Connected, Bounded Degree, Triangle Avoidance Games

Nishali Mehta*
The Ohio State University
nishali@math.ohio-state.edu

Ákos Seress*
The Ohio State University
Centre for Mathematics of Symmetry and Computation
University of Western Australia
akos@math.ohio-state.edu

Submitted: May 31, 2010; Accepted: Sept 6, 2011; Published: Sep 26, 2011
Mathematics Subject Classification: 05C57

Abstract

We consider variants of the triangle-avoidance game first defined by Harary and rediscovered by Hajnal a few years later. A graph game begins with two players and an empty graph on n vertices. The two players take turns choosing edges within K_n, building up a simple graph. The edges must be chosen according to a set of restrictions $\mathcal{R}$. The winner is the last player to choose an edge that does not violate any of the restrictions in $\mathcal{R}$. For fixed n and $\mathcal{R}$, one of the players has a winning strategy. For a pair of games where $\mathcal{R}$ includes bounded degree, connectedness, and triangle-avoidance, we determine the winner for all values of n.

1 Introduction

Two players, $\mathcal{A}$ and $\mathcal{B}$, begin with an empty graph on n vertices, where $n \geq 3$. Player $\mathcal{A}$ goes first, choosing an edge between two vertices. The two players take turns choosing edges within K_n, building up a simple graph. The edges have to be chosen according to a set of restrictions $\mathcal{R}$. The winner is the last player to choose an edge that does not violate any of the restrictions in $\mathcal{R}$. For fixed n and $\mathcal{R}$, one of the players has a winning strategy. Let $\Gamma_{\mathcal{R}}(n)$ represent the game with restriction set $\mathcal{R}$ on n vertices. Let $f_{\mathcal{R}} : \mathbb{N}\backslash\{1, 2\} \to \{\mathcal{A}, \mathcal{B}\}$ be the function defined by

*Research partially supported by the NSF and by ARC Grant DP1096525.

$$f_{\mathcal{R}}(n) \quad = \quad \text{winner of } \Gamma_{\mathcal{R}}(n).$$

We will look at games with one or more of the following restrictions and determine the winner for all values of n.

Let $G(V, E) \leq K_n$ be the graph made up of all edges chosen so far in the game. Let $u, v \in V, u \neq v, (u, v) \notin E$.

Restriction B2 (Bounded Degree)**.** *If $max\{deg_G(u), deg_G(v)\} \geq 2$, then the edge (u, v) cannot be chosen.*

Restriction B3 (Bounded Degree)**.** *If $max\{deg_G(u), deg_G(v)\} \geq 3$, then the edge (u, v) cannot be chosen.*

Restriction T (Triangle Avoidance)**.** *If $\exists w \in V\backslash\{u, v\}$ such that both edges $(u, w), (v, w) \in E$, then the edge (u, v) cannot be chosen.*

Let M be the upper bound on the vertex degree, e.g. $M = 3$ if $\mathcal{R}$ includes Restriction $B3$. When no upper bound is specified, $M = n - 1$.

Restriction C (Connectedness)**.** *If $deg_G(u) = 0, deg_G(v) = 0$ and there is $w \in V$ such that $0 < deg_G(w) < M$, then the edge (u, v) cannot be chosen.*

In this paper we prove the following two theorems:

Theorem 4.5. *For $n \geq 5, f_{\{B3,C\}}(n) = \mathcal{B} \iff n \equiv 2 \pmod 4$.*

Theorem 5.6. *For $n \geq 12, f_{\{B3,T,C\}}(n) = \mathcal{B} \iff n \equiv 1, 2 \pmod 4$.*

The games with $\mathcal{R} = \{B2, C\}$ and $\mathcal{R} = \{B2, T, C\}$ are relatively simple. As a result, it is not too difficult to show the following.

Proposition. *For $n \geq 4$, $f_{\{B2,C\}}(n) = \mathcal{A} \iff n \equiv 0 (mod\ 2)$. For $n \geq 3$, $f_{\{B2,T,C\}}(n) = \mathcal{A} \iff n \equiv 2 (mod\ 4)$.*

In an accompanying paper [5], we give a complete analysis of the four games with $\mathcal{R} \subset \{B2, B3, T\}$ and $|\mathcal{R} \cap \{B2, B3\}| = 1$.

2 Background

The triangle avoidance game $\Gamma_{\{T\}}(n)$, suggested by Frank Harary [3] and six years later by András Hajnal, remains open in the general case. The function $f_{\{T\}}(n)$ is known for values of $n \leq 15$. The winners for $n \leq 12$ were computed by Cater, Harary, and Robinson [1]. Pralat [6] computed the winners for $n = 13, 14, 15$. Gordinowicz and Pralat [2] computed the winner for $n = 16$.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f_{\{T\}}(n)$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$

The connected version of the triangle avoidance game, $\Gamma_{\{T,C\}}(n)$, was solved by Seress [7]. He proved $f_{\{T,C\}}(n) = \mathcal{A} \iff n \equiv 0 \pmod 2$.

This paper and the accompanying paper by Mehta and Seress [5], are based on the thesis work by Mehta [4].

3 Definitions

When $|V| = n$, write $V = \{v_1, \ldots, v_n\}$. Let e_i denote the ith edge chosen and $E_k = \{e_1, \ldots, e_k\}$. For given n and k, let $G_{n,k}$ be the graph on vertex set V with edge set E_k. The graphs $G = G_{n,k}$ we consider will always satisfy $deg_G(v) \leq 3$ for all vertices $v \in V$.

We make the following convention, simplifying the description of the games. The ordering of V defines a natural lexicographic ordering $<$ of the edges of the complete graph K_n. Given a graph $G_{n,k}$, if the next player chooses the edge e_{k+1} then there is no edge $e < e_{k+1}$ such that $G_{n,k+1} \simeq G_{n,k} \cup e$.

A vertex v_i is said to be *out of play* if, $\forall\ v_j \in V \backslash \{v_i\}$ such that $(v_i, v_j) \notin E$, choosing the edge (v_i, v_j) will violate a restriction in $\mathcal{R}$. A vertex v_i that is not out of play is *in play*. In each game mentioned here, once a vertex is out of play it remains out of play for the rest of the game.

We make a second convention, further simplifying the description. Given a graph $G_{n,k}$, if the next player chooses the edge e_{k+1} then there is no edge $e < e_{k+1}$ such that game play for the remainder of the game is identical in $G_{n,k+1}$ and $G_{n,k} \cup e$. (This situation can occur if the induced subgraphs on the vertices still in play are isomorphic, even if the graphs themselves are not.)

The choice of the kth edge e_k will be called the *kth round*.

Define sets $Z, P, T \subset V$ on a graph G according to vertex degree:

$$\begin{aligned} Z(G) &= \{v \in V(G) : deg_G(v) = 0\}, & z(G) &= |Z(G)| \\ P(G) &= \{v \in V(G) : deg_G(v) = 1\}, & p(G) &= |P(G)| \\ T(G) &= \{v \in V(G) : deg_G(v) = 2\}, & t(G) &= |T(G)|. \end{aligned}$$

4 $\mathcal{R} = \{B3, C\}$

In the game $\Gamma_{\{B3,C\}}(n)$, we define graph classes $\mathcal{K}_i(z)$ for i=1,2,3,4, where $z \geq 0$ is an integer. One player can maintain that, for particular values of k, the graph $G_{n,k} \in \mathcal{K}_i(z)$ for some i. That is, this player does not stay within $\mathcal{K}_i(z)$ after each of his turns, but regularly returns to it after a few rounds. Following this strategy until the end of the game forces a win.

Define sets of graphs $\mathcal{K}_i(z)$ for i=1,2,3,4 as follows:

1. For any integer $z \geq 0$, $G(V,E) \in \mathcal{K}_1(z) \iff z(G) = z, p(G) = 0$, and $t(G) = 2$, with $T(G) = \{u_1, u_2\}$, $(u_1, u_2) \in E$.

2. For any integer $z \geq 0$, $G(V,E) \in \mathcal{K}_2(z) \iff z(G) = z, p(G) = 0$, and $t(G) = 1$.

3. For any integer $z \geq 0$, $G(V,E) \in \mathcal{K}_3(z) \iff z(G) = z, p(G) = 1$, and $t(G) = 0$.

4. For any integer $z \geq 0$, $G(V,E) \in \mathcal{K}_4(z) \iff z(G) = z, p(G) = 1$, and $t(G) = 1$, with $u_1 \in P(G)$, $u_2 \in T(G)$, $(u_1, u_2) \in E$.

Lemma 4.1. *For $n \geq 3$, in $\Gamma_{\{B3,C\}}(n)$ if either player chooses an edge that creates a graph in $\mathcal{K}_1(z)$ with $z \equiv 0 \pmod 4$ then that player has a winning strategy.*

Proof. Let $n \geq 3$. Suppose the edges $e_1, \ldots, e_{k-1}$ have already been chosen. Suppose player $\mathcal{A}$ chooses the kth edge so that $G_{n,k} \in \mathcal{K}_1(z)$ with $z \equiv 0 \pmod 4$. Then $T(G_{n,k}) = \{u_1, u_2\}$ with $(u_1, u_2) \in E_k$. The strategy described here will work for $\mathcal{B}$ as well. We proceed by induction on z.

Base case: $z = 0$. All vertices are out of play except u_1 and u_2, which are already connected by an edge. Since no new edge can be chosen, $\mathcal{A}$ wins.

Assume for induction that the statement holds when $z = 4m$ for some $m \geq 0$. Suppose $z = 4(m+1)$. Let $Z(G_{n,k}) = \{w_1, \ldots, w_{4m+4}\}$. Each vertex $v \notin (Z(G_{n,k}) \cup T(G_{n,k}))$ has $deg_{G_{n,k}}(v) = 3$ and is out of play.

Up to game play equivalence, $\mathcal{B}$ must choose (u_1, w_1). $\mathcal{A}$ chooses (u_2, w_1). $\mathcal{B}$ must choose (w_1, w_2). $\mathcal{A}$ chooses (w_2, w_3). $\mathcal{B}$ has two choices: (w_2, w_4) or (w_3, w_4). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Now vertices u_1, u_2, w_1, w_2 are out of play, $z(G_{n,k+6}) = 4m$, $p(G_{n,k+6}) = 0$, and $t(G_{n,k+6}) = 2$ with $T(G_{n,k+6}) = \{w_3, w_4\}$, $(w_3, w_4) \in E_{k+6}$. Thus $G_{n,k+6} \in \mathcal{K}_1(4m)$, so $\mathcal{A}$ wins by induction. □

Lemma 4.2. *For $n \geq 3$, in $\Gamma_{\{B3,C\}}(n)$ if either player chooses an edge that creates a graph in $\mathcal{K}_2(z)$ with $z \equiv 0 \pmod 4$ then that player has a winning strategy.*

Proof. Let $n \geq 3$. Suppose the edges $e_1, \ldots, e_{k-1}$ have already been chosen. Suppose player $\mathcal{A}$ chooses the kth edge so that $G_{n,k} \in \mathcal{K}_2(z)$ with $z \equiv 0 \pmod 4$. The strategy described here will work for $\mathcal{B}$ as well. We proceed by induction on z.

Base case: $z = 0$. All but one vertex are out of play, so no new edge can be chosen. Thus $\mathcal{A}$ wins.

Assume for induction that the statement holds when $z = 4m$ for some $m \geq 0$. Suppose $z = 4(m+1)$. Let $Z(G_{n,k}) = \{w_1, \ldots, w_{4m+4}\}$. Let $T(G_{n,k}) = \{u\}$. Each vertex $v \notin (Z(G_{n,k}) \cup T(G_{n,k}))$ has $deg_{G_{n,k}}(v) = 3$ and is out of play.

Up to isomorphism, $\mathcal{B}$ must choose (u, w_1). $\mathcal{A}$ chooses (w_1, w_2). $\mathcal{B}$ has two choices: (w_1, w_3) or (w_2, w_3). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. $\mathcal{B}$ must choose (w_2, w_4). $\mathcal{A}$ chooses (w_3, w_4). Now vertices u, w_1, w_2, w_3 are out of play, $z(G_{n,k+6}) = 4m$, $p(G_{n,k+6}) = 0$, and $t(G_{n,k+6}) = 1$. Thus $G_{n,k+6} \in \mathcal{K}_2(4m)$, so $\mathcal{A}$ wins by induction. □

Lemma 4.3. *For $n \geq 3$, in $\Gamma_{\{B3,C\}}(n)$ if either player chooses an edge that creates a graph in $\mathcal{K}_3(z)$ with $z \equiv 0 \pmod 4$ then that player has a winning strategy.*

Proof. Let $n \geq 3$. Suppose the edges $e_1, \ldots, e_{k-1}$ have already been chosen. Suppose player $\mathcal{A}$ chooses the kth edge so that $G_{n,k} \in \mathcal{K}_3(z)$ with $z \equiv 0 \pmod 4$. The strategy described here will work for $\mathcal{B}$ as well. We proceed by induction.

Base case: $z = 0$. All but one vertex are out of play, so no new edge can be chosen. Thus $\mathcal{A}$ wins.

Assume for induction that the statement holds when $z = 4m$ for some $m \geq 0$. Suppose $z = 4(m + 1)$. Let $Z(G_{n,k}) = \{w_1, \ldots, w_{4m+4}\}$. Let $P(G_{n,k}) = \{u\}$. Each vertex $v \notin (Z(G_{n,k}) \cup P(G_{n,k}))$ has $deg_{G_{n,k}}(v) = 3$ and is out of play.

Up to isomorphism, $\mathcal{B}$ must choose (u, w_1). $\mathcal{A}$ chooses (u, w_2). $\mathcal{B}$ has two choices: (w_1, w_3) or (w_1, w_2). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. $\mathcal{B}$ has three choices: (w_2, w_3), (w_2, w_4), or (w_3, w_4).

- If $\mathcal{B}$ chooses (w_2, w_3) then $\mathcal{A}$ chooses (w_3, w_4). Now vertices u, w_1, w_2, w_3 are out of play, $z(G_{n,k+6}) = 4m$, $p(G_{n,k+6}) = 1$, and $t(G_{n,k+6}) = 0$. Thus $G_{n,k+6} \in \mathcal{K}_3(4m)$, so $\mathcal{A}$ wins by induction.

- If $\mathcal{B}$ chooses one of (w_2, w_4) or (w_3, w_4), then $\mathcal{A}$ chooses the other edge. Now vertices u, w_1, w_2 are out of play, $z(G_{n,k+6}) = 4m$, $p(G_{n,k+6}) = 0$, and $t(G_{n,k+6}) = 2$ with $T(G_{n,k+6}) = \{w_3, w_4\}$, $(w_3, w_4) \in E_{k+6}$. Thus $G_{n,k+6} \in \mathcal{K}_1(4m)$, so $\mathcal{A}$ wins by Lemma 4.1.

□

Lemma 4.4. *For $n \geq 3$, in $\Gamma_{\{B3,C\}}(n)$ if either player chooses an edge that creates a graph in $\mathcal{K}_4(z)$ with $z \equiv 0 \pmod 4$ then that player has a winning strategy.*

Proof. Let $n \geq 3$. Suppose the edges $e_1, \ldots, e_{k-1}$ have already been chosen. Suppose player $\mathcal{A}$ chooses the kth edge so that $G_{n,k} \in \mathcal{K}_4(z)$ with $z \equiv 0 \pmod 4$. Then $P(G_{n,k}) = \{u_1\}$ and $T(G_{n,k}) = \{u_2\}$, with $(u_1, u_2) \in E_k$. The strategy described here will work for $\mathcal{B}$ as well. We proceed by induction.

Base case: $z = 0$. All vertices are out of play except u_1 and u_2, which are already connected by an edge. Since no new edge can be chosen, $\mathcal{A}$ wins.

Assume for induction that the statement holds when $z = 4m$ for some $m \geq 0$. Suppose $z = 4(m + 1)$. Let $Z(G_{n,k}) = \{w_1, \ldots, w_{4m+4}\}$. Each vertex $v \notin (Z(G_{n,k}) \cup P(G_{n,k}) \cup T(G_{n,k}))$ has $deg_{G_{n,k}}(v) = 3$ and is out of play.

Up to isomorphism, $\mathcal{B}$ has two choices: (u_1, w_1) or (u_2, w_1). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. $\mathcal{B}$ must choose (u_1, w_2). $\mathcal{A}$ chooses (w_1, w_2). $\mathcal{B}$ must choose (w_2, w_3). $\mathcal{A}$ chooses (w_3, w_4). Now vertices u_1, u_2, w_1, w_2 are out of play, $z(G_{n,k+6}) = 4m$, $p(G_{n,k+6}) = 1$, and $t(G_{n,k+6}) = 1$ with $P(G_{n,k+6}) = \{w_4\}$, $T(G_{n,k+6}) = \{w_3\}$, $(w_3, w_4) \in E_{k+6}$. Thus $G_{n,k+6} \in \mathcal{K}_4(4m)$, so $\mathcal{A}$ wins by induction. □

Theorem 4.5. *For $n \geq 5$, $f_{\{B3,C\}}(n) = \mathcal{B} \iff n \equiv 2 \pmod 4$.*

Proof. For small values of n, an exhaustive case analysis can be carried out by hand calculation.

n	3	4	5	6	7	8
$f_{\{B3,C\}}(n)$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{A}$

$n \equiv 0 \pmod 4$:

We proceed by induction on n.

Base case: $n = 8$. This was solved by hand calculation.

Assume for induction that $f_{\{B3,C\}}(n) = \mathcal{A}$ when $n = 4m$ for some $m \geq 2$. Suppose $n = 4m + 4$. $V = \{v_1, \ldots, v_{4m+4}\}$.

$\mathcal{A}$ chooses (v_1, v_2). $\mathcal{B}$ must choose (v_1, v_3). $\mathcal{A}$ chooses (v_2, v_3) to make a triangle. $\mathcal{B}$ must choose (v_1, v_4). $\mathcal{A}$ chooses (v_2, v_4). $\mathcal{B}$ has two choices: (v_3, v_4) or (v_3, v_5).

- If $\mathcal{B}$ chooses (v_3, v_4), then vertices v_1, v_2, v_3, v_4 are out of play. The remaining vertices $v_5, \ldots, v_{4m+4}$ have degree zero and it is $\mathcal{A}$'s turn. Play continues as in $\Gamma_{B3,C}(4m)$, so $\mathcal{A}$ wins by induction.

- If $\mathcal{B}$ chooses (v_3, v_5), then $\mathcal{A}$ chooses (v_4, v_5). Now vertices v_1, v_2, v_3, v_4 are out of play. $\mathcal{B}$ must choose (v_5, v_6). $\mathcal{A}$ chooses (v_6, v_7). $\mathcal{B}$ has two choices: (v_6, v_8) or (v_7, v_8). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Now v_5, v_6 are also out of play. $z(G_{4m+4,11}) = 4m - 4$, $p(G_{4m+4,11}) = 0$, and $t(G_{4m+4,11}) = 2$ with $T(G_{4m+4,11}) = \{v_7, v_8\}$, $(v_7, v_8) \in E_{11}$. Thus $G_{4m+4,11} \in \mathcal{K}_1(4m - 4)$, so $\mathcal{A}$ wins by Lemma 4.1.

$n \equiv 1 \pmod 4$:

We proceed by induction on n.

Base case: $n = 5$. This was solved by hand calculation.

Assume for induction that $f_{\{B3,C\}}(n) = \mathcal{A}$ when $n = 4m+1$ for some $m \geq 1$. Suppose $n = 4m + 5$. $V = \{v_1, \ldots, v_{4m+5}\}$.

$\mathcal{A}$ chooses (v_1, v_2). $\mathcal{B}$ must choose (v_1, v_3). $\mathcal{A}$ chooses (v_2, v_3) to make a triangle. $\mathcal{B}$ must choose (v_1, v_4). $\mathcal{A}$ chooses (v_2, v_4). $\mathcal{B}$ has two choices: (v_3, v_4) or (v_3, v_5).

- If $\mathcal{B}$ chooses (v_3, v_4), then vertices v_1, v_2, v_3, v_4 are out of play. The remaining vertices $v_5, \ldots, v_{4m+5}$ have degree zero and it is $\mathcal{A}$'s turn. Play continues as in $\Gamma_{B3,C}(4m+1)$, so $\mathcal{A}$ wins by induction.

- If $\mathcal{B}$ chooses (v_3, v_5), then $\mathcal{A}$ chooses (v_4, v_5). Now vertices v_1, v_2, v_3, v_4 are out of play. $z(G_{4m+5,7}) = 4m$, $p(G_{4m+5,7}) = 0$, and $t(G_{4m+5,7}) = 1$. Thus $G_{4m+5,7} \in \mathcal{K}_2(4m)$, so $\mathcal{A}$ wins by Lemma 4.2.

$n \equiv 2 \pmod 4$:

Suppose $n = 4m+2$ for some $m \geq 1$. $V = \{v_1, \ldots, v_{4m+2}\}$. We give a winning strategy for player $\mathcal{B}$.

$\mathcal{A}$ must choose (v_1, v_2). $\mathcal{B}$ chooses (v_1, v_3). $\mathcal{A}$ has three choices: (v_1, v_4), (v_2, v_3), or (v_2, v_4).

- If $\mathcal{A}$ chooses (v_1, v_4) then $\mathcal{B}$ chooses (v_2, v_3).

- If $\mathcal{A}$ chooses one of (v_2, v_3) or (v_2, v_4), then $\mathcal{B}$ chooses (v_1, v_4).

The three possible resulting graphs are isomorphic, so without loss of generality, consider the first case. $\mathcal{A}$ again has three choices: (v_2, v_4), (v_2, v_5), or (v_4, v_5).

- If $\mathcal{A}$ chooses (v_2, v_4) then $\mathcal{B}$ chooses (v_3, v_5).
- If $\mathcal{A}$ chooses (v_2, v_5) then $\mathcal{B}$ chooses (v_3, v_4).
- If $\mathcal{A}$ chooses (v_4, v_5) then $\mathcal{B}$ chooses (v_2, v_4).

As above, the three resulting graphs are isomorphic. Without loss of generality, consider the first case. $\mathcal{A}$ has three choices: (v_4, v_5), (v_4, v_6), or (v_5, v_6).

- If $\mathcal{A}$ chooses one of (v_4, v_5) or (v_5, v_6), then $\mathcal{B}$ chooses the other one. Now vertices v_1, v_2, v_3, v_4, v_5 are out of play. $z(G_{4m+2,8}) = 4m - 4$, $p(G_{4m+2,8}) = 1$, and $t(G_{4m+2,8}) = 0$. Thus $G_{4m+2,8} \in \mathcal{K}_3(4m - 4)$, so $\mathcal{B}$ wins by Lemma 4.3.
- If $\mathcal{A}$ chooses (v_4, v_6) then $\mathcal{B}$ chooses (v_5, v_6). Now vertices v_1, v_2, v_3, v_4 are out of play. $z(G_{4m+2,8}) = 4m - 4$, $p(G_{4m+2,8}) = 0$, and $t(G_{4m+2,8}) = 2$ with $T(G_{4m+2,8}) = \{v_5, v_6\}$, $(v_5, v_6) \in E_8$. Thus $G_{4m+2,8} \in \mathcal{K}_1(4m - 4)$, so $\mathcal{B}$ wins by Lemma 4.1.

$\boldsymbol{n \equiv 3 \pmod 4}$:
We proceed by induction on n.
Base case: $n = 7$. This was solved by hand calculation.
Assume for induction that $f_{\{B3,C\}}(n) = \mathcal{A}$ when $n = 4m+3$ for some $m \geq 1$. Suppose $n = 4m + 7$. $V = \{v_1, \ldots, v_{4m+7}\}$.

$\mathcal{A}$ chooses (v_1, v_2). $\mathcal{B}$ must choose (v_1, v_3). $\mathcal{A}$ chooses (v_2, v_3) to make a triangle. $\mathcal{B}$ must choose (v_1, v_4). $\mathcal{A}$ chooses (v_2, v_4). $\mathcal{B}$ has two choices: (v_3, v_4) or (v_3, v_5).

- If $\mathcal{B}$ chooses (v_3, v_4), then vertices v_1, v_2, v_3, v_4 are out of play. The remaining vertices $v_5, \ldots, v_{4m+7}$ have degree zero and it is $\mathcal{A}$'s turn. Play continues as in $\Gamma_{B3,C}(4m+3)$, so $\mathcal{A}$ wins by induction.
- If $\mathcal{B}$ chooses (v_3, v_5), then $\mathcal{A}$ chooses (v_4, v_5). $\mathcal{B}$ must choose (v_5, v_6). $\mathcal{A}$ chooses (v_6, v_7). Vertices v_1, v_2, v_3, v_4, v_5 are out of play. $z(G_{4m+7,9}) = 4m$, $p(G_{4m+7,9}) = 1$, and $t(G_{4m+7,9}) = 1$ with $P(G_{4m+7,9}) = \{v_7\}$ and $T(G_{4m+7,9}) = \{v_6\}$, $(v_6, v_7) \in E_9$. Thus $G_{4m+7,9} \in \mathcal{K}_4(4m)$, so $\mathcal{A}$ wins by Lemma 4.4.

□

5 $\mathcal{R} = \{B3, T, C\}$

In the game $\Gamma_{\{B3,T,C\}}$, we define graph classes $\mathcal{L}_i(z)$ for i=1,2,3, where $z \geq 0$ is an integer. For large enough n, one player can force that $G_{n,k} \in \mathcal{L}_1(z)$ early in the game. A few rounds later, the same player can force that $G_{n,k} \in (\mathcal{L}_2(z) \cup \mathcal{L}_3(z))$. From here, this player then maintains a periodic method of gameplay, continuously returning to graphs in $\mathcal{L}_2(z) \cup \mathcal{L}_3(z)$ after a fixed number of rounds. Following this strategy until the end of the game forces a win.

For any graph $G(V,E)$, define the set $F(G)$ to be the set of edges in E^c that will create a triangle if chosen. That is,

$$F(G) = \{(v_i, v_j) \notin E | \exists w \in V \text{ such that } (v_i, w), (v_j, w) \in E\}.$$

Define sets of graphs $\mathcal{L}_i(z)$ for i=1,2,3 as follows:

1. For any integer $z \geq 0$, $G(V,E) \in \mathcal{L}_1(z) \iff z(G) = z$ and ONE of the following holds:
   (a) $p(G) = 1$, $t(G) = 1$, and if $u_1 \in P(G)$, $u_2 \in T(G)$, then $(u_1, u_2) \in F(G)$,
   (b) $p(G) = 1$, $t(G) = 1$, and if $u_1 \in P(G)$, $u_2 \in T(G)$, then $(u_1, u_2) \notin (E \cup F(G))$,
   (c) $p(G) = 0$, $t(G) = 3$, and, for some ordering of $T(G)$, if $u_1, u_2, u_3 \in T(G)$, then $(u_1, u_2) \in E$ and $(u_1, u_3), (u_2, u_3) \in F(G)$, or
   (d) $p(G) = 0$, $t(G) = 3$, and, for some ordering of $T(G)$, if $u_1, u_2, u_3 \in T(G)$, then $(u_1, u_2) \in E$, $(u_1, u_3) \in F(G)$, and $(u_2, u_3) \notin (E \cup F(G))$.

2. For any integer $z \geq 0$, $G(V,E) \in \mathcal{L}_2(z) \iff z(G) = z$ and ONE of the following holds:
   (a) $p(G) = 1$, $t(G) = 1$, and if $u_1 \in P(G)$, $u_2 \in T(G)$, then $(u_1, u_2) \in F(G)$,
   (b) $p(G) = 0$, $t(G) = 2$, and if $u_1, u_2 \in T(G)$, then $(u_1, u_2) \in E$, or
   (c) $p(G) = 0$, $t(G) = 2$, and if $u_1, u_2 \in T(G)$, then $(u_1, u_2) \in F(G)$.

3. For any integer $z \geq 0$, $G(V,E) \in \mathcal{L}_3(z) \iff z(G) = z$ and ONE of the following holds:
   (a) $p(G) = 1$ and $t(G) = 0$, or
   (b) $p(G) = 0$, $t(G) = 3$, and, for some ordering of $T(G)$, if $u_1, u_2, u_3 \in T(G)$, then (u_1, u_2), $(u_1, u_3) \in E$.

In the proof of the following Lemma, the set $\mathcal{L}_i(z)(j)$ refers to those graphs in $\mathcal{L}_i(z)$ that result from case (j). For example, the set $\mathcal{L}_1(z)(a)$ consists of graphs in $\mathcal{L}_1(z)$ such that case (a) is satisfied: For any integer $z \geq 0$, $G(V,E) \in \mathcal{L}_1(z)(a) \iff z(G) = z$, $p(G) = 1$, $t(G) = 1$, and if $u_1 \in P(G)$, $u_2 \in T(G)$, then $(u_1, u_2) \in F(G)$.

Lemma 5.1. *In $\Gamma_{\{B3,T,C\}}(n)$, $\mathcal{A}$ can choose edges $e_1, e_3, \ldots, e_k$ so that:*

- *when $n \geq 11$ and $k = 15$, $G_{n,15} \in \mathcal{L}_1(n-11)$, or*
- *when $n \geq 15$ and $k = 21$, $G_{n,21} \in \mathcal{L}_1(n-15)$.*

Proof. Let $n \geq 11$. $\mathcal{A}$ chooses (v_1, v_2). Up to isomorphism, $\mathcal{B}$ must choose (v_1, v_3). $\mathcal{A}$ chooses (v_1, v_4). $\mathcal{B}$ must choose (v_2, v_5). $\mathcal{A}$ chooses (v_2, v_6). $\mathcal{B}$ has two choices: (v_3, v_5) or (v_3, v_7). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Now $\mathcal{B}$ has six choices: (v_4, v_5), (v_4, v_6), (v_4, v_8), (v_5, v_8), (v_6, v_7), or (v_6, v_8).

- If $\mathcal{B}$ chooses one of (v_4, v_5) or (v_4, v_8), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has two choices: (v_6, v_7) or (v_6, v_9). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. $\mathcal{B}$ has five choices: (v_7, v_8), (v_7, v_{10}), (v_8, v_9), (v_8, v_{10}), and (v_9, v_{10}).
  - If $\mathcal{B}$ chooses one of (v_7, v_8) or (v_8, v_{10}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has two choices: (v_9, v_{10}) or (v_9, v_{11}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. The resulting graph is in $\mathcal{L}_1(n-11)(a)$.
  - If $\mathcal{B}$ chooses one of (v_7, v_{10}) or (v_8, v_9), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has three choices: (v_8, v_{10}), (v_8, v_{11}), or (v_{10}, v_{11}). If $\mathcal{B}$ chooses (v_8, v_{10}), $\mathcal{A}$ chooses (v_9, v_{11}). If $\mathcal{B}$ chooses (v_8, v_{11}), $\mathcal{A}$ chooses (v_9, v_{10}). If $\mathcal{B}$ chooses (v_{10}, v_{11}), $\mathcal{A}$ chooses (v_8, v_{10}). In each case, the resulting graph is in $\mathcal{L}_1(n-11)(b)$.
  - If $\mathcal{B}$ chooses (v_9, v_{10}), then $\mathcal{A}$ chooses (v_7, v_{10}). $\mathcal{B}$ has three choices: (v_8, v_9), (v_8, v_{11}), or (v_9, v_{11}). If $\mathcal{B}$ chooses one of (v_8, v_9) or (v_8, v_{11}), $\mathcal{A}$ chooses the other edge. If $\mathcal{B}$ chooses (v_9, v_{11}), $\mathcal{A}$ chooses (v_8, v_{10}). In each case, the resulting graph is in $\mathcal{L}_1(n-11)(b)$.
- If $\mathcal{B}$ chooses one of (v_4, v_6) or (v_5, v_8), then $\mathcal{A}$ chooses the other edge.

  If $\mathcal{B}$ chooses one of (v_6, v_7) or (v_6, v_8), then $\mathcal{A}$ chooses (v_5, v_8).

  The resulting three graphs have:

  1. $n-8$ vertices of degree 0, two vertices of degree 1, and two vertices of degree 2,
  2. the two vertices of degree 2 are connected by an edge, and
  3. for each pair of vertices v_i and v_j of degree 1 or 2, not both of degree 2, the distance between v_i and v_j is at least 3.

  Gameplay is identical in each case, so we may assume the first case. $\mathcal{B}$ has four choices: (v_4, v_7), (v_4, v_9), (v_7, v_8), or (v_7, v_9).

  - If $\mathcal{B}$ chooses (v_4, v_7), then $\mathcal{A}$ chooses (v_6, v_9).

    If $\mathcal{B}$ chooses (v_4, v_9), then $\mathcal{A}$ chooses (v_6, v_7).

    If $\mathcal{B}$ chooses (v_7, v_9), then $\mathcal{A}$ chooses (v_4, v_7).

    The resulting three graphs have:

    1. $n-9$ vertices of degree 0, two vertices of degree 1, and one vertex of degree 2, and

2. for each pair of vertices v_i and v_j of degree 1 or 2, the distance between v_i and v_j is at least 3.

Gameplay is identical in each case, so we may consider the first case. $\mathcal{B}$ has four choices: (v_7, v_8), (v_7, v_{10}), (v_8, v_9), or (v_8, v_{10}).

* If $\mathcal{B}$ chooses one of (v_7, v_8) or (v_8, v_{10}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has two choices: (v_9, v_{10}) or (v_9, v_{11}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. The resulting graph is in $\mathcal{L}_1(n-11)(a)$.
* If $\mathcal{B}$ chooses one of (v_7, v_{10}) or (v_8, v_9), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has three choices: (v_8, v_{10}), (v_8, v_{11}), or (v_{10}, v_{11}). If $\mathcal{B}$ chooses (v_8, v_{10}), $\mathcal{A}$ chooses (v_9, v_{11}). If $\mathcal{B}$ chooses (v_8, v_{11}), $\mathcal{A}$ chooses (v_9, v_{10}). If $\mathcal{B}$ chooses (v_{10}, v_{11}), $\mathcal{A}$ chooses (v_8, v_{10}). In each case, the resulting graph is in $\mathcal{L}_1(n-11)(b)$.

– If $\mathcal{B}$ chooses (v_7, v_8), then $\mathcal{A}$ chooses (v_4, v_9). $\mathcal{B}$ has five choices: (v_6, v_7), (v_6, v_{10}), (v_7, v_9), (v_7, v_{10}), or (v_9, v_{10}).

* If $\mathcal{B}$ chooses (v_6, v_7), then $\mathcal{A}$ chooses (v_8, v_{10}). $\mathcal{B}$ has two choices: (v_9, v_{10}) or (v_9, v_{11}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. The resulting graph is in $\mathcal{L}_1(n-11)(a)$.
* If $\mathcal{B}$ chooses one of (v_6, v_{10}) or (v_7, v_9), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has three choices: (v_8, v_{10}), (v_8, v_{11}), or (v_{10}, v_{11}). If $\mathcal{B}$ chooses (v_8, v_{10}), $\mathcal{A}$ chooses (v_9, v_{11}). If $\mathcal{B}$ chooses (v_8, v_{11}), $\mathcal{A}$ chooses (v_9, v_{10}). If $\mathcal{B}$ chooses (v_{10}, v_{11}), $\mathcal{A}$ chooses (v_8, v_{10}). In each case, the resulting graph is in $\mathcal{L}_1(n-11)(b)$.
* If $\mathcal{B}$ chooses one of (v_7, v_{10}) or (v_9, v_{10}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has four choices: (v_6, v_8), (v_6, v_{10}), (v_6, v_{11}), or (v_9, v_{11}).
  · If $\mathcal{B}$ chooses one of (v_6, v_8) or (v_9, v_{11}), then $\mathcal{A}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-11)(a)$.
  · If $\mathcal{B}$ chooses one of (v_6, v_{10}) or (v_6, v_{11}), then $\mathcal{A}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-11)(b)$.

Now let $n \geq 15$. $\mathcal{A}$ chooses (v_1, v_2). Up to isomorphism, $\mathcal{B}$ must choose (v_1, v_3). $\mathcal{A}$ chooses (v_1, v_4). $\mathcal{B}$ must choose (v_2, v_5). $\mathcal{A}$ chooses (v_3, v_5). $\mathcal{B}$ has four choices: (v_2, v_6), (v_4, v_5), (v_4, v_6), or (v_5, v_6).

- If $\mathcal{B}$ chooses one of (v_2, v_6) or (v_4, v_6), then $\mathcal{A}$ chooses the other edge. Now $\mathcal{B}$ has two choices: (v_3, v_6) or (v_3, v_7).
  – If $\mathcal{B}$ chooses (v_3, v_6), then $\mathcal{A}$ chooses (v_4, v_7). Call this graph $G^1_{(1)}$.
  – If $\mathcal{B}$ chooses (v_3, v_7), then $\mathcal{A}$ chooses (v_4, v_5). This graph is isomorphic to $G^1_{(1)}$.
- If $\mathcal{B}$ chooses (v_4, v_5), then $\mathcal{A}$ chooses (v_2, v_6). Now $\mathcal{B}$ has three choices: (v_3, v_6), (v_3, v_7), or (v_6, v_7).

- If $\mathcal{B}$ chooses (v_3, v_6), then $\mathcal{A}$ chooses (v_4, v_7). This graph is isomorphic to $G^1_{(1)}$.
  - If $\mathcal{B}$ chooses (v_3, v_7), then $\mathcal{A}$ chooses (v_4, v_6). This graph is isomorphic to $G^1_{(1)}$.
  - If $\mathcal{B}$ chooses (v_6, v_7), then $\mathcal{A}$ chooses (v_3, v_6). This graph is isomorphic to $G^1_{(1)}$.

- If $\mathcal{B}$ chooses (v_5, v_6), then $\mathcal{A}$ chooses (v_4, v_6). Now $\mathcal{B}$ has two choices: (v_2, v_7) or (v_4, v_7). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Call this graph $G^1_{(2)}$.

We can now examine the two graphs $G^1_{(1)}$ and $G^1_{(2)}$.

Case 1.1: $G_{n,9} = G^1_{(1)}$. $\mathcal{B}$ has three choices: (v_5, v_7), (v_5, v_8), or (v_7, v_8).

- If $\mathcal{B}$ chooses (v_5, v_7), then $\mathcal{A}$ chooses (v_7, v_8). Call this graph $G^2_{(1)}$.

- If $\mathcal{B}$ chooses one of (v_5, v_8) or (v_7, v_8), then $\mathcal{A}$ chooses the other edge. Call this graph $G^2_{(2)}$.

Case 1.2: $G_{n,9} = G^1_{(2)}$. $\mathcal{B}$ has three choices: (v_3, v_7), (v_3, v_8), or (v_6, v_8).

- If $\mathcal{B}$ chooses one of (v_3, v_7) or (v_6, v_8), then $\mathcal{A}$ chooses the other edge. This graph is isomorphic to $G^2_{(1)}$.

- If $\mathcal{B}$ chooses (v_3, v_8), then $\mathcal{A}$ chooses (v_6, v_9). Call this graph $G^2_{(3)}$.

Now we look at the three graphs $G^2_{(i)}$, for $i = 1, 2, 3$.

Case 2.1: $G_{n,11} = G^2_{(1)}$. $\mathcal{B}$ must choose (v_8, v_9). $\mathcal{A}$ chooses (v_8, v_{10}). $\mathcal{B}$ must choose (v_9, v_{11}). $\mathcal{A}$ chooses (v_{10}, v_{11}). $\mathcal{B}$ has two choices: (v_9, v_{12}) or (v_{11}, v_{12}).

If $\mathcal{B}$ chooses (v_9, v_{12}), $\mathcal{A}$ chooses (v_{11}, v_{13}).

If $\mathcal{B}$ chooses (v_{11}, v_{12}), $\mathcal{A}$ chooses (v_9, v_{13}).

The resulting two graphs are isomorphic, so without loss of generality we may consider the first case. $\mathcal{B}$ has five choices: (v_{10}, v_{12}), (v_{10}, v_{14}), (v_{12}, v_{13}), (v_{12}, v_{14}), or (v_{13}, v_{14}).

- If $\mathcal{B}$ chooses one of (v_{10}, v_{12}) or (v_{12}, v_{14}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has two choices: (v_{13}, v_{14}) or (v_{13}, v_{15}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. The resulting graph is in $\mathcal{L}_1(n-15)(a)$.

- If $\mathcal{B}$ chooses one of (v_{10}, v_{14}) or (v_{12}, v_{13}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has three choices: (v_{12}, v_{14}), (v_{12}, v_{15}), or (v_{14}, v_{15}).

  If $\mathcal{B}$ chooses (v_{12}, v_{14}), then $\mathcal{A}$ chooses (v_{13}, v_{15}).

  If $\mathcal{B}$ chooses (v_{12}, v_{15}), then $\mathcal{A}$ chooses (v_{13}, v_{14}).

  If $\mathcal{B}$ chooses (v_{14}, v_{15}), then $\mathcal{A}$ chooses (v_{12}, v_{14}).

  The resulting three graphs are in $\mathcal{L}_1(n-15)(b)$.

- If $\mathcal{B}$ chooses (v_{13}, v_{14}), then $\mathcal{A}$ chooses (v_{10}, v_{14}). $\mathcal{B}$ has three choices: (v_{12}, v_{13}), (v_{12}, v_{15}), or (v_{13}, v_{15}).

  If $\mathcal{B}$ chooses one of (v_{12}, v_{13}) or (v_{12}, v_{15}), then $\mathcal{A}$ chooses the other edge.

  If $\mathcal{B}$ chooses (v_{13}, v_{15}), then $\mathcal{A}$ chooses (v_{12}, v_{14}).

  The resulting two graphs are in $\mathcal{L}_1(n-15)(b)$.

Case 2.2: $G_{n,11} = G^2_{(2)}$. $\mathcal{B}$ must choose (v_7, v_9). $\mathcal{A}$ chooses (v_8, v_{10}). $\mathcal{B}$ has two choices: (v_9, v_{10}) or (v_9, v_{11}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. $\mathcal{B}$ has two choices: (v_{10}, v_{12}) or (v_{11}, v_{12}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. $\mathcal{B}$ must choose (v_{11}, v_{13}). $\mathcal{A}$ chooses (v_{12}, v_{14}). $\mathcal{B}$ has two choices: (v_{13}, v_{14}) or (v_{13}, v_{15}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. The resulting graph is in $\mathcal{L}_1(n-15)(a)$.

Case 2.3: $G_{n,11} = G^2_{(3)}$. $\mathcal{B}$ has four choices: (v_7, v_8), (v_7, v_{10}), (v_8, v_9), or (v_8, v_{10}).

Subcase 2.3.1: If $\mathcal{B}$ chooses one of (v_7, v_8) or (v_8, v_9), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ must choose (v_9, v_{10}). $\mathcal{A}$ chooses (v_{10}, v_{11}). $\mathcal{B}$ has two choices: (v_{10}, v_{12}) or (v_{11}, v_{12}).

If $\mathcal{B}$ chooses (v_{10}, v_{12}), then $\mathcal{A}$ chooses (v_{11}, v_{13}).

If $\mathcal{B}$ chooses (v_{11}, v_{12}), then $\mathcal{A}$ chooses (v_{10}, v_{13}).

The resulting two graphs are isomorphic, so without loss of generality we may consider the first case. $\mathcal{B}$ has four choices: (v_{11}, v_{14}), (v_{12}, v_{13}), (v_{12}, v_{14}), or (v_{13}, v_{14}).

- If $\mathcal{B}$ chooses one of (v_{11}, v_{14}) or (v_{12}, v_{13}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has four choices: (v_{12}, v_{14}), (v_{12}, v_{15}), (v_{13}, v_{15}), or (v_{14}, v_{15}).
  - If $\mathcal{B}$ chooses one of (v_{12}, v_{14}) or (v_{13}, v_{15}), then $\mathcal{A}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-15)(b)$.
  - If $\mathcal{B}$ chooses one of (v_{12}, v_{15}) or (v_{14}, v_{15}), then $\mathcal{A}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-15)(c)$.
- If $\mathcal{B}$ chooses (v_{12}, v_{14}), then $\mathcal{A}$ chooses (v_{11}, v_{14}). $\mathcal{B}$ has four choices: (v_{12}, v_{13}), (v_{12}, v_{15}), (v_{13}, v_{15}), or (v_{14}, v_{15}).
  - If $\mathcal{B}$ chooses one of (v_{12}, v_{13}) or (v_{13}, v_{15}), then $\mathcal{A}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-15)(b)$.
  - If $\mathcal{B}$ chooses (v_{12}, v_{15}), then $\mathcal{A}$ chooses (v_{13}, v_{15}). The resulting graph is in $\mathcal{L}_1(n-15)(c)$.
  - If $\mathcal{B}$ chooses (v_{14}, v_{15}), then $\mathcal{A}$ chooses (v_{13}, v_{15}). The resulting graph is in $\mathcal{L}_1(n-15)(d)$.
- If $\mathcal{B}$ chooses (v_{13}, v_{14}), then $\mathcal{A}$ chooses (v_{12}, v_{13}). $\mathcal{B}$ has two choices: (v_{11}, v_{15}) or (v_{14}, v_{15}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. The resulting graph is in $\mathcal{L}_1(n-15)(d)$.

Subcase 2.3.2: If $\mathcal{B}$ chooses (v_7, v_{10}), then $\mathcal{A}$ chooses (v_8, v_{11}).

If $\mathcal{B}$ chooses (v_8, v_{10}), then $\mathcal{A}$ chooses (v_7, v_{11}).

The two resulting graphs are isomorphic, so without loss of generality we may consider the first case. $\mathcal{B}$ has six choices: (v_8, v_9), (v_8, v_{12}), (v_9, v_{10}), (v_9, v_{11}), (v_9, v_{12}), or (v_{11}, v_{12}).

- If $\mathcal{B}$ chooses one of (v_8, v_9) or (v_9, v_{12}), then $\mathcal{A}$ chooses the other edge. Call this graph $G^3_{(1)}$.
- If $\mathcal{B}$ chooses one of (v_8, v_{12}) or (v_9, v_{10}), then $\mathcal{A}$ chooses the other edge. Call this graph $G^3_{(2)}$.
- If $\mathcal{B}$ chooses one of (v_9, v_{11}) or (v_{11}, v_{12}), then $\mathcal{A}$ chooses the other edge. Call this graph $G^3_{(3)}$.

All that remains is for us to look at the three graphs $G^3_{(i)}$, for $i = 1, 2, 3$.

Case 3.1: $G_{n,15} = G^3_{(1)}$. $\mathcal{B}$ has two choices: (v_{10}, v_{11}) or (v_{10}, v_{13}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. $\mathcal{B}$ has five choices: (v_{11}, v_{13}), (v_{11}, v_{14}), (v_{12}, v_{13}), (v_{12}, v_{14}), or (v_{13}, v_{14}).

- If $\mathcal{B}$ chooses one of (v_{11}, v_{13}) or (v_{13}, v_{14}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has two choices: (v_{12}, v_{14}) or (v_{12}, v_{15}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. The resulting graph is in $\mathcal{L}_1(n-15)(a)$.
- If $\mathcal{B}$ chooses one of (v_{11}, v_{14}) or (v_{12}, v_{13}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has three choices: (v_{12}, v_{14}), (v_{12}, v_{15}), or (v_{14}, v_{15}).
  - If $\mathcal{B}$ chooses (v_{12}, v_{14}), then $\mathcal{A}$ chooses (v_{13}, v_{15}). The resulting graph is in $\mathcal{L}_1(n-15)(b)$.
  - If $\mathcal{B}$ chooses one of (v_{12}, v_{15}) or (v_{14}, v_{15}), then $\mathcal{A}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-15)(d)$.
- If $\mathcal{B}$ chooses (v_{12}, v_{14}), then $\mathcal{A}$ chooses (v_{11}, v_{14}). $\mathcal{B}$ has three choices: (v_{12}, v_{13}), (v_{12}, v_{15}), or (v_{13}, v_{15}).
  - If $\mathcal{B}$ chooses (v_{12}, v_{13}), then $\mathcal{A}$ chooses (v_{13}, v_{15}). The resulting graph is in $\mathcal{L}_1(n-15)(b)$.
  - If $\mathcal{B}$ chooses one of (v_{12}, v_{15}) or (v_{13}, v_{15}), then $\mathcal{A}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-15)(d)$.

Case 3.2: $G_{n,15} = G^3_{(2)}$. $\mathcal{B}$ has three choices: (v_9, v_{11}), (v_9, v_{13}), or (v_{11}, v_{13}).

Subcase 3.2.1: If $\mathcal{B}$ chooses one of (v_9, v_{11}) or (v_{11}, v_{13}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has five choices: (v_{10}, v_{12}), (v_{10}, v_{14}), (v_{12}, v_{13}), or (v_{12}, v_{14}).

- If $\mathcal{B}$ chooses one of (v_{10}, v_{12}) or (v_{12}, v_{14}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has two choices: (v_{13}, v_{14}) or (v_{13}, v_{15}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. The resulting graph is in $\mathcal{L}_1(n-15)(a)$.

- If $\mathcal{B}$ chooses one of (v_{10}, v_{14}) or (v_{12}, v_{13}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has three choices: (v_{12}, v_{14}), (v_{12}, v_{15}), or (v_{14}, v_{15}).

  If $\mathcal{B}$ chooses (v_{12}, v_{14}), then $\mathcal{A}$ chooses (v_{13}, v_{15}).

  If $\mathcal{B}$ chooses (v_{12}, v_{15}), then $\mathcal{A}$ chooses (v_{13}, v_{14}).

  If $\mathcal{B}$ chooses (v_{14}, v_{15}), then $\mathcal{A}$ chooses (v_{12}, v_{14}).

  The resulting three graphs are in $\mathcal{L}_1(n-15)(b)$.

Subcase 3.2.2: If $\mathcal{B}$ chooses (v_9, v_{13}), then $\mathcal{A}$ chooses (v_{10}, v_{11}). $\mathcal{B}$ has five choices: (v_{11}, v_{13}), (v_{11}, v_{14}), (v_{12}, v_{13}), (v_{12}, v_{14}), or (v_{13}, v_{14}).

- If $\mathcal{B}$ chooses one of (v_{11}, v_{13}) or (v_{13}, v_{14}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has two choices: (v_{12}, v_{14}) or (v_{12}, v_{15}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. The resulting graph is in $\mathcal{L}_1(n-15)(a)$.

- If $\mathcal{B}$ chooses one of (v_{11}, v_{14}) or (v_{12}, v_{13}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has three choices: (v_{12}, v_{14}), (v_{12}, v_{15}), or (v_{14}, v_{15}).

  - If $\mathcal{B}$ chooses (v_{12}, v_{14}), then $\mathcal{A}$ chooses (v_{13}, v_{15}). The resulting graph is in $\mathcal{L}_1(n-15)(b)$.
  - If $\mathcal{B}$ chooses one of (v_{12}, v_{15}) or (v_{14}, v_{15}), then $\mathcal{A}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-15)(d)$.

- If $\mathcal{B}$ chooses (v_{12}, v_{14}), then $\mathcal{A}$ chooses (v_{11}, v_{14}). $\mathcal{B}$ has three choices: (v_{12}, v_{13}), (v_{12}, v_{15}), or (v_{13}, v_{15}).

  - If $\mathcal{B}$ chooses (v_{12}, v_{13}), then $\mathcal{A}$ chooses (v_{13}, v_{15}). The resulting graph is in $\mathcal{L}_1(n-15)(b)$.
  - If $\mathcal{B}$ chooses one of (v_{12}, v_{15}) or (v_{13}, v_{15}), then $\mathcal{A}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-15)(d)$.

Case 3.3: $G_{n,15} = G^3_{(3)}$. $\mathcal{B}$ has five choices: (v_8, v_{10}), (v_8, v_{13}), (v_{10}, v_{12}), (v_{10}, v_{13}), or (v_{12}, v_{13}).

Subcase 3.3.1: If $\mathcal{B}$ chooses one of (v_8, v_{10}) or (v_{10}, v_{13}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has five choices: (v_9, v_{13}), (v_9, v_{14}), (v_{12}, v_{13}), (v_{12}, v_{14}), or (v_{13}, v_{14}).

- If $\mathcal{B}$ chooses one of (v_9, v_{13}) or (v_{13}, v_{14}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has two choices: (v_{12}, v_{14}) or (v_{12}, v_{15}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. The resulting graph is in $\mathcal{L}_1(n-15)(a)$.

- If $\mathcal{B}$ chooses one of (v_9, v_{14}) or (v_{12}, v_{13}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has three choices: (v_{12}, v_{14}), (v_{12}, v_{15}), or (v_{14}, v_{15}).

  - If $\mathcal{B}$ chooses (v_{12}, v_{14}), then $\mathcal{A}$ chooses (v_{13}, v_{15}). The resulting graph is in $\mathcal{L}_1(n-15)(b)$.

  - If $\mathcal{B}$ chooses one of (v_{12}, v_{15}) or (v_{14}, v_{15}), then $\mathcal{A}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-15)(d)$.

- If $\mathcal{B}$ chooses (v_{12}, v_{14}), then $\mathcal{A}$ chooses (v_9, v_{14}). $\mathcal{B}$ has three choices: (v_{12}, v_{13}), (v_{12}, v_{15}), or (v_{13}, v_{15}).

  - If $\mathcal{B}$ chooses (v_{12}, v_{13}), then $\mathcal{A}$ chooses (v_{13}, v_{15}). The resulting graph is in $\mathcal{L}_1(n-15)(b)$.
  - If $\mathcal{B}$ chooses one of (v_{12}, v_{15}) or (v_{13}, v_{15}), then $\mathcal{A}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-15)(d)$.

Subcase 3.3.2: If $\mathcal{B}$ chooses (v_8, v_{13}), then $\mathcal{A}$ chooses (v_9, v_{10}). $\mathcal{B}$ has five choices: (v_{10}, v_{12}), (v_{10}, v_{14}), (v_{12}, v_{13}), or (v_{12}, v_{14}).

- If $\mathcal{B}$ chooses one of (v_{10}, v_{12}) or (v_{12}, v_{14}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has two choices: (v_{13}, v_{14}) or (v_{13}, v_{15}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. The resulting graph is in $\mathcal{L}_1(n-15)(a)$.

- If $\mathcal{B}$ chooses one of (v_{10}, v_{14}) or (v_{12}, v_{13}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has three choices: (v_{12}, v_{14}), (v_{12}, v_{15}), or (v_{14}, v_{15}).

  If $\mathcal{B}$ chooses (v_{12}, v_{14}), then $\mathcal{A}$ chooses (v_{13}, v_{15}).

  If $\mathcal{B}$ chooses (v_{12}, v_{15}), then $\mathcal{A}$ chooses (v_{13}, v_{14}).

  If $\mathcal{B}$ chooses (v_{14}, v_{15}), then $\mathcal{A}$ chooses (v_{12}, v_{14}).

  The resulting three graphs are in $\mathcal{L}_1(n-15)(b)$.

Subcase 3.3.3: If $\mathcal{B}$ chooses one of (v_{10}, v_{12}) or (v_{12}, v_{13}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has five choices: (v_8, v_{10}), (v_8, v_{13}), (v_8, v_{14}), (v_{10}, v_{14}), or (v_{13}, v_{14}).

- If $\mathcal{B}$ chooses (v_8, v_{10}), then $\mathcal{A}$ chooses (v_9, v_{14}). $\mathcal{B}$ has two choices: (v_{13}, v_{14}) or (v_{13}, v_{15}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. The resulting graph is in $\mathcal{L}_1(n-15)(a)$.

- If $\mathcal{B}$ chooses (v_8, v_{13}), then $\mathcal{A}$ chooses (v_9, v_{14}). $\mathcal{B}$ has three choices: (v_{10}, v_{14}), (v_{10}, v_{15}), or (v_{14}, v_{15}).

  If $\mathcal{B}$ chooses (v_{10}, v_{14}), then $\mathcal{A}$ chooses (v_{13}, v_{15}).

  If $\mathcal{B}$ chooses (v_{10}, v_{15}), then $\mathcal{A}$ chooses (v_{13}, v_{14}).

  If $\mathcal{B}$ chooses (v_{14}, v_{15}), then $\mathcal{A}$ chooses (v_{10}, v_{14}).

  The resulting three graphs are in $\mathcal{L}_1(n-15)(b)$.

- If $\mathcal{B}$ chooses (v_8, v_{14}), then $\mathcal{A}$ chooses (v_9, v_{10}). $\mathcal{B}$ has two choices: (v_{13}, v_{14}) or (v_{13}, v_{15}). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. The resulting graph is in $\mathcal{L}_1(n-15)(a)$.

- If $\mathcal{B}$ chooses one of (v_{10}, v_{14}) or (v_{13}, v_{14}), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has three choices: (v_8, v_{13}), (v_8, v_{15}), or (v_{13}, v_{15}).

  If $\mathcal{B}$ chooses (v_8, v_{13}), then $\mathcal{A}$ chooses (v_9, v_{15}).

  If $\mathcal{B}$ chooses (v_8, v_{15}), then $\mathcal{A}$ chooses (v_9, v_{13}).

  If $\mathcal{B}$ chooses (v_{13}, v_{15}), then $\mathcal{A}$ chooses (v_8, v_{14}).

  The resulting three graphs are in $\mathcal{L}_1(n-15)(b)$.

□

Lemma 5.2. *In $\Gamma_{\{B3,T,C\}}(n)$, $\mathcal{B}$ can choose edges $e_2, e_4, \ldots, e_k$ so that:*

- *when $n \geq 9$ and k=12, $G_{n,12} \in \mathcal{L}_1(n-9)$, or*
- *when $n \geq 13$ and k=18, $G_{n,18} \in \mathcal{L}_1(n-13)$.*

Proof. Let $n \geq 9$. Up to isomorphism, $\mathcal{A}$ must choose (v_1, v_2). $\mathcal{B}$ chooses (v_1, v_3). $\mathcal{A}$ has two choices: (v_1, v_4) or (v_2, v_4).

- If $\mathcal{A}$ chooses (v_1, v_4), then $\mathcal{B}$ chooses (v_2, v_5).
- If $\mathcal{A}$ chooses (v_2, v_4), then $\mathcal{B}$ chooses (v_1, v_5).

The two resulting graphs are isomorphic, so without loss of generality we may consider the first case. $\mathcal{A}$ has four choices: (v_2, v_6), (v_3, v_5), (v_3, v_6), or (v_5, v_6).

- If $\mathcal{A}$ chooses one of (v_2, v_6) or (v_3, v_5), then $\mathcal{B}$ chooses the other edge.

  If $\mathcal{A}$ chooses (v_3, v_6), then $\mathcal{B}$ chooses (v_3, v_5).

  The two resulting graphs are isomorphic, so without loss of generality, we may consider the first case. $\mathcal{A}$ has four choices: (v_3, v_6), (v_3, v_7), (v_4, v_6), or (v_4, v_7).

  - If $\mathcal{A}$ chooses (v_3, v_6), then $\mathcal{B}$ chooses (v_5, v_7). Call this graph $G_{(1)}$.
  - If $\mathcal{A}$ chooses one of (v_3, v_7) or (v_4, v_6), then $\mathcal{B}$ chooses the other edge. Call this graph $G_{(2)}$.
  - If $\mathcal{A}$ chooses (v_4, v_7), then $\mathcal{B}$ chooses (v_3, v_7). This graph is isomorphic to $G_{(2)}$.

- If $\mathcal{A}$ chooses (v_5, v_6), then $\mathcal{B}$ chooses (v_3, v_5). $\mathcal{A}$ has three choices: (v_2, v_7), (v_4, v_6), or (v_4, v_7).

  - If $\mathcal{A}$ chooses one of (v_2, v_7) or (v_4, v_7), then $\mathcal{B}$ chooses the other edge. This graph is isomorphic to $G_{(2)}$ above.
  - If $\mathcal{A}$ chooses (v_4, v_6), then $\mathcal{B}$ chooses (v_2, v_7). Call this graph $G_{(3)}$.

We can now examine the three graphs $G_{(i)}$ for $i = 1, 2, 3$.
Case 1: $G_{n,8} = G_{(1)}$. $\mathcal{A}$ has four choices: (v_4, v_6), (v_4, v_7), (v_4, v_8), or (v_6, v_8).

- If $\mathcal{A}$ chooses one of (v_4, v_6) or (v_4, v_8), then $\mathcal{B}$ chooses the other edge. $\mathcal{A}$ has two choices: (v_7, v_8) or (v_7, v_9). $\mathcal{B}$ chooses whichever edge $\mathcal{A}$ did not. The resulting graph is in $\mathcal{L}_1(n-11)(a)$.

- If $\mathcal{A}$ chooses one of (v_4, v_7) or (v_6, v_8), then $\mathcal{B}$ chooses the other edge. $\mathcal{A}$ has three choices: (v_4, v_8), (v_4, v_9), or (v_8, v_9).

  If $\mathcal{A}$ chooses (v_4, v_8), $\mathcal{B}$ chooses (v_7, v_9).

  If $\mathcal{A}$ chooses (v_4, v_9), $\mathcal{B}$ chooses (v_7, v_8).

  If $\mathcal{A}$ chooses (v_8, v_9), $\mathcal{B}$ chooses (v_4, v_8).

  In each case, the resulting graph is in $\mathcal{L}_1(n-11)(b)$.

Case 2: $G_{n,8} = G_{(2)}$. $\mathcal{A}$ has seven choices: (v_4, v_5), (v_4, v_7), (v_4, v_8), (v_5, v_8), (v_6, v_7), (v_6, v_8), or (v_7, v_8).

- If $\mathcal{A}$ chooses one of (v_4, v_5) or (v_6, v_8), then $\mathcal{B}$ chooses the other edge. $\mathcal{A}$ has two choices: (v_7, v_8) or (v_7, v_9). $\mathcal{B}$ chooses whichever edge $\mathcal{A}$ did not. The resulting graph is in $\mathcal{L}_1(n-11)(a)$.

- If $\mathcal{A}$ chooses one of (v_4, v_7) or (v_5, v_8), then $\mathcal{B}$ chooses the other edge. $\mathcal{A}$ has three choices: (v_6, v_8), (v_6, v_9), or (v_8, v_9).

  If $\mathcal{A}$ chooses (v_6, v_8), $\mathcal{B}$ chooses (v_7, v_9).

  If $\mathcal{A}$ chooses (v_6, v_9), $\mathcal{B}$ chooses (v_7, v_8).

  If $\mathcal{A}$ chooses (v_8, v_9), $\mathcal{B}$ chooses (v_6, v_8).

  In each case, the resulting graph is in $\mathcal{L}_1(n-11)(b)$.

- If $\mathcal{A}$ chooses (v_4, v_8), then $\mathcal{B}$ chooses (v_5, v_8). $\mathcal{A}$ has three choices: (v_6, v_7), (v_6, v_9), or (v_7, v_9).

  If $\mathcal{A}$ chooses (v_6, v_7), $\mathcal{B}$ chooses (v_7, v_9).

  If $\mathcal{A}$ chooses (v_6, v_9), $\mathcal{B}$ chooses (v_7, v_8).

  If $\mathcal{A}$ chooses (v_7, v_9), $\mathcal{B}$ chooses (v_6, v_7).

  In each case, the resulting graph is in $\mathcal{L}_1(n-11)(b)$.

- If $\mathcal{A}$ chooses (v_6, v_7), then $\mathcal{B}$ chooses (v_4, v_8). $\mathcal{A}$ has three choices: (v_5, v_8), (v_5, v_9), or (v_8, v_9).

  If $\mathcal{A}$ chooses (v_5, v_8), $\mathcal{B}$ chooses (v_7, v_9).

  If $\mathcal{A}$ chooses (v_5, v_9), $\mathcal{B}$ chooses (v_7, v_8).

  If $\mathcal{A}$ chooses (v_8, v_9), $\mathcal{B}$ chooses (v_5, v_8).

  In each case, the resulting graph is in $\mathcal{L}_1(n-11)(b)$.

- If $\mathcal{A}$ chooses (v_7, v_8), then $\mathcal{B}$ chooses (v_4, v_7). $\mathcal{A}$ has three choices: (v_5, v_8), (v_5, v_9), or (v_8, v_9).

  If $\mathcal{A}$ chooses (v_5, v_8), $\mathcal{B}$ chooses (v_6, v_9).

  If $\mathcal{A}$ chooses (v_5, v_9), $\mathcal{B}$ chooses (v_6, v_8).

  If $\mathcal{A}$ chooses (v_8, v_9), $\mathcal{B}$ chooses (v_5, v_8).

  In each case, the resulting graph is in $\mathcal{L}_1(n-11)(b)$.

Case 3: $G_{n,8} = G_{(3)}$. $\mathcal{A}$ has five choices: (v_3, v_7), (v_3, v_8), (v_4, v_7), (v_4, v_8), or (v_7, v_8).

- If $\mathcal{A}$ chooses one of (v_3, v_7) or (v_7, v_8), then $\mathcal{B}$ chooses the other edge. $\mathcal{A}$ has three choices: (v_4, v_8), (v_4, v_9), or (v_8, v_9).

  If $\mathcal{A}$ chooses (v_4, v_8), $\mathcal{B}$ chooses (v_6, v_9).

  If $\mathcal{A}$ chooses (v_4, v_9), $\mathcal{B}$ chooses (v_6, v_8).

  If $\mathcal{A}$ chooses (v_8, v_9), $\mathcal{B}$ chooses (v_4, v_8).

  In each case, the resulting graph is in $\mathcal{L}_1(n-11)(b)$.

- If $\mathcal{A}$ chooses one of (v_3, v_8) or (v_4, v_7), then $\mathcal{B}$ chooses the other edge. $\mathcal{A}$ has three choices: (v_6, v_8), (v_6, v_9), or (v_8, v_9).

  If $\mathcal{A}$ chooses (v_6, v_8), $\mathcal{B}$ chooses (v_7, v_9).

  If $\mathcal{A}$ chooses (v_6, v_9), $\mathcal{B}$ chooses (v_7, v_8).

  If $\mathcal{A}$ chooses (v_8, v_9), $\mathcal{B}$ chooses (v_6, v_8).

  In each case, the resulting graph is in $\mathcal{L}_1(n-11)(b)$.

- If $\mathcal{A}$ chooses one of (v_4, v_8), then $\mathcal{B}$ chooses (v_3, v_8). $\mathcal{A}$ has three choices: (v_6, v_7), (v_6, v_9), or (v_7, v_9).

  If $\mathcal{A}$ chooses (v_6, v_7), $\mathcal{B}$ chooses (v_7, v_9).

  If $\mathcal{A}$ chooses (v_6, v_9), $\mathcal{B}$ chooses (v_7, v_8).

  If $\mathcal{A}$ chooses (v_7, v_9), $\mathcal{B}$ chooses (v_6, v_7).

  In each case, the resulting graph is in $\mathcal{L}_1(n-11)(b)$.

Now let $n \geq 13$. Up to isomorphism, $\mathcal{A}$ must choose (v_1, v_2). $\mathcal{B}$ chooses (v_1, v_3). $\mathcal{A}$ has two choices: (v_1, v_4) or (v_2, v_4).

- If $\mathcal{A}$ chooses (v_1, v_4), then $\mathcal{B}$ chooses (v_2, v_5).

- If $\mathcal{A}$ chooses (v_2, v_4), then $\mathcal{B}$ chooses (v_1, v_5).

The two resulting graphs are isomorphic, so without loss of generality we may assume the first case. $\mathcal{A}$ has four choices: (v_2, v_6), (v_3, v_5), (v_3, v_6), or (v_5, v_6).

- If $\mathcal{A}$ chooses (v_2, v_6), then $\mathcal{B}$ chooses (v_3, v_7).

  If $\mathcal{A}$ chooses (v_3, v_6), then $\mathcal{B}$ chooses (v_2, v_7).

  If $\mathcal{A}$ chooses (v_5, v_6), then $\mathcal{B}$ chooses (v_2, v_7).

  The three resulting graphs are isomorphic, so without loss of generality we may assume the first case. $\mathcal{A}$ has eight choices: (v_3, v_5), (v_3, v_8), (v_4, v_5), (v_4, v_7), (v_4, v_8), (v_5, v_7), (v_5, v_8), or (v_7, v_8).

  – If $\mathcal{A}$ chooses one of (v_3, v_5) or (v_4, v_5), then $\mathcal{B}$ chooses the other edge. Call this graph $G^1_{(1)}$.
  – If $\mathcal{A}$ chooses one of (v_3, v_8) or (v_5, v_7), then $\mathcal{B}$ chooses the other edge. Call this graph $G^1_{(2)}$.
  – If $\mathcal{A}$ chooses (v_4, v_7), then $\mathcal{B}$ chooses (v_5, v_7). Call this graph $G^1_{(3)}$.
  – If $\mathcal{A}$ chooses one of (v_4, v_8) or (v_7, v_8), then $\mathcal{B}$ chooses the other edge. Call this graph $G^1_{(4)}$.
  – If $\mathcal{A}$ chooses (v_5, v_8), then $\mathcal{B}$ chooses (v_5, v_7). This graph is isomorphic to $G^1_{(2)}$.

- If $\mathcal{A}$ chooses (v_3, v_5), then $\mathcal{B}$ chooses (v_4, v_5). $\mathcal{A}$ must choose (v_2, v_6). Then $\mathcal{B}$ chooses (v_3, v_7). This graph is isomorphic to $G^1_{(1)}$.

Now we need to consider the four graphs $G^1_{(i)}$ for $i = 1, 2, 3, 4$.
Case 1.1: $G_{n,8} = G^1_{(1)}$. $\mathcal{A}$ has four choices: (v_4, v_6), (v_4, v_8), (v_6, v_7), or (v_6, v_8).

- If $\mathcal{A}$ chooses one of (v_4, v_6) or (v_6, v_7), then $\mathcal{B}$ the other edge. Call this graph $G^2_{(1)}$.

- If $\mathcal{A}$ chooses (v_4, v_8), then $\mathcal{B}$ chooses (v_6, v_9).

  If $\mathcal{A}$ chooses (v_6, v_8), then $\mathcal{B}$ chooses (v_4, v_9).

  The two resulting graphs are isomorphic, so without loss of generality we may consider the first case. Call this graph $G^2_{(2)}$.

Case 1.2: $G_{n,8} = G^1_{(2)}$. $\mathcal{A}$ has seven choices:

- If $\mathcal{A}$ chooses (v_4, v_5), then $\mathcal{B}$ chooses (v_7, v_9).

  If $\mathcal{A}$ chooses (v_5, v_8), then $\mathcal{B}$ chooses (v_7, v_9).

  If $\mathcal{A}$ chooses (v_6, v_9), then $\mathcal{B}$ chooses (v_6, v_7).

  The three resulting graphs are isomorphic, so we may consider the first case. Call this graph $G^2_{(3)}$.

- If $\mathcal{A}$ chooses (v_4, v_6), then $\mathcal{B}$ chooses (v_7, v_9).

  If $\mathcal{A}$ chooses one of (v_4, v_9) or (v_5, v_9), then $\mathcal{B}$ chooses the other edge.

  If $\mathcal{A}$ chooses (v_6, v_8), then $\mathcal{B}$ chooses (v_5, v_9).

  The three resulting graphs satisfy:

1. $(z(G), p(G), t(G)) = (n-9, 2, 3)$,
2. if $P(G) = \{u_1, u_2\}$ and $T(G) = \{u_3, u_4, u_5\}$, then, up to isomorphism, $(u_3, u_4) \in E(G)$, $(u_1, u_5), (u_4, u_5) \in F(G)$, and all other pairs are in $(E(G) \cup F(G))^c$.

Gameplay is identical in each case, so we may consider the first case. Call this graph $G^2_{(4)}$.

Case 1.3: $G_{n,8} = G^1_{(3)}$. $\mathcal{A}$ has four choices: (v_3, v_6), (v_3, v_8), (v_5, v_8), or (v_6, v_8).

- If $\mathcal{A}$ chooses (v_3, v_6), then $\mathcal{B}$ chooses (v_4, v_6). This graph is isomorphic to $G^2_{(1)}$.
- If $\mathcal{A}$ chooses (v_3, v_8), then $\mathcal{B}$ chooses (v_5, v_9). This graph is isomorphic to $G^2_{(3)}$.
- If $\mathcal{A}$ chooses (v_5, v_8), then $\mathcal{B}$ chooses (v_3, v_9). This graph is isomorphic to $G^2_{(3)}$.
- If $\mathcal{A}$ chooses (v_6, v_8), then $\mathcal{B}$ chooses (v_6, v_9). Call this graph $G^2_{(5)}$.

Case 1.4: $G_{n,8} = G^1_{(4)}$. $\mathcal{A}$ has five choices: (v_3, v_5), (v_3, v_9), (v_5, v_7), (v_5, v_9), or (v_7, v_9).

- If $\mathcal{A}$ chooses one of (v_3, v_5) or (v_5, v_9), then $\mathcal{B}$ chooses the other edge. Call this graph $G^2_{(6)}$.
- If $\mathcal{A}$ chooses (v_3, v_9), then $\mathcal{B}$ chooses (v_8, v_9). This graph is isomorphic to $G^2_{(5)}$.
- If $\mathcal{A}$ chooses (v_5, v_7), then $\mathcal{B}$ chooses (v_3, v_9). This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n-9, 2, 3)$,
  2. if $P(G) = \{u_1, u_2\}$ and $T(G) = \{u_3, u_4, u_5\}$, then, up to isomorphism, $(u_3, u_4) \in E(G)$, $(u_1, u_5), (u_4, u_5) \in F(G)$, and all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^2_{(4)}$.
- If $\mathcal{A}$ chooses (v_7, v_9), then $\mathcal{B}$ chooses (v_4, v_9). This graph is isomorphic to $G^2_{(5)}$.

We now need to look at the graphs $G^2_{(i)}$ for $1 \leq i \leq 6$.

Case 2.1: $G_{n,10} = G^2_{(1)}$. $\mathcal{A}$ must choose (v_7, v_8). $\mathcal{B}$ chooses (v_8, v_9). Call this graph $G^3_{(1)}$.

Case 2.2: $G_{n,10} = G^2_{(2)}$. $\mathcal{A}$ has six choices: (v_6, v_7), (v_6, v_{10}), (v_7, v_8), (v_7, v_9), (v_7, v_{10}), or (v_9, v_{10}).

- If $\mathcal{A}$ chooses one of (v_6, v_7) or (v_7, v_{10}), then $\mathcal{B}$ chooses the other edge. Call this graph $G^3_{(2)}$.

- If $\mathcal{A}$ chooses one of (v_6, v_{10}) or (v_7, v_8), then $\mathcal{B}$ chooses the other edge. Call this graph $G^3_{(3)}$.
- If $\mathcal{A}$ chooses one of (v_7, v_9) or (v_9, v_{10}), then $\mathcal{B}$ chooses the other edge. Call this graph $G^3_{(4)}$.

Case 2.3: $G_{n,10} = G^2_{(3)}$. $\mathcal{A}$ has four choices: (v_4, v_6), (v_4, v_{10}), (v_6, v_8), or (v_6, v_{10}).

- If $\mathcal{A}$ chooses one of (v_4, v_6) or (v_6, v_{10}), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 10, 3, 0)$, and
  2. for each pair $u_1, u_2 \in P(G)$, $(u_1, u_2) \notin (E(G) \cup F(G))$.

  Gameplay is identical to that of $G^3_{(2)}$.

- If $\mathcal{A}$ chooses one of (v_4, v_{10}) or (v_6, v_8), then $\mathcal{B}$ chooses the other edge. Call this graph $G^3_{(5)}$.

Case 2.4: $G_{n,10} = G^2_{(4)}$. $\mathcal{A}$ has twelve choices: (v_4, v_5), (v_4, v_8), (v_4, v_9), (v_4, v_{10}), (v_5, v_8), (v_5, v_{10}), (v_6, v_8), (v_6, v_9), (v_6, v_{10}), (v_8, v_9), (v_8, v_{10}), or (v_9, v_{10}).

- If $\mathcal{A}$ chooses one of (v_4, v_5) or (v_6, v_{10}), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 10, 3, 0)$, and
  2. for each pair $u_1, u_2 \in P(G)$, $(u_1, u_2) \notin (E(G) \cup F(G))$.

  Gameplay is identical to that of $G^3_{(2)}$.

- If $\mathcal{A}$ chooses one of (v_4, v_8) or (v_5, v_{10}), then $\mathcal{B}$ chooses the other edge.

  If $\mathcal{A}$ chooses (v_4, v_9), then $\mathcal{B}$ chooses (v_5, v_{10}).

  If $\mathcal{A}$ chooses (v_4, v_{10}), then $\mathcal{B}$ chooses (v_5, v_{10}).

  If $\mathcal{A}$ chooses (v_6, v_8), then $\mathcal{B}$ chooses (v_5, v_{10}).

  If $\mathcal{A}$ chooses (v_6, v_9), then $\mathcal{B}$ chooses (v_4, v_{10}).

  In each case, the resulting graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 10, 2, 2)$,
  2. if $T(G) = \{u_1, u_2\}$, then $(u_1, u_2) \in F(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical in each case, so we may consider the first one. Call this graph $G^3_{(6)}$.

- If $\mathcal{A}$ chooses one of (v_5, v_8) or (v_8, v_{10}), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n-10, 2, 2)$,
  2. if $T(G) = \{u_1, u_2\}$, then $(u_1, u_2) \in E(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^3_{(5)}$.

- If $\mathcal{A}$ chooses (v_8, v_9), then $\mathcal{B}$ chooses or (v_5, v_{10}).

  If $\mathcal{A}$ chooses (v_9, v_{10}), then $\mathcal{B}$ chooses or (v_5, v_{10}).

  These two graphs are isomorphic, so we may consider the first case. Call this graph $G^3_{(7)}$.

Case 2.5: $G_{n,10} = G^2_{(5)}$. $\mathcal{A}$ has three choices: (v_3, v_8), (v_3, v_{10}), or (v_8, v_{10}).

- If $\mathcal{A}$ chooses one of (v_3, v_8) or (v_8, v_{10}), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n-10, 2, 2)$,
  2. if $T(G) = \{u_1, u_2\}$, then $(u_1, u_2) \in F(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^3_{(6)}$.

- If $\mathcal{A}$ chooses (v_3, v_{10}), then $\mathcal{B}$ chooses (v_4, v_{10}). Call this graph $G^3_{(8)}$.

Case 2.6: $G_{n,10} = G^2_{(6)}$. $\mathcal{A}$ has six choices: (v_4, v_6), (v_4, v_{10}), (v_6, v_8), (v_6, v_9), (v_6, v_{10}), or (v_8, v_{10}).

- If $\mathcal{A}$ chooses one of (v_4, v_6) or (v_6, v_{10}), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n-10, 2, 2)$,
  2. if $T(G) = \{u_1, u_2\}$, then $(u_1, u_2) \in E(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^3_{(5)}$.

- If $\mathcal{A}$ chooses one of (v_4, v_{10}) or (v_6, v_8), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n-10, 2, 2)$,

2. if $T(G) = \{u_1, u_2\}$, then $(u_1, u_2) \in F(G)$, and
3. all other pairs are in $(E(G) \cup F(G))^c$.

Gameplay is identical to that of $G^3_{(6)}$.

- If $\mathcal{A}$ chooses one of (v_6, v_9) or (v_8, v_{10}), then $\mathcal{B}$ chooses the other edge. Call this graph $G^3_{(9)}$.

Now let us consider the graphs $G^3_{(i)}$ for $1 \leq i \leq 9$.

Case 3.1: $G_{n,12} = G^3_{(1)}$. $\mathcal{A}$ has two choices: (v_8, v_{10}) or (v_9, v_{10}).

If $\mathcal{A}$ chooses (v_8, v_{10}), then $\mathcal{B}$ chooses (v_9, v_{11}).

If $\mathcal{A}$ chooses (v_8, v_{10}), then $\mathcal{B}$ chooses (v_9, v_{11}).

The two resulting graphs are isomorphic, so we may consider the first case. Call this graph $G^4_{(1)}$.

Case 3.2: $G_{n,12} = G^3_{(2)}$. $\mathcal{A}$ has two choices: (v_8, v_9) or (v_8, v_{11}). $\mathcal{B}$ chooses whichever edge $\mathcal{A}$ did not. Call this graph $G^4_{(2)}$.

Case 3.3: $G_{n,12} = G^3_{(3)}$. $\mathcal{A}$ has three choices: (v_7, v_9), (v_7, v_{11}), or (v_9, v_{11}).

- If $\mathcal{A}$ chooses one of (v_7, v_9) or (v_9, v_{11}), then $\mathcal{B}$ chooses the other edge. Call this graph $G^4_{(3)}$.
- If $\mathcal{A}$ chooses (v_7, v_{11}), then $\mathcal{B}$ chooses (v_8, v_9). This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 11, 2, 1)$,
  2. if $P(G) = \{u_1, u_2\}$ and $T(G) = \{u_3\}$, then, up to isomorphism, $(u_1, u_3) \in F(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^4_{(2)}$.

Case 3.4: $G_{n,12} = G^3_{(4)}$. $\mathcal{A}$ has five choices: (v_6, v_8), (v_6, v_{11}), (v_8, v_{10}), (v_8, v_{11}), or (v_{10}, v_{11}).

- If $\mathcal{A}$ chooses one of (v_6, v_8) or (v_8, v_{11}), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 11, 2, 1)$,
  2. if $P(G) = \{u_1, u_2\}$ and $T(G) = \{u_3\}$, then, up to isomorphism, $(u_1, u_3) \in F(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^4_{(2)}$.

- If $\mathcal{A}$ chooses (v_6, v_{11}), then $\mathcal{B}$ chooses (v_7, v_8). This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 11, 2, 1)$, and
  2. if $P(G) = \{u_1, u_2\}$ and $T(G) = \{u_3\}$, then all pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^4_{(3)}$.

- If $\mathcal{A}$ chooses one of (v_8, v_{10}) or (v_{10}, v_{11}), then $\mathcal{B}$ chooses the other edge. Call this graph $G^4_{(4)}$.

Case 3.5: $G_{n,12} = G^3_{(5)}$. $\mathcal{A}$ has four choices: (v_6, v_9), (v_6, v_{11}), (v_9, v_{10}), or (v_9, v_{11}).

- If $\mathcal{A}$ chooses (v_6, v_9), then $\mathcal{B}$ chooses (v_8, v_{11}).

  If $\mathcal{A}$ chooses (v_9, v_{11}), then $\mathcal{B}$ chooses (v_6, v_9).

  In each case, the resulting graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 11, 2, 1)$, and
  2. if $P(G) = \{u_1, u_2\}$ and $T(G) = \{u_3\}$, then all pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^4_{(3)}$.

- If $\mathcal{A}$ chooses one of (v_6, v_{11}) or (v_9, v_{10}), then $\mathcal{B}$ chooses the other edge. Call this graph $G^4_{(5)}$.

Case 3.6: $G_{n,12} = G^3_{(6)}$. $\mathcal{A}$ has four choices: (v_6, v_9), (v_6, v_{11}), (v_9, v_{10}), or (v_9, v_{11}).

- If $\mathcal{A}$ chooses (v_6, v_9), then $\mathcal{B}$ chooses (v_8, v_{11}).

  If $\mathcal{A}$ chooses (v_6, v_{11}), then $\mathcal{B}$ chooses (v_8, v_9).

  In each case, the resulting graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 11, 2, 1)$, and
  2. if $P(G) = \{u_1, u_2\}$ and $T(G) = \{u_3\}$, then all pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^4_{(3)}$.

- If $\mathcal{A}$ chooses one of (v_9, v_{10}) or (v_9, v_{11}), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 11, 1, 3)$,
  2. if $P(G) = \{u_1,\}$ and $T(G) = \{u_2, u_3, u_4\}$, then, up to isomorphism, $(u_1, u_2), (u_3, u_4) \in F(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

Gameplay is identical to that of $G^4_{(4)}$.

Case 3.7: $G_{n,12} = G^3_{(7)}$. $\mathcal{A}$ has four choices: (v_4, v_8), (v_4, v_{10}), (v_4, v_{11}), or (v_{10}, v_{11}).

- If $\mathcal{A}$ chooses (v_4, v_8), then $\mathcal{B}$ chooses (v_6, v_{11}).

  If $\mathcal{A}$ chooses (v_4, v_{11}), then $\mathcal{B}$ chooses (v_6, v_8).

  In each case, the resulting graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 11, 2, 1)$, and
  2. if $P(G) = \{u_1, u_2\}$ and $T(G) = \{u_3\}$, then all pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^4_{(3)}$.

- If $\mathcal{A}$ chooses (v_4, v_{11}), then $\mathcal{B}$ chooses (v_6, v_8).

  If $\mathcal{A}$ chooses (v_{10}, v_{11}), then $\mathcal{B}$ chooses (v_4, v_{10}).

  In each case, the resulting graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 11, 1, 3)$,
  2. if $P(G) = \{u_1\}$ and $T(G) = \{u_2, u_3, u_4\}$, then, up to isomorphism, $(u_2, u_3) \in E(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical in each case, so we can consider the first case. Call this graph $G^4_{(6)}$.

Case 3.8: $G_{n,12} = G^3_{(8)}$. $\mathcal{A}$ has four choices: (v_5, v_8), (v_5, v_{10}), (v_5, v_{11}), or (v_8, v_{11}).

- If $\mathcal{A}$ chooses one of (v_5, v_8) or (v_8, v_{11}), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 11, 2, 1)$, and
  2. if $P(G) = \{u_1, u_2\}$ and $T(G) = \{u_3\}$, then all pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^4_{(3)}$.

- If $\mathcal{A}$ chooses (v_5, v_{10}), then $\mathcal{B}$ chooses (v_8, v_{11}). This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 11, 2, 1)$,
  2. if $P(G) = \{u_1, u_2\}$ and $T(G) = \{u_3\}$, then, up to isomorphism, $(u_1, u_3) \in E(G)$, $(u_2, u_3) \in F(G)$, and
  3. $(u_1, u_3) \in (E(G) \cup F(G))^c$.

Gameplay is identical to that of $G^4_{(1)}$.

- If $\mathcal{A}$ chooses (v_5, v_{11}), then $\mathcal{B}$ chooses (v_8, v_{10}). This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 11, 2, 1)$,
  2. if $P(G) = \{u_1, u_2\}$ and $T(G) = \{u_3\}$, then, up to isomorphism, $(u_1, u_3) \in F(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^4_{(2)}$.

Case 3.9: $G_{n,12} = G^3_{(9)}$. $\mathcal{A}$ has five choices: (v_4, v_6), (v_4, v_{11}), (v_6, v_{10}), (v_6, v_{11}), or (v_{10}, v_{11}).

- If $\mathcal{A}$ chooses (v_4, v_6), then $\mathcal{B}$ chooses (v_7, v_{11}).

  If $\mathcal{A}$ chooses (v_4, v_{11}), then $\mathcal{B}$ chooses (v_6, v_7).

  In each case, the resulting graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 11, 2, 1)$, and
  2. if $P(G) = \{u_1, u_2\}$ and $T(G) = \{u_3\}$, then all pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^4_{(3)}$.

- If $\mathcal{A}$ chooses (v_6, v_{10}), then $\mathcal{B}$ chooses (v_4, v_9). Call this graph $G^4_{(7)}$.

- If $\mathcal{A}$ chooses one of (v_6, v_{11}) or (v_{10}, v_{11}), then $\mathcal{B}$ chooses the other edge. Call this graph $G^4_{(8)}$.

Now we consider the graphs $G^4_{(i)}$ for $1 \le i \le 8$.
Case 4.1: $G_{n,14} = G^4_{(1)}$. $\mathcal{A}$ has four choices: (v_9, v_{12}), (v_{10}, v_{11}), (v_{10}, v_{12}), or (v_{11}, v_{12}).

- If $\mathcal{A}$ chooses one of (v_9, v_{12}) or (v_{10}, v_{11}), then $\mathcal{B}$ chooses the other edge.

  If $\mathcal{A}$ chooses (v_{10}, v_{12}), then $\mathcal{B}$ chooses (v_9, v_{12}).

  In each case, the resulting graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 12, 1, 2)$,
  2. if $P(G) = \{u_1\}$ and $T(G) = \{u_2, u_3\}$, then, up to isomorphism, $(u_2, u_3) \in E(G)$, $(u_1, u_2) \in F(G)$, and
  3. $(u_1, u_3) \in (E(G) \cup F(G))^c$.

  Gameplay is identical in each case, so we may consider the first case. Call this graph $G^5_{(1)}$.

- If $\mathcal{A}$ chooses (v_{11}, v_{12}), then $\mathcal{B}$ chooses (v_{10}, v_{11}). Call this graph $G^5_{(2)}$.

Case 4.2: $G_{n,14} = G^4_{(2)}$. $\mathcal{A}$ has five choices: (v_9, v_{10}), (v_9, v_{12}), (v_{10}, v_{11}), (v_{10}, v_{12}), or (v_{11}, v_{12}).

- If $\mathcal{A}$ chooses one of (v_9, v_{10}) or (v_{10}, v_{12}), then $\mathcal{B}$ chooses the other edge. Call this graph $G^5_{(3)}$.
- If $\mathcal{A}$ chooses one of (v_9, v_{12}) or (v_{10}, v_{11}), then $\mathcal{B}$ chooses the other edge.

  If $\mathcal{A}$ chooses (v_{11}, v_{12}), then $\mathcal{B}$ chooses (v_9, v_{12}).

  In each case, the resulting graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 12, 1, 2)$,
  2. if $P(G) = \{u_1\}$ and $T(G) = \{u_2, u_3\}$, then $(u_2, u_3) \in E(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical in each case, so we may consider the first case. Call this graph $G^5_{(4)}$.

Case 4.3: $G_{n,14} = G^4_{(3)}$. $\mathcal{A}$ has four choices: (v_8, v_{10}), (v_8, v_{12}), (v_{10}, v_{11}), or (v_{10}, v_{12}).

- If $\mathcal{A}$ chooses one of (v_8, v_{10}) or (v_{10}, v_{12}), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 12, 2, 0)$, and
  2. if $P(G) = \{u_1, u_2\}$, then $(u_1, u_2) \in (E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^5_{(3)}$.

- If $\mathcal{A}$ chooses one of (v_8, v_{12}) or (v_{10}, v_{11}), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 12, 1, 2)$,
  2. if $P(G) = \{u_1\}$ and $T(G) = \{u_2, u_3\}$, then $(u_2, u_3) \in E(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^5_{(4)}$.

Case 4.4: $G_{n,14} = G^4_{(4)}$. $\mathcal{A}$ has five choices: (v_6, v_8), (v_6, v_{11}), (v_6, v_{12}), (v_8, v_{12}), or (v_{11}, v_{12}).

- If $\mathcal{A}$ chooses (v_6, v_8), then $\mathcal{B}$ chooses (v_7, v_{12}).

  If $\mathcal{A}$ chooses (v_6, v_{12}), then $\mathcal{B}$ chooses (v_7, v_8).

  In each case, the resulting graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 12, 2, 0)$, and
  2. if $P(G) = \{u_1, u_2\}$, then $(u_1, u_2) \in (E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^5_{(3)}$.

- If $\mathcal{A}$ chooses (v_6, v_{11}), then $\mathcal{B}$ chooses (v_7, v_{12}). Call this graph $G^5_{(5)}$.

- If $\mathcal{A}$ chooses one of (v_8, v_{12}) or (v_{11}, v_{12}), then $\mathcal{B}$ chooses the other edge. Call this graph $G^5_{(6)}$.

Case 4.5: $G_{n,14} = G^4_{(5)}$. $\mathcal{A}$ has five choices: (v_8, v_9), (v_8, v_{12}), (v_9, v_{11}), (v_9, v_{12}), or (v_{11}, v_{12}).

- If $\mathcal{A}$ chooses (v_8, v_9), then $\mathcal{B}$ chooses (v_{10}, v_{12}). This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 12, 2, 0)$, and
  2. if $P(G) = \{u_1, u_2\}$, then $(u_1, u_2) \in (E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^5_{(3)}$.

- If $\mathcal{A}$ chooses one of (v_8, v_{12}) or (v_9, v_{11}), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 12, 1, 2)$,
  2. if $P(G) = \{u_1\}$ and $T(G) = \{u_2, u_3\}$, then $(u_2, u_3) \in F(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^5_{(5)}$.

- If $\mathcal{A}$ chooses one of (v_9, v_{12}) or (v_{11}, v_{12}), then $\mathcal{B}$ chooses the other edge. Call this graph $G^5_{(7)}$.

Case 4.6: $G_{n,14} = G^4_{(6)}$. $\mathcal{A}$ has six choices: (v_8, v_{10}), (v_8, v_{11}), (v_8, v_{12}), (v_{10}, v_{11}), (v_{10}, v_{12}), or (v_{11}, v_{12}).

- If $\mathcal{A}$ chooses (v_8, v_{10}), then $\mathcal{B}$ chooses (v_9, v_{12}).

  If $\mathcal{A}$ chooses (v_8, v_{12}), then $\mathcal{B}$ chooses (v_9, v_{10}).

  In each case, the resulting graph satisfies:

1. $(z(G), p(G), t(G)) = (n-12, 2, 0)$, and
2. if $P(G) = \{u_1, u_2\}$, then $(u_1, u_2) \in (E(G) \cup F(G))^c$.

Gameplay is identical to that of $G^5_{(3)}$.

- If $\mathcal{A}$ chooses one of (v_8, v_{11}) or (v_{10}, v_{12}), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n-12, 1, 2)$,
  2. if $P(G) = \{u_1\}$ and $T(G) = \{u_2, u_3\}$, then $(u_2, u_3) \in F(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^5_{(5)}$.

- If $\mathcal{A}$ chooses one of (v_{10}, v_{11}) or (v_{11}, v_{12}), then $\mathcal{B}$ chooses the other edge. This graph satisfies:

  1. $(z(G), p(G), t(G)) = (n-12, 1, 2)$,
  2. if $P(G) = \{u_1\}$ and $T(G) = \{u_2, u_3\}$, then $(u_2, u_3) \in E(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^5_{(4)}$.

Case 4.7: $G_{n,14} = G^4_{(7)}$. $\mathcal{A}$ must choose (v_7, v_{11}). Then $\mathcal{B}$ chooses (v_{10}, v_{12}). This graph satisfies:

1. $(z(G), p(G), t(G)) = (n-12, 2, 0)$, and
2. if $P(G) = \{u_1, u_2\}$, then $(u_1, u_2) \in (E(G) \cup F(G))^c$.

Gameplay is identical to that of $G^5_{(3)}$.

Case 4.8: $G_{n,14} = G^4_{(8)}$. $\mathcal{A}$ has seven choices: (v_4, v_9), (v_4, v_{11}), (v_4, v_{12}), (v_9, v_{10}), (v_9, v_{12}), (v_{10}, v_{12}), or (v_{11}, v_{12}).

- If $\mathcal{A}$ chooses (v_4, v_9), then $\mathcal{B}$ chooses (v_7, v_{12}).

  If $\mathcal{A}$ chooses (v_4, v_{12}), then $\mathcal{B}$ chooses (v_7, v_9).

  In each case, the resulting graph satisfies:

  1. $(z(G), p(G), t(G)) = (n-12, 1, 2)$,
  2. if $P(G) = \{u_1\}$ and $T(G) = \{u_2, u_3\}$, then $(u_2, u_3) \in E(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^5_{(4)}$.

- If $\mathcal{A}$ chooses one of (v_4, v_{11}) or (v_9, v_{12}), then $\mathcal{B}$ chooses the other edge.

  If $\mathcal{A}$ chooses one of (v_9, v_{10}) or (v_{11}, v_{12}), then $\mathcal{B}$ chooses the other edge.

  In each case, the resulting graph satisfies:

  1. $(z(G), p(G), t(G)) = (n - 12, 1, 2)$,
  2. if $P(G) = \{u_1\}$ and $T(G) = \{u_2, u_3\}$, then $(u_2, u_3) \in F(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical to that of $G^5_{(5)}$.

- If $\mathcal{A}$ chooses (v_{10}, v_{12}), then $\mathcal{B}$ chooses (v_9, v_{12}). Call this graph $G^5_{(8)}$.

Finally, we look at the graphs $G^5_{(i)}$ for $1 \leq i \leq 8$.
Case 5.1: $G_{n,16} = G^5_{(1)}$. $\mathcal{A}$ has four choices: (v_{10}, v_{12}), (v_{10}, v_{13}), (v_{11}, v_{13}), or (v_{12}, v_{13}).

- If $\mathcal{A}$ chooses one of (v_{10}, v_{12}) or (v_{11}, v_{13}), then $\mathcal{B}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-13)(b)$.

- If $\mathcal{A}$ chooses one of (v_{10}, v_{13}) or (v_{12}, v_{13}), then $\mathcal{B}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-13)(c)$.

Case 5.2: $G_{n,16} = G^5_{(2)}$. $\mathcal{A}$ has two choices: (v_9, v_{13}) or (v_{12}, v_{13}). $\mathcal{B}$ chooses whichever edge $\mathcal{A}$ did not. The resulting graph is in $\mathcal{L}_1(n-13)(d)$.
Case 5.3: $G_{n,16} = G^5_{(3)}$. $\mathcal{A}$ has two choices: (v_{11}, v_{12}) or (v_{11}, v_{13}). $\mathcal{B}$ chooses whichever edge $\mathcal{A}$ did not. The resulting graph is in $\mathcal{L}_1(n-13)(a)$.
Case 5.4: $G_{n,16} = G^5_{(4)}$. $\mathcal{A}$ has three choices: (v_{10}, v_{12}), (v_{10}, v_{13}), or (v_{12}, v_{13}).

- If $\mathcal{A}$ chooses (v_{10}, v_{12}), then $\mathcal{B}$ chooses (v_{11}, v_{13}). The resulting graph is in $\mathcal{L}_1(n-13)(b)$.

- If $\mathcal{A}$ chooses one of (v_{10}, v_{13}) or (v_{12}, v_{13}), then $\mathcal{B}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-13)(d)$.

Case 5.5: $G_{n,16} = G^5_{(5)}$. $\mathcal{A}$ has three choices: (v_8, v_{12}), (v_8, v_{13}), or (v_{12}, v_{13}).
If $\mathcal{A}$ chooses (v_8, v_{12}), then $\mathcal{B}$ chooses (v_{11}, v_{13}).
If $\mathcal{A}$ chooses (v_8, v_{13}), then $\mathcal{B}$ chooses (v_{11}, v_{12}).
If $\mathcal{A}$ chooses (v_{12}, v_{13}), then $\mathcal{B}$ chooses (v_8, v_{12}).
In each case, the resulting graph is in $\mathcal{L}_1(n-13)(b)$.
Case 5.6: $G_{n,16} = G^5_{(6)}$. $\mathcal{A}$ has three choices: (v_6, v_{11}), (v_6, v_{13}), or (v_{11}, v_{13}).
If $\mathcal{A}$ chooses (v_6, v_{11}), then $\mathcal{B}$ chooses (v_7, v_{13}).
If $\mathcal{A}$ chooses (v_6, v_{13}), then $\mathcal{B}$ chooses (v_7, v_{11}).
If $\mathcal{A}$ chooses (v_{11}, v_{13}), then $\mathcal{B}$ chooses (v_6, v_{12}).
In each case, the resulting graph is in $\mathcal{L}_1(n-13)(b)$.
Case 5.7: $G_{n,16} = G^5_{(7)}$. $\mathcal{A}$ has four choices: (v_8, v_{10}), (v_8, v_{12}), (v_8, v_{13}), or (v_{11}, v_{13}).

- If $\mathcal{A}$ chooses one of (v_8, v_{10}) or (v_{11}, v_{13}), then $\mathcal{B}$ chooses the other edge. The resulting graph is in $\mathcal{L}_1(n-13)(a)$.

- If $\mathcal{A}$ chooses (v_8, v_{12}), then $\mathcal{B}$ chooses (v_{10}, v_{13}).

  If $\mathcal{A}$ chooses (v_8, v_{13}), then $\mathcal{B}$ chooses (v_{10}, v_{11}).

  In each case, the resulting graph is in $\mathcal{L}_1(n-13)(b)$.

Case 5.8: $G_{n,16} = G^5_{(8)}$. $\mathcal{A}$ has two choices: (v_4, v_{11}) or (v_4, v_{13}).
If $\mathcal{A}$ chooses (v_4, v_{11}), then $\mathcal{B}$ chooses (v_7, v_{13}).
If $\mathcal{A}$ chooses (v_4, v_{13}), then $\mathcal{B}$ chooses (v_7, v_{11}).
In each case, the resulting graph is in $\mathcal{L}_1(n-13)(b)$. □

Lemma 5.3. *For $n \geq 3$, in $\Gamma_{\{B3,T,C\}}(n)$, if either player chooses an edge e_k that creates a graph $G_{n,k} \in \mathcal{L}_1(z)$, then*

- *when $z \geq 1$, that player can choose an edge e_{k+2} so that $G_{n,k+2} \in (\mathcal{L}_2(z-1) \cup \mathcal{L}_3(z-1))$, or*

- *when $z \geq 4$, that player can choose edges $e_{k+2}, e_{k+4}, e_{k+6}$ so that $G_{n,k+6} \in (\mathcal{L}_2(z-4) \cup \mathcal{L}_3(z-4))$.*

Proof. Let $n \geq 3$. Suppose the edges $e_1, \ldots, e_{k-1}$ have already been chosen. Suppose player $\mathcal{A}$ chooses the kth edge so that $G_{n,k} \in \mathcal{L}_1(z)$ with $z \geq 1$. Let $w_1 \in Z(G_{n,k})$. Each vertex $v \notin (Z(G_{n,k}) \cup P(G_{n,k}) \cup T(G_{n,k}))$ has $deg_{G_{n,k}}(v) = 3$ and is out of play. The strategy described here will work for $\mathcal{B}$ as well.

$\boldsymbol{G_{n,k} \in \mathcal{L}_1(z)(a)}$:

Let $P(G_{n,k}) = \{u_1\}$ and $T(G_{n,k}) = \{u_2\}$ with $(u_1, u_2) \in F(G_{n,k})$. Up to isomorphism, $\mathcal{B}$ has two choices: (u_1, w_1) or (u_2, w_1). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Now u_2 is out of play, $z(G_{n,k+2}) = z(G_{n,k}) - 1 = z - 1$, $p(G_{n,k+2}) = 0$, $T(G_{n,k+2}) = \{u_1, w_1\}$, and $(u_1, w_1) \in E_{k+2}$. So $G_{n,k+2} \in \mathcal{L}_2(z-1)(b)$.

$\boldsymbol{G_{n,k} \in \mathcal{L}_1(z)(b)}$:

Let $P(G_{n,k}) = \{u_1\}$ and $T(G_{n,k}) = \{u_2\}$ with $(u_1, u_2) \notin (E_k \cup F(G_{n,k}))$. Up to isomorphism, $\mathcal{B}$ has three choices: (u_1, u_2), (u_1, w_1), or (u_2, w_1).

- If $\mathcal{B}$ chooses (u_1, u_2) then $\mathcal{A}$ chooses (u_1, w_1). Now u_1, u_2 are out of play, $z(G_{n,k+2}) = z(G_{n,k}) - 1 = z - 1$, $P(G_{n,k+2}) = \{w_1\}$, and $t(G_{n,k+2}) = 0$. So $G_{n,k+2} \in \mathcal{L}_3(z-1)(a)$.

- If $\mathcal{B}$ chooses one of (u_1, w_1) or (u_2, w_1), then $\mathcal{A}$ chooses the other edge. This case is identical to the above case $G_{n,k} \in \mathcal{L}_1(z)(a)$ and $G_{n,k+2} \in \mathcal{L}_2(z-1)(b)$.

$\boldsymbol{G_{n,k} \in \mathcal{L}_1(z)(c)}$:

Let $T(G_{n,k}) = \{u_1, u_2, u_3\}$ with $(u_1, u_2) \in E_k$ and (u_1, u_3), $(u_2, u_3) \in F(G_{n,k})$. Note that $p(G_{n,k}) = 0$. Up to isomorphism, $\mathcal{B}$ has two choices: (u_1, w_1) or (u_3, w_1). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Now u_1, u_3 are out of play, $z(G_{n,k+2}) = z(G_{n,k}) - 1 = z - 1$, $p(G_{n,k+2}) = 0$, $T(G_{n,k+2}) = \{u_2, w_1\}$, and $(u_2, w_1) \in F(G_{n,k+2})$. So $G_{n,k+2} \in \mathcal{L}_2(z-1)(c)$.

$\boldsymbol{G_{n,k} \in \mathcal{L}_1(z)(d)}$:

Let $u_1, u_2, u_3 \in T(G)$, with $(u_1, u_2) \in E$, $(u_1, u_3) \in F(G)$, and $(u_2, u_3) \notin (E \cup F(G))$. Note that $p(G_{n,k}) = 0$. Up to isomorphism, $\mathcal{B}$ has four choices: (u_1, w_1), (u_2, u_3), (u_2, w_1), or (u_3, w_1).

- If $\mathcal{B}$ chooses one of (u_1, w_1) or (u_2, u_3), then $\mathcal{A}$ chooses the other edge. Now u_1, u_2, u_3 are out of play, $z(G_{n,k+2}) = z(G_{n,k}) - 1 = z - 1$, $P(G_{n,k+2}) = \{w_1\}$, and $t(G_{n,k+2}) = 0$. So $G_{n,k+2} \in \mathcal{L}_3(z-1)(a)$.

- If $\mathcal{B}$ chooses (u_2, w_1), then $\mathcal{A}$ chooses (u_3, w_1). Now u_2, u_3 are out of play, $z(G_{n,k+2}) = z(G_{n,k}) - 1 = z - 1$, $p(G_{n,k+2}) = 0$, $T(G_{n,k+2}) = \{u_1, w_1\}$, and $(u_1, w_1) \in F(G_{n,k+2})$. So $G_{n,k+2} \in \mathcal{L}_2(z-1)(c)$.

- If $\mathcal{B}$ chooses (u_3, w_1), then $\mathcal{A}$ chooses (u_1, w_1). Now u_1, u_3 are out of play, $z(G_{n,k+2}) = z(G_{n,k}) - 1 = z - 1$, $p(G_{n,k+2}) = 0$, $T(G_{n,k+2}) = \{u_2, w_1\}$, and $(u_2, w_1) \in F(G_{n,k+2})$. So $G_{n,k+2} \in \mathcal{L}_2(z-1)(c)$.

Now suppose player $\mathcal{A}$ chooses the kth edge so that $G_{n,k} \in \mathcal{L}_1(z)$ with $z \geq 4$. Let $w_1, w_2, w_3, w_4 \in Z(G_{n,k})$.

$\boldsymbol{G_{n,k} \in \mathcal{L}_1(z)(a)}$:

$\mathcal{A}$ chooses the edge e_{k+2} as above. Now $G_{n,k+2} \in \mathcal{L}_2(z-1)(b)$ with $z(G_{n,k+2}) = z - 1$, $p(G_{n,k+2}) = 0$, $T(G_{n,k+2}) = \{u_1, w_1\}$, and $(u_1, w_1) \in E_{k+2}$. Up to game play equivalence, $\mathcal{B}$ must choose (u_1, w_2). Then $\mathcal{A}$ chooses (w_1, w_3). Now $\mathcal{B}$ has two choices: (w_2, w_3) or (w_2, w_4). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Now u_1, w_1, w_2 are out of play, $z(G_{n,k+6}) = z(G_{n,k}) - 4 = z - 4$, $P(G_{n,k+6}) = \{w_4\}$, $T(G_{n,k+6}) = \{w_3\}$, and $(w_3, w_4) \in F(G_{n,k+6})$. So $G_{n,k+6} \in \mathcal{L}_2(z-4)(a)$.

$\boldsymbol{G_{n,k} \in \mathcal{L}_1(z)(b)}$:

$\mathcal{A}$ chooses the edge e_{k+2} as above. Now we have two cases.

1. $G_{n,k+2} \in \mathcal{L}_3(z-1)(a)$ with $z(G_{n,k+2}) = z - 1$, $P(G_{n,k+2}) = \{w_1\}$, and $t(G_{n,k+2}) = 0$. Up to isomorphism, $\mathcal{B}$ must choose (w_1, w_2). Then $\mathcal{A}$ chooses (w_1, w_3). $\mathcal{B}$ must choose (w_2, w_4). $\mathcal{A}$ chooses (w_3, w_4). Now w_1 is out of play, $z(G_{n,k+6}) = z(G_{n,k}) - 4 = z - 4$, $p(G_{n,k+6}) = 0$, $T(G_{n,k+6}) = \{w_2, w_3, w_4\}$, and $(w_2, w_4), (w_3, w_4) \in E_{k+6}$. So $G_{n,k+6} \in \mathcal{L}_3(z-4)(b)$.

2. $G_{n,k+2} \in \mathcal{L}_2(z-1)(b)$ with $z(G_{n,k+2}) = z - 1$, $p(G_{n,k+2}) = 0$, $T(G_{n,k+2}) = \{u_1, w_1\}$, and $(u_1, w_1) \in E_{k+2}$. This case is identical to $G_{n,k+2}$ in $\mathcal{L}_1(z)(a)$ above and $\mathcal{A}$ follows the same pattern.

$\boldsymbol{G_{n,k} \in \mathcal{L}_1(z)(c)}$:

$\mathcal{A}$ chooses the edge e_{k+2} as above. Now $G_{n,k+2} \in \mathcal{L}_2(z-1)(c)$ with $z(G_{n,k+2}) = z - 1$, $p(G_{n,k+2}) = 0$, $T(G_{n,k+2}) = \{u_2, w_1\}$, and $(u_2, w_1) \in F(G_{n,k+2})$. Up to game play equivalence, $\mathcal{B}$ must choose (u_2, w_2). Then $\mathcal{A}$ chooses (w_1, w_3). Now $\mathcal{B}$ has two choices: (w_2, w_3) or (w_2, w_4). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Now u_1, w_1, w_2 are out of play, $z(G_{n,k+6}) = z(G_{n,k}) - 4 = z - 4$, $P(G_{n,k+6}) = \{w_4\}$, $T(G_{n,k+6}) = \{w_3\}$, and $(w_3, w_4) \in F(G_{n,k+6})$. So $G_{n,k+6} \in \mathcal{L}_2(z-4)(a)$.

$\boldsymbol{G_{n,k} \in \mathcal{L}_1(z)(d)}$:

$\mathcal{A}$ chooses the edge e_{k+2} as above. Now we have three cases.

1. $G_{n,k+2} \in \mathcal{L}_3(z-1)(a)$ with $z(G_{n,k+2}) = z-1$, $P(G_{n,k+2}) = \{w_1\}$, and $t(G_{n,k+2}) = 0$. This case is identical to $G_{n,k+2}$ in the first case of $\mathcal{L}_1(z)(b)$ above and $\mathcal{A}$ follows the same pattern.

2. $G_{n,k+2} \in \mathcal{L}_2(z-1)(c)$ with $z(G_{n,k+2}) = z-1$, $p(G_{n,k+2}) = 0$, $T(G_{n,k+2}) = \{u_1, w_1\}$, and $(u_1, w_1) \in F(G_{n,k+2})$. This case is identical to $G_{n,k+2}$ in $\mathcal{L}_1(z)(c)$ above and $\mathcal{A}$ follows the same pattern.

3. $G_{n,k+2} \in \mathcal{L}_2(z-1)(c)$ with $z(G_{n,k+2}) = z-1$, $p(G_{n,k+2}) = 0$, $T(G_{n,k+2}) = \{u_2, w_1\}$, and $(u_2, w_1) \in F(G_{n,k+2})$. This case is identical to $G_{n,k+2}$ in $\mathcal{L}_1(z)(c)$ above and $\mathcal{A}$ follows the same pattern.

□

Lemma 5.4. *For $n \geq 3$, in $\Gamma_{\{B3,T,C\}}(n)$, if either player chooses an edge e_k that creates a graph $G_{n,k} \in \mathcal{L}_2(z)$ for some $z \equiv 0 \pmod 4$, then that player has a winning strategy.*

Proof. Let $n \geq 3$. Suppose the edges $e_1, \ldots, e_{k-1}$ have already been chosen. Suppose player $\mathcal{A}$ chooses the kth edge so that $G_{n,k} \in \mathcal{L}_2(z)$ with $z \equiv 0 \pmod 4$. The strategy described here will work for $\mathcal{B}$ as well. We proceed by induction.

Base case: z=0. In each graph in $\mathcal{L}_2(0)$, no new edge can be chosen. Since $\mathcal{A}$ chose the last edge, $\mathcal{A}$ wins.

Assume for induction that if $\mathcal{A}$ chooses an edge e_k that creates a graph $G_{n,k} \in \mathcal{L}_2(4m)$ for some $m \geq 0$, then $\mathcal{A}$ has a winning strategy. Suppose $z = 4(m+1)$. Let $w_1, w_2, w_3, w_4 \in Z(G_{n,k})$.

$\boldsymbol{G_{n,k} \in \mathcal{L}_2(z)(a)}$:

Let $P(G_{n,k}) = \{u_1\}$ and $T(G_{n,k}) = \{u_2\}$ with $(u_1, u_2) \in F(G_{n,k})$. Up to isomorphism, $\mathcal{B}$ has two choices: (u_1, w_1) or (u_2, w_1). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Up to game play equivalence, $\mathcal{B}$ must choose (u_1, w_2). Then $\mathcal{A}$ chooses (w_1, w_3). Now $\mathcal{B}$ has two choices: (w_2, w_3) or (w_2, w_4). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Now u_1, u_2, w_1, w_2 are out of play, $z(G_{n,k+6}) = z(G_{n,k}) - 4 = 4m$, $P(G_{n,k+6}) = \{w_4\}$, $T(G_{n,k+6}) = \{w_3\}$, and $(w_3, w_4) \in F(G_{n,k+6})$. Thus $G_{n,k+6} \in \mathcal{L}_2(4m)(a)$ and $\mathcal{A}$ wins by induction.

$\boldsymbol{G_{n,k} \in \mathcal{L}_2(z)(b)}$:

Let $T(G_{n,k}) = \{u_1, u_2\}$ with $(u_1, u_2) \in E_k$. Note that $p(G_{n,k}) = 0$. Up to game play equivalence, $\mathcal{B}$ must choose (u_1, w_1). $\mathcal{A}$ chooses (u_2, w_2). $\mathcal{B}$ has two choices: (w_1, w_2) or (w_1, w_3). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. $\mathcal{B}$ again has two choices: (w_2, w_4) or (w_3, w_4). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Now u_1, u_2, w_1, w_2 are out of play, $z(G_{n,k+6}) = z(G_{n,k}) - 4 = 4m$, $p(G_{n,k+6}) = 0$, $T(G_{n,k+6}) = \{w_3, w_4\}$, and $(w_3, w_4) \in E_{k+6}$. Thus $G_{n,k+6} \in \mathcal{L}_2(4m)(b)$ and $\mathcal{A}$ wins by induction.

$\boldsymbol{G_{n,k} \in \mathcal{L}_2(z)(c)}$:

Let $T(G_{n,k}) = \{u_1, u_2\}$ with $(u_1, u_2) \in F(G_{n,k})$. Note that $p(G_{n,k}) = 0$. Up to game play equivalence, $\mathcal{B}$ must choose (u_1, w_1). $\mathcal{A}$ chooses (u_2, w_2). Now this case is identical to the above case $G_{n,k} \in \mathcal{L}_2(z)(b)$ and $\mathcal{A}$ wins. □

Lemma 5.5. *For $n \geq 3$, in $\Gamma_{\{B3,T,C\}}(n)$, if either player chooses an edge e_k that creates a graph $G_{n,k} \in \mathcal{L}_3(z)$ for some $z \equiv 0 \pmod 8$, then that player has a winning strategy.*

Proof. Let $n \geq 3$. Suppose the edges $e_1, \ldots, e_{k-1}$ have already been chosen. Suppose player $\mathcal{A}$ chooses the kth edge so that $G_{n,k} \in \mathcal{L}_3(z)$ with $z \equiv 0 \pmod 8$. The strategy described here will work for $\mathcal{B}$ as well. We proceed by induction.

Base case: z=0. In each graph in $\mathcal{L}_3(0)$, no new edge can be chosen. Since $\mathcal{A}$ chose the last edge, $\mathcal{A}$ wins.

Assume for induction that if $\mathcal{A}$ chooses an edge e_k that creates a graph $G_{n,k} \in \mathcal{L}_3(8m)$ for some $m \geq 0$, then $\mathcal{A}$ has a winning strategy. Suppose $z = 8(m+1)$. Let $w_1, w_2, \ldots, w_8 \in Z(G_{n,k})$.

$\boldsymbol{G_{n,k} \in \mathcal{L}_3(z)(a)}$:

Let $P(G_{n,k}) = \{u_1\}$. Note that $t(G_{n,k}) = 0$. Up to isomorphism, $\mathcal{B}$ must choose (u_1, w_1). $\mathcal{A}$ chooses (u_1, w_2). $\mathcal{B}$ must choose (w_1, w_3). $\mathcal{A}$ chooses (w_2, w_3). $\mathcal{B}$ has two choices: (w_1, w_4) or (w_3, w_4).

- If $\mathcal{B}$ chooses (w_1, w_4) then $\mathcal{A}$ chooses (w_3, w_5).
- If $\mathcal{B}$ chooses (w_3, w_4) then $\mathcal{A}$ chooses (w_1, w_5).

The two resulting graphs are isomorphic, so consider the first case. Now $\mathcal{B}$ has five choices: (w_2, w_4), (w_2, w_6), (w_4, w_5), (w_4, w_6), or (w_5, w_6).

- If $\mathcal{B}$ chooses one of (w_2, w_4) or (w_4, w_6), then $\mathcal{A}$ chooses the other edge.

  $\mathcal{B}$ has two choices: (w_5, w_6) or (w_5, w_7). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. $\mathcal{B}$ again has two choices: (w_6, w_8) or (w_7, w_8). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Now $u_1, w_1, w_2, \ldots, w_6$ are out of play, $z(G_{n,k+12}) = z(G_{n,k}) - 8 = 8m$, $p(G_{n,k+12}) = 0$, $T(G_{n,k+12}) = \{w_7, w_8\}$, and $(w_7, w_8) \in E_{k+12}$. Thus $G_{n,k+12} \in \mathcal{L}_2(8m)(b)$ and $\mathcal{A}$ wins by Lemma 5.4.

- If $\mathcal{B}$ chooses one of (w_2, w_6) or (w_4, w_5), then $\mathcal{A}$ chooses the other edge.

  If $\mathcal{B}$ chooses (w_5, w_6), then $\mathcal{A}$ chooses (w_2, w_6).

  The two resulting graphs are isomorphic, so consider the first case. $\mathcal{B}$ has three choices: (w_4, w_6), (w_4, w_7), or (w_6, w_7).

  - If $\mathcal{B}$ chooses (w_4, w_6) then $\mathcal{A}$ chooses (w_5, w_7).
  - If $\mathcal{B}$ chooses (w_4, w_7) then $\mathcal{A}$ chooses (w_5, w_6).
  - If $\mathcal{B}$ chooses (w_6, w_7) then $\mathcal{A}$ chooses (w_4, w_6).

In each case, the resulting graph is in $\mathcal{L}_1(8m+1)(b)$ and gameplay is identical, so without loss of generality we may consider the first case. $\mathcal{B}$ again has three choices: (w_6, w_7), (w_6, w_8), or (w_7, w_8).

- If $\mathcal{B}$ chooses (w_6, w_7) then $\mathcal{A}$ chooses (w_7, w_8). Now u_1, w_1, w_2, $\ldots$, w_7 are out of play, $z(G_{n,k+12}) = z(G_{n,k}) - 8 = 8m$, $P(G_{n,k+12}) = \{w_8\}$, and $t(G_{n,k+12}) = 0$. Thus $G_{n,k+12} \in \mathcal{L}_3(8m)(a)$ and $\mathcal{A}$ wins by induction.
- If $\mathcal{B}$ chooses one of (w_6, w_8) or (w_7, w_8), then $\mathcal{A}$ chooses the other edge. Now $u_1, w_1, w_2, \ldots, w_6$ are out of play, $z(G_{n,k+12}) = z(G_{n,k}) - 8 = 8m$, $p(G_{n,k+12}) = 0$, $T(G_{n,k+12}) = \{w_7, w_8\}$, and $(w_7, w_8) \in E_{k+12}$. Thus $G_{n,k+12} \in \mathcal{L}_2(8m)(b)$ and $\mathcal{A}$ wins by Lemma 5.4.

$\boldsymbol{G_{n,k} \in \mathcal{L}_3(z)(b)}$:

Let $T(G_{n,k}) = \{u_1, u_2, u_3\}$ with $(u_1, u_2), (u_1, u_3) \in E_k$. Note that $p(G_{n,k}) = 0$ and $(u_2, u_3) \in F(G_{n,k})$. Up to game play equivalence, $\mathcal{B}$ has two choices: (u_1, w_1) or (u_2, w_1).

- If $\mathcal{B}$ chooses (u_1, w_1) then $\mathcal{A}$ chooses (u_2, w_2).
- If $\mathcal{B}$ chooses (u_2, w_1) then $\mathcal{A}$ chooses (u_1, w_2).

The two resulting graphs are isomorphic, so consider the first case. Now $\mathcal{B}$ has five choices: (u_3, w_2), (u_3, w_3), (w_1, w_2), (w_1, w_3), or (w_2, w_3).

- If $\mathcal{B}$ chooses one of (u_3, w_2) or (w_2, w_3), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ has two choices: (w_1, w_3) or (w_1, w_4). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. $\mathcal{B}$ again has two choices: (w_3, w_5) or (w_4, w_5). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. $\mathcal{B}$ must choose (w_4, w_6). $\mathcal{A}$ chooses (w_5, w_7). $\mathcal{B}$ has two choices: (w_6, w_7) or (w_6, w_8). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Now $u_1, u_2, U_3, w_1, w_2, \ldots, w_6$ are out of play, $z(G_{n,k+12}) = z(G_{n,k}) - 8 = 8m$, $P(G_{n,k+12}) = \{w_8\}$, $T(G_{n,k+12}) = \{w_7\}$, and $(w_7, w_8) \in F(G_{n,k+12})$. Thus $G_{n,k+12} \in \mathcal{L}_2(8m)(a)$ and $\mathcal{A}$ wins by Lemma 5.4.
- If $\mathcal{B}$ chooses one of (u_3, w_3) or (w_1, w_2), then $\mathcal{A}$ chooses the other edge.

  If $\mathcal{B}$ chooses (w_1, w_3), then $\mathcal{A}$ chooses (u_3, w_3).

  In each case, the resulting graph satisfies:

  1. $(z(G), p(G), t(G)) = (8m + 4, 1, 2)$,
  2. if $P(G) = \{x_1\}$ and $T(G) = \{x_2, x_3\}$, then $(x_2, x_3) \in E(G)$, and
  3. all other pairs are in $(E(G) \cup F(G))^c$.

  Gameplay is identical in each case, so we may consider the first case. $\mathcal{B}$ has three choices: (w_1, w_3), (w_1, w_4), or (w_3, w_4).

  - If $\mathcal{B}$ chooses (w_1, w_3) then $\mathcal{A}$ chooses (w_2, w_4).

– If $\mathcal{B}$ chooses (w_1, w_4) then $\mathcal{A}$ chooses (w_2, w_3).

– If $\mathcal{B}$ chooses (w_3, w_4) then $\mathcal{A}$ chooses (w_1, w_3).

In each case, the resulting graph is in $\mathcal{L}_1(8m + 4)(b)$ and gameplay is identical, so without loss of generality we may consider the first case. $\mathcal{B}$ again has three choices: (w_3, w_4), (w_3, w_5), or (w_4, w_5).

– If $\mathcal{B}$ chooses (w_3, w_4) then $\mathcal{A}$ chooses (w_4, w_5). $\mathcal{B}$ must choose (w_5, w_6). $\mathcal{A}$ chooses (w_5, w_7). $\mathcal{B}$ must choose (w_6, w_8). $\mathcal{A}$ chooses (w_7, w_8). Now $u_1, u_2, U_3, w_1, w_2, \ldots, w_5$ are out of play, $z(G_{n,k+12}) = z(G_{n,k}) - 8 = 8m$, $p(G_{n,k+12}) = 0$, $T(G_{n,k+12}) = \{w_6, w_7, w_8\}$, and (w_6, w_8), $(w_7, w_8) \in E_{k+12}$. Thus $G_{n,k+12} \in \mathcal{L}_3(8m)(b)$ and $\mathcal{A}$ wins by induction.

– If $\mathcal{B}$ chooses one of (w_3, w_5) or (w_4, w_5), then $\mathcal{A}$ chooses the other edge. $\mathcal{B}$ must choose (w_4, w_6). $\mathcal{A}$ chooses (w_5, w_7). $\mathcal{B}$ has two choices: (w_6, w_7) or (w_6, w_8). $\mathcal{A}$ chooses whichever edge $\mathcal{B}$ did not. Now $u_1, u_2, U_3, w_1, w_2, \ldots, w_6$ are out of play, $z(G_{n,k+12}) = z(G_{n,k}) - 8 = 8m$, $P(G_{n,k+12}) = \{w_8\}$, $T(G_{n,k+12}) = \{w_7\}$, and $(w_7, w_8) \in F(G_{n,k+12})$. Thus $G_{n,k+12} \in \mathcal{L}_2(8m)(a)$ and $\mathcal{A}$ wins by Lemma 5.4.

□

Theorem 5.6. *For $n \geq 12$, $f_{\{B3,T,C\}}(n) = \mathcal{B} \iff n \equiv 1, 2 \pmod 4$.*

Proof. For small values of n, an exhaustive case analysis can be carried out by hand calculation.

n	3	4	5	6	7	8	9	10	11
$f_{\{B3,T,C\}}(n)$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{B}$

For larger values of n, we will prove a statement that is equivalent to the theorem: For n ≥ 12, $f_{\{B3,T,C\}}(n) = \mathcal{B} \iff n \equiv 1, 2, 5, 6 \pmod 8$.

$\boldsymbol{n \equiv 0 \pmod 8}$:

Let $n = 8m$ for some $m \geq 2$. By Lemma 5.1, $\mathcal{A}$ can choose edges $e_1, e_3, \ldots, e_{21}$ so that $G_{n,21} \in \mathcal{L}_1(n - 15)$. Then by Lemma 5.3, $\mathcal{A}$ can choose the edge e_{23} so that $G_{n,23} \in (\mathcal{L}_2(n - 16) \cup \mathcal{L}_3(n - 16))$. Since $n - 16 \equiv 0 \pmod 8$, by Lemmas 5.4 and 5.5, $\mathcal{A}$ wins.

$\boldsymbol{n \equiv 1 \pmod 8}$:

Let $n = 8m + 1$ for some $m \geq 2$. By Lemma 5.2, $\mathcal{B}$ can choose edges $e_2, e_4, \ldots, e_{18}$ so that $G_{n,18} \in \mathcal{L}_1(n - 13)$. Then by Lemma 5.3, $\mathcal{B}$ can choose the edges e_{20}, e_{22}, e_{24} so that $G_{n,24} \in (\mathcal{L}_2(n - 17) \cup \mathcal{L}_3(n - 17))$. Since $n - 17 \equiv 0 \pmod 8$, by Lemmas 5.4 and 5.5, $\mathcal{B}$ wins.

$\boldsymbol{n \equiv 2 \pmod 8}$:

Let $n = 8m + 2$ for some $m \geq 1$. By Lemma 5.2, $\mathcal{B}$ can choose edges $e_2, e_4, \ldots, e_{12}$ so that $G_{n,9} \in \mathcal{L}_1(n - 9)$. Then by Lemma 5.3, $\mathcal{B}$ can choose the edge e_{14} so that

$G_{n,14} \in (\mathcal{L}_2(n-10) \cup \mathcal{L}_3(n-10))$. Since $n-10 \equiv 0 \pmod 8$, by Lemmas 5.4 and 5.5, $\mathcal{B}$ wins.

$\boldsymbol{n \equiv 3 \pmod 8}$:

Let $n = 8m+3$ for some $m \geq 2$. By Lemma 5.1, $\mathcal{A}$ can choose edges $e_1, e_3, \ldots, e_{21}$ so that $G_{n,21} \in \mathcal{L}_1(n-15)$. Then by Lemma 5.3, $\mathcal{A}$ can choose the edges e_{23}, e_{25}, e_{27} so that $G_{n,27} \in (\mathcal{L}_2(n-19) \cup \mathcal{L}_3(n-19))$. Since $n-19 \equiv 0 \pmod 8$, by Lemmas 5.4 and 5.5, $\mathcal{A}$ wins.

$\boldsymbol{n \equiv 4 \pmod 8}$:

Let $n = 8m+4$ for some $m \geq 1$. By Lemma 5.1, $\mathcal{A}$ can choose edges $e_1, e_3, \ldots, e_{15}$ so that $G_{n,15} \in \mathcal{L}_1(n-11)$. Then by Lemma 5.3, $\mathcal{A}$ can choose the edge e_{17} so that $G_{n,17} \in (\mathcal{L}_2(n-12) \cup \mathcal{L}_3(n-12))$. Since $n-12 \equiv 0 \pmod 8$, by Lemmas 5.4 and 5.5, $\mathcal{A}$ wins.

$\boldsymbol{n \equiv 5 \pmod 8}$:

Let $n = 8m+5$ for some $m \geq 1$. By Lemma 5.2, $\mathcal{B}$ can choose edges $e_2, e_4, \ldots, e_{12}$ so that $G_{n,12} \in \mathcal{L}_1(n-9)$. Then by Lemma 5.3, $\mathcal{B}$ can choose the edges e_{14}, e_{16}, e_{18} so that $G_{n,18} \in (\mathcal{L}_2(n-13) \cup \mathcal{L}_3(n-13))$. Since $n-13 \equiv 0 \pmod 8$, by Lemmas 5.4 and 5.5, $\mathcal{B}$ wins.

$\boldsymbol{n \equiv 6 \pmod 8}$:

Let $n = 8m+6$ for some $m \geq 1$. By Lemma 5.2, $\mathcal{B}$ can choose edges $e_2, e_4, \ldots, e_{18}$ so that $G_{n,18} \in \mathcal{L}_1(n-13)$. Then by Lemma 5.3, $\mathcal{B}$ can choose the edge e_{20} so that $G_{n,20} \in (\mathcal{L}_2(n-14) \cup \mathcal{L}_3(n-14))$. Since $n-14 \equiv 0 \pmod 8$, by Lemmas 5.4 and 5.5, $\mathcal{B}$ wins.

$\boldsymbol{n \equiv 7 \pmod 8}$:

Let $n = 8m+7$ for some $m \geq 1$. By Lemma 5.1, $\mathcal{A}$ can choose edges $e_1, e_3, \ldots, e_{15}$ so that $G_{n,15} \in \mathcal{L}_1(n-11)$. Then by Lemma 5.3, $\mathcal{A}$ can choose the edges e_{17}, e_{19}, e_{21} so that $G_{n,21} \in (\mathcal{L}_2(n-15) \cup \mathcal{L}_3(n-15))$. Since $n-15 \equiv 0 \pmod 8$, by Lemmas 5.4 and 5.5, $\mathcal{A}$ wins. □

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