Alspach's Problem: The Case of Hamilton Cycles and 5-Cycles

Heather Jordon

Department of Mathematics
Illinois State University
Normal, IL 61790-4520

hjordon@ilstu.edu

Submitted: Oct 19, 2009; Accepted: Mar 26, 2011; Published: Apr 7, 2011
Mathematics Subject Classifications: 05C70, 05C38

Abstract

In this paper, we settle Alspach's problem in the case of Hamilton cycles and 5-cycles; that is, we show that for all odd integers $n \geq 5$ and all nonnegative integers h and t with $hn + 5t = n(n-1)/2$, the complete graph K_n decomposes into h Hamilton cycles and t 5-cycles and for all even integers $n \geq 6$ and all nonnegative integers h and t with $hn+5t = n(n-2)/2$, the complete graph K_n decomposes into h Hamilton cycles, t 5-cycles, and a 1-factor. We also settle Alspach's problem in the case of Hamilton cycles and 4-cycles.

1 Introduction

In 1981, Alspach [5] posed the following problem: Prove there exists a decomposition of K_n (n odd) or $K_n - I$ (n even) into cycles of lengths $m_1, m_2, \ldots, m_t$ whenever $3 \leq m_i \leq n$ for $1 \leq i \leq t$ and $m_1 + m_2 + \cdots + m_t = n(n-1)/2$ (number of edges in K_n) or $m_1 + m_2 + \cdots + m_t = n(n-2)/2$ (the number of edges in $K_n - I$). Results of Alspach, Gavlas, and Šajna [6, 32] settle Alspach's problem in the case that all the cycle lengths are the same, i.e., $m_1 = m_2 = \cdots = m_t = m$. The problem has attracted much interest and has also been settled for several cases in which a small number of short cycle lengths are allowed [3, 4, 9, 25, 26, 31]. Two surveys are given in [12, 18].

In [19], Caro and Yuster settle Alspach's problem for all $n \geq N(L)$ where $L = \max\{m_1, m_2, \ldots, m_t\}$ and $N(L)$ is a function of L which grows faster than exponentially. In [8], Balister improved the result of Caro and Yuster by settling the problem for all $n \geq N$, where N is a very large constant given by a linear function of L and the longest cycle length is less than about $\frac{n}{20}$. In [16], Bryant and Horsley show that there exists a sufficiently large integer N such that for all odd $n \geq N$, the complete graph

K_n decomposes into cycles of lengths $m_1, m_2, \ldots, m_t$ with $3 \leq m_i \leq n$ for $1 \leq i \leq t$ if and only if $m_1 + m_2 + \ldots + m_t = n(n-1)/2$. Bryant and Horsley also show that for any n, if all the cycle lengths are greater than about half n [15], or if the cycle lengths $m_1 \leq m_2 \leq \cdots \leq m_t$ are less than about half n with $m_t \leq 2m_{t-1}$ [16], then the decompositions exist as long as the obvious necessary conditions are satisfied. In [14], Bryant and Horsley prove the existence of decompositions of K_n for n odd into cycles for a large family of lists of specified cycle lengths, settling the problem in about 10% of the possible cases.

In [17], Bryant and Maenhaut settle Alspach's problem in the case that the cycle lengths are the shortest and longest possible, that is, decomposing K_n or $K_n - I$ into triangles and Hamilton cycles. It turns that it is not too difficult to solve Alspach's problem in the case that the cycles lengths are four and n, and we include a proof of this result in this paper for completeness (see Theorem 4.1). It is, however, more difficult to settle Alspach's problem in the case that the cycles lengths are five and n, and thus that is our main result. This problem was solved in [16] for very large odd n; here we solve it for all n. The following theorem is the main result of this paper.

Theorem 1.1

(a) *For all odd integers $n \geq 5$ and nonnegative integers h and t, the graph K_n can be decomposed into h Hamilton cycles and t 5-cycles if and only if $hn+5t = n(n-1)/2$.*

(b) *For all even integers $n \geq 6$ and nonnegative integers h and t, the graph K_n can be decomposed into h Hamilton cycles, t 5-cycles, and a 1-factor if and only if $hn + 5t = n(n-2)/2$.*

Other authors have considered the problem of decomposing the complete graph into m-cycles and some other subgraph or subgraphs. In [24], for $m \geq 3$ and k odd, El-Zanati et al. decompose K_{2mx+k} into $\frac{k-1}{2}$ Hamilton cycles and m-cycles (except in the case that $k = 3$ and $m = 5$). In [27], Horak et al. decompose K_n into α triangle factors (a 2-factor where each component is a triangle) and β Hamilton cycles for several infinite classes of orders n. In [30], Rees gives necessary and sufficient conditions for a decomposition of K_n into α triangle factors and β 1-factors. In [22, 34], the authors consider the problem of finding the total number of triangles in 2-factorizations of K_n. In [1, 23], the problem of finding the total number of 4-cycles in 2-factorizations of K_n or $K_n - I$ is considered. In [2], Adams et al. found necessary and sufficient conditions for a decomposition of the complete graph into 5-cycle factors and 1-factors. In [11], Bryant considers the problem of finding decompositions of K_n into 2-factors in which most of the 2-factors are Hamilton cycles and the remaining 2-factors are any specified 2-regular graphs on n vertices.

Our methods involve circulant graphs and difference constructions. In Section 2, we give some basic definitions while the proof of Theorem 1.1 is given in Section 4. In Section 3, decompositions of specific circulant graphs are given which will aid in proving our main result.

2 Definitions and Preliminaries

For a positive integer n, let $[1, n]$ denote the set $\{1, 2, \ldots, n\}$. Throughout this paper, K_n denotes the complete graph on n vertices, $K_n - I$ denotes the complete graph on n vertices with the edges of a 1-factor I (a 1-regular spanning subgraph) removed, and C_m denotes the m-cycle $(v_1, v_2, \ldots, v_m)$. An n-cycle in a graph with n vertices is called a *Hamilton* cycle. A *decomposition* of a graph G is a set $\{H_1, H_2, \ldots, H_k\}$ of subgraphs of G such that every edge of G belongs to exactly one H_i for some i with $1 \leq i \leq k$.

For $x \not\equiv 0 \pmod{n}$, the *modulo n length* of an integer x, denoted $|x|_n$, is defined to be the smallest positive integer y such that $x \equiv y \pmod{n}$ or $x \equiv -y \pmod{n}$. Note that for any integer $x \not\equiv 0 \pmod{n}$, it follows that $|x|_n \in [1, \lfloor \frac{n}{2} \rfloor]$. If L is a set of modulo n lengths, the *circulant graph* $\langle L \rangle_n$ is the graph with vertex set $\mathbb{Z}_n$, the integers modulo n, and edge set $\{\{i, j\} \mid |i - j|_n \in L\}$. Observe that $K_n \cong \langle [1, \lfloor \frac{n}{2} \rfloor] \rangle_n$. An edge $\{i, j\}$ in a graph with vertex set $\mathbb{Z}_n$ is called an *edge of length* $|i - j|_n$.

Let $n > 0$ be an integer and suppose there exists an ordered m-tuple $(d_1, d_2, \ldots, d_m)$ satisfying each of the following:

(i) d_i is an integer for $i = 1, 2, \ldots, m$;

(ii) $|d_i|_n \neq |d_j|_n$ for $1 \leq i < j \leq m$;

(iii) $d_1 + d_2 + \cdots + d_m \equiv 0 \pmod{n}$; and

(iv) $d_1 + d_2 + \cdots + d_r \not\equiv d_1 + d_2 + \cdots + d_s \pmod{n}$ for $1 \leq r < s \leq m$.

Then $(0, d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_{m-1})$ is an m-cycle in the graph $\langle \{|d_1|_n, |d_2|_n, \ldots, |d_m|_n\} \rangle_n$ and $\{(i, i + d_1, i + d_1 + d_2, \ldots, i + d_1 + d_2 + \cdots + d_{m-1}) \mid i = 0, 1, \ldots, n - 1\}$ is a decomposition of $\langle \{|d_1|_n, |d_2|_n, \ldots, |d_m|_n\} \rangle_n$ into m-cycles, where all entries are taken modulo n. An m-tuple satisfying (i)-(iv) is called a *modulo n difference m-tuple*, it *corresponds* to the starter m-cycle $(0, d_1, d_1 + d_2, \ldots, d_1 + d_2 + \ldots + d_{m-1})$, it *uses* edges of lengths $|d_1|_n, |d_2|_n, \ldots, |d_m|_n$, and it *generates* a decomposition of $\langle \{|d_1|_n, |d_2|_n, \ldots, |d_m|_n\} \rangle_n$ into m-cycles. A *modulo n m-cycle difference set of size t*, or an *m-cycle difference set of size t* when the value of n is understood, is a set consisting of t modulo n difference m-tuples that use edges of distinct lengths $\ell_1, \ell_2, \ldots, \ell_{tm}$; the m-cycles corresponding to the difference m-tuples generate a decomposition of $\langle \{\ell_1, \ell_2, \ldots, \ell_{tm}\} \rangle_n$ into m-cycles. Difference m-tuples are studied in [13] where necessary and sufficient conditions are given for a partition of the set $[1, mt]$, where $m \geq 3$ and $t \geq 1$, into t difference m-tuples. In this paper, we will use difference triples to construct difference 5-tuples. Difference triples have been studied extensively and can be constructed from *Langford sequences*.

A *Langford sequence of order t and defect d* is a sequence $L = (\ell_1, \ell_2, \ldots, \ell_{2t})$ of $2t$ integers satisfying the conditions

(L1) for every $k \in [d, d + t - 1]$ there exists exactly two elements $\ell_i, \ell_j \in L$ such that $\ell_i = \ell_j = k$, and

(L2) if $\ell_i = \ell_j = k$ with $i < j$, then $j - i = k$.

A *hooked Langford sequence of order* t *and defect* d is a sequence $L = (\ell_1, \ell_2, \ldots, \ell_{2t+1})$ of $2t+1$ integers satisfying conditions (L1) and (L2) above and

(L3) $\ell_{2t} = 0$.

Simpson [33] gave necessary and sufficient conditions for the existence of a Langford sequence of order t and defect d.

Theorem 2.1 (Simpson [33]) *There exists a Langford sequence of order* t *and defect* d *if and only if*

(1) $t \geq 2d - 1$, and

(2) $t \equiv 0, 1 \pmod 4$ and d is odd, or $t \equiv 0, 3 \pmod 4$ and d is even.

There exists a hooked Langford sequence of order t *and defect* d *if and only if*

(1) $t(t - 2d + 1) + 2 \geq 0$, and

(2) $t \equiv 2, 3 \pmod 4$ and d is odd, or $t \equiv 1, 2 \pmod 4$ and d is even.

A Langford sequence or hooked Langford sequence of order t can be used to construct a 3-cycle difference set of size t using edges of lengths $[d, d+3t-1]$ or $[d, d+3t] \setminus \{d+3t-1\}$ respectively, providing a decomposition of $\langle [d, d+3t-1] \rangle_n$ for all $n \geq 2(d+3t-1)+1$ or $\langle [d, d+3t] \setminus \{d+3t-1\} \rangle_n$ for $n = 2(d+3t-1)+1$ and for all $n \geq 2(d+3t)+1$ into 3-cycles.

Notice that if $(d_1, d_2, \ldots, d_m)$ is a modulo n difference m-tuple with $d_1 + d_2 + \ldots + d_m = 0$, not just $d_1 + d_2 + \ldots + d_m \equiv 0 \pmod n$, then $(d_1, d_2, \ldots, d_m)$ is a modulo w difference m-tuple for all $w \geq M/2+1$ where $M = |d_1| + |d_2| + \cdots + |d_m|$. In the literature, difference triples obtained from Langford sequences (and hooked Langford sequences) are usually written (a, b, c) with $a + b = c$. However, as it is more convenient for extending these ideas to difference m-tuples with $m > 3$, we will use the equivalent representation with c replaced by $-c$ so that $a + b + c = 0$.

In this paper, we are interested in 5-cycle difference sets that are of Langford type. For 5-cycle difference sets, we will partition the set $[5, 5t+4]$ into a 5-cycle difference set of size t if $t \equiv 0, 3 \pmod 4$ or the set $[5, 5t+5] \setminus \{5t+4\}$ into a 5-cycle difference set of size t if $t \equiv 1, 2 \pmod 4$.

Lemma 2.2 *Let* $t \geq 1$ *be an integer.*

(1) The set $[5, 5t+4]$ *can be partitioned into a* 5*-cycle difference set of size* t *if and only if* $t \equiv 0, 3 \pmod 4$.

(2) The set $[5, 5t+5] \setminus \{5t+4\}$ *can be partitioned into a* 5*-cycle difference set of size* t *if and only if* $t \equiv 1, 2 \pmod 4$.

Proof. If $t \equiv 1, 2 \pmod 4$, then $5 + 6 + \cdots + (5t + 4)$ is odd and hence it follows that no partition of $[5, 5t + 4]$ into a 5-cycle difference set of size t exists. Similarly, if $t \equiv 0, 3 \pmod 4$, then $5 + 6 + \cdots + (5t + 3) + (5t + 5)$ is odd and thus no partition of $[5, 5t + 5] \setminus \{5t + 4\}$ into a 5-cycle difference set of size t exists. Hence, it remains to partition the set $[5, 5t + 4]$ into a 5-cycle difference set of size t if $t \equiv 0, 3 \pmod 4$ and to partition the set $[5, 5t+5] \setminus \{5t+4\}$ into a 5-cycle difference set of size t if $t \equiv 1, 2 \pmod 4$.

For $t = 1$, the required difference 5-tuple is $(5, -8, 7, 6, -10)$. For $t = 2$, the required set of difference 5-tuples is $\{(5, -8, 9, 6, -12), (7, -13, 11, 10-15)\}$. For $t = 3$, the required set of difference 5-tuples is $\{(5, -8, 9, 6, -12), (10, -17, 15, 11, -19), (7, -16, 14, 13, -18)\}$. For $t = 4$, the required set of difference 5-tuples is

$$\{(5, -10, 9, 17, -21), (6, -15, 13, 18, -22), (7, -14, 11, 19, -23), (8, -16, 12, 20, -24)\}.$$

Hence, we may assume $t \geq 5$. The proof now splits into four cases depending on the congruence class of t modulo 4.

Case 1. *Suppose that* $t \equiv 0 \pmod 4$. By Theorem 2.1, there exists a Langford sequence of order t and defect 3, and let $\{(a_i, b_i, c_i) \mid 1 \leq i \leq t\}$ be a set of t difference triples using edges of lengths $[3, 3t + 2]$ constructed from such a sequence. Note that the set $[3t+5, 5t+4]$ consists of t consecutive odd integers and t consecutive even integers. Thus, partition the set $[3t + 5, 5t + 4]$ into t pairs $\{d_i, d_i + 2\}$ for $i = 1, 2, \ldots, t$.

Case 2. *Suppose that* $t \equiv 3 \pmod 4$. Note that we may assume $t \geq 7$. By Theorem 2.1, there exists a hooked Langford sequence of order t and defect 3, and let $\{(a_i, b_i, c_i) \mid 1 \leq i \leq t\}$ be a set of t difference triples using edges of lengths $[3, 3t + 3] \setminus \{3t + 2\}$ constructed from such a sequence. Note that the set $[3t + 4, 5t + 4] \setminus \{3t + 5\}$ consists of $t+1$ consecutive odd integers and $t-1$ consecutive even integers. Thus, partition the set $[3t + 4, 5t + 4] \setminus \{3t + 5\}$ into t sets $\{d_i, d_i + 2\}$ for $i = 1, 2, \ldots, t$.

Case 3. *Suppose that* $t \equiv 1 \pmod 4$. Note that we may assume $t \geq 5$. By Theorem 2.1, there exists a Langford sequence of order t and defect 3, and let $\{(a_i, b_i, c_i) \mid 1 \leq i \leq t\}$ be a set of t difference triples using edges of lengths $[3, 3t + 2]$ constructed from such a sequence. Note that the set $[3t + 5, 5t + 5] \setminus \{5t + 4\}$ consists of $t + 1$ consecutive even integers and $t-1$ consecutive odd integers. Thus, partition the set $[3t+5, 5t+5] \setminus \{5t+4\}$ into t sets $\{d_i, d_i + 2\}$ for $i = 1, 2, \ldots, t$.

Case 4. *Suppose that* $t \equiv 2 \pmod 4$. Note that we may assume $t \geq 6$. By Theorem 2.1, there exists a hooked Langford sequence of order t and defect 3, and let $\{(a_i, b_i, c_i) \mid 1 \leq i \leq t\}$ be a set of t difference triples using edges of lengths $[3, 3t + 3] \setminus \{3t + 2\}$ constructed from such a sequence. Note that the set $[3t + 4, 5t + 5] \setminus \{3t + 5, 5t + 4\}$ consists of t consecutive odd integers and t consecutive even integers. Thus, partition the set $[3t + 4, 5t + 5] \setminus \{3t + 5, 5t + 4\}$ into t sets $\{d_i, d_i + 2\}$ for $i = 1, 2, \ldots, t$.

For each congruence class of $t \geq 5$ modulo 4, let $X = [x_{i,j}]$ be the $t \times 5$ array such that

$$X = \begin{bmatrix} a_1+2 & c_1-2 & b_1+2 & d_1 & -(d_1+2) \\ a_2+2 & c_2-2 & b_2+2 & d_2 & -(d_2+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_t+2 & c_t-2 & b_t+2 & d_t & -(d_t+2) \end{bmatrix}.$$

Construct the required set of t difference 5-tuples from the rows of X using the ordering $(x_{i,1}, x_{i,2}, x_{i,4}, x_{i,3}, x_{i,5})$ for $i = 1, 2, \ldots, t$. ■

In decomposing K_n or $K_n - I$ into Hamilton cycles and 5-cycles, the most difficult case is when $5 \mid n$, and we use m-extended Langford sequences. For an integer $m \geq 1$, an *m-extended Langford sequence of order t and defect d* is a sequence $EL_m = (\ell_1, \ell_2, \ldots, \ell_{2t+1})$ of $2t+1$ integers satisfying (L1) and (L2) above, and

(E1) $\ell_m = 0$.

Clearly, a m-extended Langford sequence of order t and defect d provides a 3-cycle difference set of size t using edges of lengths $[d, d+3t] \setminus \{d-1+t+m\}$. A *hooked m-extended Langford sequence of order t and defect d* is a sequence $HEL_m = (\ell_1, \ell_2, \ldots, \ell_{2t+2})$ of $2t+2$ integers satisfying conditions (L1), (L2), and (E1) above, and

(E2) $\ell_{2t+1} = 0$.

A hooked m-extended Langford sequence of order t and defect d provides a 3-cycle difference set of size t using edges of lengths $[d, d+3t+1] \setminus \{d-1+t+m, d+3t\}$. The following theorem provides necessary and sufficient conditions for the existence of m-extended Langford sequences with defect 1 [7] and defect 2 [29], and hooked m-extended Langford sequences with defect 1 [28] (as a consequence of a more general result).

Theorem 2.3 (Baker [7], Linek and Jiang [28], Linek and Shalaby [29]) *For $m \geq 1$,*

- *an m-extended Langford sequence of order t and defect 1 exists if and only if m is odd and $t \equiv 0, 1 \pmod 4$, or m is even and $t \equiv 2, 3 \pmod 4$;*
- *an m-extended Langford sequence of order t and defect 2 exists if and only if m is odd and $t \equiv 0, 3 \pmod 4$, or m is even and $t \equiv 1, 2 \pmod 4$; and*
- *a hooked m-extended Langford sequence of t and defect 1 exists if and only if m is even and $t \equiv 0, 1 \pmod 4$, or m is odd and $t \equiv 2, 3 \pmod 4$.*

3 Decompositions of Some Special Circulant Graphs

In this section, we decompose some special circulant graphs into various combinations of 5-cycles and Hamilton cycles. These decompositions will be used to prove our main result. Our first few lemmas concern the decomposition of certain circulant graphs into Hamilton cycles.

In [10], Bermond et al. proved that any 4-regular connected Cayley graph on a finite abelian group can be decomposed into two Hamilton cycles. Note that $\langle\{s,t\}\rangle_n$ is a connected 4-regular graph if $s,t < \frac{n}{2}$ and $\gcd(s,t,n) = 1$. We will need the following special case of their result.

Lemma 3.1 (Bermond, Favaron, Mahéo [10]) *For integers s, t, and n with $s < t < \frac{n}{2}$ and $\gcd(s,t,n) = 1$, the graph $\langle\{s,t\}\rangle_n$ can be decomposed into two Hamilton cycles.*

In [20, 21], Dean established the following result for 6-regular connected circulant graphs.

Lemma 3.2 (Dean [20, 21]) *For integers r, s, t, and n with $r < s < t < \frac{n}{2}$, $\gcd(r,s,t,n) = 1$, and either n is odd or $\gcd(x,n) = 1$ for some $x \in \{r,s,t\}$, the graph $\langle\{r,s,t\}\rangle_n$ can be decomposed into three Hamilton cycles.*

Using the previous two lemmas, we obtain the following result, whose proof is very similar to the corresponding result in [17]. Note that when n is even, the graph $\langle\{\frac{n}{2} - 2, \frac{n}{2}\}\rangle_n$ may be disconnected.

Lemma 3.3

(1) For each odd integer $n \geq 5$ and each integer x with $1 \leq x \leq \frac{n-1}{2}$, the graphs $\langle[x, \frac{n-1}{2}]\rangle_n$ and $\langle[x, \frac{n-1}{2}] \setminus \{x+1\}\rangle_n$ decompose into Hamilton cycles.

(2) For each even integer $n \geq 6$ and

(a) for each integer x with $1 \leq x \leq \frac{n}{2} - 1$, the graph $\langle[x, \frac{n}{2}]\rangle_n$ decomposes into Hamilton cycles and a 1-factor; and

(b) for each integer x with $1 \leq x \leq \frac{n}{2} - 3$, the graph $\langle[x, \frac{n-1}{2}] \setminus \{x+1\}\rangle_n$ decomposes into Hamilton cycles and a 1-factor.

Proof. Suppose first $n \geq 5$ is an odd integer and let x be an integer such that $1 \leq x \leq \frac{n-1}{2}$. By Lemmas 3.1 and 3.2, the result will follow if we partition each set $[x, \frac{n-1}{2}]$ and $[x, \frac{n-1}{2}] \setminus \{x+1\}$ into a combination of singletons $\{s\}$ with $\gcd(s,n) = 1$, pairs $\{s,t\}$ with $\gcd(s,t,n) = 1$, and triples $\{r,s,t\}$ with $\gcd(r,s,t,n) = 1$. Before continuing, note that $\gcd(\frac{n-1}{2}, n) = 1$.

First, let $D = [x, \frac{n-1}{2}]$. Partition D into consecutive pairs if $|D|$ is even or into consecutive pairs and the set $\{\frac{n-1}{2}\}$ if $|D|$ is odd. Next, let $D = [x, \frac{n-1}{2}] \setminus \{x+1\}$. If $|D|$ is odd, then partition D into $\{x, x+2, \frac{n-1}{2}\}$ and consecutive pairs. If $|D|$ is even, then partition D into $\{x, \frac{n-1}{2}\}$ and consecutive pairs.

Now let $n \geq 6$ be an even integer. In this case, we seek to decompose the desired graph into Hamilton cycles and a 1-factor. In most cases, the 1-factor will be the graph $\langle\{\frac{n}{2}\}\rangle_n$; however, we will also need to decompose the graph $\langle\{\frac{n}{2} - 1, \frac{n}{2}\}\rangle_n$ into a Hamilton cycle and 1-factor. If $n \equiv 0 \pmod 4$, then $\gcd(\frac{n}{2}-1,n)=1$ so that $\langle\{\frac{n}{2}-1\}\rangle_n$ is a Hamilton cycle and $\langle\{\frac{n}{2}\}\rangle_n$ is a 1-factor. If $n \equiv 2 \pmod 4$, then $\langle\{\frac{n}{2}-1,\frac{n}{2}\}\rangle_n \cong C_{\frac{n}{2}} \times K_2$, which clearly has a Hamilton cycle whose removal leaves a 1-factor. Thus, when n is even, the result will follow by the previous observation and Lemmas 3.1 and 3.2 if we partition the set $[x,\frac{n}{2}]$ or the set $[x,\frac{n}{2}]\setminus\{x+1\}$ into a combination of singletons $\{s\}$ with $\gcd(s,n)=1$, pairs $\{s,t\}$ with $\gcd(s,t,n)=1$, and triples $\{r,s,t\}$ with $\gcd(r,s,t,n)=1$ and $\gcd(x,n)=1$ for some $x\in\{r,s,t\}$, and possibly $\{\frac{n}{2}-1,\frac{n}{2}\}$ or $\{\frac{n}{2}\}$.

Let x be an integer with $1\le x\le \frac{n}{2}-1$ and let $D=[x,\frac{n}{2}]$. Partition D into consecutive pairs if $|D|$ is even (necessarily including the set $\{\frac{n}{2}-1,\frac{n}{2}\}$) or into consecutive pairs and $\{\frac{n}{2}\}$ if $|D|$ is odd.

Next, let x be an integer with $1\le x\le\frac{n}{2}-3$ and let $D=[x,\frac{n}{2}]\setminus\{x+1\}$. Observe that $|D|=\frac{n}{2}-x\ge 3$. Suppose first that $|D|$ is even. Thus $\frac{n}{2}$ and x are either both even or both odd and, since $|D|\ge 4$, we have $x+2<\frac{n}{2}-1$. If $\frac{n}{2}$ and x are both even, then $n\equiv 0 \pmod 4$ so that we may partition D into $\{x,x+2,\frac{n}{2}\}$, $\{\frac{n}{2}\}$, and consecutive pairs. If $\frac{n}{2}$ and x are both odd, then partition D into $\{x,x+2\}$, $\{\frac{n}{2}-1,\frac{n}{2}\}$, and consecutive pairs.

Finally, suppose $|D|$ is odd; thus $\frac{n}{2}$ and x are of opposite parity. If $\frac{n}{2}$ is even (hence $n\equiv 0 \pmod 4$) and x is odd, then partition D into $\{x,\frac{n}{2}-1\}$, $\{\frac{n}{2}\}$, and consecutive pairs. Now suppose $\frac{n}{2}$ is odd (hence $n\equiv 2 \pmod 4$) and x is even. We consider the case $|D|=3$ separately, that is, $D=\{\frac{n}{2}-3,\frac{n}{2}-1,\frac{n}{2}\}$. Consider the graph $\langle\{\frac{n}{2}-3,\frac{n}{2}-1,\frac{n}{2}\}\rangle_n$. Note that each of the graphs $\langle\{\frac{n}{2}-1\}\rangle_n$ and $\langle\{\frac{n}{2}-3\}\rangle_n$ consists of two vertex-disjoint $\frac{n}{2}$-cycles, consisting of the even and odd integers, respectively, in $\mathbb{Z}_n$. Let $C_1=\langle\{\frac{n}{2}-3\}\rangle_n\setminus\{\{\frac{n}{2}+3,0\},\{3,\frac{n}{2}\}\}\cup\{\{0,\frac{n}{2}\},\{3,\frac{n}{2}+3\}\}$. Note that $\frac{n}{2}$ odd and each of $\frac{n}{2}-1$ and $\frac{n}{2}-3$ even ensures that C_1 is, in fact, a Hamilton cycle. Similarly, $C_2=\langle\{\frac{n}{2}-1\}\rangle_n\setminus\{\{2,\frac{n}{2}+1\},\{1,\frac{n}{2}+2\}\}\cup\{\{2,\frac{n}{2}+2\},\{1,\frac{n}{2}+1\}\}$ is a Hamilton cycle. Since $\langle\{\frac{n}{2}-3,\frac{n}{2}-1,\frac{n}{2}\}\rangle_n\setminus(E(C_1)\cup E(C_2))$ is a 1-regular graph, the desired conclusion follows. Now assume $|D|\ge 5$ so that $x+2<\frac{n}{2}-2$. Note also that $\gcd(\frac{n}{2}-2,n)=1$. Partition D into $\{x,x+2,\frac{n}{2}-2\}$, $\{\frac{n}{2}-1,\frac{n}{2}\}$, and consecutive pairs. The desired result now follows. ■

Combining the previous lemma with Lemma 2.2, we obtain the following result.

Corollary 3.4

(1) For each odd integer $n\ge 11$ and for each $s=0,1,\ldots,\lfloor\frac{n-9}{10}\rfloor$, the graph $\langle[5,\frac{n-1}{2}]\rangle_n$ decomposes into sn 5-cycles and $\frac{n-1}{2}-5s-4$ Hamilton cycles.

(2) For each even integer $n\ge 14$ and for each $s=0,1,\ldots,\lfloor\frac{n-14}{10}\rfloor$, the graph $\langle[5,\frac{n}{2}]\rangle_n$ decomposes into sn 5-cycles, $\frac{n}{2}-5s-5$ Hamilton cycles, and a 1-factor.

Proof. First, let $n\ge 11$ be an odd integer. Let s be a nonnegative integer such that $s\le\lfloor\frac{n-9}{10}\rfloor$. Note that this implies $5s+4\le\frac{n-1}{2}$. Clearly, if $s=0$, then Lemma 3.3 gives the desired result. Thus, we may assume $s\ge 1$. Lemma 2.2 gives a partition of $[5,5s+4]$ or $[5,5s+5]\setminus\{5s+4\}$ into a 5-cycle difference set of size s which can be used to construct a decomposition of the graph $\langle[5,5s+4]\rangle_n$ or the graph $\langle[5,5s+5]\setminus\{5s+4\}\rangle_n$ into sn 5-cycles. If $5s+4<\frac{n-1}{2}$, then since both graphs $\langle[5s+5,\frac{n-1}{2}]\rangle_n$ and $\langle[5s+4,\frac{n-1}{2}]\setminus\{5s+5\}\rangle_n$ can be decomposed into $\frac{n-1}{2}-5s-4$ Hamilton cycles, respectively, by Lemma 3.3, the result follows.

Now, let $n\geq 14$ be an even integer. Let s be a nonnegative integer such that $s\leq\lfloor\frac{n-14}{10}\rfloor$. Note that this implies $5s+4\leq\frac{n}{2}-3$. Clearly, if $s=0$, then Lemma 3.3 gives the desired result. Thus, we may assume $s\geq 1$. Lemma 2.2 gives a partition of $[5,5s+4]$ or $[5,5s+5]\setminus\{5s+4\}$ into a 5-cycle difference set of size s which can be used to construct a decomposition of the graph $\langle[5,5s+4]\rangle_n$ or the graph $\langle[5,5s+5]\setminus\{5s+4\}\rangle_n$ into sn 5-cycles. Since $5s+4\leq\frac{n}{2}-3$, both graphs $\langle[5s+5,\frac{n}{2}]\rangle_n$ and $\langle[5s+4,\frac{n}{2}]\setminus\{5s+5\}\rangle_n$ can be decomposed into $\frac{n}{2}-5s-5$ Hamilton cycles and a 1-factor, respectively, by Lemma 3.3, the result follows. ■

The next result concerns decomposing a circulant graph with a very specific edge set into various combinations of 5-cycles and Hamilton cycles.

Lemma 3.5 *For $n\equiv 0 \pmod 5$ with $n\geq 10$ and for each $j=0,1,2,3,4$, the graph $\langle[1,4]\rangle_n$ can be decomposed into $jn/5$ 5-cycles and $4-j$ Hamilton cycles.*

Proof. Let $n\equiv 0 \pmod 5$ with $n\geq 10$. Suppose first $j=0$. Then decompose the graphs $\langle\{1,2\}\rangle_n$ and $\langle\{3,4\}\rangle_n$ into two Hamilton cycles each using Lemma 3.1.

Next, suppose $j=1$. Let $S_1=\{(i,i+2,i+4,i+6,i+3)\mid i\equiv 0 \pmod 5, i\in\mathbb{Z}_n\}$ and note that S_1 is a set of $n/5$ edge-disjoint 5-cycles in $\langle\{2,3\}\rangle_n$. Next, define the path $P_i: i,i-2,i-4,i-1,i+2,i+5$. Observe that the last vertex of P_i is the first vertex of P_{i+5}. Thus, $C=P_0\cup P_5\cup P_{10}\cup\cdots\cup P_{n-5}$ is a Hamilton cycle, and every edge of $\langle\{2,3\}\rangle_n$ belongs to a 5-cycle in S_1 or is on the Hamilton cycle C. The desired conclusion follows since the graph $\langle\{1,4\}\rangle_n$ can be decomposed into two Hamilton cycles by Lemma 3.1.

Now suppose $j=2$. Consider the set S_1 as defined above and let $S_2=\{(i,i+1,i+2,i+3,i+4)\mid i\equiv 0 \pmod 5, i\in\mathbb{Z}_n\}$. Observe that $S_1\cup S_2$ is a set of $2n/5$ edge-disjoint 5-cycles. Next, define the paths $P_i: i,i-1,i-4,i-2,i+2,i+5$ and $Q_i: i,i-2,i-6,i-3,i+1,i+5$. As above, note that the last vertex of P_i (respectively Q_i) is the first vertex of P_{i+5} (respectively Q_{i+5}). Thus, $C_1=P_0\cup P_5\cup P_{10}\cup\cdots\cup P_{n-5}$ and $C_2=Q_0\cup Q_5\cup Q_{10}\cup\cdots\cup Q_{n-5}$ are two edge-disjoint Hamilton cycles, and every edge of $\langle[1,4]\rangle_n$ belongs to a 5-cycle in $S_1\cup S_2$ or is on one of the Hamilton cycles C_1 and C_2.

Now suppose $j=3$. Let S_1 and S_2 be defined as above and let $S_3=\{(i,i-1,i+2,i-2,i-4)\mid i\equiv 0 \pmod 5, i\in\mathbb{Z}_n\}$. Observe that $S_1\cup S_2\cup S_3$ is a set of $3n/5$ edge-disjoint 5-cycles. Next, define the path $P_i: i,i-3,i+1,i+4,i+8,i+10$. Note that the last vertex of P_i is the first vertex of P_{i+10}, where all subscripts are taken modulo n. Suppose first n is odd, that is, suppose $n\equiv 5 \pmod{10}$. Then $C=P_0\cup P_{10}\cup P_{20}\cup\cdots\cup P_{n-5}\cup P_5\cup P_{15}\cup\cdots\cup P_{n-10}$ is a Hamilton cycle, and note that every edge of $\langle[1,4]\rangle_n$ belongs to a 5-cycle in $S_1\cup S_2\cup S_3$ or is on the Hamilton cycle C. Now suppose n is even. The desired set of $3n/5$ 5-cycles is given by $S_1\setminus\{(0,2,4,6,3),(5,7,9,11,8)\}\cup S_2\setminus\{(0,1,2,3,4),(5,6,7,8,9)\}\cup S_3\setminus\{(5,4,7,3,1)\}\cup\{(0,1,4,6,3),(0,4,7,5,2),(1,3,2,4,5),(3,5,8,9,7),(6,7,8,11,9)\}$. Next, we form the Hamilton cycle C by $C = P_0 \cup P_{10} \cup P_{20} \cup \cdots \cup P_{n-10} \cup P_5 \cup P_{15} \cup \cdots \cup P_{n-5} \setminus \{\{1,4\},\{2,5\},\{6,9\},\{3,5\}\} \cup \{\{1,2\},\{3,4\},\{5,6\},\{5,9\}\}$ and note that every edge of the graph $\langle[1,4]\rangle_n$ belongs to one of the $3j/5$ 5-cycles or is on the Hamilton cycle C.

Finally, suppose $j = 4$. Let S_1 and S_2 be defined as above, and let $S_3 = \{(i, i-1, i+3, i+1, i-3) \mid i \equiv 0 \pmod 5, i \in \mathbb{Z}_n\}$ and $S_4 = \{(i, i-2, i+2, i-1, i-4) \mid i \equiv 0 \pmod 5, i \in \mathbb{Z}_n\}$. Observe that $S_1 \cup S_2 \cup S_3 \cup S_4$ is a set of $4n/5$ edge-disjoint 5-cycles. ■

In [13], Bryant et al. give the following sufficient condition for a circulant $\langle L\rangle_n$ with prescribed edge set L to be decomposed into m-cycles.

Theorem 3.6 (Bryant, Gavlas, Ling [13]) *For $t \geq 1$ and $m \geq 3$,*

(1) the graph $\langle[1, mt]\rangle_n$ can be decomposed into m-cycles for all $n \geq 2mt + 1$ when $mt \equiv 0, 3 \pmod 4$; and

(2) the graph $\langle[1, mt+1] \setminus \{mt\}\rangle_n$ can be decomposed into m-cycles for $n = 2mt + 1$ and for all $n \geq 2mt + 3$ when $mt \equiv 1, 2 \pmod 4$.

4 Main Results

We begin with the case of Hamilton cycles and 4-cycles.

Theorem 4.1

(a) For all odd integers $n \geq 5$ and nonnegative integers h and t, the graph K_n can be decomposed into h Hamilton cycles and t 4-cycles if and only if $hn + 4t = n(n-1)/2$.

(b) For all even integers $n \geq 4$ and nonnegative integers h and t, the graph K_n can be decomposed into h Hamilton cycles, t 4-cycles, and a 1-factor if and only if $hn + 4t = n(n-2)/2$.

Proof. Suppose first that $n \geq 5$ is an odd integer. Clearly, if K_n decomposes into h Hamilton cycles and t 4-cycles, then $hn + 4t = \frac{n(n-1)}{2}$. Therefore, suppose h and t are nonnegative integers with $hn + 4t = \frac{n(n-1)}{2}$. Then $4t = n\left(\frac{n-1}{2} - h\right)$ and $4 \nmid n$ implies $h \equiv \frac{n-1}{2} \pmod 4$. If $h = 0$, then $\frac{n-1}{2} \equiv 0 \pmod 4$ and the result follows by Theorem 3.6, and if $t = 0$, the result clearly follows since K_n has a Hamilton decomposition. Thus, we may assume $h \geq 1$ and $t \geq 1$ so that $n \geq 11$ and $\frac{n-1}{2} - h \geq 4$. Using Theorem 3.6, decompose $\langle[1, \frac{n-1}{2} - h]\rangle_n$ into 4-cycles. Next, using Lemma 3.3, decompose $\langle[\frac{n-1}{2} - h + 1, \frac{n-1}{2}]\rangle_n$ into h Hamilton cycles.

Now suppose $n \geq 4$ is even. Clearly, if K_n decomposes into h Hamilton cycles, t 4-cycles and a 1-factor, then $hn + 4t = \frac{n(n-2)}{2}$. Therefore, suppose h and t are nonnegative integers with $hn + 4t = \frac{n(n-2)}{2}$. Suppose first $n \equiv 2 \pmod 4$. Then $4t = n\left(\frac{n-2}{2} - h\right)$ and $4 \nmid n$ implies $\frac{n-2}{2} - h$ is even so that h is also even. Let $\frac{n-2}{2} - h = 2k$. Note that $k \leq \frac{n-2}{4}$. If $t = 0$, the result clearly follows since K_n decomposes into Hamilton cycles and a 1-factor. Thus, we may assume $t \geq 1$. For each $\ell = 0, 2, \ldots, k-1$, the set $\{(i, i + \frac{n-2}{4} - \ell, i + \frac{n}{2}, i + \frac{3n-2}{4} - \ell) \mid i = 0, 1, \ldots, \frac{n}{2} - 1\}$ is a decomposition of $\langle \{\frac{n-2}{4} - \ell, \frac{n+2}{4} + \ell\} \rangle_n$ into 4-cycles thereby yielding a decomposition of $\langle [\frac{n-2}{4} - k + 1, \frac{n+2}{4} + k - 1] \rangle_n$ into 4-cycles. Using Lemma 3.3, decompose $\langle [\frac{n+2}{4} + k, \frac{n}{2}] \rangle_n$ into $\frac{n-2}{4} - k$ Hamilton cycles and a 1-factor. Let $D = [1, \frac{n-2}{4} - k]$. If $|D|$ is even, then partition D into consecutive pairs and apply Lemma 3.1 to obtain a decomposition of $\langle [1, \frac{n-2}{4} - k] \rangle_n$ into $\frac{n-2}{4} - k$ Hamilton cycles. If $|D|$ is odd, then partition D into $\{1\}$ and consecutive pairs and apply Lemma 3.1 to obtain a decomposition of $\langle [1, \frac{n-2}{4} - k] \rangle_n$ into $\frac{n-2}{4} - k$ Hamilton cycles.

Finally suppose $n \equiv 0 \pmod 4$. If $n = 4$, the result is obvious and thus we may assume $n \geq 8$. Let $\frac{n-2}{2} - h = 2k + j$ for some nonnegative integer k and $j \in \{0, 1\}$. Note that $k \leq \frac{n-2}{4}$. For each $\ell = 1, 2, \ldots, k$, the set $\{(i, i + \frac{n}{4} - \ell, i + \frac{n}{2}, i + \frac{3n}{4} - \ell) \mid i = 0, 1, \ldots, \frac{n}{2} - 1\}$ is a decomposition of $\langle \{\frac{n}{4} - \ell, \frac{n}{4} + \ell\} \rangle_n$ into 4-cycles thereby yielding a decomposition of $\langle \{\frac{n}{4} - k, \frac{n}{4} - k + 1, \ldots, \frac{n}{4} - 1, \frac{n}{4} + 1, \frac{n}{4} + 2, \ldots, \frac{n}{4} + k\} \rangle_n$ into 4-cycles. Using Lemma 3.3, decompose $\langle [\frac{n}{4} + k + 1, \frac{n}{2}] \rangle_n$ into $\frac{n}{4} - k - 1$ Hamilton cycles and a 1-factor. Let $D = [1, \frac{n}{4} - k - 1]$. Suppose first $j = 1$. Then $\langle \{\frac{n}{4}\} \rangle_n$ consists of $\frac{n}{4}$ vertex-disjoint 4-cycles. If $|D|$ is even, then partition D into consecutive pairs and apply Lemma 3.1 to obtain a decomposition of $\langle [1, \frac{n}{4} - k - 1] \rangle_n$ into $\frac{n}{4} - k - 1$ Hamilton cycles. If $|D|$ is odd, then partition D into $\{1\}$ and consecutive pairs and apply Lemma 3.1 to obtain a decomposition of $\langle [1, \frac{n}{4} - k - 1] \rangle_n$ into $\frac{n}{4} - k - 1$ Hamilton cycles. Finally, suppose $j = 0$. If $|D|$ is odd, partition $D \cup \{\frac{n}{4}\}$ into $\{1, \frac{n}{4}\}$ and consecutive pairs and apply Lemma 3.1 to obtain a decomposition of $\langle [1, \frac{n}{4} - k - 1] \cup \{\frac{n}{4}\} \rangle_n$ into $\frac{n}{4} - k$ Hamilton cycles. If $|D|$ is even, partition $D \cup \{\frac{n}{4}\}$ into $\{1, 2, \frac{n}{4}\}$ and consecutive pairs and apply Lemmas 3.1 and 3.2 to obtain a decomposition of $\langle [1, \frac{n}{4} - k - 1] \cup \{\frac{n}{4}\} \rangle_n$ into $\frac{n}{4} - k$ Hamilton cycles. ■

We now consider the case of Hamilton cycles and 5-cycles. The proof of this result when n is odd splits into 2 cases, namely when $5 \nmid n$ (Lemma 4.2) and when $5 \mid n$ (Lemma 4.3).

Lemma 4.2 *For all odd integers $n \geq 5$ with $5 \nmid n$ and for all nonnegative integers h and t with $hn + 5t = \frac{n(n-1)}{2}$, there exists a decomposition of the graph K_n into h Hamilton cycles and t 5-cycles .*

Proof. Let $n \geq 5$ be an odd integer with $5 \nmid n$. Let h and t be nonnegative integers with $hn + 5t = \frac{n(n-1)}{2}$. Then $5t = n\left(\frac{n-1}{2} - h\right)$ and $5 \nmid n$ implies $h \equiv \frac{n-1}{2} \pmod 5$. If $h = 0$, the result follows by [31] and if $t = 0$, the result clearly follows since K_n has a Hamilton decomposition. Thus, we may assume $h \geq 1$ and $t \geq 1$ so that $n \geq 13$ and $\frac{n-1}{2} - h \geq 5$.

Suppose first that $h = 1$. Then $h \equiv \frac{n-1}{2} \pmod 5$ implies $n \equiv 3 \pmod{10}$, say $n = 10k + 3$ for some positive integer k. We now proceed by considering the congruence class of $5k$ modulo 4. Suppose $5k \equiv 0, 3 \pmod 4$. Using Theorem 3.6, decompose $\langle [1, 5k] \rangle_n$ into kn 5-cycles and note that $\langle \{5k+1\} \rangle_n$ is a Hamilton cycle. Now suppose $5k \equiv 1, 2 \pmod 4$. Now we wish to decompose $\langle [1, 5k+1] \setminus \{5k-1\} \rangle_n$ into 5-cycles. In using Theorem 3.6 to do decompose $\langle [1, 5k+1] \setminus \{5k\} \rangle_n$ into 5-cycles, the approach is very similar to the one used in the proof of Lemma 2.2. Therefore, without loss of generality, we may assume one of the difference 5-tuples used in the decomposition is $(1, -2, 3, 5k-1, -(5k+1))$. Note that for $n = 10k+3$, edge length $5k+1$ is the same as length $5k+2$ and thus replace this 5-tuple with $(1, -2, 3, 5k, -(5k+2))$ to decompose $\langle [1, 5k+1] \setminus \{5k-1\} \rangle_n$ into 5-cycles. Finally, $\gcd(10k+3, 5k-1) = 1$ implies $\langle \{5k-1\} \rangle_n$ is a Hamilton cycle.

Now assume $h \geq 2$. Using Theorem 3.6, decompose $\langle [1, \frac{n-1}{2} - h] \rangle_n$ or $\langle [1, \frac{n-1}{2} - h + 1] \setminus \{\frac{n-1}{2} - h\} \rangle_n$ into 5-cycles depending on the congruence class of $\frac{n-1}{2} - h$ modulo 4. Next, using Lemma 3.3, decompose either $\langle [\frac{n-1}{2} - h + 1, \frac{n-1}{2}] \rangle_n$ or $\langle [\frac{n-1}{2} - h, \frac{n-1}{2}] \setminus \{\frac{n-1}{2} - h + 1\} \rangle_n$, as appropriate, into h Hamilton cycles. ■

We now show that that K_n can be decomposed into all possible combinations of Hamilton cycles and 5-cycles when n is an odd multiple of 5.

Lemma 4.3 *For all $n \equiv 5 \pmod{10}$ with $n \geq 5$ and for all nonnegative integers h and t with $hn + 5t = \frac{n(n-1)}{2}$, there exists a decomposition of the graph K_n into h Hamilton cycles and t 5-cycles.*

Proof. Let $n \equiv 5 \pmod{10}$, say $n = 10k + 5$ for some nonnegative integer k. Let h and t be nonnegative integers with $hn + 5t = \frac{n(n-1)}{2}$. If $h = 0$, the result follows by [31] and if $t = 0$, the result clearly follows since K_n has a Hamilton decomposition. Thus, we may assume $h \geq 1$ and $t \geq 1$.

We begin by handling a few specials cases of n. The result is obviously true for $n = 5$. Now consider $n = 15$. If $h = 1$, then $\langle \{2\} \rangle_{15}$ is a Hamilton cycle, $\langle \{3\} \rangle_{15}$ is a 2-regular subgraph of K_{15} consisting of three 5-cycles, and the difference 5-tuple $(1, -4, 5, 6, -8)$ will give a decomposition of $\langle \{1, 4, 5, 6, 7\} \rangle_{15}$ into 15 5-cycles. If $h = 2$, then the difference 5-tuple $(1, -2, 3, 4, -6)$ will give a decomposition of $\langle \{1, 2, 3, 4, 6\} \rangle_{15}$ into 15 5-cycles and Lemma 3.1 gives a decomposition of $\langle \{5, 7\} \rangle_{15}$ into two Hamilton cycles. For $h = 3, 4, 5, 6$, Lemma 3.2 gives a decomposition of $\langle \{5, 6, 7\} \rangle_{15}$ into three Hamilton cycles and applying Lemma 3.5 with $j = 7 - h$ gives the desired result. Thus, we may now assume $k \geq 2$.

Case 1. *Suppose $h = 1$.* We now partition the set $[1, 5k+1] \setminus \{2k+1\}$ when $k \equiv 0, 1 \pmod 4$ or the set $[1, 5k+1] \setminus \{4k+2\}$ when $k \equiv 2, 3 \pmod 4$ into 5-cycle difference tuples. The decomposition then follows since $\langle \{2k+1\} \rangle_{10k+5}$ or $\langle \{4k+2\} \rangle_{10k+5}$ is a 2-regular graph consisting of $2k+1$ 5-cycles and $\langle \{5k+2\} \rangle_{10k+5}$ is a Hamilton cycle. In what follows, for each case of k modulo 4, we construct a $k \times 5$ array $X = [x_{i,j}]$ such that $\{|x_{i,j}| \mid 1 \leq i \leq k, 1 \leq j \leq 5\} = [1, 5k+1] \setminus \{2k+1\}$ when $k \equiv 0, 1 \pmod 4$ or $\{|x_{i,j}| \mid 1 \leq i \leq k, 1 \leq j \leq 5\} = [1, 5k+1] \setminus \{4k+2\}$ when $k \equiv 2, 3 \pmod 4$. The required set of k difference 5-tuples can be constructed directly from the rows of X using the ordering $(x_{i,1}, x_{i,2}, x_{i,4}, x_{i,3}, x_{i,5})$ for $i = 1, 2, \ldots, k$.

Suppose $k \equiv 0, 1 \pmod 4$. By Theorem 2.3, there exists a hooked $(k+1)$-extended Langford sequence of order $k-1$ and defect 1, and let $\{(a_i, b_i, c_i) \mid 1 \leq i \leq k-1\}$ be a set of $k-1$ difference triples using edges of lengths $[1, 3k-1] \setminus \{2k, 3k-2\}$ constructed from such a sequence. Partition the set $[3k+2, 5k+1]$ of $2k$ consecutive integers into k

sets $\{d_i, d_i+1\}$ for $i = 1, 2, \ldots, k$. Let

$$X = \begin{bmatrix} a_1+1 & c_1-1 & b_1+1 & d_1 & -(d_1+1) \\ a_2+1 & c_2-1 & b_2+1 & d_2 & -(d_2+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k-1}+1 & c_{k-1}-1 & b_{k-1}+1 & d_{k-1} & -(d_{k-1}+1) \\ 1 & -(3k+1) & 3k-1 & d_k+1 & -d_k \end{bmatrix}.$$

Suppose $k \equiv 2 \pmod 4$. By Theorem 2.1, there exists a Langford sequence of order $k-1$ and defect 1, and let $\{(a_i, b_i, c_i) \mid 1 \le i \le k-1\}$ be a set of $k-1$ difference triples using edges of lengths $[1, 3k-3]$ constructed from such a sequence. Note that the sets $[3k-1, 4k]$ and $[4k+4, 5k+1]$ contain $k+2$ and $k-2$ consecutive integers respectively. Since $k \equiv 2 \pmod 4$, the set $[3k-1, 4k] \cup [4k+4, 5k+1]$ can be partitioned into into k sets $\{d_i, d_i+1\}$ for $i = 1, 2, \ldots, k$. Let

$$X = \begin{bmatrix} a_1+1 & c_1-1 & b_1+1 & d_1 & -(d_1+1) \\ a_2+1 & c_2-1 & b_2+1 & d_2 & -(d_2+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k-1}+1 & c_{k-1}-1 & b_{k-1}+1 & d_{k-1} & -(d_{k-1}+1) \\ 1 & -(4k+3) & 4k+1 & d_k+1 & -d_k \end{bmatrix}.$$

Suppose $k \equiv 3 \pmod 4$. By Theorem 2.1, there exists a hooked Langford sequence of order $k-1$ and defect 1, and let $\{(a_i, b_i, c_i) \mid 1 \le i \le k-1\}$ be a set of $k-1$ difference triples using edges of lengths $[1, 3k-2] \setminus \{3k-3\}$ constructed from such a sequence. Note that the sets $[3k+1, 4k+1]$ and $[4k+3, 5k+1]$ contain $k+1$ and $k-1$ consecutive integers respectively. Since $k \equiv 3 \pmod 4$, the set $[3k+1, 4k+1] \cup [4k+3, 5k+1]$ can be partitioned into into k sets $\{d_i, d_i+1\}$ for $i = 1, 2, \ldots, k$. Let

$$X = \begin{bmatrix} a_1+1 & c_1-1 & b_1+1 & d_1 & -(d_1+1) \\ a_2+1 & c_2-1 & b_2+1 & d_2 & -(d_2+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k-1}+1 & c_{k-1}-1 & b_{k-1}+1 & d_{k-1} & -(d_{k-1}+1) \\ 1 & -3k & 3k-2 & d_k+1 & -d_k \end{bmatrix}.$$

CASE 2. *Suppose $h = 2$.* For $k \equiv 0, 3 \pmod 4$, using Theorem 3.6, decompose $\langle [1, 5k] \rangle_{10k+5}$ into 5-cycles and using Theorem 3.1, decompose $\langle \{5k+1, 5k+2\} \rangle_{10k+5}$ into Hamilton cycles. For $k \equiv 1, 2 \pmod 4$, using Theorem 3.6, decompose $\langle [1, 5k+1] \setminus \{5k\} \rangle_{10k+5}$ into 5-cycles and using Theorem 3.1, decompose $\langle \{5k, 5k+2\} \rangle_{10k+5}$ into Hamilton cycles.

CASE 3. *Suppose $h \ge 3$.* Let $h = 5\ell + m$ for some integer m with $0 \le m \le 4$ and observe that since $h \ge 3$, at least one of ℓ and m is positive. Note that $t = (k-\ell)n + \frac{(2-m)n}{5}$. For

$m = 0, 1, 2$, using Lemma 3.5, decompose $\langle[1,4]\rangle_n$ into $\frac{(2-m)n}{5}$ 5-cycles and $2+m$ Hamilton cycles. Note that $5\ell - 2 \geq 3$, and thus use Corollary 3.4 to decompose $\langle[5, 5k+2]\rangle_n$ into $(k-\ell)n$ 5-cycles and $5\ell - 2$ Hamilton cycles.

Now consider $m = 3$ and $m = 4$. Since $t \geq 1$ and $m \geq 3$, it follows that $5k + 2 > 5\ell + m \geq 5\ell + 3$. So, $5(k-\ell-1)+4 > 0$. Since $k-\ell-1$ is an integer and $5(k-\ell-1)+4 > 0$, it must be that $k - \ell - 1 \geq 0$. Using Lemma 3.5, decompose $\langle[1,4]\rangle_n$ into $\frac{(7-m)n}{5}$ 5-cycles and $m-3$ Hamilton cycles. Using Corollary 3.4, decompose $\langle[5, 5k+2]\rangle_n$ into $(k-\ell-1)n$ 5-cycles and $5\ell + 3$ Hamilton cycles. ■

We now consider the case when n is even. As in the case when n is odd, the proof of our main result for n even is split into two cases, when $5 \nmid n$ (Lemma 4.4) and when $5 \mid n$ (Lemma 4.5).

Lemma 4.4 *For all even $n \geq 6$ with $5 \nmid n$ and for all nonnegative integers h and t with $hn + 5t = \frac{n(n-2)}{2}$, there exists a decomposition of the graph K_n into h Hamilton cycles, t 5-cycles, and a 1-factor.*

Proof. Let $n \equiv m \pmod{10}$ with $n \geq 6$ even and $5 \nmid n$. Let h and t be nonnegative integers with $hn + 5t = \frac{n(n-2)}{2}$. Then $5t = n\left(\frac{n-2}{2} - h\right)$ and $5 \nmid n$ implies $h \equiv \frac{n-2}{2} \pmod 5$. If $h = 0$, the result follows by [32] and if $t = 0$, the result clearly follows since $K_n - I$ has a Hamilton decomposition. Thus, we may assume $h \geq 1$ and $t \geq 1$ so that $n \geq 14$ and $\frac{n-2}{2} - h \geq 5$.

Suppose first that $h = 1$. Then $h \equiv \frac{n-2}{2} \pmod 5$ implies $n \equiv 4 \pmod{10}$, say $n = 10k+4$ for some positive integer k. We now proceed by considering the congruence class of $5k$ modulo 4. Suppose $5k \equiv 0, 3 \pmod 4$. Using Theorem 3.6, decompose $\langle[1, 5k]\rangle_n$ into kn 5-cycles and note that $\langle\{5k+1, 5k+2\}\rangle_n \cong C_{n/2} \times K_2$ which decomposes into a Hamilton cycle and a 1-factor. Now suppose $5k \equiv 1, 2 \pmod 4$. Using Theorem 3.6, decompose $\langle[1, 5k+1] \setminus \{5k\}\rangle_n$ into kn 5-cycles. If $5k \equiv 1 \pmod 4$, then $\gcd(5k, 10k+4) = 1$, so that $\langle\{5k\}\rangle_n$ is the required Hamilton cycle and $\langle\{5k+2\}\rangle_n$ is a 1-factor. If $5k \equiv 2 \pmod 4$, then $\langle\{5k, 5k+2\}\rangle_n \ncong C_{n/2} \times K_2$ so that we need a different approach. In using Theorem 3.6 to decompose $\langle[1, 5k+1] \setminus \{5k\}\rangle_n$ into 5-cycles, the approach is very similar to the one used in Lemma 2.2. Therefore, without loss of generality, we may assume one of the difference 5-tuples used in the decomposition is $(1, -2, 3, 5k-1, -(5k+1))$ so that $(0, 1, -1, 5k-2, 5k+1)$, $(5k, 5k+1, 5k-1, 10k-2, 10k+1)$, and $(5k-1, 5k, 5k-2, 10k-3, 10k)$ are three 5-cycles in the decomposition. We will use these three 5-cycles along with the graph $\langle\{5k\}\rangle_n$ to create one Hamilton cycle and, necessarily, three 5-cycles. The Hamilton cycle C is given by $C = \langle\{5k\}\rangle_n \setminus \{\{0, 5k\}, \{5k+1, 10k+1\}\} \cup \{\{0, 5k+1\}, \{5k, 10k+1\}\}$ and the three 5-cycles are $(5k-2, 10k-3, 10k, 5k-1, 5k+1)$, $(-1, 1, 0, 5k, 5k-2)$, and $(5k+1, 10k+1, 10k-2, 5k-1, 5k)$. The required 1-factor is $\langle\{5k+2\}\rangle_n$.

Now suppose $h \geq 2$ so that $\frac{n-2}{2} - h \leq \frac{n}{2} - 3$. Using Theorem 3.6, decompose $\langle[1, \frac{n-2}{2} - h]\rangle_n$ or $\langle[1, \frac{n-2}{2} - h + 1] \setminus \{\frac{n-2}{2} - h\}\rangle_n$ into 5-cycles depending on the congruence class of $\frac{n-2}{2} - h$ modulo 4. Next, using Lemma 3.3, decompose either $\langle[\frac{n-2}{2} - h + 1, \frac{n}{2}]\rangle_n$ or $\langle[\frac{n-2}{2} - h, \frac{n}{2}] \setminus \{\frac{n-2}{2} - h + 1\}\rangle_n$, as appropriate, into h Hamilton cycles and a 1-factor. ■

We now show that that K_n can be decomposed into all possible combinations of Hamilton cycles and 5-cycles when n is an even multiple of 5.

Lemma 4.5 *For all* $n \equiv 0 \pmod{10}$ *with* $n \geq 10$ *and for all nonnegative integers* h *and* t *with* $hn + 5t = \frac{n(n-2)}{2}$, *there exists a decomposition of the graph* K_n *into* h *Hamilton cycles,* t *5-cycles, and a 1-factor.*

Proof. Let $n \equiv 0 \pmod{10}$, say $n = 10k$ for some nonnegative integer k. Let h and t be nonnegative integers with $hn + 5t = \frac{n(n-2)}{2}$. If $h = 0$, the result follows by [32] and if $t = 0$, the result clearly follows since $K_n - I$ has a Hamilton decomposition (where I is a 1-factor). Thus, we may assume $h \geq 1$ and $t \geq 1$.

We begin by handling a few special cases of n. In each case, $\langle\{\frac{n}{2}\}\rangle_n$ will be the 1-factor. Clearly, Lemma 3.5 handles the case when $n = 10$. Now consider $n = 20$. For $h = 1$, from the proof of Lemma 3.5, the graph $\langle\{2,3\}\rangle_{20}$ decomposes into a Hamilton cycle and four 5-cycles, $\langle\{4\}\rangle_{20}$ and $\langle\{8\}\rangle_{20}$ are each 2-regular subgraphs of K_{20} consisting of four 5-cycles, and the difference 5-tuple $(1,-5,6,7,-9)$ will give a decomposition of $\langle\{1,5,6,7,9\}\rangle_{20}$ into 20 5-cycles. For $h = 2$, as before, $\langle\{4\}\rangle_{20}$ and $\langle\{8\}\rangle_{20}$ are each 2-regular subgraphs of K_{20} consisting of four 5-cycles, the difference 5-tuple $(3,-6,5,7,-9)$ will give a decomposition of $\langle\{3,5,6,7,9\}\rangle_{20}$ into 20 5-cycles and the graph $\langle\{1,2\}\rangle_{20}$ decomposes into two Hamilton cycles by Lemma 3.1. For $h = 3$, $\langle\{8\}\rangle_{20}$ is a 2-regular subgraph of K_{20} consisting of four 5-cycles, the difference 5-tuple $(3,-6,5,7,-9)$ will give a decomposition of $\langle\{3,5,6,7,9\}\rangle_{20}$ into 20 5-cycles, and the graph $\langle\{1,2,4\}\rangle_{20}$ decomposes into three Hamilton cycles by Lemma 3.2. For $h = 4$, $\langle\{8\}\rangle_{20}$ is a 2-regular subgraph of K_{20} consisting of four 5-cycles, the graph $\langle[1,4]\rangle_{20}$ decomposes into 5-cycles by Lemma 3.5, and each of the graphs $\langle\{5,6\}\rangle_{20}$ and $\langle\{7,9\}\rangle_{20}$ decomposes into two Hamilton cycles by Lemma 3.1. For $h = 5$, the graph $\langle[1,4]\rangle_{20}$ decomposes into 5-cycles by Lemma 3.5, each of the graphs $\langle\{5,6\}\rangle_{20}$ and $\langle\{7,8\}\rangle_{20}$ decomposes into two Hamilton cycles by Lemma 3.1, and $\langle\{9\}\rangle_{20}$ is a Hamilton cycle. For $h \geq 6$, apply Lemma 3.5 with $j = 9 - h$ to the graph $\langle[1,4]\rangle_{20}$, and decompose $\langle[5,9]\rangle_{20}$ into Hamilton cycles as in the case when $h = 5$.

Thus, we may now assume $n \geq 30$. Let $h = 5\ell + m$ for some integer m with $0 \leq m \leq 4$. Observe that since $h \geq 1$, at least one of ℓ and m is positive. Note that $t = (k - \ell - 1)n + \frac{(4-m)n}{5}$. Since $t \geq 1$, it follows that $5k - 1 > 5\ell + m \geq 5\ell$. So, $5(k - \ell - 1) + 4 > 0$. Since $k - \ell - 1$ is an integer and $5(k - \ell - 1) + 4 > 0$, it must be that $k - \ell - 1 \geq 0$. Suppose first that $\ell \geq 1$. Since $n \geq 20$, we have $k - \ell - 1 \leq \lfloor\frac{n-14}{10}\rfloor$. Using Lemma 3.5, decompose $\langle[1,4]\rangle_n$ into $\frac{(4-m)n}{5}$ 5-cycles and m Hamilton cycles, and using Corollary 3.4, decompose $\langle[5,5k]\rangle_n$ into $(k - \ell - 1)n$ 5-cycles, 5ℓ Hamilton cycles and a 1-factor. Thus, it remains to consider the case when $\ell = 0$. In this case, $\langle\{5k\}\rangle_{10k}$ will be the 1-factor. Suppose first that $k \equiv 0,1 \pmod 4$. Using Lemma 3.5, decompose $\langle[1,4]\rangle_n$ into $\frac{(4-m)n}{5}$ 5-cycles and m Hamilton cycles, and using Lemma 2.2, decompose $\langle[5,5k-1]\rangle_n$ into $(k-1)n$ 5-cycles and note that $\langle\{5k\}\rangle_n$ is a 1-factor. Now suppose $k \equiv 2,3 \pmod 4$. We begin by finding a partition of the set $\{1\} \cup [4, 5k-1] \setminus \{2k, 4k\}$ into difference 5-tuples.

For $k \equiv 2,3 \pmod 4$ with $k \geq 3$, by Theorem 2.3, there exists a $(k-1)$-extended Langford sequence of order $k - 2$ and defect 2, and let $\{(a_i, b_i, c_i) \mid 1 \leq i \leq k-2\}$ be a set of $k-2$ difference triples using edges of lengths $[2,3k-4]\setminus\{2k-2\}$ constructed from such a sequence.

For $k\equiv 2 \pmod 4$, the set $[3k+1,4k-1]$ has $\frac{k}{2}+1$ consecutive odd integers and $\frac{k}{2}$ consecutive even integers, and the set $[4k+1,5k-1]$ has $\frac{k}{2}$ consecutive odd integers and $\frac{k}{2}-1$ consecutive even integers. Thus, we may partition set $[3k+1,4k-1]\cup[4k+1,5k-1]$ into $\{4k-2,4k+1\}$ and $k-1$ pairs $\{d_i,d_i+2\}$ for $i=1,2,\ldots,k-1$. Without loss of generality, we may assume $d_{k-1}=5k-3$. Let $X=[x_{i,j}]$ be the $(k-1)\times 5$ array

$$X = \begin{bmatrix} a_1+2 & c_1-2 & b_1+2 & d_1 & -(d_1+2) \\ a_2+2 & c_2-2 & b_2+2 & d_2 & -(d_2+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k-2}+2 & c_{k-2}-2 & b_{k-2}+2 & d_{k-2} & -(d_{k-2}+2) \\ 1 & -(4k+1) & 4k-2 & 5k-1 & -(5k-3) \end{bmatrix}.$$

For $k\equiv 3 \pmod 4$, the set $[3k+1,4k-1]$ has $\frac{k+1}{2}$ consecutive odd integers and $\frac{k+1}{2}$ consecutive even integers, and the set $[4k+1,5k-1]$ has $\frac{k-1}{2}$ consecutive odd integers and $\frac{k-1}{2}$ consecutive even integers. Thus, we may partition set $[3k+1,4k-1]\cup[4k+1,5k-1]$ into $\{4k+1,4k+2\}$ and $k-1$ pairs $\{d_i,d_i+2\}$ for $i=1,2,\ldots,k-1$. Without loss of generality, we may assume $d_{k-1}=5k-4$. Let $X=[x_{i,j}]$ be the $(k-1)\times 5$ array

$$X = \begin{bmatrix} a_1+2 & c_1-2 & b_1+2 & d_1 & -(d_1+2) \\ a_2+2 & c_2-2 & b_2+2 & d_2 & -(d_2+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k-2}+2 & c_{k-2}-2 & b_{k-2}+2 & d_{k-2} & -(d_{k-2}+2) \\ 1 & -(4k+1) & 4k+2 & 5k-4 & -(5k-2) \end{bmatrix}.$$

The required set of $k-1$ difference 5-tuples can be constructed directly from the rows of X using the ordering $(x_{i,1},x_{i,2},x_{i,4},x_{i,3},x_{i,5})$ for $i=1,2,\ldots,k-1$.

Suppose first $h=1$. From the proof of Lemma 3.5, the graph $\langle\{2,3\}\rangle_{10k}$ decomposes into a Hamilton cycle and 5-cycles. Also, observe that each of the graphs $\langle\{2k\}\rangle_{10k}$ and $\langle\{4k\}\rangle_{10k}$ is a 2-regular graph consisting of $2k$ 5-cycles. From above, the graph $\langle\{1\}\cup[4,5k-1]\setminus\{2k,4k\}\rangle_{10k}$ decomposes into 5-cycles.

For $h=2$, using Lemma 3.1, decompose $\langle\{2,3\}\rangle_{10k}$ into two Hamilton cycles, each of the graphs $\langle\{2k\}\rangle_{10k}$ and $\langle\{4k\}\rangle_{10k}$ is a 2-regular graph consisting of $2k$ 5-cycles, and the graph $\langle\{1\}\cup[4,5k-1]\setminus\{2k,4k\}\rangle_{10k}$ decomposes into 5-cycles.

For $h=3$ and $k\equiv 3 \pmod 4$, using Lemma 3.2, decompose $\langle\{1,2,2k\}\rangle_{10k}$ into three Hamilton cycles, the graph $\langle\{4k\}\rangle_{10k}$ is a 2-regular graph consisting of $2k$ 5-cycles, and replacing the last row of X with $[\ 3 \quad -(4k+2) \quad 4k+1 \quad 5k-4 \quad -(5k-2)\ ]$ gives a decomposition of the graph $\langle[3,5k-1]\setminus\{2k,4k\}\rangle_{10k}$ into 5-cycles. For $k\equiv 2 \pmod 4$, note that $\gcd(10k,5k-1)=1$ and using Lemma 3.2, decompose $\langle\{4k+1,5k-3,5k-1\}\rangle_{10k}$ into three Hamilton cycles, the graph $\langle\{2k\}\rangle_{10k}$ is a 2-regular graph consisting of $2k$ 5-cycles, and replacing the last row of X with $[\ 1 \quad -2 \quad 3 \quad 4k-2 \quad -4k\ ]$ gives a decomposition of the graph $\langle[1,5k-2]\setminus\{2k,4k+1,5k-3\}\rangle_{10k}$ into 5-cycles.

For $h = 4$ and $k \equiv 2 \pmod 4$, using Lemma 2.2, decompose $\langle [1, 5(k-1)] \rangle_{10k}$ into 5-cycles and using Lemma 3.1, decompose $\langle \{5k-4, 5k-3\} \rangle_{10k}$ and $\langle \{5k-2, 5k-1\} \rangle_{10k}$ into Hamilton cycles. For $k \equiv 3 \pmod 4$, note that $\gcd(10k, 5k-4) = 1$ and $\gcd(10k, 5k-2) = 1$. Thus, using Lemma 3.1, decompose the graphs $\langle \{2k, 5k-4\} \rangle_{10k}$ and $\langle \{4k+1, 5k-2\} \rangle_{10k}$ into Hamilton cycles and replacing the last row of X with $[\ 1 \quad -2 \quad 3 \quad 4k \quad -(4k+2)\ ]$ gives a decomposition of the graph $\langle [1, 5k-1] \setminus \{2k, 4k+1, 5k-4, 5k-2\} \rangle_{10k}$ into 5-cycles. ■

Theorem 1.1 now follows from Lemmas 4.2, 4.3, 4.4, and 4.5.

References

[1] P. Adams, E. J. Billington, I. J. Dejter, C. C. Lindner, The number of 4-cycles in 2-factorizations of K_{2n} minus a 1-factor. *Discrete Math.* **220** (2000), 1–11.

[2] P. Adams, D. Bryant, S. El-Zanati, and H. Gavlas, Factorizations of the complete graph into C_5-factors and 1-factors. *Graphs Combin.* **19** (2003), 289–296.

[3] P. Adams, D. Bryant, and A. Khodkar, 3, 5-Cycle decompositions. *J. Combin. Des.* **6** (1998), 91–110.

[4] P. Adams, D. Bryant, and A. Khodkar, On Alspach's conjecture with two even cycle lengths. *Discrete Math.* **223** (2000), 1–12.

[5] B. Alspach, Research problems, Problem 3, *Discrete Math.* **36** (1981), 333.

[6] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$. *J. Combin. Theory Ser. B* **81** (2001), 77–99.

[7] C. A. Baker, Extended Skolem sequences. *J. Combin. Des.* **3** (1995), 363-379.

[8] P. Balister, On the Alspach conjecture. *Comin. Probab. Comput.* **10** (2001), 95–125.

[9] P. Balister, Packing circuits in K_n. *Combin. Probab. Comput.* **10** (2001), 463-499.

[10] J.-C. Bermond, O. Favaron, and M. Mahéo, Hamiltonian decomposition of Cayley graphs of degree 4. *J. Combin. Theory Ser. B* **46** (1989), 142–153.

[11] D. Bryant, Hamilton cycle rich two-factorizations of complete graphs. *J. Combin. Des.* **12** (2004), 147–155.

[12] D. Bryant, Cycle decompositions of complete graphs, in *Surveys in Combinatorics 2007*, A. Hilton and J. Talbot (Editors); London Mathematical Society Lecture Notes Seris **346**, Proceedings of the 21st British combinatorial conference, Cambridge University Press, 2007, pp. 67–97.

[13] D. Bryant, H. Gavlas, A. C.-H. Ling, Skolem-type difference sets for cycle systems. *Electron. J. Combin.* **10** (2003), #38, 1–12.

[14] D. Bryant and D. Horsley, Packing cycles in complete graphs. *J. Combin. Theory Ser. B* **98** (2008), 1014–1037.

[15] D. Bryant and D. Horsley, Decompositions of complete graphs into long cycles. *Bull. Lond. Math. Soc.* **41** (2009), 927–934.

[16] D. Bryant and D. Horsley, An asymptotic solution to the cycle decomposition problem for complete graphs. *J. Combin. Theory Ser. A* **117** (2010) 1258–1284.

[17] D. Bryant and B. Maenhaut, Decompositions of complete graphs into triangles and hamilton cycles. *J. Combin. Des.* **12** (2004), 221–232.

[18] D. Bryant and C. Rodger, "Cycle Decompositions" in *The CRC Handbook of Combinatorial Designs, Second Edition* C. J. Colbourn and J. H. Dinitz (eds), Chapman & Hall/CRC Press, Boca Raton FL, (2007).

[19] Y. Caro and R. Yuster, List decomposition of graphs. *Discrete Math.* **243** (2002), 67–77.

[20] M. Dean, On Hamilton cycle decomposition of 6-regular circulant graphs. *Graphs Combin.* **22** (2006), 331–340.

[21] M. Dean, Hamilton cycle decomposition of 6-regular circulants of odd order. *J. Combin. Des.* **15** (2007), 91–97.

[22] I. J. Dejter, F. Franek, E. Mendelsohn, and A. Rosa, Triangles in 2-factorizations. *J. Graph Theory* **26** (1997), 83–94.

[23] I. J. Dejter, C. Lindner, and A. Rosa, The number of 4-cycles in 2-factorizations of K_n. *J. Combin. Math. Combin. Comput.* **28** (1998), 101–112.

[24] S. I. El-Zanati, K. King, J. Mudrock, J. Witkowski, On decomposing complete graphs into n-cycles and Hamilton cycles. Submitted.

[25] K. Heinrich, P. Horak, and A. Rosa, On Alspach's conjecture. *Discrete Math.* **77** (1989), 97–121.

[26] A. J. W. Hilton and M. Johnson, Cycle decompositions of the complete graph. *Ars Combin.* **81** (2006), 311–324.

[27] P. Horak, R. Nedela, and A. Rosa, The Hamilton-Waterloo problem: the case of Hamilton cycles and triangle-factors. *Discrete Math.* **284** (2004), 181–188.

[28] V. Linek and Z. Jiang, Extended Langford sequences with small defects. *J. Combin. Theory Ser. A* **84** (1998), 38-54.

[29] V. Linek and N. Shalaby, The existence of (p, q)-extended Rosa sequences. *Discrete Math.* **308** (2008), 1583-1602.

[30] R. Rees, Uniformly resolvable pairwise balanced designs with blocksizes two and three. *J. Combin. Theory Ser. A* **45** (1987), 207–225.

[31] A. Rosa, On the cyclic decompositions of the complete graph into polygons with an odd number of edges. *Časopis Pěst. Math.* **91** (1966), 53–63.

[32] M. Šajna, Cycle Decompositions III: Complete graphs and fixed length cycles. *J. Combin. Des.* **10** (2002), 27–78.

[33] J.E. Simpson, Langford sequences: perfect and hooked. *Discrete Math.* **44** (1983), 97–104.

[34] Q. Sui and B. Du, The number of triangles in 2-factorizations. *J. Combin. Des.* **14** (2006), 277–289.