_s$ and satisfies a recurrence relation of the form (2.1). The proof of Theorem 2.2 provides us a method to construct a solution to (1.2). Given an arbitrary shift set S, let S_R be the boundary set defined by (2.3). For each $(\boldsymbol{\alpha},\beta) \in S \setminus S_R$, let $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)$ be an arbitrary function. For each $(\boldsymbol{\alpha},\beta) \in S_R$, we set

$$\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k) = -\sum_{(\boldsymbol{\alpha}',\beta')\in S\setminus S_R} A_{\boldsymbol{\alpha},\beta,\boldsymbol{\alpha}',\beta'}(\boldsymbol{n},k)\lambda_{\boldsymbol{\alpha}',\beta'}(\boldsymbol{n},k),$$

where $A_{\boldsymbol{\alpha},\beta,\boldsymbol{\alpha}',\beta'}(\boldsymbol{n},k)$ is given as in (2.4) and (2.5). Then these $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k)$ form a solution to (1.2).

Denote the free functions $\lambda_{\boldsymbol{\alpha},\beta}(\boldsymbol{n},k), (\boldsymbol{\alpha},\beta) \in S \setminus S_R$ by $\lambda_1, \lambda_2, \ldots$. Suppose that the equations corresponding to (1.5) can be ordered such that the first equation involves only λ_1 and there is exactly one term containing λ_1, the second equation involves only λ_1, λ_2 and there is exactly one term containing λ_2, and so on. Then we can express λ_i's as linear combinations of $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$. Finally, substituting these expressions for λ_i into the rest equations of (1.5), we obtain a system of linear equations on $c_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(\boldsymbol{n})$. We call such a split system a *partially λ-free split system.*

We illustrate the method by a summation involving Stirling numbers.

Example 4.1 *Find a recurrence relation for the sum [7, Identity (6.26)]*

$$f(n,m) = \sum_{k=-n}^{m} \binom{m-n}{m+k}\binom{m+n}{n+k} S_1(m+k,k).$$

Let

$$F(n,m,k) = S_1(m+k,k) \quad \text{and} \quad G(n,m,k) = \binom{m-n}{m+k}\binom{m+n}{n+k}.$$

Then

$$F(n,m,k) = -(m+k-1)F(n,m-1,k) + F(n,m,k-1). \tag{4.1}$$

Taking

$$S = \{(0,0,0),(0,1,0),(0,0,1)\},$$

we see that $S_R = \{(0,1,0),(0,0,1)\}$ and there is exactly one free function $\lambda_{0,0,0}(n,m,k)$. Denote the function by $\lambda(k)$ for short. We have

$$\lambda_{0,1,0}(n,m,k) = (m+k-1)\lambda(k) \quad \text{and} \quad \lambda_{0,0,1}(n,m,k) = -\lambda(k).$$

Then (1.5) becomes

$$\begin{cases} (m+k-1)\lambda(k)G(n,m,k) = \sum_i c_i(n,m)G(n-i,m-1,k) \\ \lambda(k)G(n,m,k) - \lambda(k+1)G(n,m,k+1) = \sum_j d_j(n,m)G(n-j,m,k). \end{cases}$$

From the first equation, we derive that

$$\lambda(k)G(n,m,k) = \sum_i c_i(n,m)G(n-i,m-1,k)/(m+k-1).$$

Substituting this expression into the second equation, we obtain

$$\sum_i c_i(n,m)\left(\frac{G(n-i,m-1,k)}{m+k-1} - \frac{G(n-i,m-1,k+1)}{m+k}\right) \\ = \sum_j d_j(n,m)G(n-j,m,k).$$

By setting $i,j \in \{-1,0,1\}$, we find a non-trivial solution

$$\begin{aligned} &c_{-1} = 0, \quad c_0 = \frac{(m-n)(m-n-1)}{m+n-1}C, \quad c_1 = (m-n)C, \\ &d_{-1} = \frac{n-m}{m+n-1}C, \quad d_0 = -\frac{2n-1}{m+n-1}C, \quad d_1 = C, \end{aligned}$$

and

$$\lambda(k) = \frac{(m+k)(-m+k)}{(n+k-1)(m+n-1)(m+n)}C,$$

where C is an arbitrary constant with respect to k. Hence, the sum $f(n,m)$ satisfies the recurrence relation

$$\begin{aligned} &(m-n)(m-n-1)f(n,m-1) + (m-n)(m+n-1)f(n-1,m-1) \\ &\quad - (2n-1)f(n,m) + (n-m)f(n+1,m) + (m+n-1)f(n-1,m) = 0. \quad \blacksquare \end{aligned}$$

We conclude by an example involving Stirling numbers and harmonic numbers.

Example 4.2 *Let* $H_k = \sum_{i=1}^{k} \frac{1}{i}$ *be the* k*-th harmonic number. Find a recurrence relation for the following sum [9]*

$$f(m,n) = \sum_{k=1}^{m} H_{m-k}(m-k)!(-1)^{m-k+1}\binom{m}{k-1}S_1(k-1,n).$$

Let $F(m,n,k) = S_1(k-1,n)$ and

$$G(m,n,k) = H_{m-k}(m-k)!(-1)^{m-k+1}\binom{m}{k-1}.$$

Then

$$F(m,n,k) = F(m,n-1,k-1) - (k-2)F(m,n,k-1).$$

Taking

$$S = \{(0,0,0),\ (0,0,1)\ (0,1,0),\ (0,1,1),\ (0,2,1)\},$$

we see that

$$S_R = \{(0,0,1),\ (0,1,1),\ (0,2,1)\}.$$

Hence, we have two free functions $\lambda_{0,0,0}(m,n,k)$ and $\lambda_{0,1,0}(m,n,k)$. Denote them by $\lambda(k)$ and $\mu(k)$ for short. We have

$$\lambda_{0,0,1}(m,n,k) = (k-2)\lambda(k), \quad \lambda_{0,1,1}(m,n,k) = -\lambda(k) + (k-2)\mu(k),$$

and $\lambda_{0,2,1}(m,n,k) = -\mu(k)$. Then (1.5) becomes

$$\begin{cases} -\mu(k+1)G(m,n,k+1) = \sum_i c_i(m,n)G(m-i,n-2,k), \\ \mu(k)G(m,n,k) + (-\lambda(k+1) + (k-1)\mu(k+1))G(m,n,k+1) \\ \qquad\qquad\qquad = \sum_j d_j(m,n)G(m-j,n-1,k), \\ \lambda(k)G(m,n,k) + (k-1)\lambda(k+1)G(m,n,k+1) \\ \qquad\qquad\qquad = \sum_\ell e_\ell(m,n)G(m-\ell,n,k). \end{cases}$$

Set $i,j,\ell \in \{-1,0,1\}$, express H_{m-k+t} in terms of H_{m-k}, and compare the coefficients of H_{m-k}. We find a non-trivial solution

$$\begin{aligned} &c_{-1} = 0, \quad c_0 = 0, \quad c_1 = C, \\ &d_{-1} = 0, \quad d_0 = -2C, \quad d_1 = -2(m-1)C, \\ &e_{-1} = C, \quad e_0 = (2m-1)C, \quad e_1 = (m-1)^2C, \end{aligned}$$

and

$$\mu(k) = -\frac{(k-1)C}{m}, \quad \lambda(k) = \frac{(k-2)H_{m-k} - 2m + k - 2}{mH_{m-k}(m-k+2)}(k-1)C,$$

where C is an arbitrary constant with respect to k. Therefore, the sum $f(m,n)$ satisfies the recurrence relation

$$(2m-1)f(m,n)+f(m-1,n-2)-2f(m,n-1)-2(m-1)f(m-1,n-1) \\ +f(m+1,n)+(m-1)^2f(m-1,n)=S_1(m-1,n).$$

The non-homogenous part $S_1(m-1,n)$ comes from the boundary values. ∎

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