Two Characterizations of Hypercubes

Juhani Nieminen, Matti Peltola and Pasi Ruotsalainen

Department of Mathematics, University of Oulu
University of Oulu, Faculty of Technology, Mathematics Division,
P.O. Box 4500, 90014 Oulun yliopisto, Finland

ujuhani.nieminen@ ee.oulu.fi; matti.peltola@ee.oulu.fi;
pasi.ruotsalainen@ee.oulu.fi

Submitted: Jun 11, 2008; Accepted: Apr 20, 2011; Published: Apr 29, 2011
Mathematics Subject Classification: 05C75

Abstract

Two characterizations of hypercubes are given: 1) A graph is a hypercube if and only if it is antipodal and bipartite $(0,2)$-graph. 2) A graph is an n-hypercube if and only if there are n pairs of prime convexes, the graph is a prime convex intersection graph, and each intersection of n prime convexes (no one of which is from the same pair) is a vertex.

1 Introduction

Hypercubes constitute a very remarkable class of graphs especially for transmitting communication and therefore each characterization of hypercubes offers a new point of view to use and construct hypercubes.

The class of $(0,2)$-graphs is a subclass of strongly regular graphs studied in the theory of combinatorial design. It was introduced in [6] and intensively studied in [3] and in [4].

We begin with some basic properties of antipodal graphs (see also [7]). Then we prove that a graph is a hypercube if and only if it is an antipodal, bipartite $(0,2)$-graph. This characterization gives another characterization of hypercubes by using prime convex intersection graphs.

The graphs $G = (V, E)$ considered here are finite, connected and undirected without loops and multiple edges. The set V is the set of vertices and E the set of edges in G. A shortest $u-v$ path is called a $u-v$ geodesic and $d(u,v)$ is its length. The interval $[u,v]$ is the set of all vertices locating on any $u-v$ geodesic. By $N(u)$ we denote the set of neighbours of u, i.e $N(u)=\{v|d(u,v)=1\}$ and by $deg(u)$ the cardinality of $N(u)$. The diameter of a graph G is denoted by $diam(G)=max\{d(u,v)|u,v\in V\}$. A graph $G=(V,E)$ is called antipodal, if for every vertex u there exists - a necessarily unique - vertex u' called the antipode of u, such that $[u,u']=V$, see [1] and [5].

2 Bipartite and antipodal graphs

We give a sufficient conditions for an antipodal bipartite graph to be a regular one. First we give two basic properties of antipodal graphs (see also [7]).

Lemma 1 *An antipodal graph $G=(V,E)$ is bipartite if and only if for any two adjacent vertices u and v of G intervals $[u,v']$ and $[v,u']$ constitute a partition of V.*

Proof. Assume first that G is bipartite. If u and v are adjacent vertices then let $V_u=\{w\in V|d(u,w)<d(v,w)\}=\{w\in V|d(u,w)+1=d(v,w)\}$ and $V_v=\{w\in V|d(v,w)<d(u,w)\}=\{w\in V|d(v,w)+1=d(u,w)\}$. Since G is bipartite, the vertex sets V_u and V_v constitute a partition of V. Moreover, since $[v,v']=V$, it follows that for every $w\in V_u$ we have $w\in[u,v']$, thus $V_u=[u,v']$. Analogously $V_v=[v,u']$.

Assume that $[u,v']\cup[v,u']=V$ and $[u,v']\cap[v,u']=\emptyset$ for any two adjacent vertices u and v of V. Let $a\in V$, $V_1=\{x\in V|d(a,x)$ is odd$\}$ and $V_2=\{x\in V|d(a,x)$ is even$\}$. Assume first, that there are two adjacent vertices $x,y\in V_2$. Let $d(a,x)=2m$ and $d(a,y)=2n$. If $m<n$, then, because x and y are adjacent, we have $2n=d(a,y)\leq d(a,x)+d(x,y)=1+2m<2n$; a contradiction. The case $n<m$ is analogous, and accordingly, $n=m$ and $d(a,y)=d(a,x)$. If $a\in[x,y']$, then $d(x,y')=d(x,a)+d(a,y')=d(a,y)+d(a,y')=diam(G)$, whence $x'=y'$, This is a contradiction, since the antipode of a vertex is unique and thus $a\notin[x,y']$. Similarly we see that $a\notin[x',y]$, and accordingly, $a\notin[x,y']\cup[y,x']$, which is a contradiction. Thus the assumption that there are two adjacent vertices in V_2 is false and any two vertices $x,y\in V_2$ are nonadjacent. Similarly we can prove that any two vertices x,y of V_1 are nonadjacent. Thus G is bipartite.

Lemma 2 *If a graph $G=(V,E)$ is antipodal, then vertices u and v of G are adjacent if and only if u' and v' are adjacent. Moreover $deg(u)=deg(u')$ for all $u\in V$.*

Proof. Assume that u and v are adjacent vertices. It suffices to prove, that u' and v' are adjacent. Assume on the contrary, that there exists a vertex z on the $u'-v'$ geodesic such that $z \neq u', v'$. Because $u' \in [v, v']$ and $z \in [u', v']$, there exists a $v-v'$ geodesic through u' and z, and thus $d(v, v') = d(v, u')+d(u', z)+d(z, v') \geq d(v, u')+2$. On the other hand, because $v \in [u, u']$ and because v is adjacent to u, we have $d(u, u') = 1+d(v, u')$. Because G is antipodal, we have $d(v, v') = d(u, u') = diam(G)$ and thus $d(v, v') = d(u, u') = 1 + d(v, u') \geq d(v, u') + 2$; a contradiction.

The following theorem gives a sufficient condition for an antipodal bipartite graph to be regular.

Theorem 3 *Let $G = (V, E)$ be an antipodal bipartite graph. If for any two adjacent vertices u and v of G there exists a (graph) isomorphism f from the subgraph G_1 induced by $[v, u']$ onto the subgraph G_2 induced by $[u, v']$ such that $f(x) = y$ for any two adjacent vertices $x \in G_1$ and $y \in G_2$, then G is regular.*

Proof. Let u and v be adjacent vertices. According to the isomorphism f, $f(u) = v$ and $deg(u) = deg(v)$. Because u' and v' also are adjacent, $u' \in [v, u']$ and $v' \in [u, v']$, we have $f(u') = v'$ and $deg(u') = deg(v')$. This implies, by Lemma 2, that $deg(u') = deg(u) = deg(v) = deg(v')$, for any two adjacent vertices u and v. If z is adjacent to u, then by repeating the proof above we have $deg(z) = deg(u)$, and further, $deg(z) = deg(u) = deg(v)$. Because G is connected, the result follows.

3 Spherical graphs

A graph $G = (V, E)$ is spherical if for any two vertices u and v and for any $z \in [u, v]$ there exists precisely one vertex z_1 such that $[z, z_1] = [u, v]$. In [6] a graph $G = (V, E)$ is defined as a $(0, \lambda)$-graph if any two distinct vertices u and v have exactly λ common neighbours or none at all. Clearly in a $(0, 2)$-graph for any two vertices u and v such that $d(u, v) = 2$ and for any $z \in [u, v]$ there exists precisely one vertex z_1 such that $[z, z_1] = [u, v]$.

Theorem 4 *A graph $G = (V, E)$ is a hypercube if and only if G is an antipodal and bipartite $(0, 2)$-graph.*

As noted in [2] and [7], a graph G is a hypercube if and only if G is spherical and bipartite. We aim to substitute this condition for a graph to be spherical with a weaker condition of antipodality and with a local condition for a graph to be a $(0, 2)$-graph. Thus we have

Corollary 5 *A bipartite graph is spherical if and only if it is an antipodal* $(0,2)$*-graph.*

Proof of Theorem 4. A hypercube is clearly an antipodal and bipartite $(0,2)$-graph. For the converse proof we assume that $G=(V,E)$ is an antipodal bipartite $(0,2)$-graph. Let u and v be adjacent vertices, whence $[u,v']\cup[v,u']=V$ and $[u,v']\cap[v,u']=\emptyset$.

We first prove the following result.

Claim. *If* $u,u_1,u_2,...,u_n$ *is a geodesic on* $[u,v']$*, then for any* u_i*,* $i=1,2,\dots,n$ *there exists a unique vertex* $v_i\in[v,u']$ *such that* $v_i\in N(u_i)$*,* $v,v_1,v_2,...,v_n$ *is a geodesic on* $[v,u']$ *and* $N(v_i)\cap[u,v']=\{u_i\}$*.*

Proof of Claim. We proceed by induction on n. If $n=1$, then u_1 is adjacent to u and $d(u_1,v)=2$. Because $u\in[u_1,v]$ and G is a $(0,2)$-graph, there exists a vertex v_1 such that $[u,v_1]=[v,u_1]$. Clearly the path u,v,v_1 is a geodesic. If $v_1\in[u,v']$, there exists a geodesic $u,v,v_1,\dots,v'$, which is a contradiction. Thus $v_1\in[v,u']$. Let w be a vertex such that $w\in N(u_1)\cap[v,u']$, $w\neq v_1$. If $w\in N(v_1)$, there is a triangle u_1,v_1,w,u_1, which is a contradiction since G is bipartite. If v,u,u_1,w is a geodesic, then $u\in[v,u']$, because $w\in[v,u']$; a contradiction. If $d(v,w)=2$, there exists a vertex $w_1\in[v,u']$ such that w,w_1,v is a geodesic. This implies an odd cycle v,u,u_1,w,w_1,v, which is a contradiction. Thus $d(v,w)=1$ and then $w\in[v,u_1]$. This contradicts the fact that G is a $(0,2)$-graph, $d(u,v_1)=d(u,w)=2$, and $u,v_1,w\in[v,u_1]$. Thus the Claim holds for $n=1$.

Assume now, that the Claim holds for all $n\leq k-1$. Let $u,u_1,u_2,\dots,u_k$ be a geodesic on $[u,v']$. By the induction assumption there exists a geodesic $v,v_1,v_2,\dots,v_{k-1}$ such that $v_i\in N(u_i)\cap[v,u']$ and $N(v_i)\cap[u,v']=\{u_i\}$ for all $i=1,2,\dots,k-1$.

Because $d(u_k,v_{k-1})=2$, $u_{k-1}\in[u_k,v_{k-1}]$, and because G is a $(0,2)$-graph, there exists a vertex $v_k\neq u_{k-1}$ adjacent to u_k and v_{k-1}. By induction assumption $\{u_{k-1}\}=N(v_{k-1})\cap[u,v']$, and thus we have $v_k\in[v,u']$. Because $d(v,v_{k-1})=k-1$ and v_k is adjacent to v_{k-1}, we have $k-2\leq d(v,v_k)\leq k$. If $d(v,v_k)=k-2$, there exists a geodesic $u,v,v_1,...,v_k,u_k$, whence $v\in[u,v']$, which is a contradiction. If $d(v,v_k)=k-1$, there exists a cycle $u,u_1,u_2,...,u_k,v_k,v_{k-1},...,v,u$ such that the length of the cycle is $k+1+(k-1)+1=2k+1$, a contradiction. Thus $d(v,v_k)=k$ and $v,v_1,...,v_{k-1},v_k$ is a geodesic.

If there exists a vertex $z\in N(u_k)\cap[v,u']$, $z\neq v_k$, then $z\notin N(v_k)$, because otherwise u_k,v_k,z,u_k is a triangle. Thus $d(v_k,z)=2$. Because $d(u_{k-1},z)=2$, $u_k\in[u_{k-1},z]$, and G is a $(0,2)$-graph, there exists a vertex w such that $[u_k,w]=[u_{k-1},z]$. Clearly $w\in N(u_{k-1})\cap N(z)$. Two cases arise (i) $w\in[v,u']$ (ii) $w\in[u,v']$.

(i) If $w\in[v,u']$, then by induction assumption $w=v_{k-1}$, and $z,v_k,u_{k-1}\in[u_k,v_{k-1}]$. Because $d(u_{k-1},z)=d(u_{k-1},v_k)=2$ and G is a $(0,2)$-graph we have a contradiction.

(ii) Since $w \in N(u_{k-1})$ and $d(u, u_{k-1}) = k-1$, the relation $w \in N(u_{k-1})$ implies $k-2 \leq d(u,w) \leq k$. If $d(u,w) = k-2$, then $z \in N(w) \cap [v, u']$ and the induction assumption imply $d(v,z) = k-2$. Because z, u_k, v_k is a geodesic and $d(v, v_k) = k$, the vertex u_k is on the $v, v_1, ..., v_k$ geodesic. Now, by the relation $v_k \in [v, u']$ we have $u_k \in [v, u']$, which is a contradiction. Because $z \in N(w) \cap [v, u']$ and the induction assumption, the assumption $d(u, w) = k-1$ implies $d(v, z) = k-1$. But then there exists a cycle $u, u_1, ..., u_k, z, ..., v, u$ of length $k+1+k-1+1 = 2k+1$; a contradiction. Since w is adjacent to u_{k-1}, the relation $d(u,w) = k$ implies that the path $u, u_1, ..., u_{k-1}, w$ is a geodesic. By the first part of the proof of the Claim, there exists a geodesic $v, z_1, z_2, ..., z_{k-1}, z$ on $[v, u']$ such that $z_i \in N(u_i)$ for all $i = 1, 2, \ldots k-1$. By induction assumption, $z_{k-1} = v_{k-1}$. This implies that $u_{k-1}, z, v_k \in [v_{k-1}, u_k]$ and $d(u_{k-1}, z) = d(u_{k-1}, v_k) = 2$ which is a contradiction, because G is a $(0,2)$-graph and $d(v_{k-1}, u_k) = 2$. If we assume that there exists $z \in [u, v'] \cap N(v_k)$, $z \neq u_k$, then by symmetry this yields a contradiction. Thus the Claim holds for all n.

To prove Theorem 4, we proceed by induction on $|V|$ of G. If $|V| = 4$, then clearly G is Q_2. Assume that the theorem holds for $|V| \leq k$, and let u and v be adjacent vertices. By the Claim, the subgraphs induced by $[u, v']$ and $[v, u']$ are isomorphic, and moreover, G is isomorphic to the graph $Q_1 \times G_0$ where G_0 is isomorphic to the subgraph induced by $[u, v']$. By the induction assumption, it suffices to prove that the subgraph G_1 of G induced by $[u, v']$ is an antipodal and bipartite $(0,2)$-graph.

Because G is bipartite, the subgraph G_1 is also bipartite. It follows from the Claim that every $x-y$ geodesic, $x, y \in [u, v']$, contains p vertices of $[v, u']$, where p is zero or an even number. Thus if $d(x,y) = 2$ and $x, y \in [u, v']$, then $[x, y]$ does not contain any vertices of $[v, u']$. Thus $[x, y] \subseteq [u, v']$ and G_1 is a $(0,2)$-graph, since G is a $(0,2)$-graph.

Assume that $u_1 \in [u, v']$. By the Claim there is a unique vertex $v_1 \in N(u_1) \cap [v, u']$ and thus $v_1' \in [u, v']$. To prove the antipodality of G_1 it suffices to prove that $[u_1, v_1'] = [u, v']$.

Assume first, that $z \in [u, v']$. Since $z \in [v_1, v_1']$, there exists a geodesic $v_1, z_1, \cdots, z_{n-1}, z, \cdots, v_1'$. Since $z, u_1 \in [u, v']$ and v_1 is adjacent to u_1, there exists, by the Claim, a geodesic $u_1, w_1, w_2, \cdots, w_{n-1}, z, \cdots, v_1'$ such that $w_i = z_i$, if $z_i \in [u, v']$ and $w_i \in N(z_i) \cap [u, v']$, if $z_i \in [v, u']$. By the Claim, the vertex w_i is unique for all $i = 1, \ldots, n-1$, and thus $z \in [u_1, v_1']$.

To prove the another inclusion we assume that there exists a geodesic $u_1, \cdots, z_1, z_2, \cdots, z_l, \cdots, v_1'$, such that $z_1, z_l \in [u, v']$ and $z_2, \ldots z_{l-1} \in [v, u']$. We may assume now that the $u_1 - z_1$ geodesic does not contain any vertex of $[v', u]$. If $l = 3$, then

$z_1, z_l \in N(z_2) \cap [v, u']$, which contradicts the Claim. Thus $z_3 \in [v, u']$. Because G is a $(0, 2)$-graph and $d(z_1, z_3) = 2$, there exists a vertex $w_2 \in N(z_1) \cap N(z_3)$, $w_2 \neq z_2$. By the Claim, $N(z_1) \cap [v, u'] = \{z_2\}$, and thus $w_2 \in [u, v']$. By repeating the process above we conclude, that there exists a geodesic $z_1, w_2, w_3, \cdots, w_{l-1}$, such that $w_i \in N(z_i) \cap [u, v']$ for all $i = 2, 3, \ldots, l-1$. Since $z_l, w_{l-1} \in N(z_{l-1}) \cap [u, v']$, the Claim implies $w_{l-1} = z_l$, which contradicts the fact that the path $z_1, z_2 \cdots, z_l$ goes along a $u_1 - v_1'$ geodesic. Thus any vertex on the $u_1 - v_1'$ geodesic is in the interval $[u, v']$. This completes the proof.

4 Prime convex intersection characterization

A nonempty vertex set $A \subset V$ is a convex, if $x, y \in A$ and z on an $x - y$ geodesic imply that $z \in A$. Clearly a nonempty intersection $A \cap B$ of two convexes is a convex, too. By $< D >$ we denote the least convex containing the vertex set D: $< D >= \bigcap\{C | C \text{ is a convex and } D \subset C\}$. The least convex containing the vertices x and y is briefly denoted by $< x, y >$. In general, the convex $< x, y >$ also contains vertices not on an $x-y$ geodesic and hence $[x, y] \subset < x, y >$. In the covering graph of a finite distributive lattice, $[x, y] =< x, y >$ for each two vertices x and y, and thus in each (finite) hypercube $< x, y > \subset [x, y]$ for all $x, y \in V$. A convex $P \neq V$ is called prime if also the set $V \setminus P = \bar{P}$ is a convex. A graph G is a prime convex intersection graph if each convex C of G is the intersection of prime convexes containing C: $C = \bigcap\{P | P \text{ is a prime convex and } C \subset P\}$. For example, all nontrivial trees and all nontrivial complete graphs are prime convex intersection graphs. As the definition shows, prime convexes exist in pairs and this property is used in our characterization

Theorem 6 *A graph G is an $n-$cube Q_n if and only if*

(i) there are n disjoint pairs of prime convexes in G;

(ii) G is a prime convex intersection graph;

(iii) each intersection of n prime convexes, no one of which is from the same pair, is a vertex of G.

Proof. Let G have the properties $(i) - (iii)$. By Theorem 4 proving that G is a hypercube it suffices to prove that G is an antipodal and bipartite $(0, 2)$-graph. We prove this by sequence of claims given and proved below.

Claim 1. *Each prime convex P_i of G is maximal, i.e. there is no prime convex P_j in G containing P_i properly.*

Proof of Claim 1. Assume that $P_1 \subset P_2$ with $P_1 \neq P_2$. By (iii), there exists a vertex a of G such that $\{a\} = P_1 \cap P_2 \cap P_3 \cap ... \cap P_n$. Because $P_1 \subset P_2$, we have

$P_1 \cap \bar{P}_2 = \emptyset$ and thus $\emptyset = P_1 \cap \bar{P}_2 \cap P_3 \cap ... \cap P_n$ contradicts the property (*iii*), and the Claim 1 follows.

If ab is an edge of G, $a \in P_i$ and $b \in \bar{P}_i$, we say that the pair $P_i, \bar{P}_i$ of prime convexes *cuts off* the edge ab.

Claim 2. *If a is a vertex of G, then each pair $P_i, \bar{P}_i, i = 1, ..., n$ cuts off exactly one edge incident to a.*

Proof of Claim 2. Because G is a prime convex intersection graph and the vertex a is obtained as an intersection of prime convexes, the pairs of prime convexes of G must cut off each edge incident to a. If there is a pair of prime convexes, say $P_1, \bar{P}_1$, such that $a \in P_1$ but the pair does not cut off any edge incident to a, then the vertex a has an intersection representation $\{a\} = \bigcap\{P_i | i = 2, 3, .., n\}$, and because $a \in P_1$, a also has the representation $\{a\} = \bigcap\{P_i | i = 1, 2, ..., n\}$. On the other hand, $a \notin \bar{P}_1$, and thus $\bar{P}_1 \cap P_2 \cap ... \cap P_n = \emptyset$, which contradicts (*iii*), and thus each pair of prime convexes cuts off at least one edge incident to a. Let now $a \in P_1$ and the pair $P_1, \bar{P}_1$ cut off at least two edges ab_1 and ab_2. If the edge b_1b_2 does not exist in G, then one of the $b_1 - b_2$ geodesics goes through a, and thus $a \in \bar{P}_1$; a contradiction. Thus we assume that the edge b_1b_2 exists in G and the vertices a, b_1, b_2 induce a complete subgraph K_3 of G. In order to obtain the vertex b_1 as an intersection of prime convexes, there must be a pair cutting off all edges incident to b_1 or to b_2 in K_3. Assume that the pair $P_2, \bar{P}_2$ cuts off the edges ab_1 and b_1b_2 and $b_1 \in \bar{P}_2$, and thus the edge ab_1 is cutted off by at least two pairs $P_1, \bar{P}_1$ and $P_2, \bar{P}_2$. The relation $P_2 \subset \bar{P}_1$ is a contradiction because each prime convex is maximal as stated above. Assume that $P_2 \not\subset \bar{P}_1$, i.e. there is a vertex c belonging to the intersection $P_1 \cap P_2$. Because G is a prime convex intersection graph and $\{a, b_1\}$ is a convex as a set of end vertices of an edge, we have $\{a, b_1\} = \bigcap\{P_i | a, b_1 \in P_i\}$. Thus $\{a\} = \bar{P}_2 \cap (\bigcap\{P_i | a, b_1 \in P_i\}) = P_1 \cap \bar{P}_2 \cap (\bigcap\{P_i | a, b_1 \in P_i\})$ containing by (*iii*) at most n prime convexes from n disjoint pairs of prime convexes. Now $P_1 \cap P_2 \cap (\bigcap\{P_i | a, b_1 \in P_i\}) = \emptyset$ because the vertex $c \in P_1 \cap P_2$ cannot belong to the intersection $\{a, b_1\} = \bigcap\{P_i | a, b_1 \in P_i\}$. This contradicts (*iii*), and the Claim 2 holds.

Claim 2 also implies that each vertex of G has degree n i.e. G is regular of degree n.

Claim 3. *The graph G is bipartite.*

Proof of Claim 3. If a prime convex P_i cuts off an edge of a cycle it must cut off also another edge of the cycle (i.e. cut off the cycle), because P_i induces a connected subgraph of G. If there is an odd cycle in G there also is a least odd cycle C without chords in G. If P_i cuts off an edge of C, it also cuts off an opposite edge because otherwise P_i or $\bar{P}_i$ is not a convex. Each edge of C must be cutted off by prime convexes; if there is an uncutted edge ab then its endpoints do not have a prime convex intersection representation which contradicts (ii). Because C is odd, one edge must be cutted off twice, which contradicts the proof of the Claim 2 and Claim 3 holds.

Claim 4. *The graph G is antipodal.*

Proof of Claim 4. If $\{a\} = P_1 \cap P_2 \cap ... \cap P_n$ we denote the set $\bar{P}_1 \cap \bar{P}_2 \cap ... \cap \bar{P}_n$ by $\{a'\}$ and call a' the complement of a. Let $d(a, a') = l$. By the definitions of a and a', no prime convex of G can contain simultaneously the vertices a and a'. Moreover, any prime convex P_k cannot simultaneously cut off (at least) two edges of an $x - x'$ geodesic: if P_k cuts off the edges e_1e_2 and e_re_s of an $x - x'$ geodesic such that $e_2, e_r \in P_k$ and $e_1, e_s \in \bar{P}_k$, then $e_2, e_r \in \bar{P}_k$, because e_2 and e_r are on the $e_1 - e_s$ geodesic. In order to obtain each vertex of an $x - x'$ geodesic as prime convex intersection, each edge of that $x - x'$ geodesic must be cutted off exactly once. If $l < n$, there is a pair $P_j, \bar{P}_j$ not cutting of any edge of $x - x'$ geodesic, and thus $x, x' \in P_j$ or $x, x' \in \bar{P}_j$, which is a contradiction. Because each edge of $x - x'$ geodesic must be cutted off, the relation $l > n$ cannot hold, and thus we have $l = n$. By the definition of a', there is for any vertex a exactly one vertex a'. If $x \neq y$ for two vertices x and y of G, there must be at least one pair, say $P_t, \bar{P}_t$, such that $y \in P_t$ and $x \notin P_t$. This means that $y \notin \bar{P}_t$ and $x \in \bar{P}_t$, whence $y' \in \bar{P}_t$, $x' \in P_t$, and $x' \neq y'$. Hence for any vertex a of G there is a unique vertex a' such that $d(a, a') = n$.

The vertices in P_1 induce a connected subgraph G^1_{n-1} of the graph G which we shall here denote by G_n ($G = G_n$). If P_i is a prime convex of G_n, there are $n-1$ pairs of prime convexes $P_1 \cap P_i, P_1 \cap \bar{P}_i$ $(i = 2, 3, ..., n)$ in G^1_{n-1}. This proves (i) for G^1_{n-1}. If a is vertex in P_1, then we have by (iii) of G_n the representation $\{a\} = P_1 \cap P_2 \cap P_3 \cap ... \cap P_n = (P_1 \cap P_2) \cap (P_1 \cap P_3) \cap ... \cap (P_1 \cap P_n)$, which is a representation of a as an intersection of $n - 1$ prime convexes of G^1_{n-1}. Thus each prime convex intersection representation of a convex in G^1_{n-1} can be obtained from the corresponding representation in G_n. This shows that G^1_{n-1} has the properties (ii) and (iii). The result holds also if P_1 is replaced by any of the prime convexes $P_i, \bar{P}_i (i = 2, 3, ..., n)$ and $\bar{P}_1$. Because G^1_{n-1} has the same properties as $G = G_n$, we can deduce from G^1_{n-1} (and from all other graphs deduced from G_n by the same way as G^1_{n-1}) graph G^{12}_{n-2}. The graph G^{12}_{n-2} is induced by the vertices in the set $P_1 \cap P_2$ and there are $n-2$ pairs $(P_1 \cap P_2) \cap P_i, (P_1 \cap P_2) \cap \bar{P}_i$ $(i = 3, 4, ..., n)$ of prime convexes in G^{12}_{n-2}. As above, we see, that G^{12}_{n-2} has the properties (i) $-$ (iii). By continuing this process, the vertex set $P_1 \cap P_2 \cap ... \cap P_{n-2}$ induces the graph $G^{12...n-2}_2$ having two pairs $(P_1 \cap ... \cap P_{n-2}) \cap P_j, (P_1 \cap ... \cap P_{n-2}) \cap \bar{P}_j$ $(j = n - 1, n)$ of prime convexes and having the properties (i) $-$ (iii) of the theorem. Thus the properties proved to hold for $G = G_n$ also hold for $G^{12...n-2}_2$, and accordingly, $G^{12...n-2}_2$ is regular of degree 2 and there are four vertices obtained as intersections of the prime convexes: $\{c_1\} = (P_1 \cap ... \cap P_{n-2}) \cap P_{n-1} \cap (P_1 \cap ... \cap P_{n-2}) \cap P_n$, $\{c_2\} = (P_1 \cap ... \cap P_{n-2}) \cap P_{n-1} \cap (P_1 \cap ... \cap P_{n-2}) \cap \bar{P}_n$, $\{c_3\} = (P_1 \cap ... \cap P_{n-2}) \cap \bar{P}_{n-1} \cap (P_1 \cap ... \cap P_{n-2}) \cap P_n$ and $\{c_4\} = (P_1 \cap ... \cap P_{n-2}) \cap \bar{P}_{n-1} \cap (P_1 \cap ... \cap P_{n-2}) \cap \bar{P}_n$. By the definition, $c_4 = c_1'$ and $c_3 = c_2'$. Now the proof of the properties of G_n implies that $d(c_1, c_1') = 2 = d(c_2, c_2')$ in $G_2^{12...n-2}$. These properties imply that $G_2^{12...n-2}$ is a four-cycle, where $[x, x'] = V(G_2^{12...n-2})$ for each vertex x of $G_2^{12...n-2}$. By using induction, we assume that $[x, x'] = V(G_{n-j}^{12...j})$, $j = 1, 2, ..., n-2$ and thus $[x, x'] = V(G_{n-1}^1)$ for each vertex x of G_{n-1}^1, and prove that $[x, x'] = V(G_n)$ for each vertex x of G_n. Let the vertex x have the representation $\{x\} = P_1 \cap P_2 \cap P_3 \cap ... \cap P_n$ in G_n and thus the same vertex has in the graph G_{n-1}^1 induced by the vertex set P_1 the representation $\{x\} = (P_1 \cap P_2) \cap (P_1 \cap P_3) \cap ... \cap (P_1 \cap P_n) = P_1 \cap (P_2 \cap P_3 \cap ... \cap P_n)$. The complement c of x in the graph G_{n-1}^1 is by the definition $(P_1 \cap \bar{P}_2) \cap (P_1 \cap \bar{P}_3) \cap ... \cap (P_1 \cap \bar{P}_n) = P_1 \cap (\bar{P}_2 \cap \bar{P}_3 \cap ... \cap \bar{P}_n) = \{c\}$. By the induction assumption, $[x, c] = V(G_{n-1}^1)$ and by the proof above $d(x, c) = n-1$ in G_{n-1}^1 as well as in G_n. The representations $\{x'\} = \bar{P}_1 \cap \bar{P}_2 \cap \bar{P}_3 \cap ... \cap \bar{P}_n$ and $\{c\} = P_1 \cap (\bar{P}_2 \cap \bar{P}_3 \cap ... \bar{\cap} P_n)$ show that $x', c \in \bar{P}_2 \cap \bar{P}_3 \cap ... \cap \bar{P}_n$. Because a pair of prime convexes cuts off exactly one edge incident to a vertex of G_n, the prime convex P_1 (as well as $\bar{P}_1$) cuts off the edge $x'c$ and $\bar{P}_2 \cap \bar{P}_3 \cap ... \cap \bar{P}_n = \{x', c\}$. Because $d(x, x') = n = d(x, c) + 1$ in G_n, we see that each vertex of $[x, c] = V(G_{n-1}^1) = P_1$ is on an $x - x'$ geodesic in G_n. By repeating the proof for the subgraph $G_{n-1}^{\bar{1}}$ induced by $\bar{P}_1$ in G_n we see that each vertex in $V(G_{n-1}^{\bar{1}}) = \bar{P}_1$ is on an $x - x'$ geodesic in G_n. Accordingly, $[x, x'] = P_1 \cup \bar{P}_1 = V(G_n)$ for each vertex x and its complement in G_n and thus the graph $G = G_n$ is antipodal such that the complement x' is also the antipode of x and Claim 4 holds.

Claim 5. *G is a $(0, 2)$-graph.*

Proof of Claim 5. Let x and y be two vertices of G such that $d(x, y) = 2$. Because G is a prime convex intersection graph, $< x, y >= \bigcap\{P_i | x, y \in P_i\}$. As seen above, each nonempty intersection of prime convexes induces a subgraph G_{xy} of G having the properties $(i) - (iii)$. The vertex x has a complement/an antipode x' in G_{xy} such that $[x, x'] = V(G_{xy})$ and thus the vertex y is on an $x - x'$ geodesic in G_{xy} as well as in G. If $y \neq x'$, then $d(x, x') > d(x, y) = 2$, and we can cut off the vertex x' from G_{xy} by using prime convexes containing x'. As a result we obtain a smaller convex containing x and y, which is a contradiction. Thus $x' = y$, $d(x, x') = 2$, and G_{xy} is a 4-cycle (without chords) of G. In a 4-cycle there exists for a vertex z a unique vertex z_1 with $[z, z_1]$ containg all vertices of the 4-cycle. Hence the Claim 5 holds and so the conditions $(i) - (iii)$ imply an n-hypercube by Theorem 4.

Conversely, let G be an n-hypercube Q_n. As well known, $Q_n = Q_1 \times Q_{n-1}$, which means that Q_n is obtained by combining two complementary $n-1$-hypercubes Q_{n-1}^1

and Q^2_{n-1} such that if f is an isomorphism between these two $n-1$-hypercubes then each vertex x of Q^1_{n-1} is joined by an edge to its image vertex $f(x)$ in Q^2_{n-1}. As known, an n-hypercube is a prime convex intersection graph where the vertex sets $V(Q^1_{n-1}), V(Q^2_{n-1})$ of each pair of complementary $n-1$-hypercubes Q^1_{n-1} and Q^2_{n-1} constitute a pair of prime convexes. As well known, there are n disjoint pairs of complementary $n-1$-hypercubes in an $n-$hypercube. Thus (i) and (ii) hold in the graph of Q_n. It is also known that the intersection of two $n-1$-hypercubes of an $n-$hypercube is an $n-2$-hypercube or an empty graph (when the $n-1$-hypercubes are the pair of complementary $n-1$-hypercubes of Q_n). We prove the validity of (iii) of Q_n by induction on the dimension of the hypercube. One can see by inspection that (iii) holds for Q_2 and we assume that (iii) holds for all $n-1$-hypercubes. Let $V(Q_{n-1}(j))$ $j=1,...,n$ be n prime convexes of Q_n, no one of which is from the same pair and consider the intersection $\bigcap\{V(Q_{n-1}(j))|j=1,...,n\}$. We can write this intersection as follows: $\bigcap\{V(Q_{n-1}(j))|j=1,...,n\} = (V(Q_{n-1}(1)) \cap V(Q_{n-1}(n))) \cap (V(Q_{n-1}(2)) \cap V(Q_{n-1}(n))) \cap (V(Q_{n-1}(3)) \cap V(Q_{n-1}(n))) \cap ... \cap (V(Q_{n-1}(n-1)) \cap V(Q_{n-1}(n)))$, where each set $V(Q_{n-1}(j)) \cap V(Q_{n-1}(n))$ $(j=1,...,n-1)$ is the vertex set of an $n-2-$hypercube/prime convex in the $n-1-$hypercube induced by $V(Q_{n-1}(n))$ in Q_n. By the assumption, this intersection in the $n-1-$hypercube is a vertex of $Q_{n-1}(n)$ which is contained in Q_n. This proves the assertion for Q_n and the characterization follows.

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