Ramanujan Type Congruences for a Partition Function

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Abstract

We investigate the arithmetic properties of a certain function $b(n)$ given by $\sum_{n=0}^{\infty} b(n)q^n = (q;q)_\infty^{-2}(q^2;q^2)_\infty^{-2}$. One of our main results is $b(9n+7) \equiv 0 \pmod 9$.

1 Introduction

Recently, Chan [5] introduced the function $a(n)$, which arised from his study of Ramanujan's cubic continued fraction. The function $a(n)$ is defined by

$$\frac{1}{(q;q)_\infty(q^2;q^2)_\infty} = \sum_{n=0}^{\infty} a(n)q^n.$$

Throughout this paper, we assume $|q| < 1$ and we adopt the customary notation

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1 - aq^{n-1}).$$

There are many similar properties between $a(n)$ and the standard partition function $p(n)$, see [5–9, 11] for examples. One of the nice results of $a(n)$ is the generating function of $a(3n+2)$ obtained by Chan [5], which states that

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^4}. \tag{1.1}$$

This identity was also proven by Baruah and Ojah [1] using the 3-dissections for $\varphi(-q)^{-1}$ and $\psi(q)^{-1}$, and by Cao [4] applying the 3-dissection for $(q;q)_\infty(q^2;q^2)_\infty$. We will give another proof based on identities of cubic theta functions in Section 2.

Later, Kim [10] studied the following function $\bar{a}(n)$ counting the number of overcubic partitions of n,

$$\sum_{n=0}^{\infty} \bar{a}(n)q^n = \frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty(q^2;q^2)_\infty}. \tag{1.2}$$

In this paper, we are interested in the function $b(n)$ defined by

$$\frac{1}{(q;q)_\infty^2(q^2;q^2)_\infty^2} = \sum_{n=0}^{\infty} b(n)q^n. \tag{1.3}$$

Our main aim is to present certain arithmetic properties for $b(n)$. In Section 3, we will prove the following Ramanujan type congruence modulo 9, that is, for any $n \geq 0$,

$$b(9n+7) \equiv 0 \pmod 9. \tag{1.4}$$

We also establish two Ramanujan type congruences modulo 5 and 7 by using two classical identities, that is, for any $n \geq 0$,

$$b(5n+4) \equiv 0 \pmod 5, \tag{1.5}$$

and

$$b(7n+2) \equiv b(7n+3) \equiv b(7n+4) \equiv b(7n+6) \equiv 0 \pmod 7. \tag{1.6}$$

2 Preliminaries

In this section, we use cubic theta functions to obtain a 3-dissection of $(q;q)_\infty^{-1}(q^2;q^2)_\infty^{-1}$, which reproduces Chan's identity, and then give a number of facts that will be used in the next section.

Now, let us recall the definition of cubic theta functions $A(q), B(q), C(q)$ due to Borwein et al. [3], namely,

$$\begin{aligned}
A(q) &= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \\
B(q) &= \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \quad \omega = \exp(2\pi i/3), \\
C(q) &= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n}.
\end{aligned}$$

Borwein et al. [3] established the following relations which are useful to our proofs.

Lemma 2.1.

$$A(q) = A(q^3) + 2qC(q^3), \tag{2.1}$$

$$B(q) = A(q^3) - qC(q^3), \tag{2.2}$$

$$C(q) = 3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}, \tag{2.3}$$

$$A(q)A(q^2) = B(q)B(q^2) + qC(q)C(q^2). \tag{2.4}$$

We now derive the 3-dissection for $(q;q)_\infty^{-1}(q^2;q^2)_\infty^{-1}$.

Theorem 2.1. *We have*

$$\frac{1}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{A(q^6)(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^4(q^6;q^6)_\infty^3} + \frac{qA(q^3)(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^4} + \frac{3q^2(q^9;q^9)^3(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}. \tag{2.5}$$

Proof. By (2.3), we see that

$$\frac{q}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{qC(q)C(q^2)}{9(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}. \tag{2.6}$$

From (2.1), (2.2) and (2.4), we find that

$$\begin{aligned} qC(q)C(q^2) &= A(q)A(q^2) - B(q)B(q^2) \\ &= 3qC(q^3)A(q^6) + 3q^2A(q^3)C(q^6) + 3q^3C(q^3)C(q^6). \end{aligned} \tag{2.7}$$

By combining (2.6) and (2.7) together, we obtain that

$$\frac{q}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{qC(q^3)A(q^6) + q^2A(q^3)C(q^6) + q^3C(q^3)C(q^6)}{3(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3},$$

which is equivalent to (2.5) upon using (2.3) to simplify it. This completes the proof. ∎

From the above theorem, we immediately have the following corollary.

Corollary 2.1. *Identity* (1.1) *holds, and*

$$\sum_{n=0}^{\infty} a(3n)q^n = \frac{A(q^2)(q^3;q^3)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^3}, \tag{2.8}$$

$$\sum_{n=0}^{\infty} a(3n+1)q^n = \frac{A(q)(q^6;q^6)_\infty^3}{(q;q)_\infty^3(q^2;q^2)_\infty^4}. \tag{2.9}$$

Now, recall that Ramanujan theta functions $\varphi(q)$ and $\psi(q)$ which are defined as

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

We need several properties of these two functions stated as the following lemmas.

Lemma 2.2.

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty},$$

$$\psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}.$$

Proof. The above two identities are consequence of Jacobi's triple product identity. See [2, p.11] for the detail. ∎

Lemma 2.3.

$$\psi(q) = P(q^3) + q\psi(q^9), \tag{2.10}$$

$$\varphi(-q) = \varphi(-q^9) - 2qQ(q^3), \tag{2.11}$$

where

$$P(q) = \frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^2}{(q;q)_\infty (q^6;q^6)_\infty}$$

and

$$Q(q) = \frac{(q;q)_\infty (q^6;q^6)_\infty^2}{(q^2;q^2)_\infty (q^3;q^3)_\infty}.$$

Proof. With series manipulations, applying Jacobi's product identity, it is not hard to derive above identites and the detail is omitted here. ∎

Lemma 2.4. *If $r_8(n)$ and $t_8(n)$ are given by*

$$\varphi(q)^8 = \sum_{n=1}^{\infty} r_8(n) q^n,$$

$$\psi(q)^8 = \sum_{n=0}^{\infty} t_8(n) q^n.$$

Then

$$r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3 \equiv (-1)^n \sum_{d|n} (-1)^d d \pmod 3, \tag{2.12}$$

$$t_8(n) = \sum_{\substack{d|n+1 \\ d \text{ odd}}} \left(\frac{n+1}{d}\right)^3 \equiv \sum_{\substack{d|n+1 \\ d \text{ odd}}} \frac{n+1}{d} \pmod 3. \tag{2.13}$$

Proof. There are many proofs of the above facts, see, for example, [2, p.70, p.139]. ∎

Lemma 2.5. *For any positive prime p,*

$$(q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod p. \tag{2.14}$$

Proof. The above fact is easily obtained by the following elementary fact

$$(1-q)^p \equiv 1 - q^p \pmod p,$$

and we omit the detail here. ∎

With above lemmas, we can now move to the goal of proving the desired congruences.

3 Ramanujan Type Congruence Modulo $5, 7$ and 9

In this section, we shall first use 3-dissection (2.5) to investigate the behavior of $b(3n+1)$ modulo 9 which yields the desired congruence of $b(9n+7)$. After that, we will apply Jacobi's identity to derive the congruence modulo 5 and use an identity of Ramanujan to establish the congruence modulo 7.

Theorem 3.1. *For any $n \geq 0$, we have*

$$b(9n+7) \equiv 0 \pmod{9}. \tag{3.1}$$

Note that the result in the above theorem is best possible, in the sense that the modulus, 3, cannot be replaced by a higher power of 3.
Proof. By Theorem 2.1, we see that

$$\sum_{n=0}^{\infty} b(n)q^n = \left(\frac{A(q^6)(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^4 (q^6;q^6)_\infty^3} + \frac{qA(q^3)(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^4} + \frac{3q^2(q^9;q^9)^3(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^4 (q^6;q^6)_\infty^4} \right)^2.$$

If we extract those terms from both sides of the above identity in which the power of q is congruent to 1 modulo 3, we easily obtain that

$$\sum_{n=0}^{\infty} b(3n+1)q^{3n+1} \equiv \frac{2qA(q^3)A(q^6)(q^9;q^9)_\infty^3 (q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^7 (q^6;q^6)_\infty^7} \pmod{9}.$$

By dividing both sides of the above identity by q, replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} b(3n+1)q^n \equiv \frac{2A(q)A(q^2)(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^7 (q^2;q^2)_\infty^7} \pmod{9}. \tag{3.2}$$

Now we need the following result

$$(q^3;q^3)_\infty^3 \equiv (q;q)_\infty^9 \pmod{9},$$

which is obtained from the elementary fact

$$(1-q)^9 \equiv (1-q^3)^3 \pmod{9}.$$

By applying the above result in (3.2), we find that

$$\sum_{n=0}^{\infty} b(3n+1)q^n \equiv 2A(q)A(q^2)(q;q)_\infty^2 (q^2;q^2)_\infty^2 \pmod{9}. \tag{3.3}$$

By Lemma 2.2, we see that

$$(q;q)_\infty (q^2;q^2)_\infty = \varphi(-q)\psi(q).$$

By Lemma 2.3, we obtain

$$\begin{aligned}(q;q)_\infty(q^2;q^2)_\infty &= \big(P(q^3)+q\psi(q^9)\big)\big(\varphi(-q^9)-2qQ(q^3)\big)\\ &= P(q^3)\varphi(-q^9)-qP(q^3)Q(q^3)-2q^2Q(q^3)\psi(q^9). \end{aligned}\tag{3.4}$$

Here the last equality is established by the fact that

$$P(q)Q(q)=(q^3;q^3)_\infty(q^6;q^6)_\infty=\varphi(-q^3)\psi(q^3).$$

Now by (2.1), we have

$$\begin{aligned}A(q)A(q^2) &= \big(A(q^3)+2qC(q^3)\big)\big(A(q^6)+2q^2C(q^6)\big)\\ &\equiv A(q^3)A(q^6)+2qA(q^6)C(q^3)+2q^2A(q^3)C(q^6) \pmod 9. \end{aligned}\tag{3.5}$$

By substituting (3.4) and (3.5) into (3.3), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} b(3n+1)q^n &\equiv \big(A(q^3)A(q^6)+2qA(q^6)C(q^3)+2q^2A(q^3)C(q^6)\big)\times\\ &\quad \big(P(q^3)\varphi(-q^9)-qP(q^3)Q(q^3)-2q^2Q(q^3)\psi(q^9)\big)^2 \pmod 9.\end{aligned}$$

Extracting those terms of form q^{3n+2}, dividing by q^2 and replacing q^3 by q, yields that

$$\begin{aligned}\sum_{n=0}^{\infty} b(9n+7)q^n &\equiv A(q)A(q^2)\big(P(q)^2Q(q)^2-4P(q)Q(q)\varphi(-q^3)\psi(q^3)\big)\\ &+ 2A(q)C(q^2)\big(P(q)^2\varphi(-q^3)^2+4qP(q)Q(q)^2\psi(q^3)\big)\\ &+ 2A(q^2)C(q)\big(-2P(q)^2Q(q)\varphi(-q^3)+4qQ(q)^2\psi(q^3)^2\big) \pmod 9.\end{aligned}$$

Now noticing that $A(q)\equiv 1 \pmod 3$, since

$$A(q)=1+6\sum_{n=0}^{\infty}\left\{\frac{q^{3n+1}}{1-q^{3n+1}}-\frac{q^{3n+2}}{1-q^{3n+2}}\right\},$$

by [3] and $C(q)\equiv 0 \pmod 3$ by (2.3), using the relation that $P(q)Q(q)=\varphi(-q^3)\psi(q^3)$, we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} b(9n+7)q^n &\equiv -3P(q)^2Q(q)^2+2C(q^2)P(q)^2\varphi(-q^3)^2+8qC(q^2)P(q)Q(q)^2\psi(q^3)\\ &- 4C(q)P(q)^2Q(q)\varphi(-q^3)+8qC(q)Q(q)^2\psi(q^3)^2 \pmod 9.\end{aligned}$$

It is easy to check that

$$C(q^2)P(q)^2\varphi(-q^3)^2=C(q)P(q)^2Q(q)\varphi(-q^3)=3\frac{(q^2;q^2)_\infty(q^3;q^3)_\infty^8}{(q;q)_\infty^2(q^6;q^6)_\infty}$$

and

$$qC(q^2)P(q)Q(q)^2\psi(q^3) = qC(q)Q(q)^2\psi(q^3)^2 = 3q\frac{(q;q)_\infty(q^6;q^6)_\infty^8}{(q^2;q^2)_\infty^2(q^3;q^3)_\infty}.$$

Thus, to prove $b(9n+7) \equiv 0 \pmod 9$, it only needs to prove that

$$-(q^3;q^3)_\infty^2(q^6;q^6)_\infty^2 + \frac{(q^2;q^2)_\infty(q^3;q^3)_\infty^8}{(q;q)_\infty^2(q^6;q^6)_\infty} + q\frac{(q;q)_\infty(q^6;q^6)_\infty^8}{(q^2;q^2)_\infty^2(q^3;q^3)_\infty} \equiv 0 \pmod 3,$$

which is equivalent to

$$q\frac{(q^2;q^2)_\infty^{16}}{(q;q)_\infty^8} + \frac{(q;q)_\infty^{16}}{(q^2;q^2)_\infty^8} \equiv 1 \pmod 3.$$

By the product formulae for $\varphi(-q)$ and $\psi(q)$, the above congruence can be rewritten as the following form

$$q\psi(q)^8 + \varphi(-q)^8 \equiv 1 \pmod 3.$$

By the definitions of $r_8(n)$ and $t_8(n)$, one can show that the above identity is equivalent to

$$(-1)^n r_8(n) + t_8(n-1) \equiv 0 \pmod 3$$

for all $n \geq 1$. This, in return, is equivalent to

$$\sum_{d|n}(-1)^d d + \sum_{\substack{d|n \\ d\,odd}} \frac{n}{d} \equiv 0 \pmod 3. \tag{3.6}$$

for all $n \geq 1$. To establish (3.6), let $n = 2^r n_1$, where $(2, n_1) = 1$. Then

$$\begin{aligned}
\sum_{d|n}(-1)^d d + \sum_{\substack{d|n \\ d\,odd}} \frac{n}{d} &= \sum_{k=1}^{r} 2^k \sum_{d|n_1} d - \sum_{d|n_1} d + \sum_{d|n_1} \frac{2^r n_1}{d} \\
&= \left(\sum_{k=1}^{r} 2^k - 1 + 2^r\right) \sum_{d|n_1} d \\
&= 3(2^r - 1) \sum_{d|n_1} d.
\end{aligned}$$

This completes the congruence relation (3.6) and the proof is complete. ∎

Now we turn to prove the following theorem.

Theorem 3.2. *For any $n \geq 0$, we have*

$$b(5n+4) \equiv 0 \pmod 5 \tag{3.7}$$

and

$$b(7n+2) \equiv b(7n+3) \equiv b(7n+4) \equiv b(7n+6) \equiv 0 \pmod 7. \tag{3.8}$$

Proof. By applying the case $p=5$ in Lemma 2.5, we obtain

$$\sum_{n=0}^{\infty} b(n)q^n \equiv \frac{(q;q)_\infty^3 (q^2;q^2)_\infty^3}{(q^5;q^5)_\infty (q^{10};q^{10})_\infty} \pmod{5}.$$

Thus, to prove that $b(5n+4)$ is congruent to 0 modulo 5, we only need to show that the coefficient of q^{5n+4} in the function $(q;q)_\infty^3 (q^2;q^2)_\infty^3$ is a multiple of 5. Using Jacobi's identity [2, p.14], namely,

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2},$$

we have

$$\begin{aligned}(q;q)_\infty^3 (q^2;q^2)_\infty^3 &= \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{m(m+1)/2} \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} (2m+1)(2n+1) q^{m(m+1)/2+n(n+1)}.\end{aligned}$$

If $m(m+1)/2 + n(n+1)$ is congruent to 4 modulo 5, we must have $m \equiv n \equiv 2 \pmod{5}$, that is, $2m+1$ and $2n+1$ are both divided by 5. This establishes the congruence (3.7).

Now we turn to the congruence modulo 7. By applying the case $p=7$ in Lemma 2.5, we get

$$\begin{aligned}\sum_{n=0}^{\infty} b(n)q^n &\equiv \frac{(q;q)_\infty^5}{(q^2;q^2)_\infty^2 (q^7;q^7)_\infty} \pmod{7} \\ &= \frac{\varphi(-q)^2 (q;q)_\infty}{(q^7;q^7)_\infty}.\end{aligned}$$

By an identity of Ramanujan [2, p.20], namely,

$$\varphi(-q^2)^2 (q^2;q^2)_\infty = \sum_{n=-\infty}^{\infty} (6n+1) q^{3n^2+n},$$

we find that

$$\sum_{n=0}^{\infty} b(n)q^n \equiv \frac{1}{(q^7;q^7)_\infty} \sum_{n=-\infty}^{\infty} (6n+1) q^{(3n^2+n)/2} \pmod{7}.$$

Since there are no integer n with $(3n^2+n)/2$ congruent to $3, 4$, or 6 modulo 7, it follows that

$$b(7n+3) \equiv b(7n+4) \equiv b(7n+6) \equiv 0 \pmod{7}.$$

If $(3n^2+n)/2 \equiv 2 \pmod{7}$ holds, then n should be congruent to 1 modulo 7, that is, $6n+1$ is a multiple of 7. This yields that $b(7n+2) \equiv 0 \pmod{7}$, and we complete the proof. ∎

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