Composition matrices, $(\mathbf{2}+\mathbf{2})$-free posets and their specializations

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Abstract

In this paper we present a bijection between composition matrices and $(\mathbf{2}+\mathbf{2})$-free posets. This bijection maps partition matrices to factorial posets, and induces a bijection from upper triangular matrices with non-negative entries having no rows or columns of zeros to unlabeled $(\mathbf{2}+\mathbf{2})$-free posets. Chains in a $(\mathbf{2}+\mathbf{2})$-free poset are shown to correspond to entries in the associated composition matrix whose hooks satisfy a simple condition. It is shown that the action of taking the dual of a poset corresponds to reflecting the associated composition matrix in its anti-diagonal. We further characterize posets which are both $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free by certain properties of their associated composition matrices.

Keywords: (2+2)-free poset; interval orders; composition matrix; dual poset; bijection

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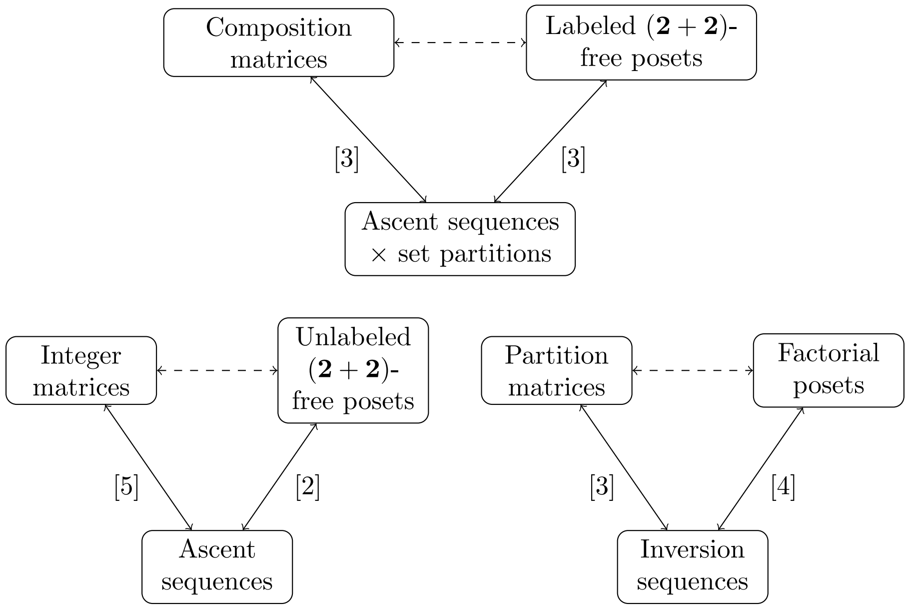


Figure 1: Overview of known correspondences.

1 Introduction

The recent introduction of bivincular patterns has unearthed surprising new connections between several combinatorial objects. Relaxing some parts of the definitions of these objects led to yet more new connections between supersets of these structures. Figure 1 summarizes the correspondences between three of these objects (at three different levels). Black lines indicate that a bijection has been proven and the label on a line indicates the paper in which this has been achieved. A dashed line indicates that no direct bijection between the corresponding objects is known.

This paper completes these pictures by presenting and proving bijections for the dashed lines. We do this first at the 'highest' level, namely mapping labelled $(\mathbf{2}+\mathbf{2})$-free posets to composition matrices. We then show that restrictions of this general map bijections from unlabeled $(\mathbf{2}+\mathbf{2})$-free posets to integer matrices, and from factorial posets to partition matrices. The bijection we present here is much simpler and more direct than the previously known bijections involving ascent sequences [2, 3, 5], which are defined recursively using case distinctions. In particular, our description of the correspondence between posets and matrices is simpler than a description obtained by mechanical composition of previously known bijections, which allows us to point out some additional properties of this correspondence. For instance, we show that the composition matrices of a poset and its dual are related by transposition along the anti-diagonal, and that the $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free posets (also known as *semiorders*) correspond precisely to matrices avoiding a north-east chain of length 2.

For the special case of unlabeled $(\mathbf{2}+\mathbf{2})$-free posets with no indistinguishable elements, our correspondence between posets and matrices coincides with the long-known notion of *characteristic matrix* of an interval order, introduced by Fishburn [6, Chpt. 2]. Although Fishburn's original definition is different from ours, it is easy to check that they are indeed equivalent.

2 $(\mathbf{2}+\mathbf{2})$-free posets and composition matrices

In this section we present and prove a bijection between $(\mathbf{2}+\mathbf{2})$-free posets and composition matrices.

We call an upper triangular matrix M a *composition matrix* on the set $[n] = \{1, \ldots, n\}$ if the entries of M are subsets of $[n]$, where all the non-empty entries form a partition of $[n]$. and there are no rows or columns that contain only empty sets. Let M_n be the set of composition matrices on $[n]$.

A partially ordered set $P = (P, \preceq_P)$ is called $(\mathbf{2}+\mathbf{2})$*-free* if it contains no induced subposet that is isomorphic to $\mathbf{2}+\mathbf{2}$, the union of two disjoint 2-element chains. An equivalent characterization of $(\mathbf{2}+\mathbf{2})$-free posets is the following [1]: a poset $(P, \preceq_P)$ is $(\mathbf{2}+\mathbf{2})$-free if and only if the set of downsets $\{D(x)\}_{x\in P}$ may be linearly ordered by inclusion. Here, for an element $x \in P$, the set $D(x) = \{y \in P : y \prec_P x\}$ denotes the *downset of* x. For the purposes of this paper, and without loss of generality, we will always assume that as a set P equals $[n]$. Let P_n be the collection of $(\mathbf{2}+\mathbf{2})$-free posets with labels in the set $[n]$.

Throughout this paper, when there is no confusion about the poset in question, we will replace $\preceq_P$ with $\preceq$ and so forth.

In order to discuss $(\mathbf{2}+\mathbf{2})$-free posets more precisely, we will make use of the following definitions.

Suppose $P = ([n], \preceq) \in \mathsf{P}_n$. If $|\{D(x) : x \in P\}| = m + 1$ then we will write $d(P) = m$. Let $D(P) = (D_0, \ldots, D_{d(P)})$ be the sequence of different downsets of P, linearly ordered by inclusion, i.e., $\emptyset = D_0 \subsetneq \cdots \subsetneq D_{d(P)}$. An element $x \in P$ with $D(x) = D_i$ for a certain i will be said to lie on *level* i of P. We use $L_i = L_i(P)$ to denote the set of elements on level i of P, and set $L(P) = (L_0, \ldots, L_{d(P)})$. The sequence $L(P)$ will be referred to as the *level sequence* of P. Note that every poset $P \in \mathsf{P}_n$ is uniquely described by listing the sequences $D(P)$ and $L(P)$.

Define $D_{d(P)+1} := [n]$ and set $K_j := D_{j+1} \backslash D_j$ for all $0 \le j \le d(P)$.

Definition 1. *Given* $(P, \preceq) \in \mathsf{P}_n$. *Let* $M = \Gamma(P)$ *be the* $(d(P) + 1) \times (d(P) + 1)$*-matrix over the powerset of* $[n]$ *where the entry in the* i^{th} *row and* j^{th} *column of* M *is given as* $M_{ij} := L_{i-1} \cap K_{j-1}$ *for all* $1 \le i, j \le d(P) + 1$.

Example 2. *Let* $P = ([8], \preceq) \in \mathsf{P}_8$ *with the following relations:* $4, 8 \prec 2$*;* $3, 4, 8 \prec 5 \prec 7 \prec 1, 6$. *There are five different downsets:* $D_0 = \emptyset$*,* $D_1 = \{4, 8\}$*,* $D_2 = \{3, 4, 8\}$*,* $D_3 = \{3, 4, 5, 8\}$*, and* $D_4 = \{3, 4, 5, 7, 8\}$. *From these we form the sets* $K_0 = \{4, 8\}$*,* $K_1 = \{3\}$*,* $K_2 = \{5\}$*,* $K_3 = \{7\}$*, and* $K_4 = \{1, 2, 6\}$. *The elements on each level are*

$L_0 = \{3,4,8\}$, $L_1 = \{2\}$, $L_2 = \{5\}$, $L_3 = \{7\}$, *and* $L_4 = \{1,6\}$. *From this we get*

$$M = \Gamma(P) = \begin{bmatrix} \{4,8\} & \{3\} & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \{2\} \\ \emptyset & \emptyset & \{5\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \{7\} & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \{1,6\} \end{bmatrix}.$$

Note that the matrix $\Gamma(P)$ which was constructed in the above example is a composition matrix on $[8]$. This is no coincidence and turns out to be true in general.

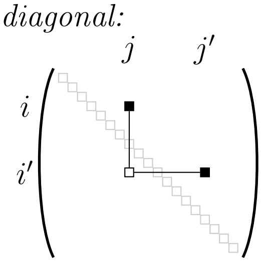
Lemma 3. *Let* $P \in \mathsf{P}_n$. *Then* $\Gamma(P) \in \mathsf{M}_n$.

Proof. Let $P = ([n], \preceq) \in \mathsf{P}_n$ and set $M = \Gamma(P)$. The union of the elements in the $(i+1)^{st}$ row of M is L_i. Since the set L_i is always non-empty, it follows that the $(i+1)^{st}$ row cannot consist of only empty sets. The union of the elements in the $(j+1)^{st}$ column of M is K_j. Since every non-trivial poset contains at least one maximal element, $K_{d(P)}$ is non-empty. For $j < d(P)$ it holds that $D_j \subsetneq D_{j+1}$ which implies that $K_j = D_{j+1} \backslash D_j$ is non-empty. Hence each column of M contains at least one non-empty set.

It remains to show that the entries of M partition the set $[n]$ which is the case if and only if every element $x \in [n]$ appears in exactly one entry of M. Since both $\{L_i\}$ and $\{K_i\}$ are partitions of the set $[n]$, it follows that no two rows (or columns) have common elements. This implies that all the non-empty entries form a partition of $[n]$.

Suppose that $x \in D_j$ for some j. Then x cannot be above level $j-1$ in P, i.e. $x \notin L_j, L_{j+1}, \ldots$. This means that the sets $L_i \cap D_j = \emptyset$ for all $i \geq j \implies L_i \cap K_j = M_{i+1,j} = \emptyset$ for all $i \geq j$, giving us that $M = \Gamma(P)$ is upper triangular. Thus $M = \Gamma(P) \in \mathsf{M}_n$. □

Definition 4. *Let* $R_n = \{(i,j) : 1 \leq i \leq j \leq n\}$. *Given* $(i,j), (i',j') \in R_n$ *let us write* $(i,j) \rightsquigarrow (i',j')$ *if* $j < i'$. *This is equivalent to the 'hook' between both entries having its bottom-left corner below the main diagonal:*



Lemma 5. *Let* $P \in \mathsf{P}_n$ *and* $M = \Gamma(P)$. *Suppose that* $x, y \in P$ *with* $x \in M_{ij}$ *and* $y \in M_{i'j'}$. *Then* $x \prec y$ *if and only if* $(i,j) \rightsquigarrow (i',j')$.

Proof. Let $P = ([n], \preceq) \in \mathsf{P}_n$, $d(P) = m$ and $M = \Gamma(P)$. We have the two sequences of sets associated with P;

$$L(P) = (L_0, L_1, \ldots, L_m) \quad \text{and} \quad D(P) = (D_0, D_1, \ldots, D_m).$$

Since $x \in M_{ij} = (D_j \setminus D_{j-1}) \cap L_{i-1}$ we have $x \in D_r$ iff $r \geq j$. On the other hand $y \in M_{i'j'} = (D'_j \setminus D_{j'-1}) \cap L_{i'-1}$, i.e., y is on level $i'-1$ and thus $D(y) = D_{i'-1}$. Finally, $x \prec y \iff x \in D(y) = D_{i'-1} \iff i'-1 \geq j$, that is, $x \rightsquigarrow y$. □

For $M \in \mathsf{M}_n$ and $x \in [n]$, if $x \in M_{ij}$ we use $\zeta(x) = \zeta_M(x) = (i, j)$ to denote the position in which x occurs in M.

Definition 6. *Let $M \in M_n$ be an $(m+1) \times (m+1)$ composition matrix. Define a poset $\Phi(M) = ([n], \preceq)$ as follows: for $x, y \in [n]$, let $x \prec y$ if and only if $\zeta_M(x) \rightsquigarrow \zeta_M(y)$.*

It is clear that $\Phi(M)$ is indeed a poset. Our goal is to show that the mapping Φ is the inverse of Γ.

Example 7. *Let M be the matrix*

$$\begin{bmatrix} \{5\} & \emptyset & \{3,6\} & \emptyset \\ \emptyset & \{1,8,9\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \{4,7\} \\ \emptyset & \emptyset & \emptyset & \{2\} \end{bmatrix}.$$

This gives the poset $P = \Phi(M)$ on $[9]$ with order relations:

$$5 \prec 1, 8, 9 \prec 2, 4, 7; \quad 3, 6 \prec 2.$$

In analogy to the notion of downset it is common to consider the so-called *upset* of an element $x \in P$. More precisely, if $(P, \preceq)$ is a poset and $x \in P$ then $U(x) := \{y \in P \mid y \succ x\}$ is referred to as the *upset* of x. Elements $x, y \in P$ are called *indistinguishable* if they have the same downsets and upsets, i.e., $D(x) = D(y)$ and $U(x) = U(y)$.

Upsets and downsets of the poset $\Phi(M)$ can be easily described in terms of the matrix M, as shown in the next lemma.

Lemma 8. *Let $M \in \mathsf{M}_n$ be a composition matrix with m rows. Let $R_i \subseteq [n]$ be the union of the cells in the i-th row of M, and let C_j be the union of the cells in its j-th column. Let $D(x)$ and $U(x)$ denote the downset and upset of an element x in the poset $\Phi(M)$. The following holds.*

- *For $x \in M_{ij}$, we have $D(x) = \bigcup_{k=1}^{i-1} C_k$ and $U(x) = \bigcup_{k=j+1}^{m} R_k$.*
- *For two elements $x, y \in [n]$, we have $D(x) = D(y)$ iff x and y appear in the same row of M, and $U(x) = U(y)$ iff x and y appear in the same column.*
- *Two elements $x, y \in [n]$ are indistinguishable in $\Phi(M)$ iff they belong to the same cell of M.*
- *The poset $\Phi(M)$ is $(\mathbf{2}+\mathbf{2})$-free.*

Proof. The first claim follows directly from the definition of Φ. The second claim is a consequence of the first one, together with the fact that every row and every column of M has at least one nonempty cell. The third claim is an immediate consequence of the second one. Finally, to prove the fourth claim, we first observe that the first claim of the lemma implies that any two downsets of $\Phi(M)$ are comparable by inclusion. A poset whose downsets are linearly ordered by inclusion is $(\mathbf{2}+\mathbf{2})$-free. $\square$

Theorem 9. $\Gamma\colon \mathsf{P}_n \to \mathsf{M}_n$ *is a bijection and* Φ *is its inverse.*

Proof. Lemma 5 shows that $\Phi(\Gamma(P)) = P$ for any $P \in \mathsf{P}_n$. It remains to verify that Φ is injective. Let $M \in \mathsf{M}_n$ be a composition matrix, with R_i and C_j defined as in Lemma 8. Let $P = \Phi(M)$. From Lemma 8, we see that the number of rows of M is equal to the number of distinct downsets in P. Suppose that P has $m+1$ distinct downsets $D_0 \subset D_1 \subset \cdots \subset D_m$, and define $D_{m+1} = [n]$. Lemma 8 then implies that $C_j = D_j \setminus D_{j-1}$ for any $j \in [m+1]$. Similarly, the sets R_i are determined by the upsets of P. Since the matrix M can be uniquely reconstructed from the sets $\{R_i;\ i \in [m+1]\}$ and $\{C_j;\ j \in [m+1]\}$, we conclude that P determines M uniquely, hence Φ is injective. □

For any poset P let $P^\star$ be the dual of P, i.e., $P^\star$ is defined by the equivalence $x \prec^\star_P y \iff y \prec_P x$.

Given an $(a \times a)$-matrix M, let $\mathsf{trans}(M)$ be the matrix defined by

$$\mathsf{trans}(M)_{ij} = M_{a+1-j,a+1-i}.$$

The matrix $\mathsf{trans}(M)$ is obtained by reflecting M in its anti-diagonal and we call $\mathsf{trans}(M)$ the *anti-transpose* of M.

We conclude this section by pointing out that poset duality corresponds to matrix anti-transpose.

Theorem 10. *Let* $P \in \mathsf{P}_n$ *and* $M = \Gamma(P)$. *Then* $\Gamma(P^\star) = \mathsf{trans}(M)$.

Proof. From the definition of Φ, we immediately see that for any $M \in \mathsf{M}_n$, $\Phi(\mathsf{trans}(M))$ is the dual of the poset $\Phi(M)$. Since Γ is the inverse of Φ, the theorem follows. □

3 Special classes of $(\mathbf{2}+\mathbf{2})$-free posets

3.1 Unlabeled $(\mathbf{2}+\mathbf{2})$-free posets

Let $P \in \mathsf{P}_n$ be a poset, let $M \in \mathsf{M}_n$ be a composition matrix and let $\pi : [n] \to [n]$ be a permutation. Define $\pi(P)$ to be the poset obtained from P by replacing each label i with the label $\pi(i)$ and let $\pi(M)$ to be the matrix obtained from M by changing each value i into $\pi(i)$. Notice that $\pi(\Gamma(P)) = \Gamma(\pi(P))$.

We say that two posets $P, Q \in \mathsf{P}_n$ are *isomorphic*, denoted by $P \sim Q$, if there is a permutation π such that $Q = \pi(P)$. Let $[P]$ denote the isomorphism class of a poset $P \in \mathsf{P}_n$. Let U_n be the set of equivalence classes of posets from P_n. The elements of U_n are often referred to as *unlabeled* $(\mathbf{2}+\mathbf{2})$*-free posets*.

Let $\mathsf{card}(M)$ be the integer matrix whose the entry at a position (i,j) is equal to the cardinality of the set M_{ij}.

Lemma 11. *Let* $P, Q \in \mathsf{P}_n$. *Then* $P \sim Q$ *iff* $\mathsf{card}(\Gamma(P)) = \mathsf{card}(\Gamma(Q))$.

Proof. If $P \sim Q$ then there exists a permutation $\pi\colon [n] \to [n]$ such that $P = \pi(Q)$. This implies that $\mathsf{card}(\Gamma(P)) = \mathsf{card}(\Gamma(\pi(Q))) = \mathsf{card}(\pi(\Gamma(Q)))$. Since π only changes the entries of the sets in $\Gamma(Q)$, and not their cardinality, we have $\mathsf{card}(\pi(\Gamma(Q))) = \mathsf{card}(\Gamma(Q))$. Hence $\mathsf{card}(\Gamma(P)) = \mathsf{card}(\Gamma(Q))$.

Conversely, if $\mathsf{card}(\Gamma(P)) = \mathsf{card}(\Gamma(Q))$ then there exists a permutation $\sigma : [n] \to [n]$ such that $\Gamma(P) = \sigma(\Gamma(Q)) = \Gamma(\sigma(Q))$. Since by Theorem 9 Γ is a bijection we obtain $P = \sigma(Q)$ which implies $P \sim Q$. □

Let Int_n be the set of upper triangular matrices whose entries are non-negative integers having the property that the matrix contains no row or column of all zero entries and the sum of all entries equals n. Note that a matrix N belongs to Int_n if and only if $N = \mathsf{card}(M)$ for some $M \in \mathsf{M}_n$.

Let us note that by looking at equivalence classes we have gotten rid of the labeling of posets and we have exactly one equivalence class for each unlabeled poset on $[n]$. We now set $\Gamma'([P]) = \mathsf{card}(\Gamma(P))$. The map Γ' should be thought of as the restriction of the previously used map Γ to unlabeled posets on $[n]$.

Theorem 12. *$\Gamma'\colon \mathsf{U}_n \to \mathsf{Int}_n$ is a bijection.*

Proof. It is clear that Γ' is surjective, because for every $N \in \mathsf{Int}_n$, there is a composition matrix $M \in \mathsf{M}_n$ such that $N = \mathsf{card}(M)$, and the corresponding poset $P = \Phi(M)$ then satisfies $\Gamma'([P]) = N$.

Let us verify that Γ' is injective. Given two unlabeled posets $[P], [Q] \in \mathsf{U}_n$, we see that $\Gamma'([P]) = \Gamma'([Q])$ iff $\mathsf{card}(\Gamma(P)) = \mathsf{card}(\Gamma(Q))$ iff $P \sim Q$, by Lemma 11. Hence $[P] = [Q]$. □

Let us remark that our bijections Γ and Γ' provide the same mapping from posets to matrices as the bijections obtained by composing previously known bijections from posets to special kinds of integer sequences [2, 3] and from integer sequences to matrices [3, 5].

3.2 Factorial posets and partition matrices

Let $P = ([n], \preceq_P) \in \mathsf{P}_n$. Let FP_n be the collection of *factorial posets* in P_n, where a poset $P \in \mathsf{P}_n$ is *factorial* if for all $i, j, k \in [n]$,

$$i < j \prec_P k \implies i \prec_P k.$$

Here $<$ denotes the usual order relation on $\mathbb{N}$. These posets were introduced by Claesson and Linusson in [4]. An equivalent definition of factorial posets is the following:

Definition 13. *Let $P \in \mathsf{P}_n$ with $d(P) = m$. The poset P is factorial iff there exist integers $1 \leq a_1 < \cdots < a_m < n$ such that $D_i(P) = [1, a_i]$ for all $1 \leq i \leq m$.*

Given $M \in \mathsf{M}_n$ and $x \in [n]$, let $\mathsf{col}(x)$ denote the index of the column of M in which x appears. Following [3] we call M a *partition matrix* if $\mathsf{col}(x) < \mathsf{col}(y) \implies x < y$. Let PM_n be the set of partition matrices in M_n.

Restricting the map Γ to factorial posets we obtain the following result.

Theorem 14. *A poset $P \in \mathsf{P}_n$ is a factorial poset if and only the matrix $\Gamma(P)$ is a partition matrix. In other words, Γ can be restricted to a bijection from FP_n to PM_n.*

Proof. Suppose that P is a factorial poset, and let $M = \Gamma(P)$. Choose $x, y \in [n]$ such that $\mathsf{col}(x) < \mathsf{col}(y)$. Lemma 8 implies that P has a downset that contains x but not y. Since P is factorial, each of its downsets has the form $\{1, 2, \ldots, a_i\}$ for some a_i. This means that $x < y$ and hence $M \in \mathsf{PM}_n$. The converse implication can be verified by an analogous argument. □

3.3 $(\mathbf{2}+\mathbf{2})$ and $(\mathbf{3}+\mathbf{1})$-free posets

The posets we consider in this subsection are labeled. A poset P is $(\mathbf{3}+\mathbf{1})$-free if it contains no induced subposet that is isomorphic to $\mathbf{3}+\mathbf{1}$, the disjoint union of a 3-element chain and a singleton. Posets that are $(\mathbf{3}+\mathbf{1})$-free have been studied by several people (see for example Skandera [7]). Posets that are both $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free are also called *semiorders*. The condition for an interval order to be $(\mathbf{3}+\mathbf{1})$-free is that its intervals are all of the same length. In this section we characterize posets which are $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free in terms of their associated upper triangular matrices.

Lemma 15. *Let $P \in \mathsf{P}_n$ with $d(P) = m$ and $M = \Gamma(P)$. Given $x \in M_{ij}$, the set of elements in P which are incomparable to x are those elements (other than x) that appear in the following entries of M: $\{(i', j') \in R_{m+1} \,:\, i' \le j,\ j' \ge i\}$.*

Proof. This is a consequence of the first part of Lemma 8. □

Proposition 16. *Let $P \in \mathsf{P}_n$ with $d(P) = m$ and let $M = \Gamma(P)$. Then the following conditions are equivalent:*

(i) P contains an induced subposet isomorphic to $\mathbf{3}+\mathbf{1}$.

(ii) There exist elements $x \in M_{ij}$ and $y \in M_{i'j'}$ such that $i' < i$ and $j' > j$.

Proof. First assume that P contains an induced subposet isomorphic to $\mathbf{3}+\mathbf{1}$. Let x_1, x_2, x_3, x_4 be the elements of this occurrence where $x_1 \prec x_2 \prec x_3$ is the 3-element chain. Let $\zeta(x_k) = (i_k, j_k)$ for $1 \le k \le 4$. We then have $(i_1, j_1) \rightsquigarrow (i_2, j_2) \rightsquigarrow (i_3, j_3)$. Since x_1 and x_3 are incomparable to x_4, we deduce from Lemma 15 that $i_4 \le j_1$ and $j_4 \ge i_3$. Also, since $i_4 \le j_1$ and $(i_1, j_1) \rightsquigarrow (i_2, j_2)$ we have $i_4 < i_2$. Similarly, $j_4 \ge i_3$ and $(i_2, j_2) \rightsquigarrow (i_3, j_3)$ implies $j_4 > j_2$. Thus (ii) is satisfied with $x = x_2$ and $y = x_4$.

Now assume that (ii) holds. Since $j' \in [j+1, m+1]$ and $i' \in [1, i-1]$ we have that y is north-east of x in M. From Lemma 15 we thus have that y is incomparable to x. It follows from $i' < i \le j < j'$ that $i' \ne j, j'$ and $j' \ne i, i'$. Thus neither x nor y are in row j' or column i'. Let a be an element in column i' of M and let b be an element in row j' of M.

Since the hook of a and y lies on the diagonal, we have the a and y are incomparable. The elements y and b are incomparable for the same reason. So y is incomparable to a, x and b. The element a is in column i' of M which means that $a \prec x$, since $i' < i$.

Similarly the element b is in row j' of M which gives $x \prec b$ since $j < j'$. Combining these observations yields that $a \prec x \prec b$ and y is incomparable to a, x and b. Thus P restricted to the set $\{a, x, b, y\}$ is isomorphic to $\mathbf{3}+\mathbf{1}$, and condition (i) is satisfied. $\square$

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