Pattern avoidance by even permutations[*]

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Dedicated to Doron Zeilberger, on the occasion of his $|A_5|^{th}$ *birthday.*

Abstract

We study questions of even-Wilf-equivalence, the analogue of Wilf-equivalence when attention is restricted to pattern avoidance by permutations in the alternating group. Although some Wilf-equivalence results break when considering even-Wilf-equivalence analogues, we prove that other Wilf-equivalence results continue to hold in the even-Wilf-equivalence setting. In particular, we prove that $t(t-1)\cdots 321$ and $(t-1)(t-2)\cdots 21t$ are even-shape-Wilf-equivalent for odd t, paralleling a result (which held for all t) of Backelin, West, and Xin for shape-Wilf-equivalence. This allows us to classify the symmetric group $\mathcal{S}_4$, and to partially classify $\mathcal{S}_5$ and $\mathcal{S}_6$, according to even-Wilf-equivalence. As with transition to involution-Wilf-equivalence, some—but not all—of the classical Wilf-equivalence results are preserved when we make the transition to even-Wilf-equivalence.

Keywords: Permutation pattern, Wilf-equivalence, even-Wilf-equivalence, alternating group, even permutation.

1 Introduction

In this paper we focus on questions of Wilf-equivalence when we count only the *even* permutations (i.e., the members of the alternating subgroup) that avoid a particular pattern. In particular, we are interested in which classical Wilf-equivalence results have

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parallels when we instead consider even-Wilf-equivalence. These investigations parallel comparisons between classical Wilf-equivalence and involution-Wilf-equivalence seen in other work (e.g., [SS85, Jag03, DJMR09, JM11]).

Given a permutation $\sigma = \sigma_1 \cdots \sigma_k \in \mathcal{S}_k$, $\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_n$ is said to *contain the pattern* $\tau \in S_k$ if there is some sequence of indices $i_1 < \cdots < i_k$ such that $\pi_{i_1} \cdots \pi_{i_k}$ is order isomorphic to $\tau_1 \cdots \tau_k$. If π does not contain τ, then π is said to *avoid the pattern* τ. We let $\mathcal{S}_n(\sigma)$ denote the set of permutations avoiding σ and let $S_n(\sigma) := \#\mathcal{S}_n(\sigma)$. Two permutations σ and τ are said to be (classically) *Wilf-equivalent* if, for every positive integer n, $S_n(\sigma) = S_n(\tau)$; we then write $\sigma \equiv \tau$.

A pair of indices $i < j$ forms an *inversion* in permutation π if $\pi_i > \pi_j$. Let $\textsc{inv}(\pi)$ denote the number of inversions in π, and let $\operatorname{sgn}(\pi) = (-1)^{\textsc{inv}(\pi)}$ be the *sign* of π. If $\operatorname{sgn}(\pi) = 1$ (i.e., $\textsc{inv}(\pi)$ is even) we say that π is *even* and otherwise π is *odd*. Let $\mathcal{E}_n \subset \mathcal{S}_n$ be the set of even permutations of length n and $\mathcal{O}_n \subset \mathcal{S}_n$ be the set of odd permutations of length n. Let $\mathcal{E}_n(\sigma) = \mathcal{S}_n(\sigma) \cap \mathcal{E}_n$ be the set of even permutations avoiding σ and let $E_n(\sigma) := \#\mathcal{E}_n(\sigma)$, and similarly for $\mathcal{O}_n(\sigma)$ and $O_n(\sigma)$. We say that two permutations σ and τ are *even-Wilf-equivalent* if $E_n(\sigma) = E_n(\tau)$ for all $n \geq 0$; we then write $\sigma \equiv_{\mathcal{E}_n} \tau$. When we need contrast with classical Wilf-equivalence, we denote classical Wilf-equivalence $\sigma \equiv_{\mathcal{S}_n} \tau$.

Our main result, presented in Theorem 10, is the even-shape-Wilf-equivalence of $J_t = t(t-1)\cdots 321$ and $F_t = (t-1)(t-2)\cdots 21t$ when t is odd. This parallels the analogous result for shape-Wilf-equivalence due to Backelin, West, and Xin [BWX07], which held for all t. As corollaries of our main result, we classify the permutations in $\mathcal{S}_4$ according to $\equiv_{\mathcal{E}_n}$ and give partial classifications of $\mathcal{S}_5$ and $\mathcal{S}_6$ according to $\equiv_{\mathcal{E}_n}$; we also conjecture a number of other even-Wilf-equivalences that parallel known Wilf-equivalences.

2 Equivalences via Symmetry

In this section we present two useful lemmas connecting classical Wilf-equivalence to even-Wilf-equivalence. First we exhibit a case that shows where it is is clear that σ is *not* even-Wilf-equivalent to τ.

Lemma 1. *If* $\sigma, \tau \in \mathcal{S}_k$ *but* $\operatorname{sgn}(\sigma) \neq \operatorname{sgn}(\tau)$*, then* $\sigma \not\equiv_{\mathcal{E}_n} \tau$*.*

Proof. If σ is even and τ is odd, then $\mathcal{E}_k(\sigma) = \mathcal{E}_n \setminus \{\sigma\}$ while $\mathcal{E}_k(\tau) = \mathcal{E}_k$. Hence $E_k(\sigma) = E_k(\tau) - 1$. □

Next we consider the trivial symmetries induced by the symmetry of the square. Recall the *reverse* of $\pi = \pi_1 \pi_2 \cdots \pi_n$ is the horizontal reflection of π, denoted

$$\pi^r := \pi_n \pi_{n-1} \cdots \pi_1.$$

Similarly, the *complement* of $\pi \in \mathcal{S}_n$ is the vertical reflection

$$\pi^c := (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_n).$$

The inverse of π is denoted as usual by π^{-1}. The following lemma summarizes how these reflections affect the sign of π.

Lemma 2. *The sign of a permutation $\pi \in S_n$ is affected by reflections in the following ways:*

(a.) $\operatorname{sgn}(\pi) = \operatorname{sgn}(\pi^r)$ *if and only if* $n \equiv 0, 1 \pmod 4$.

(b.) $\operatorname{sgn}(\pi) = \operatorname{sgn}(\pi^c)$ *if and only if* $n \equiv 0, 1 \pmod 4$.

(c.) $\operatorname{sgn}(\pi) = \operatorname{sgn}(\pi^{-1})$

Proof. For each pair of indices $i < j$, $\pi_i > \pi_j$ if and only if $(\pi^r)_i < (\pi^r)_j$. That is, the reversal map swaps the sites of inversions and non-inversions. Therefore $\textsc{inv}(\pi^r) = \binom{n}{2} - \textsc{inv}(\pi)$. Since $\binom{n}{2}$ is even if and only if $n \equiv 0, 1 \pmod 4$, part (a) is proven. Part (b) is proven similarly since it is also the case that $\textsc{inv}(\pi^c) = \binom{n}{2} - \textsc{inv}(\pi)$. Part (c) follows from the fact that for any permutation, $\textsc{inv}(\pi) = \textsc{inv}(\pi^{-1})$. ☐

In the classical case, $\sigma \equiv_{\mathcal{S}_n} \sigma^r$, $\sigma \equiv_{\mathcal{S}_n} \sigma^c$, and $\sigma \equiv_{\mathcal{S}_n} \sigma^{-1}$. Parts (a) and (b) of the lemma above, however, show that even-Wilf-equivalence for σ and σ^r is not guaranteed, and similarly for σ^c. For example $123 \not\equiv_{\mathcal{E}_n} 321$, since $E_3(123) = 2$ and $E_3(321) = 3$. Part (c) confirms, however, that we still have $\sigma \equiv_{\mathcal{E}_n} \sigma^{-1}$.

The next lemma demonstrates that while we lose the equivalences from reversal and complement, we may use symmetric versions of any even-Wilf-equivalences discovered.

Lemma 3. *If $\sigma \equiv_{\mathcal{E}_n} \tau$ and $\sigma \equiv_{\mathcal{S}_n} \tau$, then $\sigma^r \equiv_{\mathcal{E}_n} \tau^r$ and $\sigma^c \equiv_{\mathcal{E}_n} \tau^c$.*

Proof. We will prove $\sigma^r \equiv_{\mathcal{E}_n} \tau^r$. The proof for $\sigma^c \equiv_{\mathcal{E}_n} \tau^c$ is analogous. First observe that if $S_n(\sigma) = S_n(\tau)$ and $E_n(\sigma) = E_n(\tau)$, then $O_n(\sigma) = O_n(\tau)$. We continue by cases. If $n \equiv 0 \text{ or } 1 \pmod 4$, then

$$E_n(\sigma^r) = E_n(\sigma) = E_n(\tau) = E_n(\tau^r),$$

where the first and third equalities follow from Lemma 2 and the second equality by our assumptions. If $n \equiv 2 \text{ or } 3 \pmod 4$, then we see

$$E_n(\sigma^r) = O_n(\sigma) = O_n(\tau) = E_n(\tau^r),$$

where again the first and third equalities follow from Lemma 2 and the second equality by the observation above. ☐

It is worth stating the following lemma regarding the trivial equivalence classes for even-Wilf-equivalences. Its proof is similar to those above and is left to the reader.

Lemma 4. *For a pattern σ, we have the following trivial equivalences:*

- $\sigma \equiv_{\mathcal{E}_n} \sigma^{-1} \equiv_{\mathcal{E}_n} \sigma^{rc} \equiv_{\mathcal{E}_n} (\sigma^{-1})^{rc}$
- $\sigma^r \equiv_{\mathcal{E}_n} \sigma^c \equiv_{\mathcal{E}_n} (\sigma^{-1})^r \equiv_{\mathcal{E}_n} (\sigma^{-1})^c$

3 Short Patterns

In this section we turn to the question of classifying patterns of a given length according to even-Wilf-equivalence.

Lemma 1 immediately implies $12 \not\equiv_{\mathcal{E}_n} 21$, which is the classification of $\mathcal{S}_2$.

Moving on to patterns of length three, we turn to observations of Simion and Schmidt [SS85]. Their enumerations of $E_n(\sigma) - O_n(\sigma)$ for each $\sigma \in \mathcal{S}_3$ imply the following equivalences:

Theorem 5 (Simion and Schmidt [SS85]). *There are two distinct even-Wilf-equivalence classes for patterns of length 3:*

- $123 \equiv_{\mathcal{E}_n} 312 \equiv_{\mathcal{E}_n} 231$
- $321 \equiv_{\mathcal{E}_n} 213 \equiv_{\mathcal{E}_n} 132$

This suggests that if $\sigma \equiv_{\mathcal{S}_n} \tau$ and $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\tau)$, then $\sigma \equiv_{\mathcal{E}_n} \tau$. This is not the case, however, as demonstrated by $1234 \not\equiv_{\mathcal{E}_n} 4321$: $E_6(1234) = 258$, while $E_6(4321) = 255$.

To classify patterns of length 4 or more, we use tools developed in the next section.

4 An Infinite Class of Non-trivial Equivalences

In this section we discuss an extension of the celebrated "prefix reversal" result for classical Wilf-equivalence, as proven by Backelin, West, and Xin [BWX07]. We follow and adapt their notation, aside from a change in convention: we reflect everything vertically. Backelin et al. state their results in terms of (permutation) matrices avoiding other (permutation) matrices. Hence the permutation 132, for example, is written as:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

They then proceed to consider pattern avoidance in non-attacking rook placements in Young diagrams. These rook placements correspond to permutation matrices with some of the southeast cells of the matrix absent.

We choose to illustrate our permutations as graphs of functions, hence our graph of 132 looks like Figure 1.

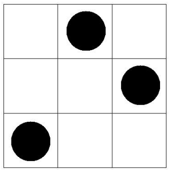

Figure 1: The graph of the permutation 132

As a result of this new convention, we orient our Young diagrams $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ such that the largest part λ_1 forms the *bottom* row of cells (boxes), then λ_2 cells lie above this bottom layer, and so on as per the French custom. Cells of λ are indexed from the lower-left corner by rows and columns, so (r, c) is the cell in the r^{th} row (increasing from the bottom) and c^{th} column (increasing from the left). Hence (r', c') is *above* (r, c) if $r' > r$ and *to the right* if $c' > c$.

A transversal of $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a permutation $\pi \in \mathcal{S}_n$ such that each point in the graph of π lies inside some cell of λ (i.e., $\pi_i^{-1} \leq \lambda_i$ for $1 \leq i \leq n$). Figure 2 illustrates that $\pi = 45321$ is a transversal of $\lambda = (5, 5, 3, 2, 2)$. Let $\mathcal{S}_\lambda$ denote all transversals of λ.

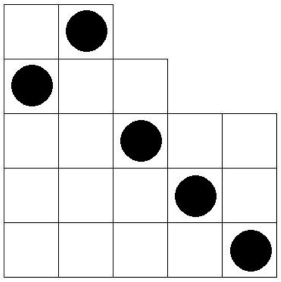

Figure 2: $\pi = 45321$ is a transversal of $\lambda = (5, 5, 5, 3, 2)$.

Pattern containment is stricter for transversals than it is for permutations. A transversal $\pi \in \mathcal{S}_\lambda$ contains $\sigma \in \mathcal{S}_k$ if there exists a subsequence $i_1 < i_2 < \cdots < i_k$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k} \sim \sigma$ and the cell $(\max\{\pi_{i_1}, \pi_{i_2}, \cdots, \pi_{i_k}\}, i_k)$ lies in λ. In other words, the rows and columns of λ containing $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ must form a full $k \times k$ square. In Figure 2 we see the transversal 45321 in $(5, 5, 5, 3, 2)$ contains 321 in the last three entries. Further, the transversal 45321 in $(5, 5, 5, 3, 2)$ avoids 231 even though the *permutation* 45321 does not. We let $\mathcal{S}_\lambda(\sigma)$ denote the set of all transversals of λ which do not contain σ, and $S_\lambda(\sigma) := \#\mathcal{S}_\lambda(\sigma)$. Two patterns σ and τ are called *shape-Wilf-equivalent* if $S_\lambda(\sigma) = S_\lambda(\tau)$ for all shapes λ; we denote this $\sigma \overset{s}{\equiv}_{\mathcal{S}_n} \tau$. Clearly shape-Wilf-equivalence implies Wilf-equivalence, since Wilf-equivalence considers only the shapes λ which are $n \times n$ squares.

We adapt these concepts for even permutations as follows. A transversal $\pi \in \mathcal{S}_\lambda$ is *even* if the underlying permutation π is even. Note that the presence/absence of an inversion is independent of λ, that is, an inversion is *not necessarily* a copy of a 21 pattern in the sense of transversals. Let $\mathcal{E}_\lambda$ be the even transversals in $\mathcal{S}_\lambda$, $\mathcal{E}_\lambda(\sigma)$ be the even transversals in λ avoiding σ, and $E_\lambda(\sigma) := \#\mathcal{E}_\lambda(\sigma)$. We may do the same for odd transversals, using $\mathcal{O}_\lambda$, $\mathcal{O}_\lambda(\sigma)$, and $O_\lambda(\sigma)$. If $E_\lambda(\sigma) = E_\lambda(\tau)$, then we say σ and τ are *even-shape-Wilf-equivalent* and we write $\sigma \overset{s}{\equiv}_{\mathcal{E}_n} \tau$.

Recall the direct sum of two permutations, $\alpha \in \mathcal{S}_k$ and $\beta \in \mathcal{S}_\ell$, is the length-$(k + \ell)$ permutation $\alpha_1\alpha_2\cdots\alpha_k(\beta_1 + k)(\beta_2 + k)\cdots(\beta_\ell + k)$. This is most easily seen as placing β above and to the right of α. Figure 3 depicts $312 \oplus 2413 = 3125746$.

We now re-state Backelin, West and Xin's Proposition 2.3 [BWX07] as a lemma.

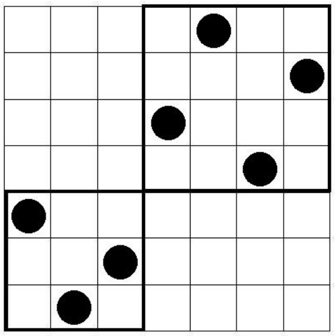

Figure 3: The direct sum $312 \oplus 2413 = 3125746$

Lemma 6 (Backelin, West, and Xin [BWX07])**.** *For patterns α and β, $\alpha \overset{s}{\equiv}_{\mathcal{S}_n} \beta$ implies $\alpha \oplus \sigma \overset{s}{\equiv}_{\mathcal{S}_n} \beta \oplus \sigma$.*

We summarize the proof here, as it will be useful for the following lemma.

Proof (summary). For any shape λ, let $f_\lambda : \mathcal{S}_\lambda(\alpha) \to \mathcal{S}_\lambda(\beta)$ be a bijection implied by the hypothesis. Now fix λ and let $\pi \in \mathcal{S}_\lambda(\alpha \oplus \sigma)$. We will color the cells of λ either white or gray by a two-step procedure, then transform within the white cells while leaving the gray cells fixed. In this way we create a bijection $\mathcal{S}_\lambda(\alpha \oplus \sigma) \to \mathcal{S}_\lambda(\beta \oplus \sigma)$. We illustrate these steps in Figure 4.

Step 1. Color cell (r, c) white if the part of π lying in the subboard above and to the right of it contains σ (as a transversal). Otherwise color (r, c) gray.

Step 2. For each point in the graph of π which lies in a gray cell, color gray the remaining cells in its row and column.

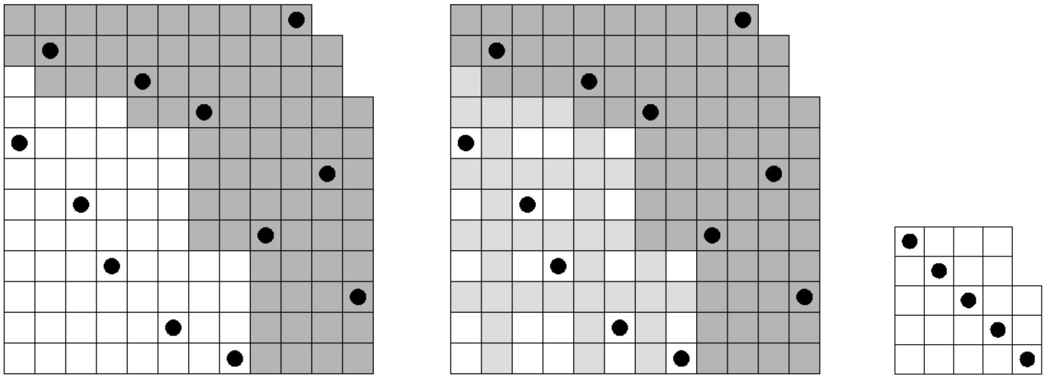

Figure 4: For $\pi = 8(11)64(10)2915(12)73$ and $\lambda = (12^9, 11^2, 10)$, steps 1 (left) and 2 (center) of the Backelin–West–Xin transformation, and the resulting $\bar{\pi} = 54321$ and $\bar{\lambda} = (5, 5, 5, 4, 4)$ (right)

Denote the white cells by $\bar{\lambda}$ and the subtransversal of π lying in $\bar{\lambda}$ by $\bar{\pi}$. By step 2, $\bar{\lambda}$ is itself a Young diagram, and further $\bar{\pi}$ is a transversal of $\bar{\lambda}$. Further, since π avoids $\alpha \oplus \sigma$, step 1 implies $\bar{\pi}$ avoids α. We apply $f_{\bar{\lambda}}$ to $\bar{\pi}$ within the white cells, and so $f_{\bar{\lambda}}(\bar{\pi})$ avoids β. Restoring the gray portions of λ and π, we return to a transversal π' of λ avoiding $\beta \oplus \sigma$. While the application of $f_{\bar{\lambda}}$ could have created or destroyed some copies of σ overall, it can be seen that re-coloring the cells of λ according to the new transversal π' leads to the same white subboard as when one colored according to π. Thus we see the map is invertible, where the inverse map applies $f_{\bar{\lambda}}^{-1}$ to the white subboard. ☐

We are now ready to state the even-shape-Wilf-equivalence analogue.

Lemma 7. *For patterns α and β, if $\alpha \overset{s}{\equiv}_{\mathcal{E}_n} \beta$ and $\alpha \overset{s}{\equiv}_{\mathcal{S}_n} \beta$ then $\alpha \oplus \sigma \overset{s}{\equiv}_{\mathcal{E}_n} \beta \oplus \sigma$.*

Proof. We will adapt notation from Lemma 6 above. Let $g_\lambda : \mathcal{E}_\lambda(\alpha) \to \mathcal{E}_\lambda(\beta)$ be a bijection implied by the hypothesis. By reasoning similar to that in Lemma 3, we see that we may also construct a bijection for odd transversals $h_\lambda : \mathcal{O}_\lambda(\alpha) \to \mathcal{O}_\lambda(\beta)$.

The map $\mathcal{E}_\lambda(\alpha \oplus \sigma) \to \mathcal{E}_\lambda(\beta \oplus \sigma)$ is constructed in the same way as for Lemma 6. Color cells of λ white or gray by the same rules, and isolate $\bar{\lambda}$ and $\bar{\pi}$. Now $\bar{\pi}$ is either even or odd, so we apply the appropriate map $g_{\bar{\lambda}}$ or $h_{\bar{\lambda}}$. Observe that these maps preserve sign and so correspond to multiplying the original π by some even permutation. Hence the image of the transversal π is also even and we have our bijection. ☐

Backelin et al. also prove $J_t \overset{s}{\equiv}_{\mathcal{S}_n} I_t$, where J_t is the decreasing permutation $t(t-1)\cdots 21$ and I_t is the increasing permutation $12\cdots t$. By Lemma 6 above, this implies the well-known "prefix reversal" maneuver for Wilf-equivalence, namely $12\cdots k \oplus \sigma \equiv_{\mathcal{S}_n} k(k-1)\cdots 1 \oplus \sigma$. They prove[1] $J_t \overset{s}{\equiv}_{\mathcal{S}_n} I_t$ via their Proposition 3.1, that $J_t \overset{s}{\equiv}_{\mathcal{S}_n} F_t$ for all $t > 0$, where $F_t = J_{t-1} \oplus 1 = (t-1)(t-2)\cdots 21t$. Iterating this proves $J_t \overset{s}{\equiv}_{\mathcal{S}_n} J_{t-k} \oplus I_k$ for all $0 \leq k \leq t$. They provide a bijection $\phi_t^* : \mathcal{S}_\lambda(F_t) \to \mathcal{S}_\lambda(J_t)$, which we will show preserves sign. Here we will construct the map only; the proof of its correctness was given by Backelin et al. [BWX07].

The map from $\mathcal{S}_\lambda(F_t)$ to $\mathcal{S}_\lambda(J_t)$ uses the following transformation. At its heart, it systematically converts all occurrences of J_t into occurrences of F_t. Suppose $\pi \in \mathcal{S}_\lambda(J_t)$. Then we apply the following algorithm:

Algorithm 8.

Step 1. *Find all occurrences of J_t in π (as a transversal). If π contains no J_t, then stop and return π.*

Step 2. *Find the smallest letter $\pi(i_1)$ such that $\pi(i_1)$ is the leftmost letter in an copy of J_t.*

Step 3. *Find the leftmost letter $\pi(i_2)$ such that $i_1 < i_2$ and there is an occurrence of J_t such that $\pi(i_1)$ and $\pi(i_2)$ are the leftmost letters.*

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[1] Backelin et al. actually provide two proofs of $J_t \overset{s}{\equiv}_{\mathcal{S}_n} I_t$. Here we discuss only their first proof.

Step 4. *Find indices $i_3 < i_4 < \cdots < i_t$ one by one as described in step 3. This yields a subpermutation $\pi(i_1)\pi(i_2)\cdots\pi(i_t)$, which is a copy of J_t, as shown in the left part of Figure 5.*

Step 5. *Form a new permutation π' by moving $\pi(i_1)$ to the ${i_t}^{th}$ position, and each other $\pi(i_j)$ to the ${i_{j-1}}^{th}$ position. Call this transformation $\theta(\pi) = \pi'$. Observe that $\pi'(i_1)\pi'(i_2)\cdots\pi'(i_t)$ is a copy of F_t. (See the right part of Figure 5.)*

Step 6. *Return to step 1.*

We denote a single application of steps 2 through 5 by $\phi_t(\pi)$. As described in steps 1 and 6, we compose ϕ_t with itself repeatedly until all copies of J_t are eliminated. We denote this repeated composition ϕ_t^*.

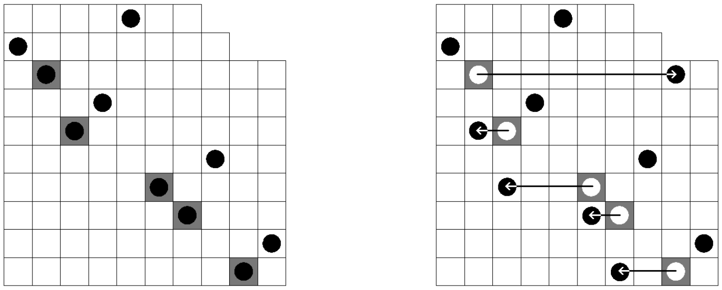

Figure 5: Selecting a copy of J_t (left) and applying the transformation θ (right)

For completeness we present the inverse map from $\mathcal{S}_\lambda(J_t)$ to $\mathcal{S}_\lambda(F_t)$. It operates on the same principle, converting copies of F_t into copies of J_t.

Algorithm 9.

Step 1. *Find all occurrences of F_t in π (as a transversal). If π contains no F_t, then stop and return π.*

Step 2. *Find the largest letter $\pi(i_t)$ such that $\pi(i_t)$ is the rightmost letter in an copy of F_t.*

Step 3. *Find the largest letter $\pi(i_{t-1})$ such that $i_{t-1} < i_t$ and there is an occurrence of F_t such that $\pi(i_{t-1})$ and $\pi(i_t)$ are the rightmost letters.*

Step 4. *Find indices $i_{t-2} > i_{t-3} > \cdots > i_1$ one by one as described in step 3. This yields a subpermutation $\pi(i_1)\pi(i_2)\cdots\pi(i_t)$, which is a copy of F_t.*

Step 5. *Form a new permutation π' by moving $\pi(i_t)$ to the ${i_1}^{th}$ position, and each other $\pi(i_j)$ to the ${i_{j+1}}^{th}$ position. Call this transformation $\theta'(\pi) = \pi'$. Observe that $\pi'(i_1)\pi'(i_2)\cdots\pi'(i_t)$ is a copy of J_t.*

Step 6. *Return to step 1.*

We denote the application of steps 2 through 5 by $\psi_t(\pi)$. We compose ψ_t with itself a certain number of times as outlined in steps 1 and 6, yielding a map $\psi_t^* : \mathcal{S}_\lambda(J_t) \to \mathcal{S}_\lambda(F_t)$. Backelin et al. then show that ϕ_t and ψ_t are inverses of one another, and hence so are ϕ_t^* and ψ_t^*.

We are now ready to prove our main result.

Theorem 10. *$J_t \overset{s}{\equiv}_{\mathcal{E}_n} F_t$ for all odd t.*

Proof. Fix t odd. The theorem follows from the claim that ϕ_t preserve sign. If ϕ_t preserves sign, then so does ϕ_t^*. Hence ϕ_t^* restricts to the map $\phi_t^* : \mathcal{E}_\lambda(F_t) \to \mathcal{E}_\lambda(J_t)$. Since ψ_t is the inverse of ϕ_t, ψ_t^* must also preserve sign and hence we have our desired bijection.

Thus it remains to show that ϕ_t preserves sign when t is odd. Careful inspection reveals that the map θ in Step 5 is merely multiplying by the cycle $(i_1\, i_2\, \cdots\, i_t)$. Since an odd cycle is an even permutation, applying θ preserves sign. □

It should be noted that ϕ_t reverses sign when t is even. This follows from the fact that θ is an even cycle and hence its application reverses sign. Since ϕ_t may be composed with itself either an even or odd number of times in the application of ϕ_t^*, however, the composition ϕ_t^* neither preserves nor reverses sign for the entirety of $\mathcal{E}_\lambda(F_t)$.

The restriction that t be odd prevents the iteration which implies $J_t \overset{s}{\equiv}_{\mathcal{S}_n} J_{t-k} \oplus I_k$ for all $0 \leq k \leq t$. Applying the theorem once gets us $J_t \overset{s}{\equiv}_{\mathcal{E}_n} J_{t-1} \oplus 1$, at which point $t-1$ is even and the theorem no longer applies. Note that the general prefix reversal result is not true for even-Wilf-equivalence: for example, $1234 \not\equiv_{\mathcal{E}_n} 4321$.

5 Classifications

This section makes the classification of 4-patterns under $\equiv_{\mathcal{E}_n}$ explicit, as well as the partial classifications of patterns of length 5 and 6.

5.1 Classification of $\mathcal{S}_4$

With Theorem 10 above and sufficient numerical computation, we may classify all patterns $\sigma \in \mathcal{S}_4$. There are eleven equivalence classes in total. The values of each $E_n(\sigma)$ are listed for $n \leq 10$ in Table 1.

In $\mathcal{S}_4$, only two non-trivial equivalences appear. Since $\sigma \equiv_{\mathcal{E}_n} \sigma^{rc}$ we get that $3214 \equiv_{\mathcal{E}_n} 1432$ and $2134 \equiv_{\mathcal{E}_n} 1243$. Applying Theorem 10 and Lemma 7, we get that $3214 = J_3 \oplus 1 \equiv_{\mathcal{E}_n} F_3 \oplus 1 = 2134$ to complete the class. The reverses of these patterns comprise the other non-trivial class, as per Lemma 3: $3421 \equiv_{\mathcal{E}_n} 4312 \equiv_{\mathcal{E}_n} 4123 \equiv_{\mathcal{E}_n} 2341$. Thus we obtain the classification shown in Table 1. Horizontal lines separate the even-Wilf classes: patterns in the same even-Wilf class appear in adjacent rows with no separating line.

σ	sgn(σ)	$E_4(\sigma)$	$E_5(\sigma)$	$E_6(\sigma)$	$E_7(\sigma)$	$E_8(\sigma)$	$E_9(\sigma)$	$E_{10}(\sigma)$
2134	-1	12	52	257	1381	7885	47181	293297
3214	-1	12	52	257	1381	7885	47181	293297
1243	-1	12	52	257	1381	7885	47181	293297
1432	-1	12	52	257	1381	7885	47181	293297
4312	-1	12	52	256	1380	7885	47181	293293
4123	-1	12	52	256	1380	7885	47181	293293
3421	-1	12	52	256	1380	7885	47181	293293
2341	-1	12	52	256	1380	7885	47181	293293
2314	1	11	51	257	1371	7742	45622	277826
1423	1	11	51	257	1371	7742	45622	277826
3124	1	11	51	257	1371	7742	45622	277826
1342	1	11	51	257	1371	7742	45622	277826
4132	1	11	51	255	1369	7742	45622	277836
3241	1	11	51	255	1369	7742	45622	277836
4213	1	11	51	255	1369	7742	45622	277836
2431	1	11	51	255	1369	7742	45622	277836
2413	-1	12	52	256	1370	7743	45623	277831
3142	-1	12	52	256	1370	7743	45623	277831
1234	1	11	51	258	1382	7879	47175	293311
4321	1	11	51	255	1379	7879	47175	293279
2143	1	11	51	256	1380	7885	47181	293301
3412	1	11	51	257	1381	7885	47181	293289
1324	-1	12	52	258	1382	7903	47393	296002
4231	-1	12	52	255	1380	7903	47393	295948

Table 1: The classification of $\mathcal{S}_4$ into $\equiv_{\mathcal{E}_n}$-classes with values of $E_n(\sigma)$ for $\sigma \in \mathcal{S}_4$ and $n \leq 10$.

5.2 Partial Classification of $\mathcal{S}_5$

The techniques of the previous section imply a partial classification of $\mathcal{S}_5$. Based on computations for $E_n(\sigma)$ for $n \leq 11$, there appear to be four non-trivial equivalence classes, listed in Table 2. Each row represents one trivial equivalence class with a chosen representative. Two rows written adjacently with no separating line are proven above to be even-Wilf-equivalent, as discussed below. Two rows written adjacently and separated by a dotted line are conjectured to be even-Wilf-equivalent based on numerical data for $n \leq 11$. Solid lines separate rows which are not even-Wilf-equivalent.

σ	$\mathrm{sgn}(\sigma)$	$E_7(\sigma)$	$E_8(\sigma)$	$E_9(\sigma)$	$E_{10}(\sigma)$	$E_{11}(\sigma)$
12345	1	2293	16662	130897	1095344	9659368
23451	1	2293	16662	130897	1095344	9659368
45312	1	2293	16662	130897	1095344	9659368
34512	1	2293	16662	130897	1095344	9659368
15432	1	2289	16662	130897	1095344	9659320
54321	1	2289	16662	130897	1095344	9659320
21354	1	2289	16662	130897	1095344	9659320
21543	1	2289	16662	130897	1095344	9659320
12354	-1	2291	16662	130907	1095344	9659344
12543	-1	2291	16662	130907	1095344	9659344
45321	-1	2291	16662	130907	1095344	9659344
34521	-1	2291	16662	130907	1095344	9659344
13524	-1	2290	16627	130145	1081965	9450267
42531	-1	2290	16627	130145	1081965	9450267

Table 2: The classification of $\mathcal{S}_5$ into $\equiv_{\mathcal{E}_n}$-classes with values of $E_n(\sigma)$ for $\sigma \in \mathcal{S}_5$ and $n \leq 11$.

The proven equivalences are each a corollary to Theorem 10 in conjunction with the symmetries in Lemmas 3 and 4. For example, $12345 \equiv_{\mathcal{E}_n} 23451$ since $12345^c = 54321$, $54321 \overset{s}{\equiv}_{\mathcal{E}_n} \phi_5^*(54321) = 43215$, and $43215^c = 23451$. Thus we see that $\pi \mapsto \phi_5^*(\pi^c)^c$ provides the bijection $\mathcal{E}_n(12345) \to \mathcal{E}_n(23451)$.

- $12345 \equiv_{\mathcal{E}_n} 23451$ (under $\phi_5^*(\pi^c)^c$)
- $45312 \equiv_{\mathcal{E}_n} 34512$ (under $\psi_3^*(\pi^c)^c$)
- $15432 \equiv_{\mathcal{E}_n} 54321$ (under $\phi_5^*(\pi)^{rc}$)
- $21354 \equiv_{\mathcal{E}_n} 21543$ (under $\psi_3^*(\pi^{rc})^{rc}$)
- $12354 \equiv_{\mathcal{E}_n} 12543$ (under $\psi_3^*(\pi^{rc})^{rc}$)
- $45321 \equiv_{\mathcal{E}_n} 34521$ (under $\psi_3^*(\pi^c)^c$)

This leaves the following conjectured equivalences:

Conjecture 11. *The following equivalences hold:*

- $12345 \equiv_{\mathcal{E}_n} 45312$
- $54321 \equiv_{\mathcal{E}_n} 21354$
- $12354 \equiv_{\mathcal{E}_n} 45321$
- $13524 \equiv_{\mathcal{E}_n} 42531$

Observe that Lemma 3 implies that the first and second conjectured equivalences follow from one another.

The second conjectured equivalence class contains all patterns of the form $J_r \oplus J_s$ for all $r + s = 5$ and $r, s \geq 0$, together with 21354. The first conjectured class contains the reverses of these. A similar pattern seems to emerge in patterns of length 7, although again conjecturally. This suggests the following more general statement:

Conjecture 12. *For odd t, $J_r \oplus J_s \equiv_{\mathcal{E}_n} J_t$ for any $r + s = t$.*

Also notice that the third and fourth conjectured equivalences in Conjecture 11 have been written in the form $\sigma \equiv_{\mathcal{E}_n} \sigma^r$. In the classical case this is trivial under the reversal map since $\mathcal{S}_n(\sigma)^r = \mathcal{S}_n(\sigma^r)$, but if $n = 3, 4 \pmod 4$, then $\mathcal{E}_n(\sigma)^r \cap \mathcal{E}_n(\sigma^r) = \emptyset$ since $\mathcal{E}_n(\sigma)^r$ contains only odd permutations by Lemma 2.

5.3 Partial Classification of $\mathcal{S}_6$

For $\mathcal{S}_6$ there are 10 non-trivial even-Wilf classes, plus two more conjectured based on numerical results. These are listed below in Table 3. As in Table 2, we separate classes known to be even-Wilf-inequivalent by solid lines, we separate by dashed lines classes that we conjecture to be even-Wilf-equivalent, and there is no separation between classes that we know to be even-Wilf-equivalent. Each of these equivalences follows from Theorem 10 and its symmetries. In the classical case, classifying the length 6 patterns required an additional result provided by Stankova and West [SW02]. They prove that $312 \overset{s}{\equiv}_{\mathcal{S}_n} 231$, which in combination with Lemma 6 provides the equivalence $312564 \equiv_{\mathcal{S}_n} 231564$. We have checked computationally for all Ferrers shapes λ which lie in an 9×9 box that $E_\lambda(312) = E_\lambda(231)$, and that $E_n(231 \oplus \alpha) = E_n(312 \oplus \alpha)$ for all $\alpha \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$ and $n \leq 11$. This naturally leads to Conjecture 13, which would imply $312564 \overset{s}{\equiv}_{\mathcal{E}_n} 231564$ (and $465312 \overset{s}{\equiv}_{\mathcal{E}_n} 465132$ by Lemma 7).

Conjecture 13. 312 *is even-shape-Wilf-equivalent to* 231.

This analogue of the Stankova-West result, combined with those discussed in the previous sections, would complete the classification of the length 6 patterns.

σ	sgn(σ)	$E_7(\sigma)$	$E_8(\sigma)$	$E_9(\sigma)$	$E_{10}(\sigma)$	$E_{11}(\sigma)$
543216	1	2501	19713	172417	1645790	16917552
432156	1	2501	19713	172417	1645790	16917552
612345	-1	2502	19713	172417	1645800	16917562
651234	-1	2502	19713	172417	1645800	16917562
213564	-1	2502	19714	172392	1644933	16895077
321564	-1	2502	19714	172392	1644933	16895077
465312	1	2501	19714	172392	1644930	16895074
465123	1	2501	19714	172392	1644930	16895074
213456	-1	2502	19714	172418	1645799	16917561
321456	-1	2502	19714	172418	1645799	16917561
654312	1	2501	19714	172418	1645791	16917553
654123	1	2501	19714	172418	1645791	16917553
213546	1	2501	19712	172417	1645814	16918707
321546	1	2501	19712	172417	1645814	16918707
645312	-1	2502	19712	172417	1645838	16918725
645123	-1	2502	19712	172417	1645838	16918725
213465	1	2501	19713	172417	1645791	16917553
321465	1	2501	19713	172417	1645791	16917553
321654	1	2501	19713	172417	1645791	16917553
564312	-1	2502	19713	172417	1645799	16917561
564123	-1	2502	19713	172417	1645799	16917561
456123	-1	2502	19713	172417	1645799	16917561
231564	1	2501	19716	172388	1644575	16882865
312564	1	2501	19716	172388	1644575	16882865
465132	-1	2502	19716	172388	1644588	16882878
465213	-1	2502	19716	172388	1644588	16882878

Table 3: The classification of $\mathcal{S}_6$ into $\equiv_{\mathcal{E}_n}$-classes with values of $E_n(\sigma)$ for $\sigma \in \mathcal{S}_6$ and $n \leq 11$.

6 Conclusions and Future Directions

In this paper we have established the foundation for a theory of even-Wilf-equivalence, parallel to the classical theory of Wilf-equivalence. As with involution-Wilf-equivalence, the general trend appears to be that results in Wilf-equivalence have weaker versions for even-Wilf-equivalence. For example, $\sigma \not\equiv_{\mathcal{E}_n} \sigma^r$ but if $\sigma \equiv_{\mathcal{E}_n} \tau$ then $\sigma^r \equiv_{\mathcal{E}_n} \tau^r$. Similarly, we have prove an $\equiv_{\mathcal{E}_n}$-analogue to Backelin et al.'s result that $J \equiv_{\mathcal{S}_n} F_t$; this requires t to be be odd. These results allow us to classify $\mathcal{S}_4$ according to even-Wilf-equivalence, and to partially classify $\mathcal{S}_k$ for larger k.

Known even-Wilf-equivalences, and the ones we conjecture above based on numerical results, suggest that even-Wilf-equivalence is a refinement of Wilf-equivalence; we thus make the following conjecture:

Conjecture 14. *If $\sigma \equiv_{\mathcal{E}_n} \tau$, then $\sigma \equiv_{\mathcal{S}_n} \tau$.*

We note that the analogue (which has not been formally conjectured, but which motivated aspects of [JM11]) for involution-Wilf-equivalence[2] remains open.

Examining the equivalence classes under $\equiv_{\mathcal{E}_n}$ suggests that even-Wilf-equivalence is a very strong condition. Table 4 summarizes the number of equivalence classes under classical and even-Wilf-equivalence. Values for the number of equivalence classes under Wilf equivalence are taken from OEIS sequence A099952 [OEI11]. Lower bounds for the even-Wilf-equivalence classes for 5- and 6-patterns are based on avoidance by permutations of length $n \leq 11$; we obtain upper bounds by assuming all conjectures above are false. Note that $312564 \overset{s}{\equiv}_{\mathcal{E}_n} 231564$ holds if and only if $465312 \overset{s}{\equiv}_{\mathcal{E}_n} 465132$ holds, so 217 is not a possible value for the bottom-right table entry.

n	1	2	3	4	5	6
Wilf-equivalence	1	1	1	3	16	91
even-Wilf-equivalence	1	1	2	11	$[35, 39]$	$\{216, 218\}$

Table 4: The number of equivalence classes for patterns of length n.

There are many more trivial equivalence classes under $\equiv_{\mathcal{E}_n}$ than in the classical case. A possible weakening of even-Wilf-equivalence is perhaps to require that $E_n(\sigma) = E_n(\tau)$ only for "most" n. For example, the results of Simion and Schmidt [SS85] imply that $E_n(123) = E_n(132)$ for any $n \neq 0 \pmod 4$. Similarly, $E_n(\sigma) = E_n(\sigma^r) = E_n(\sigma^c)$ for any $n = 0, 1 \pmod 4$. In other instances, data suggests pairs (σ, τ) such that $E_{2n}(\sigma) = E_{2n}(\tau)$ for all n. For example, the enumeration schemes in [Bax10] verify that $E_{2n}(12345) = E_{2n}(54321)$ for $n \leq 7$. An investigation into these weakened forms of equivalence may yield a classification of patterns which more closely resembles Wilf-classification.

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[2] Two patterns σ and τ are said to be involution-Wilf-equivalent if $|\mathcal{S}_n(\sigma) \cap \mathcal{I}_n| = |\mathcal{S}_n(\tau) \cap \mathcal{I}_n|$ for all n, where $\mathcal{I}_n$ is the set of involutions of length n.

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