Pentavalent symmetric graphs of order $12p$*

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Abstract

A graph is said to be *symmetric* if its automorphism group acts transitively on its arcs. In this paper, a complete classification of connected pentavalent symmetric graphs of order $12p$ is given for each prime p. As a result, a connected pentavalent symmetric graph of order $12p$ exists if and only if $p = 2$, 3, 5 or 11, and up to isomorphism, there are only nine such graphs: one for each $p = 2$, 3 and 5, and six for $p = 11$.

Keywords: symmetric graph; s-arc-transitive graph; s-transitive graph.

1 Introduction

Throughout this paper graphs are assumed to be finite and, unless stated otherwise, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [31, 35] or [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. For any $u, v \in V(X)$, denote by $\{u, v\}$ the edge incident to u and v in X, and by $X_1(v)$ the *neighborhood* of v.

A graph X is said to be *G-vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1}

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is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G,s)-*arc-transitive* or (G,s)-*regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G,s)-arc-transitive graph is said to be (G,s)-*transitive* if it is not $(G,s+1)$-arc-transitive. In particular, a $(G,1)$-arc-transitive graph is called G-*symmetric*. A graph X is simply called s-*arc-transitive*, s-*regular* or s-*transitive* if it is $(\mathrm{Aut}(X),s)$-arc-transitive, $(\mathrm{Aut}(X),s)$-regular or $(\mathrm{Aut}(X),s)$-transitive, respectively.

The problem of classifying symmetric graphs has received considerable attention over forty years, beginning with a classification of symmetric graphs of prime order [4]. Following this, by using deep group theory, all symmetric graphs of order $2p, 3p$ or qp were classified in [5, 34, 29, 30], where p, q are distinct primes. Recently, Li [22] classified vertex-primitive and vertex-biprimitive s-transitive graphs for $s \geq 4$, and Fang et al. [8] classified vertex-primitive 2-regular graphs. For more results on symmetric graphs with general valencies, see, for example, [10, 20, 21, 22]. Despite all of these efforts, however, further classifications of symmetric graphs with general valencies seem to be very difficult. For example, the classification of symmetric graphs of order $4p$ for p a prime has been considered for many years by several authors, but it still has not been achieved.

Symmetric graphs with certain valency have also been extensively studied in literature. For example, Conder and Dobcsányi [6] exhausted all cubic symmetric graphs on up to 768 vertices. Let p and q be primes. By analyzing automorphism groups of graphs, a classification of cubic symmetric graphs of order $2p^2$ was given in [11], and together with covering techniques, cubic symmetric graphs of order np or np^2 with $4 \leq n \leq 10$ were classified in [12, 13, 14, 15]. Recently, Oh [27, 28] classified cubic symmetric graphs of order $14p$ or $16p$. The classification of tetravalent s-transitive Cayley graphs on abelian groups were given in [38]. Zhou and Feng [42] gave a classification of tetravalent 1-regular graphs of order $2pq$, and they also classified tetravalent s-transitive graphs of order $4p$ or $2p^2$ in [39, 40]. For the pentavalent symmetric graphs, Li and Feng [23] classified pentavalent 1-regular graphs of square free order, and Hua and Feng [17, 18] classified pentavalent symmetric graphs of order $2pq$ or $8p$. In this paper, we classify pentavalent symmetric graphs of order $12p$ for each prime p.

2 Preliminaries

In this section, we introduce some notational conventions and preliminary results. We denote by $\mathbb{Z}_n$, F_n, D_{2n}, A_n and S_n the cyclic group of order n, the Frobenius group of order n, the dihedral group of order $2n$, the alternating group and the symmetric group of degree n, respectively. Denote by K_n the complete graph of order n, and by $K_{n,n}$ the complete bipartite graph of order $2n$.

For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G.

Proposition 2.1 ([19, Chapter I, Theorem 4.5]) *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ of H.*

The following proposition is due to Burnside.

Proposition 2.2 ([31, Theorem 8.5.3]) *Let p and q be primes, and let m and n be non-negative integers. Then every group of order $p^m q^n$ is solvable.*

From [17, Proposition 2.3] we obtain the following proposition.

Proposition 2.3 *Let p be a prime, and let G be a non-abelian simple group of order $2^i{\cdot}3^j{\cdot}5{\cdot}p$ with $1 \leq i \leq 19$ and $1 \leq j \leq 3$. Then G is one of the groups listed in Table 1.*

Table 1: Non-abelian simple $\{2, 3, 5, p\}$-groups

3-prime factor		4-prime factor	
G	Order	G	Order
A_5	$2^2 \cdot 3 \cdot 5$	A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$
A_6	$2^3 \cdot 3^2 \cdot 5$	A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
$PSL(2, 7)$	$2^3 \cdot 3 \cdot 7$	$PSL(2, 11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$
$PSL(2, 2^3)$	$2^3 \cdot 3^2 \cdot 7$	$PSL(2, 2^4)$	$2^4 \cdot 3 \cdot 5 \cdot 17$
$PSL(2, 17)$	$2^4 \cdot 3^2 \cdot 17$	$PSL(2, 19)$	$2^2 \cdot 3^2 \cdot 5 \cdot 19$
$PSL(3, 3)$	$2^4 \cdot 3^3 \cdot 13$	$PSL(2, 31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$
$PSU(3, 3)$	$2^5 \cdot 3^3 \cdot 7$	$PSL(3, 4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
$PSU(4, 2)$	$2^6 \cdot 3^4 \cdot 5$	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$
		M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$

Let X be a connected G-vertex-transitive graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [24, Theorem 9], we have the following:

Proposition 2.4 *Let X be a connected pentavalent (G, s)-arc-transitive graph for some $s \geq 1$, and let N be a normal subgroup of G with more than two orbits on $V(X)$. Then X_N is also a pentavalent symmetric graph and N is the kernel of the action of G on $V(X_N)$. Moreover, N is semiregular on $V(X)$ and G/N is an s-arc-transitive subgroup of* $\mathrm{Aut}(X_N)$.

The next proposition determines the solvable vertex stabilizers of pentavalent symmetric graphs.

Proposition 2.5 ([43, Theorem 4.1]) *Let X be a connected pentavalent (G, s)-transitive graph for some $s \geq 1$. If the vertex stabilizer G_v of a vertex $v \in V(X)$ in G is solvable then $s \leq 3$. Furthermore,*

(1) *If $s = 1$ then G_v is isomorphic to $\mathbb{Z}_5$, D_{10} or D_{20};*

(2) *If $s = 2$ then G_v is isomorphic to F_{20} or $F_{20} \times \mathbb{Z}_2$;*

(3) *If $s = 3$ then G_v is isomorphic to $F_{20} \times \mathbb{Z}_4$.*

Let X be a graph. The *standard double cover* $X^{(2)}$ of X is defined as the graph with vertex set $\{u_1, u_2 \mid u \in V(X)\}$ and edge set $\{\{u_1, v_2\}, \{u_2, v_1\} \mid \{u, v\} \in E(X)\}$. It is easy to see that $\mathrm{Aut}(X) \times \mathbb{Z}_2$ is a group of automorphisms of $X^{(2)}$. Let X be connected. Then $X^{(2)}$ is connected if and only X is not bipartite.

Proposition 2.6 *Let X be a connected G-symmetric bipartite graph of valency at least 2, and let G contain a normal subgroup N of order 2. If the involution in N interchanges the two bipartite sets of X, then X is a standard double cover of the quotient graph X_N.*

Proof. Write $N = \langle z \rangle$. Let U be a set with cardinality $|V(X)|/2$, and let $U_1 = \{u_1 \mid u \in U\}$ and $U_2 = \{u_2 \mid u \in U\}$ be the bipartite sets of X. Then z interchanges u_1 and u_2 for any $u \in U$. Since $N \trianglelefteq G$, $\{u_1, u_2\}$ cannot be an edge of X because X has valency at least 2. Take an arbitrary edge $\{u_1, v_2\}$ in X. Since $u_1^z = u_2$ and $v_2^z = v_1$, $\{u_2, v_1\}$ is also an edge of X. Clearly, the quotient graph X_N is isomorphic to the graph with vertex set U and edge set $\{\{u, v\} \mid \{u_1, v_2\} \in E(X), \{u_2, v_1\} \in E(X)\}$. Thus, $X \cong X_N^{(2)}$. □

3 Constructions of pentavalent symmetric graphs

In this section, we shall construct some pentavalent symmetric graphs. To do this, we need to introduce the so called coset graph (see [26, 33]) constructed from a finite group G relative to a subgroup H of G and a union D of some double cosets of H in G such that $D^{-1} = D$. The *coset graph* $\mathrm{Cos}(G, H, D)$ of G with respect to H and D is defined to have vertex set $[G : H]$, the set of right cosets of H in G, and edge set $\{\{Hg, Hdg\} \mid g \in G, d \in D\}$. The graph $\mathrm{Cos}(G, H, D)$ has valency $|D|/|H|$ and is connected if and only if D generates the group G. The action of G on $V(\mathrm{Cos}(G, H, D))$ by right multiplication induces a vertex-transitive automorphism group, which is arc-transitive if and only if D is a single double coset. Moreover, this action is faithful if and only if $H_G = 1$, where H_G is the largest normal subgroup of G in H. Clearly, $\mathrm{Cos}(G, H, D) \cong \mathrm{Cos}(G, H^\alpha, D^\alpha)$ for every $\alpha \in \mathrm{Aut}(G)$.

Conversely, let X be a A-vertex-transitive graph with $A \leq \mathrm{Aut}(X)$. By [33], the graph X is isomorphic to a coset graph $\mathrm{Cos}(A, H, D)$, where $H = A_u$ is the vertex stabilizer of $u \in V(X)$ in A and D consists of all elements of A which map u to one of its neighbors. It is easy to show that $H_A = 1$ and that D is a union of some double cosets of H in A satisfying $D = D^{-1}$. Assume that A is arc-transitive and that $g \in A$ interchanges u and one of its neighbors. Then $g^2 \in H$ and $D = HgH$. Furthermore, g can be chosen as a 2-element in A, and the valency of X is $|D|/|H| = |H : H \cap H^g|$. For more details regarding coset graphs, see, for example, [9, 24, 33].

Now we introduce three pentavalent symmetric graphs of order $6p$ for some prime p which were constructed in [17, Section 3].

Example 3.1 (1) *Let $T = \mathrm{PSL}(3, 4)$. Then T has two conjugacy classes of maximal subgroups isomorphic to $\mathbb{Z}_2^4 \rtimes \mathrm{A}_5$, which are fused by an automorphism g of T of order 2.*

Let $H = \mathbb{Z}_2^4 \rtimes \mathrm{A}_5$ *be a subgroup of* T*. Set* $G = \langle T, g\rangle$*, and Denote* $\mathcal{G}_{42} = \mathrm{Cos}(G, H, HgH)$*, where* 42 *means the order of* $\mathcal{G}_{42}$*. In particular,* $\mathcal{G}_{42}$ *is bipartite.*

(2) *Let* $G = \mathrm{PSL}(2, 11)$*. Then* G *has a subgroup* $H \cong D_{10}$ *and an involution* g *such that* $|HgH|/|H| = 5$ *and* $\langle H, g\rangle = G$*. Denote* $\mathcal{G}_{66} = \mathrm{Cos}(G, H, HgH)$*. In particular,* $\mathcal{G}_{66}$ *is not bipartite.*

(3) *Let* $G = \mathrm{PGL}(2, 19)$*. Then* G *has a subgroup* $H \cong \mathrm{A}_5$ *and an involution* g *such that* $|HgH|/|H| = 5$ *and* $\langle H, g\rangle = G$*. Denote* $\mathcal{G}_{114} = \mathrm{Cos}(G, H, HgH)$*. In particular,* $\mathcal{G}_{114}$ *is bipartite.*

Denote by $\mathbf{I}_{12}$ the Icosahedron graph, and by $K_{6,6} - 6K_2$ the complete bipartite graph of order 12 minus a one-factor. From [25], it is easy to see that there exists no connected pentavalent symmetric graph of order 18. Together with [17, Theorems 4.1 and 4.2], we have the following proposition.

Proposition 3.2 *Let* X *be a connected pentavalent symmetric graph of order* $6p$ *for a prime* p*. Then one of the following occurs:*

(1) $X \cong \mathbf{I}_{12}$ *and* $\mathrm{Aut}(X) \cong \mathrm{A}_5 \times \mathbb{Z}_2$ *with* $p = 2$*;*

(2) $X \cong K_{6,6} - 6K_2$ *and* $\mathrm{Aut}(X) \cong \mathrm{S}_6 \times \mathbb{Z}_2$ *with* $p = 2$*;*

(3) $X \cong \mathcal{G}_{42}$ *and* $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathrm{PSL}(3, 4))$ *with* $p = 7$*;*

(4) $X \cong \mathcal{G}_{66}$ *and* $\mathrm{Aut}(X) \cong \mathrm{PGL}(2, 11)$ *with* $p = 11$*;*

(5) $X \cong \mathcal{G}_{114}$ *and* $\mathrm{Aut}(X) \cong \mathrm{PGL}(2, 19)$ *with* $p = 19$.

In what follows, we shall construct several pentavalent symmetric graphs of order $12p$ for some prime p.

Construction I: Let $G = \mathrm{A}_6$. Take a Sylow 5-subgroup, say P, of G, and set $H = N_G(P)$. From Atlas [7, pp.4] it is easily known that $H \cong D_{10}$. Note that all involutions in G are conjugate each other. For any involution $x \in H$, we have $C_G(x) \cong D_8$. Take an element g of order 4 in $C_G(x)$. Denote $\mathcal{G}_{36} = \mathrm{Cos}(G, H, HgH)$.

Lemma 3.3 *The graph* $\mathcal{G}_{36}$ *is a connected* 2*-transitive graph of order* 36 *and* $\mathrm{Aut}(\mathcal{G}_{36}) \cong \mathrm{Aut}(\mathrm{A}_6)$*. Furthermore, every connected pentavalent symmetric graph of order* 36 *admitting* A_6 *as an arc-transitive automorphism group is isomorphic to* $\mathcal{G}_{36}$*.*

Proof. By Construction I, $g^2 = x \in H$. So, $|HgH|/|H| = 5$. From Atlas [7, pp.4] we may easily see that $\langle H, g\rangle = G$. This implies that $\mathrm{Cos}(G, H, HgH)$ is a connected pentavalent symmetric graph of order 36. By Magma [3], $\mathrm{Aut}(\mathcal{G}_{36}) \cong \mathrm{Aut}(\mathrm{A}_6)$ and $\mathcal{G}_{36}$ is 2-transitive.

Let X be a connected pentavalent symmetric graph of order 36 admitting $G = \mathrm{A}_6$ as an arc-transitive group of automorphisms. For any $v \in V(X)$, the vertex stabilizer G_v has order 10. From Proposition 2.5 it follows that $G_v \cong D_{10}$. Since all Sylow 5-subgroups of G are conjugate, their normalizers are also conjugate. This implies that G_v is conjugate to H. Without loss of generality, assume that $G_v = H$. Then $X \cong \mathrm{Cos}(G, H, HfH)$, where f is a 2-element in G such that $f^2 \in H$, $|H|/|H^f \cap H| = 5$ and $\langle f, H\rangle = G$. So, $H^f \cap H = \langle y\rangle \cong \mathbb{Z}_2$ and $f \in C_G(y)$. Since $H \cong D_{10}$, we have $y^z = x$ for some $z \in H$. Set $d = f^z$. Then $\mathrm{Cos}(G, H, HfH) = \mathrm{Cos}(G, H, HdH)$. Furthermore, $d^2 = (f^2)^z \in H$, $|H|/|H^d \cap H| = 5$, $\langle d, H\rangle = G$ and $d \in C_G(x)$. By Magma [3], if d is an involution, then $\langle d, H\rangle < G$, which is impossible. Thus, $d = g$ or g^{-1}. Again by Magma [3], $\mathrm{Cos}(G, H, HgH) \cong \mathrm{Cos}(G, H, Hg^{-1}H)$. Consequently, $X \cong \mathrm{Cos}(G, H, HgH) = \mathcal{G}_{36}$. □

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Given $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$ is a permutation group isomorphic to G, which acts regularly on the vertex set $V(\mathrm{Cay}(G, S))$. It is proved that a graph X is isomorphic to a Cayley graph on a group G if and only if $\mathrm{Aut}(X)$ has a regular subgroup isomorphic to G (see [33, Lemma 4]). A Cayley graph $\mathrm{Cay}(G, S)$ is called *normal* if $R(G)$ is a normal subgroup in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. The following pentavalent symmetric graph was first constructed in [41, Corollary 5.2].

Construction II: Let $\mathcal{G}_{60} = \mathrm{Cay}(\mathrm{A}_5, T)$ with $T = \{(1\ 4)(2\ 5), (1\ 3)(2\ 5), (1\ 3)(2\ 4), (2\ 4)(3\ 5), (1\ 4)(3\ 5)\}$. Then every connected pentavalent symmetric Cayley graph on A_5 is isomorphic to $\mathcal{G}_{60}$. Furthermore, $\mathcal{G}_{60}$ is a normal Cayley graph and $\mathrm{Aut}(\mathcal{G}_{60}) \cong \mathrm{A}_5 \rtimes D_{10}$.

Construction III: Let G be a primitive subgroup of the symmetric group S_{12} of degree 12, which is isomorphic to $\mathrm{PGL}(2, 11)$. Let T be the socle of G. Then $T \cong \mathrm{PSL}(2, 11)$. Choose the following elements in T:

$$a = (3\ 11\ 9\ 7\ 5)(4\ 12\ 10\ 8\ 6), \qquad b_1 = (1\ 12)(2\ 7)(3\ 10)(4\ 8)(5\ 11)(6\ 9),$$
$$b_2 = (1\ 12)(2\ 6)(3\ 5)(4\ 9)(7\ 10)(8\ 11), \quad b_3 = (1\ 12)(2\ 4)(3\ 8)(5\ 7)(6\ 10)(9\ 11),$$

and choose the following elements in G:

$$c = (1\ 2)(3\ 4)(5\ 12)(6\ 11)(7\ 10)(8\ 9),$$
$$d_1 = (1\ 12)(2\ 5)(3\ 6)(4\ 11)(8\ 9), \qquad d_2 = (1\ 6\ 2\ 11)(3\ 12\ 4\ 5)(7\ 8\ 10\ 9).$$

Then $T = \langle a, b_i\rangle$ and $G = \langle a, c, d_j\rangle$ for each $1 \le i \le 3$ and $1 \le j \le 2$. Set $H_1 = \langle a\rangle$ and $H_2 = \langle a, c\rangle$. Define five coset graphs as following:

$$\mathcal{G}^1_{132} = \mathrm{Cos}(T, H_1, H_1b_1H_1), \quad \mathcal{G}^2_{132} = \mathrm{Cos}(T, H_1, H_1b_2H_1), \quad \mathcal{G}^3_{132} = \mathrm{Cos}(T, H_1, H_1b_3H_1),$$
$$\mathcal{G}^4_{132} = \mathrm{Cos}(G, H_2, H_2d_1H_2), \quad \mathcal{G}^5_{132} = \mathrm{Cos}(G, H_2, H_2d_2H_2).$$

Lemma 3.4 *The graphs* $\mathcal{G}^i_{132}$ $(1 \le i \le 5)$ *are pairwise non-isomorphic connected pentavalent symmetric graphs of order* 132 *with* $\mathrm{Aut}(\mathcal{G}^1_{132}) = \mathrm{PSL}(2, 11) \times \mathbb{Z}_2$ *and* $\mathrm{Aut}(\mathcal{G}^i_{132}) = \mathrm{PGL}(2, 11)$ *for* $2 \le i \le 5$. *Furthermore, every connected pentavalent symmetric graph of order* 132 *admitting* $\mathrm{PSL}(2, 11)$ *as an arc-transitive automorphism group is isomorphic to one of* $\mathcal{G}^i_{132}$ $(1 \le i \le 3)$, *and every connected pentavalent symmetric graph of order* 132 *admitting* $\mathrm{PGL}(2, 11)$ *as an arc-transitive automorphism group is isomorphic to one of* $\mathcal{G}^i_{132}$ $(2 \le i \le 5)$.

Proof. By Magma [3], $\mathcal{G}_{132}^i$ $(1 \le i \le 5)$ are pairwise non-isomorphic connected pentavalent symmetric graphs of order 132, and $\mathrm{Aut}(\mathcal{G}_{132}^1) \cong \mathrm{PSL}(2,11) \times \mathbb{Z}_2$ and $\mathrm{Aut}(\mathcal{G}_{132}^i) \cong \mathrm{PGL}(2,11)$ for $2 \le i \le 5$.

Let X be a connected pentavalent symmetric graph of order 132 admitting T as an arc-transitive group of automorphisms. Then $T_v \cong \mathbb{Z}_5$, and $X \cong \mathrm{Cos}(T, T_v, T_v g T_v)$, where g is a 2-element in T such that $g^2 \in T_v$. Furthermore, $|T_v g T_v|/|T| = 5$ and $\langle T_v, g\rangle = T$. Since $T_v \cong \mathbb{Z}_5$, we have $g^2 = 1$. By Magma [3], g has 30 choices and let S be the set of all such involutions. By Atlas [7, pp.7], $\mathrm{Aut}(T, T_v) \cong D_{20}$ and by Magma [3], the action of $\mathrm{Aut}(T, T_v)$ on S has three orbits with b_1, b_2 and b_3 as their representatives. It follows that X is isomorphic to $\mathcal{G}_{132}^1$, $\mathcal{G}_{132}^2$ or $\mathcal{G}_{132}^3$

Now let X be a connected pentavalent symmetric graph of order 132 admitting G as an arc-transitive automorphism group. Then $G_v \cong D_{10}$ for any $v \in V(X)$. Since T is normal in G, by Proposition 2.4, T has at most two orbits on $V(X)$. If T is transitive on $V(X)$, then T is arc-transitive on X. From the argument in the above paragraph it is known that X is isomorphic to $\mathcal{G}_{132}^2$ or $\mathcal{G}_{132}^3$. In what follows, assume that T has two orbits on $V(X)$. Then X is bipartite and $G_v = T_v \cong D_{10}$. Since G has one conjugacy class of subgroups isomorphic to D_{10}, we may assume that $G_v = H_2$. So, $X \cong \mathrm{Cos}(G, H_2, H_2 g H_2)$, where $g \in G$ is a 2-element such that $g^2 \in H_2$, $|H_2 g H_2|/|H_2| = 5$ and $\langle H_2, g\rangle = G$. Set $H_2^g \cap H_2 = \langle x\rangle$. Since $H_2 \cong D_{10}$, $x = c^y$ for some $y \in H_2$. Set $d = g^y$. Then $d \in C_G(c) \cong D_{24}$ and $\mathrm{Cos}(G, H_2, H_2 g H_2) = \mathrm{Cos}(G, H_2, H_2 d H_2)$. Since $\langle H_2, d\rangle = G$, by Magma [3], g has six choices, and the resulting coset graphs corresponding to these six 2-elements form two non-isomorphic graphs. It follows that $X \cong \mathcal{G}_{132}^4$ or $\mathcal{G}_{132}^5$. □

4 Classification

In this section, we classify pentavalent symmetric graphs of order $12p$ for p a prime. Let G be a simple group and Z an abelian group. We call an extension $E = Z.G$ of Z by G a *central extension* of G if $Z \le Z(E)$. If E is perfect, that is, the derived group $E' = E$, we call E a *covering group* of G. Schur [19] proved that for every simple group G there is a unique maximal covering group M such that every covering group of G is a factor group of M. This group M is called the full covering group of G, and the center of M is called the *Schur multiplier* of G, denoted by $\mathrm{Mult}(G)$.

Theorem 4.1 *Let p be a prime. A connected pentavalent graph of order $12p$ is symmetric if and only if it is isomorphic to one of the graphs in Table 2. Furthermore, all graphs in Table 2 are pairwise non-isomorphic.*

Proof. Note that the Icosahedron graph $\mathbf{I}_{12}$ and the graph $\mathcal{G}_{66}$ (Example 3.1) are not bipartite. Their the standard double covers $\mathbf{I}_{12}^{(2)}$ and $\mathcal{G}_{66}^{(2)}$ are connected pentavalent symmetric graphs. By Magma [3], $\mathrm{Aut}(\mathbf{I}_{12}^{(2)}) \cong \mathrm{A}_5 \times \mathbb{Z}_2^2$ and $\mathrm{Aut}(\mathcal{G}_{66}^{(2)}) \cong \mathrm{PGL}(2,11) \times \mathbb{Z}_2$. Together with Constructions I, II, III and Lemmas 3.3-3.4, all graphs in Table 2 are pairwise non-isomorphic connected pentavalent symmetric graphs. To complete the proof, it suffices to prove the necessity.

Table 2: Pentavalent symmetric graphs of order $12p$

X	s-transitive	Aut(X)	Comments
$\mathbf{I}_{12}^{(2)}$	1-transitive	$A_5 \times \mathbb{Z}_2^2$	$p = 2$
$\mathcal{G}_{36}$	2-transitive	Aut(A_6)	Construction I, $p = 3$
$\mathcal{G}_{60}$	1-transitive	$A_5 \rtimes D_{10}$	Construction II, $p = 5$
$\mathcal{G}_{66}^{(2)}$	1-transitive	$\mathrm{PGL}(2,11) \times \mathbb{Z}_2$	$p = 11$
$\mathcal{G}_{132}^1$	1-transitive	$\mathrm{PSL}(2,11) \times \mathbb{Z}_2$	Construction III, $p = 11$
$\mathcal{G}_{132}^2$	1-transitive	$\mathrm{PGL}(2,11)$	Construction III, $p = 11$
$\mathcal{G}_{132}^3$	1-transitive	$\mathrm{PGL}(2,11)$	Construction III, $p = 11$
$\mathcal{G}_{132}^4$	1-transitive	$\mathrm{PGL}(2,11)$	Construction III, $p = 11$
$\mathcal{G}_{132}^5$	1-transitive	$\mathrm{PGL}(2,11)$	Construction III, $p = 11$

By [32, Section 5.1], there is a unique connected pentavalent symmetric graph of order 24, that is, the standard double cover $\mathbf{I}_{12}^{(2)}$ of the Icosahedron graph $\mathbf{I}_{12}$. In what follows we assume that $p \geq 3$. Let X be a connected pentavalent symmetric graph of order $12p$. Set $A = \mathrm{Aut}(X)$. Take $v \in V(X)$. By Weiss [36, 37], $|A_v| \mid 2^{17}\cdot 3^2\cdot 5$ and hence $|A| = |V(X)||A_v| = 2^i\cdot 3^j\cdot 5\cdot p$ with $2 \leq i \leq 19$ and $1 \leq j \leq 3$. We divide the proof into the following two cases.

Case 1 A has a solvable minimal normal subgroup.

Let N be a solvable minimal normal subgroup of A. Then N is an elementary abelian q-group with $q = 2, 3$ or p. Since X has order $12p$, by Proposition 2.4, N is semiregular on $V(X)$ and the quotient graph X_N of X relative to N is a pentavalent symmetric graph with A/N as an arc-transitive automorphism group. Clearly, the order of X_N is even and at least 6. This implies that N is isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{Z}_p$.

Assume that $N \cong \mathbb{Z}_3$ or $\mathbb{Z}_p$. Then $|X_N| = 4p$ or 12. It follows from [17, Theorem 4.1] that X_N is isomorphic to the Icosahedron graph $\mathbf{I}_{12}$ or the complete bipartite graph of order 12 minus a one-factor $K_{6,6} - 6K_2$.

Let $X_N \cong \mathbf{I}_{12}$. Then $A/N \leq \mathrm{Aut}(\mathbf{I}_{12}) \cong A_5 \times \mathbb{Z}_2$, which has a unique subgroup of order 60. Since A/N is arc-transitive on X_N, we have $60 \mid |A/N|$. Thus, A/N contains an arc-transitive subgroup $H/N \cong A_5$. Set $C = C_H(N)$, the centralizer of N in H. Then $N \leq C$, and by Proposition 2.1, $H/C \lesssim \mathrm{Aut}(N) \cong \mathbb{Z}_{p-1}$. As $H/N \cong A_5$, we have $C = H$, and hence N is in the center $Z(H)$ of H. Since H is non-solvable, its derived subgroup H' is also non-solvable. Then $1 < H'N/N \trianglelefteq H/N$, and hence $H'N/N = H/N$. If $N \leq H'$ then $H' = H$, and hence H is a covering group of A_5. However, by [19, Chapter V,Theorem 25.7], the Schur multiplier of A_5 is $\mathbb{Z}_2$, a contradiction. Thus, $N \nleq H'$. Then $N \cap H' = 1$, and hence $H = N \times H'$ with $H' \cong A_5$. If H' has more than two orbits on $V(X)$, then by Proposition 2.4, H' is semiregular on $V(X)$. This forces that $|H'| \mid 12p$ and $|H'| < 12p$, and hence H' is solvable, a contradiction. If H' has two orbits then X is bipartite with the two orbits of H' as the bipartite sets. Since $|N|$ is odd, N preserves the bipartite sets and hence H cannot be arc-transitive, a contradiction. Consequently,

H' is transitive on $V(X)$. So, $p = 5$ and H' is also regular on $V(X)$. Thus, X is an arc-transitive Cayley graph on $H' \cong \mathrm{A}_5$. By Construction II, $X \cong \mathcal{G}_{60}$.

Let $X_N \cong K_{6,6} - 6K_2$. Then $A/N \leq \mathrm{Aut}(X_N) \cong \mathrm{S}_6 \times \mathbb{Z}_2$. Since A/N is arc-transitive on X_N, we have $60 \mid |A/N|$. By Magma [3], A/N has an arc-transitive subgroup $H/N \cong \mathrm{A}_5 \times \mathbb{Z}_2$ or S_5. It follows that H/N has a normal subgroup $M/N \cong \mathrm{A}_5$. Since X_N is bipartite, M/N has two orbits on $V(X_N)$ and hence M has two orbits on $V(X)$. With a similar argument as in the above paragraph, it can be deduced that $M = M' \times N$ with $M' \cong \mathrm{A}_5$. Since M' is characteristic in M, M' is normal in H. Clearly, M' has at least two orbits on $V(X)$. If M' has more than two orbits, then by Proposition 2.4, M' is semiregular on $V(X)$, a contradiction. If M' has exactly two orbits, then $p = 5$ and $1 \neq M'_v \trianglelefteq H_v$. It follows that $5 \mid |M'_v|$, and hence $25 \mid |M'| = 60$, a contradiction.

Now assume that $N \cong \mathbb{Z}_2$. Then X_N has order $6p$. Recall that $p \geq 3$. By Proposition 3.2, X_N is isomorphic to $\mathcal{G}_{42}$, $\mathcal{G}_{66}$ or $\mathcal{G}_{114}$.

Let $X_N \cong \mathcal{G}_{42}$. Then $A/N \lesssim \mathrm{Aut}(\mathrm{PSL}(3,4))$. Since A/N is arc-transitive on X_N, we have $5{\cdot}42 \mid |A/N|$. From Atlas [7, pp.23] it can be obtained that A/N contains a normal subgroup $H/N \cong \mathrm{PSL}(3,4)$. Since $N \cong \mathbb{Z}_2$ is normal in H, we have $N \leq Z(H)$. Hence, H is isomorphic to the central extension $\mathbb{Z}_2.\mathrm{PSL}(3,4)$ of $\mathbb{Z}_2$ by $\mathrm{PSL}(3,4)$, or to $\mathrm{PSL}(3,4) \times \mathbb{Z}_2$. By Example 3.1 (1), $\mathcal{G}_{42}$ is bipartite and $\mathrm{PSL}(3,4)$ has two orbits on $V(\mathcal{G}_{42})$. This means that H has two orbits on $V(X)$ and $|H_v| = 2^6{\cdot}3{\cdot}5$. By Magma [3], $\mathbb{Z}_2.\mathrm{PSL}(3,4)$ has no subgroups of order $2^6{\cdot}3{\cdot}5$. Thus, $H \cong \mathrm{PSL}(3,4) \times \mathbb{Z}_2$. Let M be the subgroup of H isomorphic to $\mathrm{PSL}(3,4)$. Then M is characteristic in H and hence normal in A. Since H has two orbits on $V(X)$, M has at least two orbits on $V(X)$. As M is non-solvable, by Proposition 2.4, M has exactly two orbits on $V(X)$, implying that $|M_v| = 2^5{\cdot}3{\cdot}5$. However, by Magma [3], M has no subgroups of order $2^5{\cdot}3{\cdot}5$, a contradiction.

Let $X_N \cong \mathcal{G}_{66}$. Then $A/N \lesssim \mathrm{PGL}(2,11)$. Since A/N is arc-transitive on X_N, we have $5{\cdot}66 \mid |A/N|$. By Atlas [7, pp.7], A/N must contain a normal subgroup $H/N \cong \mathrm{PSL}(2,11)$. Since $N \cong \mathbb{Z}_2$ is in the center of H, we have $H = \mathrm{SL}(2,11)$ or $\mathrm{PSL}(2,11) \times \mathbb{Z}_2$. By Example 3.1 (2), H/N is arc-transitive on X_N. It follows that H is arc-transitive on X and $|H_v| = 10$. By Proposition 2.5, $H_v \cong D_{10}$. If $H \cong \mathrm{SL}(2,11)$, then H has a unique involution, say z, such that $Z(H) = \langle z \rangle$. This forces that $Z(H) \leq H_v$. However, since $Z(H)$ is characteristic in H, the normality of H in A implies that $Z(H) \trianglelefteq A$, forcing $Z(H) = 1$, a contradiction. Thus, $H \cong \mathrm{PSL}(2,11) \times \mathbb{Z}_2$. Let M be the subgroup of H isomorphic to $\mathrm{PSL}(2,11)$. Then M is normal in H. Since H is arc-transitive on X, M has at most two orbits on $V(X)$ by Proposition 2.4. If M is transitive on $V(X)$ then M is arc-transitive on X. Since $H \cong \mathrm{PSL}(2,11) \times \mathbb{Z}_2$, from Lemma 3.4 it follows that $X \cong \mathcal{G}^1_{132}$. Let M have two orbits on $V(X)$. Then X is bipartite and by Proposition 2.6, we may easily deduce that $X \cong \mathcal{G}^{(2)}_{66}$.

Let $X_N = \mathcal{G}_{114}$. Then $A/N \lesssim \mathrm{PGL}(2,19)$. From the arc-transitivity of A/N on X_N it follows that $5{\cdot}114 \mid |A/N|$. By Atlas [7, pp.11], A/N has a normal subgroup $H/N \cong \mathrm{PSL}(2,19)$. Since $N \leq Z(H)$, we have $H \cong \mathrm{SL}(2,19)$ or $\mathrm{PSL}(2,19) \times \mathbb{Z}_2$. By Example 3.1 (3), $\mathcal{G}_{114}$ is bipartite and $\mathrm{PSL}(2,19)$ has two orbits on $V(\mathcal{G}_{114})$. Thus, H has two orbits on $V(X)$, and hence $|H_v| = 60$. Noting that $\mathrm{SL}(2,19)$ has a unique involution, we have $H \cong \mathrm{PSL}(2,19) \times \mathbb{Z}_2$. Let M be the subgroup of H isomorphic to $\mathrm{PSL}(2,19)$. Then M is characteristic in H and hence normal in A. Since H has two orbits on $V(X)$, M has at least two orbits on $V(X)$. Since M is non-solvable, by Proposition 2.4, M has exactly two orbits on $V(X)$, implying $|M_v| = 30$. However, by Atlas [7, pp.11], $M \cong \mathrm{PSL}(2,19)$ has no subgroups of order 30, a contradiction.

Case 2 A has no solvable minimal normal subgroups.

For convenience, we still use N to denote a minimal normal subgroup of A. Then N is non-solvable. By Proposition 2.2, N has at least 3-prime factors and by Proposition 2.4, N has at most two orbits on $V(X)$. This implies that $|N| = 6p|N_v|$ or $12p|N_v|$.

Let $p = 3$. Then $|V(X)| = 36$ and $|A| \mid 2^{19}{\cdot}3^4{\cdot}5$. It follows that N must be a non-abelian simple group, and by Proposition 2.3, $N \cong \mathrm{A}_5$, A_6 or $\mathrm{PSU}(4,2)$. Since $18 \mid |N|$, we have $A \ncong \mathrm{A}_5$. Suppose $N = \mathrm{PSU}(4,2)$. Then $|N| = 2^6{\cdot}3^4{\cdot}5$. Since N has at most two orbits on $V(X)$, we have $|N_v| = 2^4{\cdot}3^2{\cdot}5$ or $2^5{\cdot}3^2{\cdot}5$. From Atlas [7, pp.26] it is easy to see that $N_v \cong \mathrm{S}_6$. This is clearly impossible because S_6 cannot have a permutation representation of degree 5. Thus, $N \cong \mathrm{A}_6$. If N has two orbits on $V(X)$, then N_v would be a subgroup of N of order 20, which is impossible by Atlas [7, pp.4]. Thus, N is transitive on $V(X)$. Since $5 \mid |N|$, N is arc-transitive on X. It follows from Lemma 3.3 that $X \cong \mathcal{G}_{36}$.

Let $p = 5$. Then $|V(X)| = 60$ and $|A| = 2^i{\cdot}3^j{\cdot}5^2$ with $2 \leq i \leq 19$ and $1 \leq j \leq 3$. From Proposition 2.3 it can be obtained that $N \cong \mathrm{A}_5$, $\mathrm{A}_5 \times \mathrm{A}_5$ or A_6. The normality of N in A implies that $N_v \trianglelefteq A_v$. As A_v is primitive on $X_1(v)$, either $N_v = 1$ or $5 \mid |N_v|$. Suppose $5 \mid |N_v|$. Noting that $|N| = 30|N_v|$ or $60|N_v|$, we have $N = H \times T \cong \mathrm{A}_5 \times \mathrm{A}_5$. Since $5 \mid |N_v|$, N_v is also primitive on $X_1(v)$. Let N be transitive on $V(X)$. Then the normality of H_v in N_v gives $5 \mid |H_v|$. Similarly, $5 \mid |T_v|$. As $H_v \times T_v \leq N_v$, we have $5^2 \mid |N_v|$, a contradiction. Let N have two orbits, say B_0 and B_1, on $V(X)$, and let $v \in B_0$. Then $|N_v| = 120$. By Proposition 2.5, N_v is non-solvable. Consequently, $N_v \cong \mathrm{SL}(2,5)$, S_5 or $\mathrm{A}_5 \times \mathbb{Z}_2$. By Magma [3], $\mathrm{A}_5 \times \mathrm{A}_5$ has no subgroups isomorphic to $\mathrm{SL}(2,5)$ or S_5. So, $N_v \cong \mathrm{A}_5 \times \mathbb{Z}_2$. Since $H \trianglelefteq N$, $1 < N_v \cap H \trianglelefteq N_v$. Similarly, $1 < N_v \cap T \trianglelefteq N_v$. If neither H nor T is contained in N_v, then $N_v \cap H = N_v \cap T = Z(N_v) \cong \mathbb{Z}_2$, where $Z(N_v)$ is the center of N_v. This is impossible. Thus, either H or T is contained in N_v. Without loss of generality, assume $H \leq N_v$. Since $H \trianglelefteq N$, H fixes all vertices in B_0. This forces that X is isomorphic to the complete bipartite graph $K_{5,5}$, a contradiction. Thus, $N_v = 1$. Since N has at most two orbits on $V(X)$, we must have $N \cong \mathrm{A}_5$, which is regular on $V(X)$. In this case, X is a Cayley graph on A_5, and by Construction II, $X \cong \mathcal{G}_{60}$. However, $\mathrm{Aut}(\mathcal{G}_{60}) \cong \mathrm{A}_5 \rtimes D_{10}$ has a normal subgroup isomorphic to $\mathbb{Z}_5$, contrary to our assumption.

Let $p > 5$. Then $|A| = 2^i{\cdot}3^j{\cdot}5{\cdot}p$ with $2 \leq i \leq 19$ and $1 \leq j \leq 3$, and hence N is a non-abelian simple group by Proposition 2.3. Also, from Proposition 2.4, we know that N cannot be a non-abelian simple $\{2,3,5\}$-group. Set $C = C_A(N)$. By the simplicity of N, $C \cap N = 1$. If $C \neq 1$, then C would have more than p orbits on $V(X)$ because $p \nmid |C|$. By Proposition 2.4, C is semiregular on $V(X)$, and by Proposition 2.2, C is solvable which is contrary to our assumption. Thus, $C = 1$, and hence $A \cong A/C \lesssim \mathrm{Aut}(N)$ by Proposition 2.1, that is, A is an almost simple group.

Since N is non-abelian, it is not semiregular on $V(X)$, and hence $N_v \neq 1$. Since A_v is primitive on $X_1(v)$, we have $5 \mid |N_v|$, namely, $5 \mid |N|$. Thus, N is a non-abelian simple $\{2,3,5,p\}$-group. By Table 1, N is isomorphic to one of the following groups:

$$\mathrm{A}_7, \mathrm{A}_8, \mathrm{PSL}(2,11), \mathrm{PSL}(2,16), \mathrm{PSL}(2,19), \mathrm{PSL}(2,31), \mathrm{PSL}(3,4), \mathrm{M}_{11}, \mathrm{M}_{12}.$$

Since N has at most two orbits on $V(X)$, we have $|N|/|N_v| = 12p$ or $6p$.

Suppose $N \cong \mathrm{A}_8$. Then $|N_v| = 2^4{\cdot}3{\cdot}5$ or $2^5{\cdot}3{\cdot}5$. However, by Magma [3], A_8 has no subgroups of order $2^4{\cdot}3{\cdot}5$ or $2^5{\cdot}3{\cdot}5$, a contradiction. We have a similar contradiction if $N \cong \mathrm{PSL}(2,16)$, $\mathrm{PSL}(2,19)$, $\mathrm{PSL}(2,31)$, $\mathrm{PSL}(3,4)$ or M_{12}.

Suppose $N \cong \mathrm{A}_7$. Then $|N_v| = 2{\cdot}3{\cdot}5$ or $2^2{\cdot}3{\cdot}5$. By Atlas [7, pp.10], A_7 has no subgroups of order 30. Thus, $|N_v| = 2^2{\cdot}3{\cdot}5 = 60$ and $N_v \cong \mathrm{A}_5$. It follows that N has two orbits on $V(X)$ and hence X is bipartite. Since $A \leq \mathrm{Aut}(N)$, we have $A \cong \mathrm{S}_7$. Clearly, $p = 7$ and $|V(X)| = 12 \cdot 7$. Then, $X \cong \mathrm{Cos}(A, A_v, A_v g A_v)$, where g is a 2-element in A such that $g^2 \in A_v$, $A_v \cap A_v^g \cong \mathrm{A}_4$ and $\langle A_v, g\rangle = A$. Set $H = A_v \cap A_v^g$. Then $g \in N_A(H)$. By Atlas [7, pp.10], there are two conjugacy classes of subgroups isomorphic to A_4. By Magma [3], $N_A(H) \cong \mathrm{S}_4 \times \mathrm{S}_3$ or $\mathrm{S}_4 \times \mathbb{Z}_2$, and moreover, in both cases there exists no 2-element $g \in N_A(H)$ such that $\langle A_v, g\rangle = A$, contrary to the fact that X is connected.

Suppose $N \cong \mathrm{M}_{11}$. Then $|N_v| = 2^2{\cdot}3{\cdot}5$ or $2^3{\cdot}3{\cdot}5$. By Atlas [7, pp.18], $N_v \cong \mathrm{A}_5$ or S_5. If $N_v = \mathrm{A}_5$ then N is arc-transitive on X. So, $X \cong \mathrm{Cos}(N, N_v, N_v g N_v)$, where g is a 2-element such that $g^2 \in N_v$, $N_v \cap N_v^g \cong \mathrm{A}_4$ and $\langle N_v, g\rangle = N$. Set $H = N_v \cap N_v^g$. Then $g \in N_N(H) \backslash N_N(N_v)$. By Magma [3], $N_N(H) = \mathrm{S}_4$ and for any 2-element $g \in N_N(H) \backslash N_N(N_v)$, g and N_v cannot generate N, a contradiction. If $N_v \cong \mathrm{S}_5$ then N has two orbits on $V(X)$ and X is bipartite. Since A is almost simple, we have $N < A \leq \mathrm{Aut}(N)$. However, from Atlas [7, pp.18] we know that $\mathrm{Aut}(N) = N$, a contradiction.

Thus, $N \cong \mathrm{PSL}(2,11)$. It follows that $p = 11$ and $|N_v| = 5$ or $2 \cdot 5$. If $|N_v| = 5$, then N is arc-transitive on X. By Lemma 3.4, $X \cong \mathcal{G}_{132}^2$ or $\mathcal{G}_{132}^3$. If $|N_v| = 10$, then N has two orbits on $V(X)$ and X is bipartite. Recall that A is almost simple. So, $N < A \leq \mathrm{Aut}(N)$. Since $\mathrm{Aut}(N) \cong \mathrm{PGL}(2,11)$, we have $A \cong \mathrm{PGL}(2,11)$. By Lemma 3.4, $X \cong \mathcal{G}_{132}^4$ or $\mathcal{G}_{132}^5$. This completes the proof. □

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