Fixed points and excedances in restricted permutations

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Submitted: Oct 29, 2010; Accepted: Dec 15, 2011; Published: Jan 2, 2012
Mathematics Subject Classification: 05A15, 05A05; 30E20

Abstract

Using an unprecedented technique involving diagonals of non-rational generating functions, we prove that among the permutations of length n with i fixed points and j excedances, the number of 321-avoiding ones equals the number of 132-avoiding ones, for any given i, j.

Our theorem generalizes a result of Robertson, Saracino and Zeilberger. Even though bijective proofs have later been found by the author jointly with Pak and with Deutsch, this paper contains the original analytic proof that was presented at FPSAC 2003.

1 Introduction

I met Doron Zeilberger for the first time at the 2002 Summer Meeting of the Canadian Mathematical Society in Québec. He gave one of his memorable talks, this one about joint work with Robertson and Saracino, which later appeared in [13]. The central result was, in Zeilberger's own words, "the amazing and easily-stated fact that the number of 132-avoiding derangements equals the number of 321-avoiding derangements, and even more amazingly, that the same is true if you replace 'derangements' by 'permutations with i fixed points', for any $0 \leq i \leq n$," where n is the length of the permutations in question. This beautiful result, and the fact that the proof in [13] is quite involved, motivated me to try to find generalizations and a more direct proof, encouraged by Richard Stanley, who at that time was my thesis advisor. The product is presented in this paper, namely a proof of the more general statement that the number of 132-avoiding permutations with i fixed points and j excedances equals the number of 321-avoiding permutations with i fixed points and j excedances, for any $0 \leq i, j \leq n$. One ingredient of the proof is a method to extract a diagonal of a non-rational generating function, which is used to prove an equality between generating functions. To the best of our knowledge, this is the first time that such technique is used to solve a combinatorial problem, and we expect that it can be applied to other problems in the future. Other ingredients include bijections between restricted permutations and Dyck paths, and the introduction of a new family of Dyck paths statistics which we call tunnels. These statistics have been later used in [6], together with the RSK algorithm, to give a bijective proof of our main theorem, and in [5] to prove a different generalization of Robertson, Saracino and Zeilberger's result. Because of these subsequent results, the work presented in this paper, after being accepted as a talk at FPSAC 2003, was never submitted for publication until now, on the occasion of Doron Zeilberger's 60th birthday.

2 Definitions and main theorem

Let n, m be two positive integers with $m \leq n$, and let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_m \in \mathcal{S}_m$. We say that π *contains* σ if there exist indices $i_1 < i_2 < \ldots < i_m$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$ is in the same relative order as $\sigma_1\sigma_2\cdots\sigma_m$. If π does not contain σ, we say that π is *σ-avoiding*. For example, if $\sigma = 132$, then $\pi = 24531$ contains σ, because $\pi_1\pi_3\pi_4 = 253$. However, $\pi = 42351$ is σ-avoiding.

We say that i is a *fixed point* of π if $\pi_i = i$, and that i is an *excedance* of π if $\pi_i > i$. Denote by $\mathrm{fp}(\pi)$ and $\mathrm{exc}(\pi)$ the number of fixed points and the number of excedances of π respectively. Denote by $\mathcal{S}_n(\sigma)$ the set of σ-avoiding permutations in $\mathcal{S}_n$.

For the case of patterns of length 3, it was shown by Knuth [9] that for every pattern $\sigma \in \mathcal{S}_3$, $|\mathcal{S}_n(\sigma)| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number. Several bijective proofs of this fact have been known for some time [10, 12, 14, 16].

More recently, Robertson, Saracino and Zeilberger [13] found an unexpected connection between pattern avoidance and permutation statistics, giving an interesting refinement of this result. They showed that for any $i \leq n$, the number of 321-avoiding permutations of length n with i fixed points equals the number of 132-avoiding permutations of length n with i fixed points. In this paper we prove a further refinement of this result, namely that it still holds when we fix not only the number of fixed points but also the number of excedances:

Theorem 2.1. *For any* $0 \leq i, j \leq n$,

$$|\{\pi \in \mathcal{S}_n(321) : \ \mathrm{fp}(\pi) = i, \ \mathrm{exc}(\pi) = j\}| = |\{\pi \in \mathcal{S}_n(132) : \ \mathrm{fp}(\pi) = i, \ \mathrm{exc}(\pi) = j\}|.$$

Equivalently,

$$\sum_{\pi \in \mathcal{S}_n(321)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} = \sum_{\pi \in \mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)}.$$

In the proof we will use bijections between pattern-avoiding permutations and Dyck paths. Recall that a *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ which never goes below the x-axis. Sometimes it will be convenient to encode each up-step by a letter u and each down-step by d, obtaining an encoding of the Dyck path as a *Dyck word*. We shall denote by $\mathcal{D}_n$ the set of Dyck paths of length $2n$, and by $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}_n$ the set of all Dyck paths. It is well-known that $|\mathcal{D}_n| = C_n$. If $D \in \mathcal{D}_n$, we will write $|D| = n$ to indicate the semilength of D. The generating function that enumerates Dyck paths according to their semilength is $\sum_{D\in\mathcal{D}} t^{|D|} = \sum_{n\geq 0} C_n t^n = \frac{1-\sqrt{1-4t}}{2t}$, which we denote by $C(t)$. A *peak* of a Dyck path is an up-step followed by a down-step (i.e., an occurrence of ud in the associated Dyck word). A *hill* is a peak at height 1, where the height is the y-coordinate of the top of the peak. Denote by $h(D)$ the number of hills of D. A *double rise* of a Dyck path is an up-step followed by another up-step (uu when seen as a word). Denote by $\mathrm{dr}(D)$ the number of double rises of D.

Another key definition in the paper is the diagonal of a generating function. Given a generating function $F(v,t) = \sum_{i,j} a_{i,j} v^i t^j$ in the variables v and t, the diagonal of F is the generating function $\mathrm{diag}^z_{v,t}\, F = \sum_n a_{n,n} z^n$. Some properties of diagonals and techniques to compute them are described in [15, 8].

Our proof of Theorem 2.1 is a combination of bijective combinatorics, "manipulatorics," and complex analysis. First, in Section 3 we find the generating function for 321-avoiding permutations with respect to the number of fixed points and excedances, using a bijection to Dyck paths and standard techniques. Section 4 is the central section of the paper, where we we show that the same generating function also counts 132-avoiding permutations with respect to the number of fixed points and excedances. To do this, we first use a bijection to turn the problem into the enumeration of Dyck paths with respect to some new statistics, which we call centered and left tunnels. For this enumeration, we introduce an extra variable to the generating function and we find a complicated identity satisfied by it, using combinatorial properties of Dyck paths. This identity has a unique solution (i.e., it determines the generating function), but it involves a diagonal as defined above. To solve it, we guess an expression for the solution and check that it indeed satisfies the identity, using techniques from complex analysis.

Finally, Section 5 describes other bijections between 321-avoiding permutations and Dyck paths derived from the one that we use in Section 3. These bijections provide combinatorial proofs of the equidistribution of certain statistics on Dyck paths and also on restricted permutations.

3 Counting 321-avoiding permutations according to fixed points and excedances

The goal of this section is to find an expression for the generating function

$$F_{321}(x,q,t) := \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(321)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} t^n.$$

Instead of counting fixed points and excedances directly in 321-avoiding permutations, we define a bijection ψ between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$, first suggested by Richard Stanley and used also in [5]. We give three equivalent definitions of ψ.

Given $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n(321)$, let

$$a_i = \max\{j \geq 0 : \{1,2,\ldots,j\} \subseteq \{\pi_1,\pi_2,\ldots,\pi_i\}\},$$

for each $1 \leq i \leq n$. Now build the Dyck path $\psi(\pi)$ by adjoining, for each i from 1 to n, one up-step followed by $\max\{a_i - \pi_i + 1, 0\}$ down-steps. For example, for $\pi = 23147586$ we get $a_1 = a_2 = 0$, $a_3 = 3$, $a_4 = a_5 = 4$, $a_6 = a_7 = 5$, $a_8 = 8$, and the corresponding Dyck path is given in Figure 1.

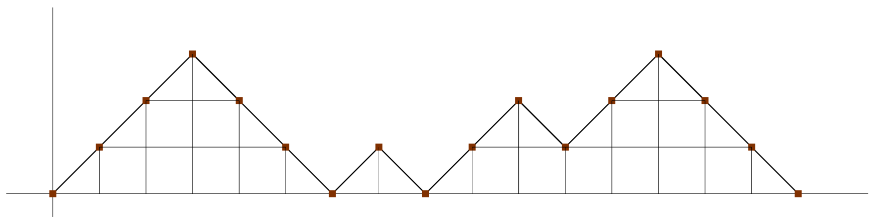

Figure 1: The Dyck path $\psi(23147586)$.

Here is an alternative way to define this bijection. A *right-to-left minimum* of π is an element π_i such that $\pi_i < \pi_j$ for all $j > i$. Let $\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_k}$ be the right-to-left minima of π, from left to right. For example, the right-to-left minima of 23147586 are $1, 4, 5, 6$. Then, $\psi(\pi)$ is precisely the path that starts with i_1 up-steps, then has, for each j from 2 to k, $\pi_{i_j} - \pi_{i_{j-1}}$ down-steps followed by $i_j - i_{j-1}$ up-steps, and finally ends with $n+1-\pi_{i_k}$ down-steps.

The third definition of ψ is the easiest one to visualize. First we represent π as an $n \times n$ array (with rows and columns numbered as in a matrix) with crosses on the squares (i, π_i). It is known [11] that a permutation is 321-avoiding if and only if both the subsequence determined by its excedances and the one determined by the remaining elements are increasing. In this array representation, excedances correspond to crosses strictly to the right of the main diagonal. The rest of the crosses are precisely the right-to-left minima. Consider the path with *down* and *right* steps along the edges of the squares that goes from the upper-left corner to the lower-right corner of the array leaving all the crosses to the right and remaining always as close to the main diagonal as possible. Then $\psi(\pi)$ can be obtained from this path just by reading an up-step every time the path goes down, and a down-step every time the path goes right. Figure 2 shows a picture of this bijection, again for $\pi = 23147586$.

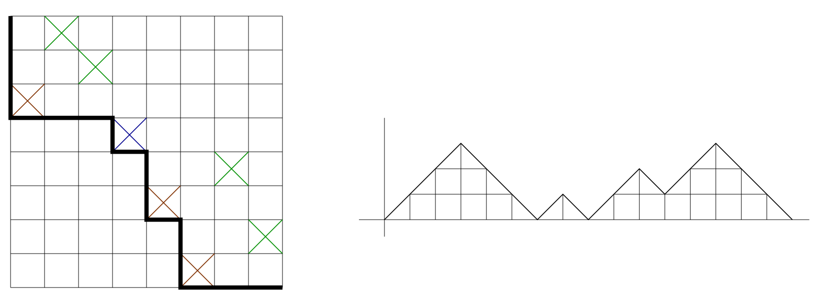

Figure 2: The bijection ψ.

It can easily be checked that ψ has the property that $\mathrm{fp}(\pi) = h(\psi(\pi))$ and $\mathrm{exc}(\pi) = \mathrm{dr}(\psi(\pi))$. Therefore, counting 321-avoiding permutations according to the number fixed points and excedances is equivalent to counting Dyck paths according to the number of hills and double rises. More precisely,

$$F_{321}(x,q,t) = \sum_{D\in\mathcal{D}} x^{h(D)} q^{\mathrm{dr}(D)} t^{|D|}.$$

We now give an equation for F_{321} using the symbolic method described in [7]. A recursive definition for the class $\mathcal{D}$ is given by the fact that every non-empty Dyck path D can be decomposed in a unique way as $D = uAdB$, where $A, B \in \mathcal{D}$. Clearly if A is empty, $h(D) = h(B) + 1$ and $\mathrm{dr}(D) = \mathrm{dr}(B)$, and otherwise $h(D) = h(B)$ and $\mathrm{dr}(D) = \mathrm{dr}(A) + \mathrm{dr}(B) + 1$. Hence, we obtain the following equation for F_{321}:

$$F_{321}(x,q,t) = 1 + t(x + q(F_{321}(1,q,t) - 1))F_{321}(x,q,t). \tag{1}$$

Substituting first $x = 1$, we obtain that

$$F_{321}(1,q,t) = \frac{1 + t(q-1) - \sqrt{1 - 2t(1+q) + t^2(1-q)^2}}{2qt}.$$

Now, solving (1) for $F_{321}(x,q,t)$ gives

$$F_{321}(x,q,t) = \frac{2}{1 + t(1+q-2x) + \sqrt{1 - 2t(1+q) + t^2(1-q)^2}}. \tag{2}$$

To conclude this section, we remark that the same method can be used to obtain the generating function counting fixed points, excedances and descents in 321-avoiding permutations. The number of descents of a 321-avoiding permutation π (i.e., indices i for which $\pi_i > \pi_{i+1}$), denoted $\mathrm{des}(\pi)$, equals the number of occurrences of uud in the Dyck word of $\psi(\pi)$. Using the same decomposition as before, we conclude that

$$\sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(321)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} p^{\mathrm{des}(\pi)} t^n = \frac{2}{1 + t(1+q-2x) + \sqrt{1 - 2t(1+q) + t^2((1+q)^2 - 4qp)}}.$$

4 Counting 132-avoiding permutations according to fixed points and excedances

Analogously to the previous section, we define

$$F_{132}(x,q,t) := \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} t^n.$$

Theorem 2.1 is equivalent to the statement $F_{321}(x,q,t) = F_{132}(x,q,t)$.

Instead of enumerating fixed points and excedances directly in 132-avoiding permutations, we use a bijection between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$ that transforms fp and exc into certain statistics on Dyck paths.

4.1 New statistics on Dyck paths

For any $D \in \mathcal{D}$, we define a *tunnel* of D to be a horizontal segment between two lattice points of D that intersects D only in these two points, and stays below D everywhere else. Tunnels are in obvious one-to-one correspondence with decompositions of the Dyck word $D = AuBdC$, where $B \in \mathcal{D}$ (no restrictions on A and C). In the decomposition, the tunnel is the segment that goes from the beginning of u to the end of d. If $D \in \mathcal{D}_n$, then D has exactly n tunnels, since such a decomposition can be given for each up-step of D.

A tunnel of $D \in \mathcal{D}_n$ is called a *centered tunnel* if the x-coordinate of its midpoint (as a segment) is n, that is, the tunnel is centered with respect to the vertical line through the middle of D. In terms of the decomposition $D = AuBdC$, this is equivalent to A and C having the same length. Denote by $\mathrm{CT}(D)$ the set of centered tunnels of D, and let $\mathrm{c}(D) = |\,\mathrm{CT}(D)|$.

A tunnel of $D \in \mathcal{D}_n$ is called a *left tunnel* if the x-coordinate of its midpoint is strictly less than n. In terms of the decomposition $D = AuBdC$, this is equivalent to the length of A being strictly smaller than the length of C. Denote by $\mathrm{l}(D)$ the number of left tunnels of D. In Figure 3, there is one centered tunnel drawn with a solid line, and four left tunnels drawn with dotted lines.

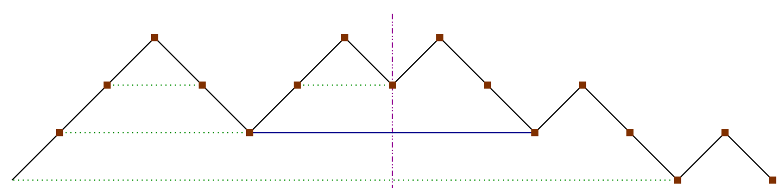

Figure 3: Centered and left tunnels.

We will use the bijection between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$ given by Krattenthaler in [10], which we denote by φ. For $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n(132)$, $\varphi(\pi)$ is obtained by reading π from left to right and adjoining for each π_j as many up-steps as necessary followed by a down-step from height h_j+1 to height h_j, where h_j is the number of elements in $\pi_{j+1}\cdots\pi_n$ which are larger than π_j. As pointed out by Reifegerste in [11], this path is easily visualized using the diagram of π obtained from the $n \times n$ array representation of π by shading, for each cross, the cell containing it and the squares that are due south and due east of it. The diagram, defined as the region that remains unshaded, is determined by the path with *left* and *down* steps that goes from the upper-right corner to the lower-left corner, leaving all the crosses to the right, and staying always as close to the diagonal connecting these two corners as possible. If we go along this path reading an up-step every time it goes left and a down-step every time it goes down, we get $\varphi(\pi)$. Figure 4 shows an example when $\pi = 67435281$.

The key property of this bijection for our purposes is that it maps fixed points to centered tunnels, and excedances to left tunnels. This can be seen using the diagram representation. First note that there is an easy way to recover a permutation $\pi \in \mathcal{S}_n(132)$

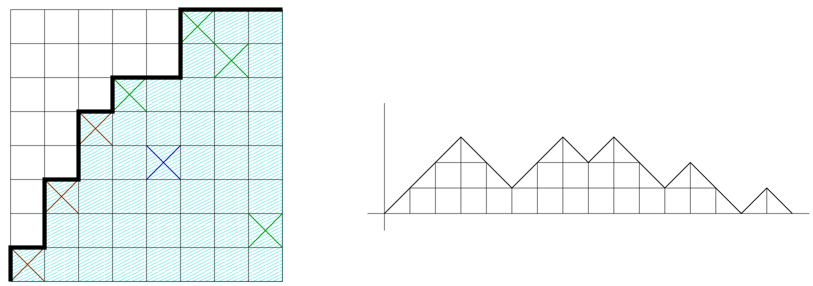

Figure 4: The bijection φ.

from its diagram: row by row, put a cross in the leftmost shaded square whose column has no other crosses. Now, instead of looking directly at $\varphi(\pi)$, consider the path from the upper-right corner to the lower-left corner of the array of π. To each cross we can associate a tunnel in a natural way. Indeed, if a cross is in position (i, j), the horizontal step in column j and the vertical step in row i produce a decomposition $\varphi(\pi) = AuBdC$, where B corresponds to the part of the path above and to the left of the cross (see Figure 5). Thus, fixed points, which are crosses on the main diagonal, give centered tunnels, and excedances, which are crosses to the right of the main diagonal, give left tunnels. It follows that $\mathrm{fp}(\pi) = \mathrm{c}(\varphi(\pi))$ and $\mathrm{exc}(\pi) = \mathrm{l}(\varphi(\pi))$. So, counting 132-avoiding permutations with respect to fixed points and excedances is equivalent to counting Dyck paths with respect to centered and left tunnels, and the generating function we want to find becomes

$$F_{132}(x, q, t) = \sum_{D \in \mathcal{D}} x^{\mathrm{c}(D)} q^{\mathrm{l}(D)} t^{|D|}.$$

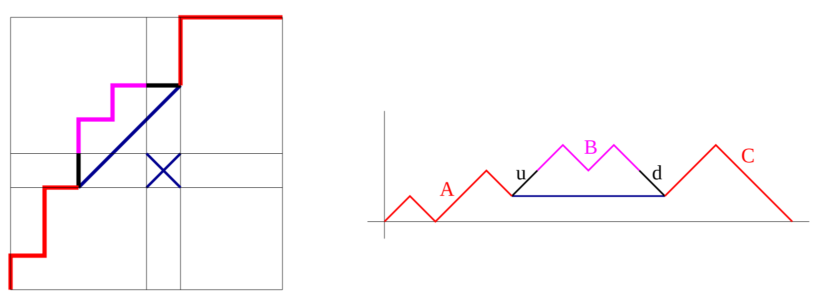

Figure 5: A cross in a 132-avoiding permutation and the corresponding tunnel in the Dyck path.

Unfortunately, the decomposition of $\mathcal{D}$ that we used to enumerate hills and double rises in Section 3 no longer works here. The reason is that if we write $D = uAdB$ with $A, B \in \mathcal{D}$, then $\mathrm{c}(A)$ and $\mathrm{c}(B)$ do not give information about $\mathrm{c}(D)$. However, it is possible use a different decomposition to count centered tunnels (but not left tunnels), obtaining an an expression for $F_{132}(x, 1, t)$.

For this purpose, we consider Dyck paths with *marked* centered tunnels. That is, we count pairs (D, S) where $D \in \mathcal{D}$ and $S \subseteq \mathrm{CT}(D)$. Each such pair is given weight

$(x-1)^{|S|}t^{|D|}$, so that for a fixed D, the sum of weights of all pairs (D,S) will be

$$\sum_{S\subseteq \mathrm{CT}(D)} (x-1)^{|S|}t^{|D|} = ((x-1)+1)^{|\,\mathrm{CT}(D)|}t^{|D|} = x^{\mathrm{c}(D)}t^{|D|},$$

which is precisely the weight that D has in $F_{132}(x,1,t)$.

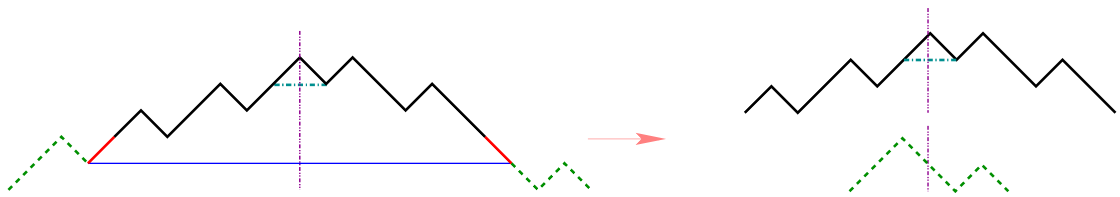

Figure 6: Decomposing Dyck paths with marked centered tunnels.

Dyck paths with no marked tunnels (i.e., pairs $(D,\emptyset)$) are enumerated by $C(t)$, the generating function for the Catalan numbers. On the other hand, for an arbitrary Dyck path D with some centered tunnel marked (i.e., a pair (D,S) with $S\neq\emptyset$), we can consider the decomposition given by the longest marked tunnel, say $D = AuBdC$. Then, AC (seen as the concatenation of Dyck words) is an arbitrary Dyck path with no marked centered tunnels, and B is an arbitrary Dyck path where some centered tunnels may be marked (Figure 6). This decomposition translates into the following equation:

$$F_{132}(x,1,t) = C(t) + (x-1)tC(t)F_{132}(x,1,t).$$

Solving it, we obtain

$$F_{132}(x,1,t) = \frac{2}{1+2t(1-x)+\sqrt{1-4t}},$$

which agrees with the expression for $F_{321}(x,1,t)$ in (2). This gives a new, simpler proof of the main result in [13], namely that $|\{\pi\in\mathcal{S}_n(321) : \mathrm{fp}(\pi)=i\}| = |\{\pi\in\mathcal{S}_n(132) : \mathrm{fp}(\pi)=i\}|$ for all $i\leq n$.

4.2 An identity involving diagonals of generating functions

To enumerate left tunnels we need a different approach. Instead of F_{132}, we will consider a more general generating function with an additional variable. First we generalize the concepts of centered and left tunnels to allow the vertical line that we use as a reference to be shifted away from the center of the Dyck path. For $D\in\mathcal{D}$ and $r\in\mathbb{Z}$, let $\mathrm{c}_r(D)$ be the number of tunnels of D whose midpoint lies on the vertical line $x=n-r$ (we call this the *reference line*). Similarly, let $\mathrm{l}_r(D)$ be the number of tunnels of D whose midpoint lies on the half-plane $x<n-r$. Notice that by definition, c_0 and l_0 are, respectively, the statistics c and l defined previously.

We add to the generating function a new variable v which marks the distance from the reference line to the actual center of the path. Define

$$G(x,q,t,v) := \sum_{n,r\geq 0} \sum_{D\in\mathcal{D}_n} x^{\mathrm{c}_r(D)} q^{\mathrm{l}_r(D)} v^r t^n,$$

and note that $G(x,q,t,0) = F_{132}(x,q,t)$.

Our next goal is to find an equation that determines $G(x,q,t,v)$. Again we use the decomposition of Dyck paths as $D = uAdB$, where $A, B \in \mathcal{D}$, with the difference that now the generating functions involve sums not only over Dyck paths but also over the possible positions of the reference line.

For the first part uAd of the decomposition we define the generating function

$$H_1(x,q,t,v) := \sum_{\substack{n_1\geq 1\\ k\geq -n_1}} \sum_{A\in\mathcal{D}_{n_1-1}} x^{\mathrm{c}_{-k}(uAd)} q^{\mathrm{c}_{-k}(uAd)} v^k t^{n_1}, \tag{3}$$

allowing the reference line, whose distance from the center is measured by k (see Figure 7), to be anywhere to the right of the beginning of the path. Similarly, for the second part B of the decomposition we define

$$H_2(x,q,t,v) := \sum_{\substack{n_2\geq 0\\ r\geq -n_2}} \sum_{B\in\mathcal{D}_{n_2}} x^{\mathrm{c}_r(B)} q^{\mathrm{l}_r(B)} v^r t^{n_2}, \tag{4}$$

allowing the reference line, whose distance from the center is measured by r, to be anywhere to the left of the end of the path.

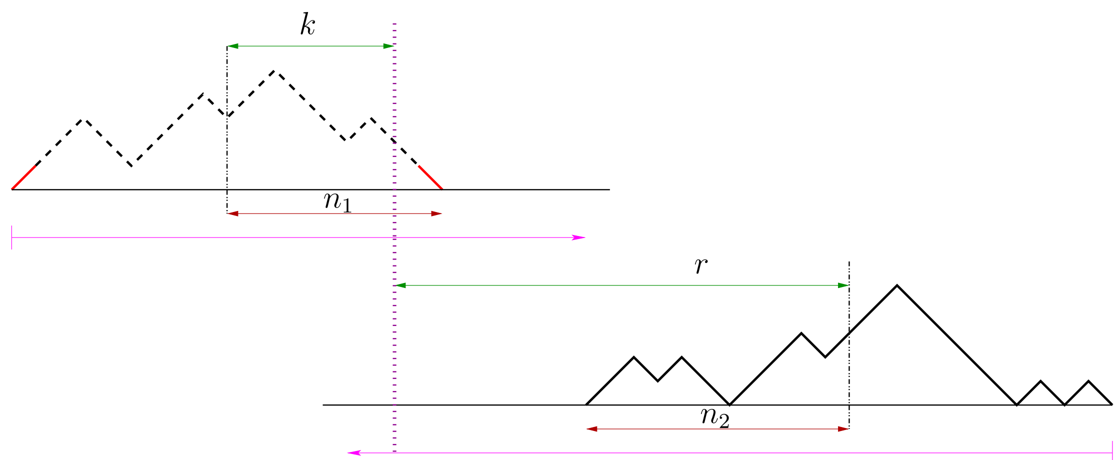

Figure 7: The generating functions H_1 and H_2.

We would like to express the generating function for paths of the form $uAdB$, where the reference line is not fixed, in terms of H_1 and H_2. The product H_1H_2 counts pairs (uAd, B), but if we draw the two paths uAd and in B next to each other making their reference lines coincide, then the end of uAd does not necessarily coincide with the beginning of B, as shown in Figure 7. However, we can correct the problem by noticing the following. The exponent k of v in H_1 indicates how far to the right the reference line is from the center of the path uAd, and similarly the exponent r of v in H_2 indicates how far to the left the reference line is from the center of the path B. Thus, in the product H_1H_2, the exponent $k+r$ of v is the distance from the center of the path uAd to the center of the path B if we draw them so that their reference lines coincide. The key observation is that the terms that correspond to an actual path $D = uAdB$, with B beginning where uAd ends, are those where the exponent of v equals the exponent $n_1 + n_2$ of t in the product H_1H_2, which is half of the sum of lengths of uAd and B (see Figure 8). As described in Section 2, the generating function consisting of only such terms is called a diagonal.

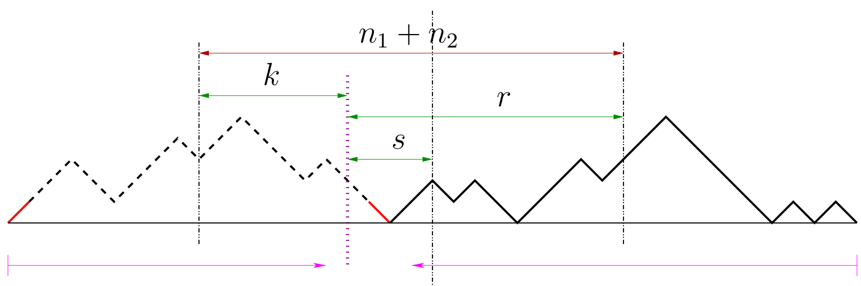

Figure 8: Terms with equal exponent in t and v.

In order to keep track of the distance s between the reference line and the center of the new path $D = uAdB$, we use an additional variable y. Considering that D starts at $(0, 0)$, the x-coordinate of its center is the exponent of t in H_1H_2, which is $n_1 + n_2$. On the other hand, the x-coordinate of the reference line is the exponent of t in H_1 plus the exponent of v in H_1, namely $n_1 + k$. Thus, the distance from the center of D to its reference line is $s = n_1 + n_2 - (n_1 + k) = n_2 - k$, that is, the exponent of t in H_2 minus the exponent of v in H_1.

We introduce this variable in the product by letting

$$P(x, q, t, v, y) := H_1(x, q, t, \frac{v}{y})H_2(x, q, ty, v).$$

If we write its series expansion in v and t as

$$P(x, q, t, v, y) = \sum_{\substack{n\geq 0\\ j\geq -n}} P_{j,n}(x, q, y)v^j t^n,$$

then the diagonal (in v and t) of P is

$$\operatorname{diag}^z_{v,t}\ P := \sum_{n\geq 0} P_{n,n}(x, q, y)z^n.$$

The above combinatorial argument implies that this diagonal equals precisely

$$H_3(x,q,z,y) := \sum_{\substack{n\geq 1\\ -n\leq r\leq n}} \sum_{D\in\mathcal{D}_n} x^{\mathrm{c}_r(D)} q^{\mathrm{l}_r(D)} y^r z^n, \tag{5}$$

where we sum over arbitrary non-empty Dyck paths D, allowing the reference line to be anywhere between the beginning and the end of the path. Let us state the obtained equation relating H_1, H_2 and H_3 as a lemma.

Lemma 4.1. *Let H_1, H_2 and H_3 be defined respectively by (3), (4), and (5). Then,*

$$\mathrm{diag}^z_{v,t}\ H_1(x,q,t,\frac{v}{y})H_2(x,q,ty,v) = H_3(x,q,z,y). \tag{6}$$

We have chosen our definitions of H_1, H_2 and H_3 to make the statement of Lemma 4.1 as simple as possible. However, for the lemma to be useful, we have to turn (6) into an equation for G by expressing these three generating functions in terms of G. This part is relatively straightforward. First we note that given $D \in \mathcal{D}_n$, if D^R is the Dyck path obtained by reflecting D over the vertical line $x = n$, we have that $\mathrm{c}_{-r}(D) = \mathrm{c}_r(D^R)$ and $\mathrm{l}_{-r}(D) = n - \mathrm{l}_r(D^R) - \mathrm{c}_r(D^R)$, since the total number of tunnels of D^R is n. Thus,

$$\sum_{n,r\geq 0} \sum_{D\in\mathcal{D}_n} x^{\mathrm{c}_{-r}(D)} q^{\mathrm{l}_{-r}(D)} v^r t^n = \sum_{n,r\geq 0} \sum_{D\in\mathcal{D}_n} \left(\frac{x}{q}\right)^{\mathrm{c}_r(D^R)} \left(\frac{1}{q}\right)^{\mathrm{l}_r(D^R)} v^r (qt)^n = G(\frac{x}{q},\frac{1}{q},qt,v). \tag{7}$$

Also, if $D \in \mathcal{D}_n$ and $r \geq n$, then $\mathrm{c}_r(D) = \mathrm{l}_r(D) = \mathrm{c}_{-r}(D) = 0$ and $\mathrm{l}_{-r}(D) = n$, so

$$\sum_{\substack{n\geq 0\\ r>n}} \sum_{D\in\mathcal{D}_n} x^{\mathrm{c}_r(D)} q^{\mathrm{l}_r(D)} v^r t^n = \sum_{\substack{n\geq 0\\ r>n}} C_n v^r t^n = \sum_{n\geq 0} C_n \frac{v^{n+1}}{1-v} t^n = \frac{v}{1-v} C(tv). \tag{8}$$

Now we can write H_1 as

$$\begin{aligned} H_1(x,q,t,v) &= \sum_{\substack{n\geq 0\\ k\geq -n-1}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{c}_{-k}(uAd)} q^{\mathrm{l}_{-k}(uAd)} v^k t^{n+1} \\ &= t\left[\sum_{\substack{n\geq 0\\ k>0}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{c}_{-k}(uAd)} q^{\mathrm{l}_{-k}(uAd)} v^k t^n + \sum_{n\geq 0} \sum_{A\in\mathcal{D}_n} x^{\mathrm{c}_0(uAd)} q^{\mathrm{l}_0(uAd)} t^n \right.\\ &\qquad \left. + \sum_{\substack{n\geq 0\\ 0<r\leq n+1}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{c}_r(uAd)} q^{\mathrm{l}_r(uAd)} v^{-r} t^n \right]. \end{aligned} \tag{9}$$

The three sums on the right hand side of (9) can be simplified as follows. Using that $\mathrm{c}_{-k}(uAd) = \mathrm{c}_{-k}(A)$ and $\mathrm{l}_{-k}(uAd) = \mathrm{l}_{-k}(A) + 1$ for $k > 0$, and equation (7), the first sum can be written as

$$q \sum_{\substack{n\geq 0\\ k>0}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{c}_{-k}(A)} q^{\mathrm{l}_{-k}(A)} v^k t^n = q \left[\sum_{\substack{n\geq 0\\ k\geq 0}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{c}_{-k}(A)} q^{\mathrm{l}_{-k}(A)} v^k t^n - \sum_{n\geq 0} \sum_{A\in\mathcal{D}_n} x^{\mathrm{c}_0(A)} q^{\mathrm{l}_0(A)} t^n \right]$$
$$= q \left[G(\frac{x}{q}, \frac{1}{q}, qt, v) - G(x,q,t,0) \right].$$

The second sum, using that $\mathrm{c}_0(uAd) = \mathrm{c}_0(A) + 1$ and $\mathrm{l}_0(uAd) = \mathrm{l}_0(A)$, becomes

$$x \sum_{n\geq 0} \sum_{A\in\mathcal{D}_n} x^{\mathrm{c}_0(A)} q^{\mathrm{l}_0(A)} t^n = xG(x,q,t,0).$$

For the third sum, we use that $\mathrm{c}_r(uAd) = \mathrm{c}_r(A)$ and $\mathrm{l}_r(uAd) = \mathrm{l}_r(A)$ for $r > 0$, together with equation (8), to write it as

$$\sum_{\substack{n\geq 0\\ r>0}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{c}_r(A)} q^{\mathrm{l}_r(A)} v^{-r} t^n - \sum_{\substack{n\geq 0\\ r>n+1}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{c}_r(A)} q^{\mathrm{l}_r(A)} v^{-r} t^n$$
$$= G(x,q,t,\frac{1}{v}) - G(x,q,t,0) - \frac{1}{v(v-1)} C(\frac{t}{v}).$$

Combining the last three equations we get

$$H_1(x,q,t,v) = t \left[q\, G(\frac{x}{q}, \frac{1}{q}, qt, v) + (x-q-1)G(x,q,t,0) + G(x,q,t,\frac{1}{v}) + \frac{1}{v(1-v)} C(\frac{t}{v}) \right]. \tag{10}$$

For H_2 and H_3, very similar arguments show that

$$H_2(x,q,t,v) = G(x,q,t,v) - G(x,q,t,0) + G(\frac{x}{q}, \frac{1}{q}, qt, \frac{1}{v}) + \frac{1}{1-v} C(\frac{qt}{v}), \tag{11}$$

$$H_3(x,q,z,y) = G(x,q,z,y) + G(\frac{x}{q}, \frac{1}{q}, qz, \frac{1}{y}) - G(x,q,z,0) + \frac{1}{1-y}[C(\frac{qz}{y}) - y\, C(zy)] - 1. \tag{12}$$

Substituting the above expressions for H_1, H_2 and H_3 in (6), we obtain an equation satisfied by G, which we call equation (6'). This identity uniquely determines G as a generating function. Indeed, because of the common factor t in expression (10) for H_1, equation (6') expresses the coefficient of z^n on the right hand side in terms of coefficients of t^i with $i < n$ in the product $H_1 H_2$. Combinatorially, this is just a consequence of the fact that the decomposition $D = uAdB$ expresses a Dyck path D in terms of strictly smaller Dyck paths.

4.3 The solution

The solution to equation (6') is given by the following formula.

Proposition 4.2. *We have*

$$G(x,q,t,v)=\frac{\dfrac{1-v+(q-1)tvC(tv)}{1-v+(q-1)tvF_{321}(1,q,t)}-(x-1)tvC(tv)}{[1-qt(F_{321}(1,q,t)-1)-xt](1-v)}. \tag{13}$$

Before proving that this expression for G satisfies equation (6'), let us show that Proposition 4.2 implies Theorem 2.1. Indeed, we have by definition

$$G(x,q,t,0)=\sum_{n\geq 0}\sum_{D\in\mathcal{D}_n}x^{\mathrm{c}_0(D)}q^{\mathrm{l}_0(D)}t^n=F_{132}(x,q,t).$$

On the other hand, Proposition 4.2 implies that

$$G(x,q,t,0)=\frac{1}{1-qt(F_{321}(1,q,t)-1)-xt}=F_{321}(x,q,t),$$

where the last equality follows from equation (1). Thus, to conclude that $F_{132}(x,q,t)=F_{321}(x,q,t)$, it only remains to prove Proposition 4.2, which we do next.

Proof. Let $\widetilde{H}_1$, $\widetilde{H}_2$ and $\widetilde{H}_3$ be the expressions obtained from (10), (11) and (12), respectively, when G is substituted with the formula given in equation (13). It suffices to check that

$$\mathrm{diag}^z_{v,t}\ \widetilde{H}_1(x,q,t,\frac{v}{y})\widetilde{H}_2(x,q,ty,v)=\widetilde{H}_3(x,q,z,y).$$

Let $\widetilde{P}(x,q,t,v,y):=\widetilde{H}_1(x,q,t,\frac{v}{y})\widetilde{H}_2(x,q,ty,v)$. A general method for obtaining diagonals of rational generating functions is described in [15, Section 6.3]. This theory, however, does not apply to our function $\widetilde{P}$, because it is not rational. In order to compute $\mathrm{diag}^z_{v,t}\ \widetilde{P}$, we will modify this technique and show that it can be extended to our particular case.

Taking $\alpha,\beta>0$ to be sufficiently small, the series expansion of $\widetilde{P}$ in v and t,

$$\widetilde{P}(x,q,t,v,y)=\sum_{\substack{n\geq 0\\ j\geq -n}}\widetilde{P}_{j,n}(x,q,y)v^jt^n=\sum_{n,i\geq 0}\widetilde{P}_{i-n,n}(x,q,y)v^i\left(\frac{t}{v}\right)^n,$$

converges for $|v|<\beta$, $|\frac{t}{v}|<\alpha$. Similarly,

$$\mathrm{diag}^z_{v,t}\ \widetilde{P}=\sum_{n\geq 0}\widetilde{P}_{n,n}(x,q,y)z^n$$

converges for $|z|$ sufficiently small. Fix such a small z with $|z|<\alpha\beta^2$. Then the series

$$\widetilde{P}(x,q,t,\frac{z}{t},y)=\sum_{\substack{n\geq 0\\ j\geq -n}}\widetilde{P}_{j,n}(x,q,y)z^jt^{n-j}$$

converges for $|\frac{z}{t}| < \beta$ and $|\frac{t^2}{z}| < \alpha$. Regarded as a function of t, it converges for t in the annulus

$$\frac{|z|}{\beta} < |t| < \sqrt{\alpha|z|}, \tag{14}$$

which is non-empty because $|z| < \alpha\beta^2$. In particular, it converges on some circle $|t| = \rho$ in the annulus. As in [8, Theorem 1], we have by Cauchy's integral theorem that

$$\operatorname{diag}^z_{v,t}\ \widetilde{P} = \frac{1}{2\pi i}\int_{|t|=\rho} \widetilde{P}(x,q,t,\frac{z}{t},y)\frac{dt}{t}. \tag{15}$$

It can be checked that the singularities of $\widetilde{P}(x,q,t,\frac{z}{t},y)/t$, as a function of t, that lie inside the circle $|t| = \rho$ are all simple poles. These poles are

$$t_1 = 0,\ t_2 = z,\ t_3 = \frac{z}{y},\ t_{4,5} = \frac{(1+q)y \pm (1-q)\sqrt{y(y-4qz)}}{2y(y+z(1-q)^2)}z,$$

$$t_{6,7} = \frac{1+q \pm (1-q)\sqrt{1-4zy}}{2(q+zy(1-q)^2)}z.$$

There are also branch points at

$$t = \pm\frac{1}{2}\sqrt{\frac{z}{y}} \text{ and } t = \pm\frac{1}{2}\sqrt{\frac{z}{qy}},$$

but they lie outside the circle in the annulus (14), for an appropriate choice of radius ρ. The remaining singularities do not depend on z and lie outside the circle.

By the Residue Theorem, the integral (15) can be computed by adding up the residues at the poles inside the circle $|t| = \rho$. All the residues are 0 except for those in t_2 and t_3. Thus,

$$\operatorname{diag}^z_{v,t}\ \widetilde{P} = \operatorname{Res}_{t=z}\ \widetilde{P}(x,q,t,\frac{z}{t},y)\frac{1}{t} + \operatorname{Res}_{t=\frac{z}{y}}\ \widetilde{P}(x,q,t,\frac{z}{t},y)\frac{1}{t}.$$

A routine computation in *Maple* shows that this sum of residues equals $\widetilde{H}_3(x,q,z,y)$ as claimed. □

5 Some other bijections involving $\mathcal{S}_n(321)$ and $\mathcal{D}_n$

Looking at permutations as arrays of crosses, as we did to define ψ, some other known bijections between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$ can easily be viewed in a systematic way, as paths with down and right steps from the upper-left corner to the lower-right corner of the permutation array. For each of these bijections, our canonical example will be $\pi = 23147586$. One such bijection, which we denote by ψ_2, was established by Billey, Jockusch and Stanley in [1, p. 361]. Consider the path that leaves the crosses corresponding to excedances to the right, and stays always as far from the main diagonal as possible (Figure 9). Then $\psi_2(\pi)$ can be obtained from it by reading an up-step every time the path goes right and a down-step every time the path goes down.

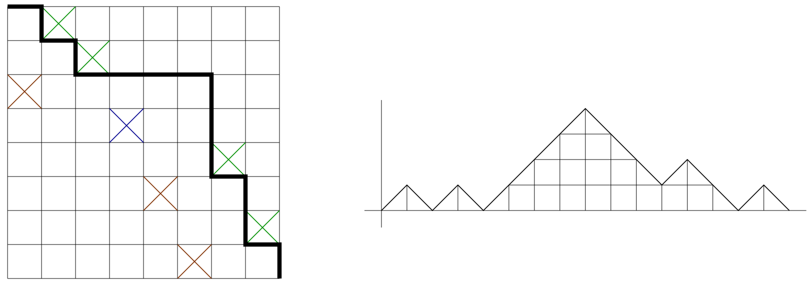

Figure 9: The bijection ψ_2.

In [10], Krattenthaler describes a bijection from $\mathcal{S}_n(123)$ to $\mathcal{D}_n$. If we omit the last step, consisting of reflecting the path over a vertical line, and compose the bijection with the reversal operation mapping a permutation $\pi_1\pi_2\cdots\pi_n$ to $\pi_n\cdots\pi_2\pi_1$, we get a bijection from $\mathcal{S}_n(321)$ to $\mathcal{D}_n$, which we denote by ψ_3. In the array representation, $\psi_3(\pi)$ corresponds, by the same trivial transformation as before, to the path that leaves all the crosses to the left and remains as close to the main diagonal as possible (see Figure 10).

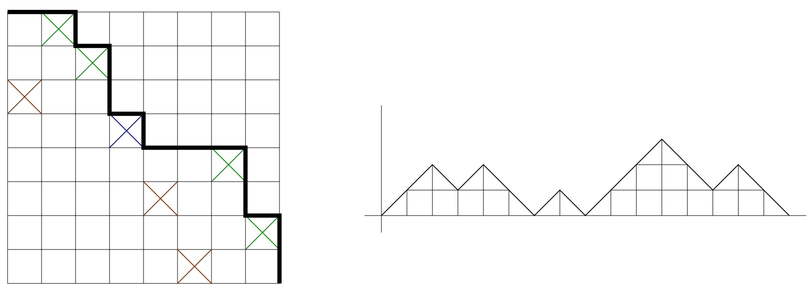

Figure 10: The bijection ψ_3.

This last bijection is related to the one from Section 3 by $\psi_3(\pi) = \psi(\pi^{-1})$. In a similar way, one can define a fourth bijection $\psi_4 : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ by $\psi_4(\pi) := \psi_2(\pi^{-1})$ (see Figure 11). In their survey of bijections between 321- and 132-avoiding permutations [2], Claesson and Kitaev mention some of the above bijections between permutations and Dyck paths.

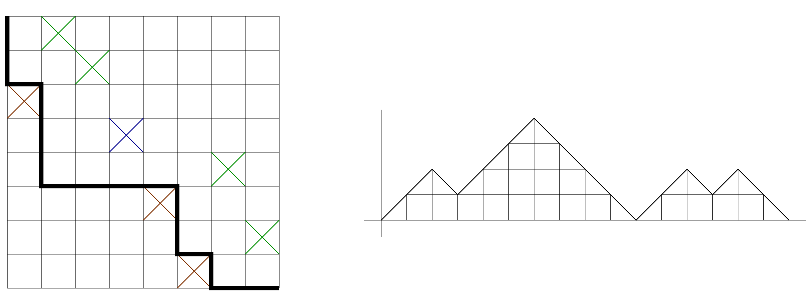

Figure 11: The bijection ψ_4.

Combining the bijections $\psi, \psi_2, \psi_3, \psi_4$ and their inverses, we get some automorphisms on Dyck paths and on 321-avoiding permutations with interesting properties. Recall that a *valley* of a Dyck path D is a down-step followed by an up-step (du in the Dyck word). Denote by $\mathrm{va}(D)$ the number of valleys of D. Denote by $p_2(D)$ the number of peaks of D of height at least 2. Clearly, both $p_2(D) + h(D)$ and $\mathrm{va}(D) + 1$ equal the total number of peaks of D. It can be checked that $\psi \circ \psi_2^{-1}$ is an involution on $\mathcal{D}_n$ with the property that $\mathrm{va}(\psi \circ \psi_2^{-1}(D)) = \mathrm{dr}(D)$ and $\mathrm{dr}(\psi \circ \psi_2^{-1}(D)) = \mathrm{va}(D)$. Indeed, this follows from the fact that excedances are sent to valleys by ψ_2 and to double rises by ψ. This bijection gives yet another proof of the symmetry of the bivariate distribution of the pair $(\mathrm{va}, \mathrm{dr})$ of statistics in Dyck paths. A different involution with this property was introduced in [3].

Another involution on $\mathcal{D}_n$ is given by $\psi \circ \psi_3^{-1}$. This one shows the symmetry of the distribution of the pair (dr, p_2), because $\mathrm{dr}(\psi \circ \psi_3^{-1}(D)) = p_2(D)$ and $p_2(\psi \circ \psi_3^{-1}(D)) = \mathrm{dr}(D)$. In addition, it preserves the number of hills, i.e., $h(\psi \circ \psi_3^{-1}(D)) = h(D)$. These properties follow from the fact that both ψ_3 and ψ send fixed points to hills, whereas excedances are sent to peaks of height at least 2 by ψ_3 and to double rises by ψ.

Finally, the involution on $\mathcal{S}_n(321)$ that maps π to $(\psi_2^{-1}(\psi(\pi)))^{-1}$ gives a combinatorial proof of the fact that the number of 321-avoiding permutations with k excedances equals the number of 321-avoiding permutations with with $k + 1$ weak excedances. Recall that i is a *weak excedance* of π if $\pi_i \geq i$. The analogous result for all permutations is well known. An implication of Theorem 2.1 is that this result is also true for 132-avoiding permutations.

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