The Bipartite Ramsey Numbers $b(C_{2m}; K_{2,2})$

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Abstract

Given bipartite graphs H_1 and H_2, the bipartite Ramsey number $b(H_1; H_2)$ is the smallest integer b such that any subgraph G of the complete bipartite graph $K_{b,b}$, either G contains a copy of H_1 or its complement relative to $K_{b,b}$ contains a copy of H_2. It is known that $b(K_{2,2}; K_{2,2}) = 5, b(K_{2,3}; K_{2,3}) = 9, b(K_{2,4}; K_{2,4}) = 14$ and $b(K_{3,3}; K_{3,3}) = 17$. In this paper we study the case H_1 being even cycles and H_2 being $K_{2,2}$, prove that $b(C_6; K_{2,2}) = 5$ and $b(C_{2m}; K_{2,2}) = m + 1$ for $m \geq 4$.

Keywords: *bipartite graph; Ramsey number; even cycle*

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. For a graph G with vertex-set $V(G)$ and edge-set $E(G)$, we denote the order and the size of G by $p(G) = |V(G)|$ and $q(G) = |E(G)|$. $\delta(G)$ and $\Delta(G)$ are the minimum degree and the maximum degree of G respectively.

Let $K_{m,n}$ be a complete m by n bipartite graph, that is, $K_{m,n}$ consists of $m+n$ vertices, partitioned into sets of size m and n, and the mn edges between them. P_k is a path on k vertices, and C_k is a cycle of length k. Let H_1 and H_2 be bipartite graphs, the bipartite Ramsey number $b(H_1; H_2)$ is the smallest integer b such that given any subgraph G of the complete bipartite graph $K_{b,b}$, either G contains a copy of H_1 or there exists a copy of H_2 in the complement of G relative to $K_{b,b}$. Obviously, we have $b(H_1; H_2) = b(H_2; H_1)$.

Beineke and Schwenk [1] showed that

$$b(K_{2,2}; K_{2,2}) = 5, \quad b(K_{2,4}; K_{2,4}) = 13, \quad b(K_{3,3}; K_{3,3}) = 17.$$

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In particular, they proved that $b(K_{2,n}; K_{2,n}) = 4n-3$ for n odd and less than 100 except possibly $n = 59$ or $n = 95$. Carnielli and Carmelo [2] proved that $b(K_{2,n}; K_{2,n}) = 4n-3$ if $4n-3$ is a prime power. They also showed that $b(K_{2,2}; K_{1,n}) = n+q$ for $q^2-q+1 \leq n \leq q^2$, where q is a prime power. Irving [6] showed that $b(K_{4,4}; K_{4,4}) \leq 48$. Hattingh and Henning [4] proved that

$$b(K_{2,2}; K_{3,3}) = 9, \quad b(K_{2,2}; K_{4,4}) = 14.$$

They also determined the values of $b(P_m; K_{1,n})$ in [5]. Faudree and Schelp [3] proved the values of $b(H_1; H_2)$ when both H_1 and H_2 are two paths.

Let G_i be the subgraph of G whose edges are in the i-th color in an r-coloring of the edges of G. If there exists an r-coloring of the edges of G such that $H_i \nsubseteq G_i$ for all $1 \leq i \leq r$, then G is said to be r-colorable to $(H_1, H_2, \ldots, H_r)$. The neighborhood of a vertex $v \in V(G)$ are denoted by $N(v) = \{u \in V(G) | uv \in E(G)\}$, and let $d(v) = |N(v)|$. G^c denotes the complement of G relative to $K_{b,b}$. $G\langle W \rangle$ denotes the subgraph of G induced by $W \subseteq V(G)$.

In this paper we study the case that H_1 being even cycles and H_2 being $K_{2,2}$, prove that $b(C_6; K_{2,2}) = 5$ and $b(C_{2m}; K_{2,2}) = m+1$ for $m \geq 4$. For the sake of convenience, let $V(K_{m,n}) = X \cup Y$, where $X = \{x_i | 1 \leq i \leq m\}$ and $Y = \{y_j | 1 \leq j \leq n\}$, and $E(K_{m,n}) = \{x_i y_j | 1 \leq i \leq m, 1 \leq j \leq n\}$.

2 The lower bounds of $b(C_{2m}; K_{2,2})$

Theorem 1. $b(C_{2m}; C_{2n}) \geq m+n-1$.

Proof. Let G_1 and G_2 be the subgraphs of $K_{m+n-2,m+n-2}$, where G_1 is a complete $m-1$ by $m+n-2$ bipartite graph, and G_2 is a complete $n-1$ by $m+n-2$ bipartite graph. And let

$$V(G_1) = X_1 \cup Y, \text{ where } X_1 = \{x_i | 1 \leq i \leq m-1\}, \ Y = \{y_i | 1 \leq i \leq m+n-2\};$$
$$V(G_2) = X_2 \cup Y, \text{ where } X_2 = \{x_i | m \leq i \leq m+n-2\}, \ Y = \{y_i | 1 \leq i \leq m+n-2\}.$$

Then we have $E(G_1) \cap E(G_2) = \emptyset$ and $E(G_1) \cup E(G_2) = E(K_{m+n-2,m+n-2})$. Note that $C_{2m} \nsubseteq G_1$ and $C_{2n} \nsubseteq G_2$. So $K_{m+n-2,m+n-2}$ is 2-colorable to (C_{2m}, C_{2n}), that is, $b(C_{2m}; C_{2n}) \geq m+n-1$. □

Setting $n = 2$ in Theorem 1, we have

Corollary 1. $b(C_{2m}; K_{2,2}) \geq m+1$.

3 The upper bounds of $b(C_{2m}; K_{2,2})(m \geq 3)$

Lemma 1. $b(C_6; K_{2,2}) \leq 5$.

Proof. We may assume that $b(C_6; K_{2,2}) > 5$, that is, $K_{5,5}$ is 2-colorable to $(C_6, K_{2,2})$. Since $K_{2,2} \nsubseteq G^c$ and $b(K_{2,2}; K_{2,2}) = 5$, we have $K_{2,2} \subseteq G$. Without loss of generality, we may assume $\{x_1y_1, y_1x_2, x_2y_2, y_2x_1\} \subseteq E(G)$.

Since $K_{2,2} \nsubseteq G^c$, there is at least one edge between $\{x_3, x_4\}$ and $\{y_3, y_4\}$, say $x_3y_3 \in E(G)$. Similarly, there is at least one edge between $\{x_4, x_5\}$ and $\{y_4, y_5\}$, say $x_4y_4 \in E(G)$. And there is at least one edge between $\{x_1, x_2\}$ and $\{y_3, y_4\}$, say $x_1y_4 \in E(G)$. Since $C_6 \nsubseteq G$, x_4 is nonadjacent to any vertex of $\{y_1, y_2\}$. Therefore since $K_{2,2} \nsubseteq G^c$, x_3 has to be adjacent to one vertex of $\{y_1, y_2\}$, say $x_3y_2 \in E(G)$. x_4 is nonadjacent to any vertex if $\{y_1, y_2, y_3\}$, since otherwise we have $C_6 \subseteq G$. And since $K_{2,2} \nsubseteq G^c$, x_5 is adjacent to at least two vertices of $\{y_1, y_2, y_3\}$. If x_5 is adjacent to both y_1 and y_3, then we have $C_6 \subseteq G$, a contradiction. Hence we have $x_5y_1, x_5y_2 \in E(G)$ or $x_5y_2, x_5y_3 \in E(G)$.
Case 1. Suppose that $x_5y_1, x_5y_2 \in E(G)$, see Fig. 1(a). Since $C_6 \nsubseteq G$, x_5 is nonadjacent to y_3 or y_4. Therefore since $K_{2,2} \nsubseteq G^c$, x_2 has to be adjacent to at least one vertex of $\{y_3, y_4\}$. In any case, we have $C_6 \subseteq G$, a contradiction.
Case 2. Suppose that $x_5y_2, x_5y_3 \in E(G)$, see Fig. 1(b). Since $C_6 \nsubseteq G$, x_5 is nonadjacent to y_1 or y_4. Therefore since $K_{2,2} \nsubseteq G^c$, x_3 has to be adjacent to at least one vertex of $\{y_1, y_4\}$. In any case, we have $C_6 \subseteq G$, a contradiction too.

Fig. 1. The two cases of $N(x_5)$

By Case 1 and 2, the assumption that $b(C_6; K_{2,2}) > 5$ does not hold. Then we have the lemma follows. □

Lemma 2. Let G be a spanning subgraph of $K_{5,5}$ and $C_8 \nsubseteq G$. If $K_{2,2} \nsubseteq G^c$, then there exists at most one vertex of X(or Y) whose degrees is at most 2.
Proof. For $1 \leq i, j \leq 5$, if $|N(x_i) \cup N(x_j)| \leq 3 (i \neq j)$, then there are at least two vertices of Y are nonadjacent to x_i or x_j, we have $K_{2,2} \subseteq G^c$. Hence we have
Claim 1. $|N(x_i) \cup N(x_j)| \geq 4$.

By way of contradiction, we assume that there exists at least two vertices of X whose degrees are at most 2, say x_1 and x_2. By Claim 1, we have $|N(x_1) \cup N(x_2)| = 4$. We may assume $N(x_1) = \{y_1, y_2\}$ and $N(x_2) = \{y_3, y_4\}$. There are two subcases depending on $N(y_5)$.
Case 1. Suppose that there is at least one vertex of $\{x_3, x_4, x_5\}$, say x_3 which is nonadjacent to y_5. By Claim 1, we have $|N(x_1) \cup N(x_3)| \geq 4$, x_3 has to be adjacent to both y_3 and y_4. Similarly we have $|N(x_2) \cup N(x_3)| \geq 4$, x_3 has to be adjacent to both y_1 and y_2. By Claim 1, we have $|N(x_1) \cup N(x_4)| \geq 4$, x_4 has to be adjacent to at lease one vertex of $\{y_3, y_4\}$, say $x_4y_3 \in E(G)$ as shown in Fig. 2(a). Since $C_8 \nsubseteq G$, x_4 is nonadjacent to y_1

or y_2. Hence we have $|N(x_2) \cup N(x_4)| \leq 3$, a contradiction to Claim 1.

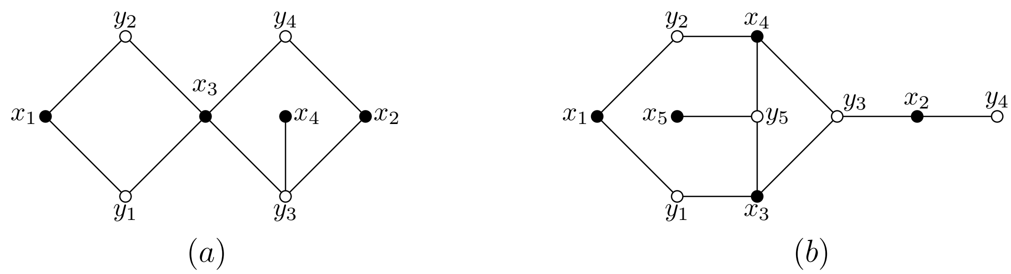

Fig. 2. The two cases of $N(y_5)$

Case 2. Suppose that each vertex of $\{x_3, x_4, x_5\}$ is adjacent to y_5. By Claim 1, we have $|N(x_1) \cup N(x_3)| \geq 4$, x_3 has to be adjacent to at lease one vertex of $\{y_3, y_4\}$, say $x_3y_3 \in E(G)$. Similarly, we have $|N(x_2) \cup N(x_3)| \geq 4$, x_3 has to be adjacent to at lease one vertex of $\{y_1, y_2\}$, say $x_3y_1 \in E(G)$. Since $K_{2,2} \nsubseteq G^c$, there is at least one edge between $\{x_4, x_5\}$ and $\{y_2, y_4\}$, say $x_4y_2 \in E(G)$. Since $C_8 \nsubseteq G$, x_4 is nonadjacent to y_4. By Claim 1, we have $|N(x_1) \cup N(x_4)| \geq 4$, x_4 has to be adjacent to y_3 as shown in Fig. 2(b). Since $C_8 \nsubseteq G$, x_5 is nonadjacent to y_1 or y_2. Hence we have $|N(x_2) \cup N(x_5)| \leq 3$, a contradiction to Claim 1.

By Case 1 and 2, the assumption does not hold. Then we have the lemma follows. □

Lemma 3. $b(C_8; K_{2,2}) \leq 5$.

Proof. We may assume that $b(C_8; K_{2,2}) > 5$, that is, $K_{5,5}$ is 2-colorable to $(C_8, K_{2,2})$, say $C_8 \nsubseteq G$ and $K_{2,2} \nsubseteq G^c$. Since $K_{2,2} \nsubseteq G^c$ and $b(C_6; K_{2,2}) \leq 5$, we have $C_6 \subseteq G$. Without loss of generality, we may assume $\{x_1y_1, y_1x_2, x_2y_2, y_2x_3, x_3y_3, y_3x_1\} \subseteq E(G)$.

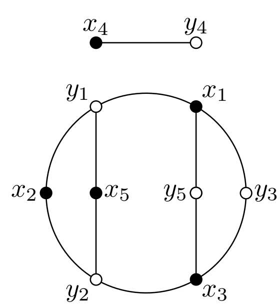

Fig. 3. No edge between $\{x_4, y_4\}$ and $V(C_6)$

Since $K_{2,2} \nsubseteq G^c$, there is at least one edge between $\{x_4, x_5\}$ and $\{y_4, y_5\}$, say $x_4y_4 \in E(G)$. Assume that x_4 is nonadjacent to any vertex of $\{y_1, y_2, y_3\}$ and y_4 is nonadjacent to any vertex of $\{x_1, x_2, x_3\}$. Then we have $d(x_4) \leq 2$ and $d(y_4) \leq 2$. Since $K_{2,2} \nsubseteq G^c$, x_5 has to be adjacent to at least two vertices of $\{y_1, y_2, y_3\}$, say $x_5y_1, x_5y_2 \in E(G)$. Similarly, y_5 has to be adjacent to at least two vertices of $\{x_1, x_2, x_3\}$. By symmetry, we may assume that $y_5x_1, y_5x_2 \in E(G)$ or $y_5x_1, y_5x_3 \in E(G)$. If $y_5x_1, y_5x_2 \in E(G)$, then $C_8 \subseteq G$, a contradiction. Hence we have $y_5x_1, y_5x_3 \in E(G)$, as shown in Fig. 3. Since $C_8 \nsubseteq G$, x_2 is nonadjacent to y_3 or y_5. So we have $d(x_2) = 2$, a contradiction to Lemma 2. Hence x_4 is adjacent to at least one vertex of $\{y_1, y_2, y_3\}$ or y_4 is adjacent to at least one vertex of $\{x_1, x_2, x_3\}$, say $x_4y_3 \in E(G)$.

Since $C_8 \nsubseteq G$, y_4 is nonadjacent to x_1 or x_3. Therefore since $K_{2,2} \nsubseteq G^c$, y_5 has to be adjacent to at least one vertex of $\{x_1, x_3\}$, say $y_5x_1 \in E(G)$. Now we consider the vertex of x_5, there are three subcases.

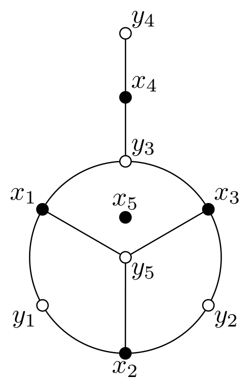

Fig. 4. x_5 being nonadjacent to y_4 or y_5

Case 1. Suppose that x_5 is nonadjacent to any vertex of $\{y_4, y_5\}$. Since $C_8 \nsubseteq G$, y_4 is nonadjacent to any vertex of $\{x_1, x_3\}$. Hence we have $d(y_4) \leq 2$. By Lemma 2, we have $d(y_5) \geq 3$. Therefore since $C_8 \nsubseteq G$, y_5 has to be adjacent to both x_2 and x_3 as shown in Fig. 4. Since $C_8 \nsubseteq G$, x_4 is nonadjacent to any vertex of $\{y_1, y_2, y_5\}$. Hence we have $d(x_4) = 2$. By Lemma 2, we have $d(x_5) \geq 3$. Hence x_5 has to be adjacent to each vertex of $\{y_1, y_2, y_3\}$, we have $C_8 \subseteq G$, a contradiction.

Case 2. Suppose that x_5 is adjacent to just one vertex of $\{y_4, y_5\}$, that is, $x_5y_4 \in E(G)$ or $x_5y_5 \in E(G)$. Suppose $x_5y_4 \in E(G)$, then $x_5y_5 \notin E(G)$. Since $C_8 \nsubseteq G$, we have $x_4y_5 \notin E(G)$. Since $K_{2,2} \nsubseteq G^c$, y_1 is adjacent to at least one vertex of $\{x_4, x_5\}$. Therefore since $C_8 \nsubseteq G$, y_1 has to be adjacent to x_4. Similarly, y_2 and y_3 have to be adjacent to x_4, see Fig. 5(a). Since $C_8 \nsubseteq G$, y_5 is nonadjacent to any vertex of $\{x_2, x_3\}$. Hence we have $d(y_5) = 1$. By Lemma 2, we have $d(y_4) \geq 3$. Hence y_4 has to be adjacent to at least one vertex of $\{x_1, x_2, x_3\}$. In any case, we have $C_8 \subseteq G$, a contradiction.

Suppose that $x_5y_5 \in E(G)$. Since $C_8 \nsubseteq G$, x_5 is nonadjacent to any vertex of $\{y_1, y_3\}$. Hence we have $d(x_5) \leq 2$. By Lemma 2, we have $d(x_3) \geq 3$. Since $C_8 \nsubseteq G$, x_3 is nonadjacent to y_4. x_3 has to be adjacent to y_1, since otherwise $K_{2,2} \subseteq G^c\langle\{x_3, x_5, y_1, y_4\}\rangle$, see Fig. 5($b$). By Lemma 2, we have $d(x_4) \geq 3$. Hence x_4 is adjacent to at least one vertex of $\{y_1, y_2\}$. In any case, since $C_8 \nsubseteq G$, y_5 is nonadjacent to x_2 or x_3. Hence we have $d(y_5) = 2$, a contradiction to Lemma 2.

Case 3. Suppose that x_5 is adjacent to each vertex of $\{y_4, y_5\}$, as shown in Fig. 6. Since $C_8 \nsubseteq G$, y_4 is nonadjacent to any vertex of $\{x_1, x_2, x_3\}$. Hence $d(y_4) = 2$. By Lemma 2, we have $d(y_2) \geq 3$. So y_2 is adjacent to at least one vertex of $\{x_1, x_4, x_5\}$. In any case, we have $C_8 \subseteq G$, a contradiction.

By Case 1-3, the assumption does not hold. Then we have the lemma follows. □

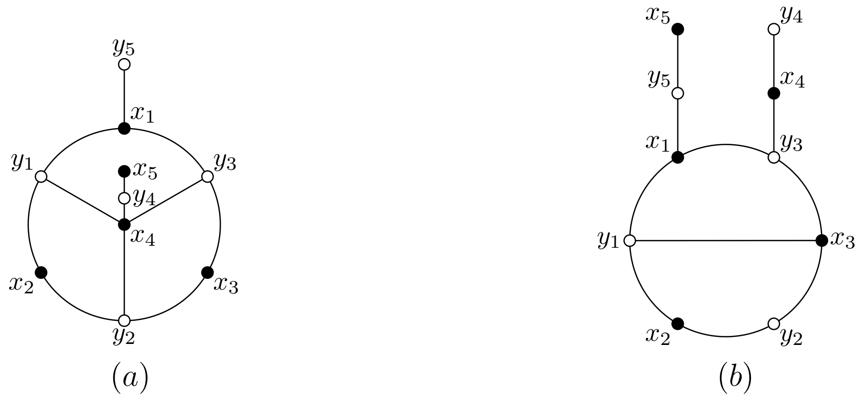

Fig. 5. x_5 being adjacent to just one of $\{y_4, y_5\}$

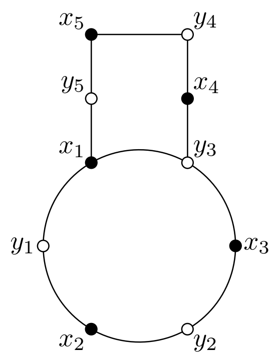

Fig. 6. x_5 being adjacent to y_4 and y_5

Lemma 4. Let G be a spanning subgraph of $K_{k+1,k+1}$ such that $C_{2k} \subseteq G$ and $x_{k+1}, y_{k+1} \notin V(C_{2k})$. If x_{k+1} and y_{k+1} are adjacent to at least $k-1$ vertices of $V(C_{2k})$ respectively, then we have $C_{2(k+1)} \subseteq G$.

Proof. Without loss of generality, let $E(C_{2k}) = \{x_1y_1, y_1x_2, \ldots, x_xy_k, y_kx_1\}$. Then x_{k+1} is adjacent to at least $k-1$ vertices of $\{y_1, y_2, \ldots, y_k\}$, say $\{x_{k+1}y_1, x_{k+1}y_2, \ldots, x_{k+1}y_{k-1}\} \subseteq E(G)$. And since y_{k+1} is adjacent to at least $k-1$ vertices of $\{x_1, x_2, \ldots, x_k\}$, y_{k+1} is nonadjacent to at most one vertex of $\{x_1, x_k\}$, say $x_ky_{k+1} \notin E(G)$. Hence we have $C_{2(k+1)} \subseteq G(x_1y_{k+1}x_2y_1x_{k+1}y_2x_3y_3, \ldots, x_ky_kx_1)$ as shown in Fig. 7. $\square$

Lemma 5. If $m \geq 4$, then $b(C_{2m}; K_{2,2}) \leq m+1$.

Proof. We will prove it by way of induction.

(1) For $m = 4$, by Lemma 3, we have the lemma holds.

(2) Suppose that $b(C_{2k}; K_{2,2}) \leq k+1$ for $k \geq 4$. We will show that $b(C_{2(k+1)}; K_{2,2}) \leq k+2$ as follows. The proof is similar to Lemma 3, however, arbitrary k makes Lemma 2 not applicable, which makes the proof more difficult.

By contradiction, we may assume that $b(C_{2(k+1)}; K_{2,2}) > k+2$, that is, $K_{k+2,k+2}$ is 2-colorable to $(C_{2(k+1)}, K_{2,2})$, say $C_{2(k+1)} \nsubseteq G$ and $K_{2,2} \nsubseteq G^c$. By the induction hy-

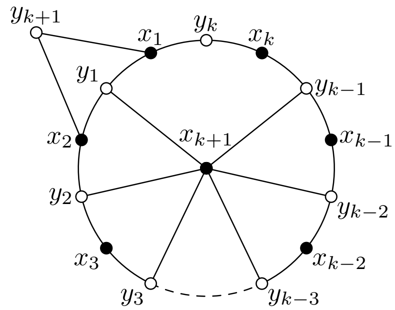

Fig. 7. The graph with a cycle of length $2(k+1)$

pothesis, we have $b(C_{2k}; K_{2,2}) \leq k+1$. Therefore since $K_{2,2} \nsubseteq G^c$, we have $C_{2k} \subseteq G$, let $E(C_{2k}) = \{x_1y_1, y_1x_2, x_2y_2, \ldots, x_ky_k, y_kx_1\}$. Firstly, we consider the four vertices not belong to $V(C_{2k})$, that is, $x_{k+1}, x_{k+2}, y_{k+1}$ and y_{k+2}.

Since $K_{2,2} \nsubseteq G^c$, there is at least one edge between $\{x_{k+1}, x_{k+2}\}$ and $\{y_{k+1}, y_{k+2}\}$, say $x_{k+1}y_{k+1} \in E(G)$. Assume that there is no edge between $\{x_{k+1}, y_{k+1}\}$ and $V(C_{2k})$. Since $K_{2,2} \nsubseteq G^c$, x_{k+2} is adjacent to at least $k-1$ vertices of $\{y_1, y_2, \ldots, y_k\}$, and y_{k+2} is adjacent to at least $k-1$ vertices of $\{x_1, x_2, \ldots, x_k\}$. By Lemma 4, we have $C_{2(k+1)} \subseteq G$, a contradiction. So there is at least one edge between $\{x_{k+1}, y_{k+1}\}$ and $V(C_{2k})$, say $x_{k+1}y_k \in E(G)$. Since $C_{2(k+1)} \nsubseteq G$, y_{k+1} is nonadjacent to x_1 or x_k. Therefore since $K_{2,2} \nsubseteq G^c$, y_{k+2} has to be adjacent to at least one vertex of $\{x_1, x_k\}$, say $y_{k+2}x_1 \in E(G)$ as shown in Fig. 8. Now we consider the edge number between $\{x_{k+2}\}$ and $\{y_{k+1}, y_{k+2}\}$, there are three subcases as follows.

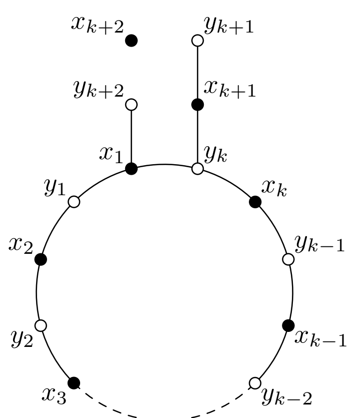

Fig. 8. The subgraph of G

Case 1. Suppose that there is no edge between $\{x_{k+2}\}$ and $\{y_{k+1}, y_{k+2}\}$. Since $K_{2,2} \nsubseteq G^c$, x_k is adjacent to at least one vertex of $\{y_{k+1}, y_{k+2}\}$. Therefore since $C_{2(k+1)} \nsubseteq G$, x_k has to be adjacent to y_{k+2}. Then both x_2 and x_{k-1} are nonadjacent to y_{k+1}, since otherwise $C_{2(k+1)} \subseteq G$. If y_{k+2} is nonadjacent to at least one vertex of $\{x_2, x_{k-1}\}$, say x_2, then x_2, x_{k+2}, y_{k+1} and y_{k+2} would construct a $K_{2,2}$ in G^c, a contradiction. Hence both x_2 and x_{k-1} have to be adjacent to y_{k+2}. x_{k+1} is nonadjacent to any vertex of $\{y_1, y_{k-1}, y_{k+2}\}$,

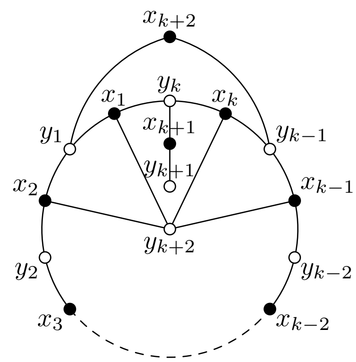

Fig. 9. x_{k+2} being nonadjacent to y_{k+1} or y_{k+2}

since otherwise $C_{2(k+1)} \subseteq G$. If x_{k+2} is nonadjacent to at least one vertex of $\{y_1, y_{k-1}\}$, say y_1, then x_{k+1}, x_{k+2}, y_1 and y_{k+2} would construct a $K_{2,2}$ in G^c, a contradiction. Hence x_{k+2} has to be adjacent to both y_1 and y_{k-1} as shown in Fig. 9. Now we have $C_{2(k+1)} \subseteq G(y_1x_1y_kx_ky_{k+2}x_2y_2, \ldots, x_{k-1}y_{k-1}x_{k+2}y_1)$, a contradiction too.
Case 2. Suppose that there is just one edge between $\{x_{k+2}\}$ and $\{y_{k+1}, y_{k+2}\}$, namely $x_{k+2}y_{k+1} \in E(G)$ or $x_{k+2}y_{k+2} \in E(G)$.
Case 2.1. Suppose that $x_{k+2}y_{k+1} \in E(G)$, then $x_{k+2}y_{k+2} \notin E(G)$. Since $C_{2(k+1)} \nsubseteq G$, we have $x_{k+1}y_{k+2}, x_{k+2}y_{k-1} \notin E(G)$. Then y_{k-1} has to be adjacent to x_{k+1}, since otherwise $x_{k+1}, x_{k+2}, y_{k-1}$ and y_{k+2} would construct a $K_{2,2}$ in G^c. Note that x_{k+1} together with $V(C_{2k}) - x_k$ construct a new cycle of length $2k$ as shown in Fig. 10(a). Since $C_{2(k+1)} \nsubseteq G$, y_{k+2} is nonadjacent to x_k or x_{k+2}. So, the proof is same as Case 1.

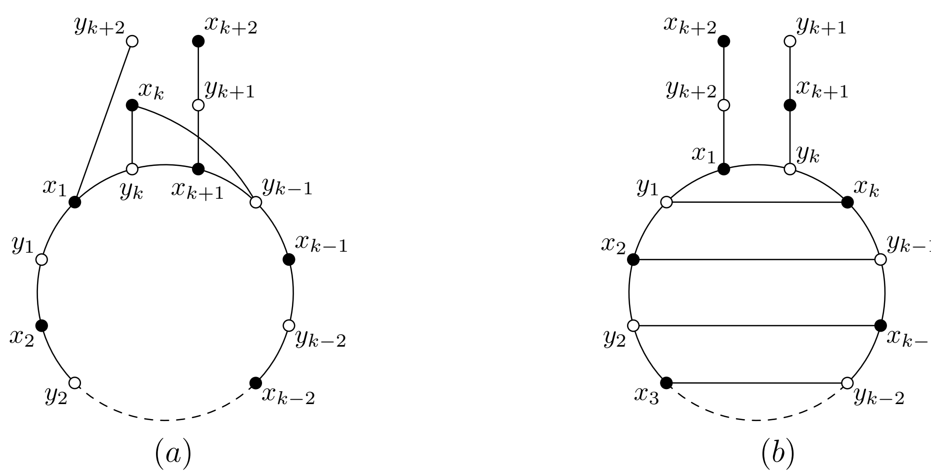

Fig. 10. x_{k+2} being adjacent to just one of $\{y_{k+1}, y_{k+2}\}$

Case 2.2. Suppose that $x_{k+2}y_{k+2} \in E(G)$. Then $x_{k+2}y_{k+1} \notin E(G)$. Since $C_{2(k+1)} \nsubseteq G$, we have $x_ky_{k+1}, x_{k+2}y_1 \notin E(G)$. Then x_k has to adjacent to y_1, since otherwise x_k, x_{k+2}, y_1 and y_{k+1} would construct a $K_{2,2}$ in G^c. Note that there exists a path of length $2(k+1)$ between x_{k+2} and $y_{k-1}(x_{k+2}y_{k+2}x_1y_kx_ky_1x_2y_2x_3y_3 \ldots x_{k-2}y_{k-2}x_{k-1}y_{k-1})$. Hence we have x_{k+2} is nonadjacent to y_{k-1}. By symmetry, y_{k+1} is nonadjacent to x_2. Therefore since $K_{2,2} \nsubseteq$

G^c, x_2 has to adjacent to y_{k-1}. Similarly, there exists a path of length $2(k+1)$ between x_{k+2} and $y_2(x_{k+2}y_{k+2}x_1y_kx_ky_1x_2y_{k-1}x_{k-1}y_{k-2}x_{k-2}\dots y_3x_3y_2)$. Hence we have x_{k+2} is nonadjacent to y_2. By symmetry, y_{k+1} is nonadjacent to x_{k-1}. Therefore since $K_{2,2} \nsubseteq G^c$, y_2 has to adjacent to x_{k-1}. So for even k, we can have $x_3y_{k-2} \in E(G), y_3x_{k-2} \in E(G), x_4y_{k-3} \in E(G), y_4x_{k-3} \in E(G), \dots, x_{\frac{k}{2}-1}y_{\frac{k}{2}+2} \in E(G), y_{\frac{k}{2}-1}x_{\frac{k}{2}+2} \in E(G), x_{\frac{k}{2}}y_{\frac{k}{2}+1} \in E(G)$ sequentially. And for odd k, we can have $x_3y_{k-2} \in E(G), y_3x_{k-2} \in E(G), x_4y_{k-3} \in E(G), y_4x_{k-3} \in E(G), \dots, x_{\frac{k-1}{2}}y_{\frac{k+3}{2}} \in E(G), y_{\frac{k-1}{2}}x_{\frac{k+3}{2}} \in E(G)$ sequentially. That is, we will add $k-2$ chords on the cycle C_{2k} as shown in Fig. 10(b).

Since $C_{2(k+1)} \nsubseteq G$, x_{k+2} is nonadjacent to any vertex of $\{y_1, y_2, \dots, y_k\}$. Therefore since $K_{2,2} \nsubseteq G^c$, x_{k+1} has to be adjacent to at least $k-1$ vertices of $\{y_1, y_2, \dots, y_k\}$. By symmetry, we have y_{k+2} has to be adjacent to at least $k-1$ vertices of $\{x_1, x_2, \dots, x_k\}$. By Lemma 4, we have $C_{2(k+1)} \subseteq G$, a contradiction.

Case 3. Suppose that there are two edges between $\{x_{k+2}\}$ and $\{y_{k+1}, y_{k+2}\}$, namely, $x_{k+2}y_{k+1}, x_{k+2}y_{k+2} \in E(G)$. Since $C_{2(k+1)} \nsubseteq G$, we have $x_{k-1}y_k, x_{k+2}y_1, x_{k+2}y_k \notin E(G)$. Then x_{k-1} has to be adjacent to y_1, since otherwise x_{k-1}, x_{k+2}, y_1 and y_k would construct a $K_{2,2}$ in G^c. By symmetry, y_2 has to be adjacent to x_k as shown in Fig. 12. Now we have $C_{2(k+1)} \subseteq G(y_{k+1}x_{k+1}y_kx_ky_2x_3\dots y_{k-2}x_{k-1}y_1x_1y_{k+2}x_{k+2}y_{k+1})$, a contradiction.

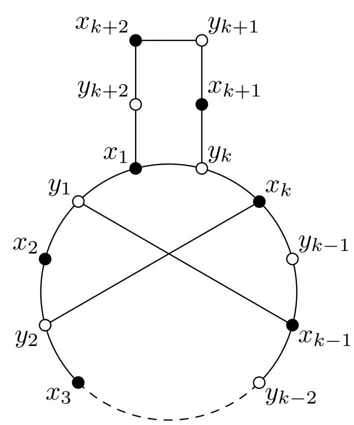

Fig. 12. x_{k+2} being adjacent to y_{k+1} and y_{k+2}

By Case 1-3, we have the assumption that $b(C_{2(k+1)}; K_{2,2}) > k+2$ does not hold. So, we have $b(C_{2(k+1)}; K_{2,2}) \leq k+2$. This completes the induction step, and the proof is finished. □

4 Main results

Setting $m = 3$ in Corollary 1, we have $b(C_6; K_{2,2}) \geq 4$. Furthermore, we can find that a $K_{4,4}$ is a disjoint sum of two subgraphs isomorphic to C_8. Hence, we have $b(C_6; K_{2,2}) \geq 5$. By results in [1], Corollary 1, Lemma 1, Lemma 3 and Lemma 5, we obtain the values of $b(C_{2m}; K_{2,2})$ as follows.

Theorem 2.

$$b(C_{2m}; K_{2,2}) = \begin{cases} 5, & m = 2 \text{ or } 3, \\ m+1, & m \geq 4. \end{cases}$$

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