The smallest one-realization of a given set

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Abstract

For any set S of positive integers, a mixed hypergraph $\mathcal{H}$ is a realization of S if its feasible set is S, furthermore, $\mathcal{H}$ is a one-realization of S if it is a realization of S and each entry of its chromatic spectrum is either 0 or 1. Jiang et al. showed that the minimum number of vertices of a realization of $\{s,t\}$ with $2 \leq s \leq t-2$ is $2t-s$. Král proved that there exists a one-realization of S with at most $|S|+2\max S-\min S$ vertices. In this paper, we determine the number of vertices of the smallest one-realization of a given set. As a result, we partially solve an open problem proposed by Jiang et al. in 2002 and by Král in 2004.

Key words: hypergraph coloring; mixed hypergraph; feasible set; chromatic spectrum; one-realization

1 Introduction

A *mixed hypergraph* on a finite set X is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of X, called the $\mathcal{C}$*-edges* and $\mathcal{D}$*-edges*, respectively. A *bi-hypergraph* is a mixed hypergraph with $\mathcal{C} = \mathcal{D}$. A sub-hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ of a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a *spanning sub-hypergraph* if $X' = X$, and $\mathcal{H}'$ is called a *derived sub-hypergraph* of $\mathcal{H}$ on X', denoted by $\mathcal{H}[X']$, when $\mathcal{C}' = \{C \in \mathcal{C} | C \subseteq X'\}$ and $\mathcal{D}' = \{D \in \mathcal{D} | D \subseteq X'\}$. Two mixed hypergraphs $\mathcal{H}_1 = (X_1, \mathcal{C}_1, \mathcal{D}_1)$ and $\mathcal{H}_2 = (X_2, \mathcal{C}_2, \mathcal{D}_2)$

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are *isomorphic* if there exists a bijection ϕ from X_1 to X_2 that preserves the incidence between vertices and edges and maps each $\mathcal{C}$-edge of $\mathcal{C}_1$ onto a $\mathcal{C}$-edge of $\mathcal{C}_2$ and maps each $\mathcal{D}$-edge of $\mathcal{D}_1$ onto a $\mathcal{D}$-edge of $\mathcal{D}_2$, and vice versa. The bijection ϕ is called an *isomorphism* from $\mathcal{H}_1$ to $\mathcal{H}_2$.

A *proper k-coloring* of $\mathcal{H}$ is a mapping from X into a set of k *colors* so that each $\mathcal{C}$-edge has two vertices with a *Common* color and each $\mathcal{D}$-edge has two vertices with *Distinct* colors. A *strict k-coloring* is a proper k-coloring using all of the k colors, and a mixed hypergraph is *k-colorable* if it has a strict k-coloring. The maximum (minimum) number of colors in a strict coloring of $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is the *upper chromatic number* $\overline{\chi}(\mathcal{H})$ (resp. *lower chromatic number* $\chi(\mathcal{H})$) of $\mathcal{H}$. The study of the colorings of mixed hypergraphs has made a lot of progress since its inception [7]. For more information, we would like refer readers to [3, 6, 8, 9].

A coloring of $\mathcal{H}$ may be viewed as a *partition* of the vertex set, where each *color class* consists of vertices assigned to the same color. Then no class contains a $\mathcal{D}$-edge, and each $\mathcal{C}$-edge meets some class in more than one vertex. Such partitions are called *feasible partitions*. So a strict n-coloring $c = \{C_1, C_2, \ldots, C_n\}$ of a mixed hypergraph means that $C_1, C_2, \ldots, C_n$ are the n color classes under c.

The set of all the values k such that $\mathcal{H}$ has a strict k-coloring is called the *feasible set* of $\mathcal{H}$, denoted by $\mathcal{F}(\mathcal{H})$. For each k, let r_k denote the number of *partitions* of the vertex set corresponding to the strict colorings of $\mathcal{H}$ with k colors. The vector $R(\mathcal{H}) = (r_1, r_2, \ldots, r_{\overline{\chi}})$ is called the *chromatic spectrum* of $\mathcal{H}$. A mixed hypergraph has a *gap at k* if its feasible set contains elements larger and smaller than k but omits k. A *gap of size g* means g consecutive gaps. If some gaps occur, the feasible set and the chromatic spectrum of $\mathcal{H}$ are said to be *broken*, and if there are no gaps then they are called *continuous* or *gap-free*. If S is a set of positive integers, we say that a mixed hypergraph $\mathcal{H}$ is a *realization* of S if $\mathcal{F}(\mathcal{H}) = S$. A mixed hypergraph $\mathcal{H}$ is a *one-realization* of S if it is a realization of S and all the entries of the chromatic spectrum of $\mathcal{H}$ are either 0 or 1. This concept was firstly introduced by Král [4].

Bujtás et al. [1] gave a necessary and sufficient condition for a set S to be the feasible set of an r-uniform mixed hypergraph. Kündgen et al. [5] found a one-realization of $\{2, 4\}$ on 6 vertices for planar hypergraphs. Jiang et al. [2] proved that a set S of positive integers is a feasible set of a mixed hypergraph if and only if $1 \notin S$ or S is an interval. They also discussed the bound on the number of vertices of a mixed hypergraph with a gap, in particular, the minimum number of vertices of a realization of $\{s, t\}$ with $2 \leq s \leq t-2$ is $2t - s$. Moreover, they also mentioned that the question of finding the minimum number of vertices in a mixed hypergraph with feasible set S of size at least 3 remains open. In [10], we obtained an upper bound on the minimum number of vertices of 3-uniform bi-hypergraphs with a given feasible set. Král [4] proved that there exists a one-realization of S with at most $|S| + 2\max S - \min S$ vertices, and proposed the following problem: what is the number of vertices of the smallest mixed hypergraph whose spectrum is equal to a given spectrum $(r_1, r_2, \ldots, r_m)$?

In this paper, we determine the number of vertices of the smallest one-realization of a given set and obtain the following result:

Theorem 1.1 *For any integers $2 \le n_s < \cdots < n_2 < n_1$, let $\delta(S)$ denote the number of vertices of the smallest one-realization of $S = \{n_1, n_2, \ldots, n_s\}$. Then*

$$\delta(S) = \begin{cases} 2n_1 - n_s, & \text{if } n_1 > n_2 + 1, \\ 2n_1 - n_s - 1, & \text{if } n_1 = n_2 + 1. \end{cases}$$

As a result, we partially solve the above open problem proposed by Jiang et al. and by Král.

2 Proof of Theorem 1.1

In this section we always assume that $S = \{n_1, n_2, \ldots, n_s\}$ is a set of integers with $2 \le n_s < \cdots < n_2 < n_1$. We first show that the number $\delta(S)$ given in Theorem 1.1 is a lower bound on the number of vertices of the smallest one-realization of S, then construct two families of mixed hypergraphs which meet the bounds.

Jiang et al. [2] discussed the bound on the number of vertices of a mixed hypergraph with a gap.

Proposition 2.1 ([2, Theorem 3]) *If $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is an s-colorable mixed hypergraph with a gap at $t-1$, then $|X| \ge 2t - s$. For $2 \le s \le t-2$, this bound is sharp.*

Lemma 2.2

$$\delta(S) \ge \begin{cases} 2n_1 - n_s, & \text{if } n_1 > n_2 + 1, \\ 2n_1 - n_s - 1, & \text{if } n_1 = n_2 + 1. \end{cases}$$

Proof. Assume that $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a one-realization of S.

Case 1. $n_1 > n_2 + 1$. Then $\mathcal{H}$ has a gap at $n_1 - 1$. By Proposition 2.1, we have $\delta(S) \ge 2n_1 - n_s$.

Case 2. $n_1 = n_2 + 1$. Suppose $|X| \le 2n_1 - (n_s + 2)$. For any strict n_1-coloring $c_1 = \{C_1, C_2, \ldots, C_{n_1}\}$ of $\mathcal{H}$, there exist at least $n_s + 2$ color classes of size one. Suppose $C_1 = \{\alpha_1\}, C_2 = \{\alpha_2\}, \ldots, C_{n_s+2} = \{\alpha_{n_s+2}\}$. For any strict n_s-coloring c_s of $\mathcal{H}$, there are the following two possible cases.

Case 2.1. There exist three vertices in $\{\alpha_1, \alpha_2, \ldots, \alpha_{n_s+2}\}$ which fall into a common color class under c_s. Suppose $\alpha_1, \alpha_2, \alpha_3$ are in a common color class under c_s. Then $\{\alpha_1, \alpha_2\}, \{\alpha_1, \alpha_3\}, \{\alpha_2, \alpha_3\} \notin \mathcal{D}$, which implies that $\{C_1 \cup C_2, C_3, \ldots, C_{n_1}\}, \{C_1 \cup C_3, C_2, C_4, \ldots, C_{n_1}\}, \{C_1, C_2 \cup C_3, C_4, \ldots, C_{n_1}\}$ are strict n_2-colorings of $\mathcal{H}$. Therefore, $\mathcal{H}$ is not a one-realization of S, a contradiction.

Case 2.2. There exist two pairs of vertices in $\{\alpha_1, \alpha_2, \ldots, \alpha_{n_s+2}\}$ each of which falls into a common color class under c_s. Suppose α_1, α_2 are in a common color class and α_3, α_4 are in common color class under c_s. Then $\{\alpha_1, \alpha_2\}, \{\alpha_3, \alpha_4\} \notin \mathcal{D}$. It follows that $\{C_1 \cup C_2, C_3, \ldots, C_{n_1}\}$ and $\{C_1, C_2, C_3 \cup C_4, C_5, \ldots, C_{n_1}\}$ are strict n_2-colorings of $\mathcal{H}$. Then $\mathcal{H}$ is not a one-realization of S, a contradiction. Hence, $\delta(S) \ge 2n_1 - n_s - 1$. □

In the rest of this section, we shall construct two families of mixed hypergraphs which meet the bound in Lemma 2.2.

For any positive integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

Construction I. For any positive integer $s \geq 2$, let

$$X^0_{n_1,\ldots,n_s} = \{(\underbrace{i, i, \ldots, i}_{s})|\ i = 1, 2, \ldots, n_s - 1\},$$

$$X^1_{n_1,\ldots,n_s} = \bigcup_{t=2}^{s} \bigcup_{j=n_t}^{n_{t-1}-1} \{(\underbrace{j, \ldots, j}_{t-1}, n_t, n_{t+1}, \ldots, n_s), (\underbrace{j, \ldots, j}_{t-1}, \underbrace{1, \ldots, 1}_{s-t+1})\}.$$

Suppose

$$\begin{aligned}
X^*_{n_1,\ldots,n_s} &= X^0_{n_1,\ldots,n_s} \cup X^1_{n_1,\ldots,n_s} \cup \{(n_1, n_2, \ldots, n_s)\},\\
\mathcal{D}^*_{n_1,\ldots,n_s} &= \{\{(x_1, x_2, \ldots, x_s), (y_1, y_2, \ldots, y_s)\}|x_i \neq y_i, i \in [s]\},\\
\mathcal{C}^*_{n_1,\ldots,n_s} &= \{\{(x_1, \ldots, x_s), (y_1, \ldots, y_s), (z_1, \ldots, z_s)\}|\ |\{x_j, y_j, z_j\}| = 2, j \in [s]\}.
\end{aligned}$$

Then $\mathcal{H}^*_{n_1,\ldots,n_s} = (X^*_{n_1,\ldots,n_s}, \mathcal{C}^*_{n_1,\ldots,n_s}, \mathcal{D}^*_{n_1,\ldots,n_s})$ is a mixed hypergraph with $2n_1 - n_s$ vertices.

Let

$$X_{n_1,\ldots,n_s} = \{(x_1, x_2, \ldots, x_s)|x_i \in [n_i], i \in [s]\},$$
$$X^s_{ij} = \{(x_1, x_2, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_s)|x_k \in [n_k], k \in [s] \setminus \{i\}\}, j \in [n_i].$$

Then, for any $i \in [s]$,

$$c^{s*}_i = \{X^*_{i1}, X^*_{i2}, \ldots, X^*_{in_i}\}$$

is a strict n_i-coloring of $\mathcal{H}^*_{n_1,\ldots,n_s}$, where $X^*_{ij} = X^*_{n_1,\ldots,n_s} \cap X^s_{ij}, j \in [n_i]$.

For the case of $s = 3, n_1 = 7, n_2 = 4, n_3 = 2$, we have

$$\begin{aligned}
X^*_{7,4,2} &= \{(1, 1, 1)\} \cup \{(2, 2, 2), (2, 2, 1), (3, 3, 2), (3, 3, 1)\}\\
&\cup\{(4, 4, 2), (4, 1, 1), (5, 4, 2), (5, 1, 1), (6, 4, 2), (6, 1, 1)\} \cup \{(7, 4, 2)\}.
\end{aligned}$$

Lemma 2.3 *$\mathcal{H}^*_{n_1,n_2}$ is a one-realization of $\{n_1, n_2\}$.*

Proof. Under any strict coloring $c = \{C_1, C_2, \ldots, C_m\}$ of $\mathcal{H}^*_{n_1,n_2}$, the vertices $(1, 1)$, $(2, 2), \ldots, (n_2, n_2)$ fall into distinct color classes. For each $i \in [n_2]$, suppose $(i, i) \in C_i$. Then, for any $i \in [n_2 - 1]$ and $j \in [n_1 - n_2 - 1]$, the $\mathcal{D}$-edge $\{(n_2 + j, n_2), (i, i)\}$ implies that $(n_2 + j, n_2) \notin C_i$ and the $\mathcal{D}$-edge $\{(n_2 + j, 1), (n_2, n_2)\}$ implies that $(n_2 + j, 1) \notin C_{n_2}$. Since $\{(1, 1), (n_2, 1), (n_2, n_2)\}$ is a $\mathcal{C}$-edge, $(n_2, 1) \in C_1 \cup C_{n_2}$.

Case 1. $(n_2, 1) \in C_1$. The fact that $\{(n_2, 1), (n_2, n_2), (n_2 + 1, n_2)\}$ is a $\mathcal{C}$-edge follows that $(n_2 + 1, n_2) \in C_{n_2}$. From the $\mathcal{C}$-edge $\{(n_2, 1), (n_2 + 1, 1), (n_2 + 1, n_2)\}$, we observe $(n_2 + 1, 1) \in C_1$. Similarly, $(n_2 + j, 1) \in C_1, (n_2 + j, n_2) \in C_{n_2}$ for any $j \in [n_1 - n_2 - 1]$ and $(n_1, n_2) \in C_{n_2}$. Therefore, $c = c^{2*}_2$.

Case 2. $(n_2, 1) \in C_{n_2}$. The $\mathcal{D}$-edge $\{(n_2, 1), (n_2+1, n_2)\}$ implies that $(n_2+1, n_2) \notin C_{n_2}$. Suppose $(n_2 + 1, n_2) \in C_{n_2+1}$. From the $\mathcal{C}$-edge $\{(n_2, 1), (n_2 + 1, 1), (n_2 + 1, n_2)\}$, we have

$(n_2+1,1) \in C_{n_2+1}$. Similarly, $(n_2+j, n_2), (n_2+j, 1) \in C_{n_2+j}$ for any $j \in [n_1-n_2-1]$ and $(n_1, n_2) \in C_{n_1}$. Therefore, $c = c_1^{2*}$.

Hence, the desired result follows. □

Theorem 2.4 *$\mathcal{H}^*_{n_1,\dots,n_s}$ is a one-realization of S.*

Proof. By Lemma 2.3, the conclusion is true for $s=2$.

Let $X' = \{(x_2, x_2, x_3, \dots, x_s) | x_j \in [n_j], j \in [s] \setminus \{1\}\}$. Then $\mathcal{H}' = \mathcal{H}^*_{n_1,\dots,n_s}[X']$ is isomorphic to $\mathcal{H}^*_{n_2,n_3,n_4,\dots,n_s}$. By induction, all the strict colorings of $\mathcal{H}'$ are as follows:

$$c'_i = \{X'_{i1}, X'_{i2}, \dots, X'_{in_i}\}, \quad i \in [s] \setminus \{1\},$$

where $X'_{ij} = X' \cap X^*_{ij}, j \in [n_i]$. For any strict coloring $c = \{C_1, \dots, C_m\}$ of $\mathcal{H}^*_{n_1,\dots,n_s}$, the vertices $(1,1,\dots,1)$, $(2,2,\dots,2)$, $\dots$, $(n_s, n_s, \dots, n_s)$ fall into distinct color classes. Without loss of generality, suppose $(i,i,\dots,i) \in C_i$ for any $i \in [n_s]$. Then there are the following two possible cases.

Case 1. $c|_{X'} = c'_2$. The $\mathcal{C}$-edge $\{(1,1,\dots,1), (n_2,1,\dots,1), (n_2,n_2,n_3,\dots,n_s)\}$ implies that $(n_2,1,\dots,1) \in C_1 \cup C_{n_2}$.

Case 1.1. $(n_2,1,\dots,1) \in C_1$. From the $\mathcal{D}$-edge $\{(1,\dots,1), (n_2+1, n_2, n_3, \dots, n_s)\}$ and the $\mathcal{C}$-edge $\{(n_2,n_2,n_3,\dots,n_s), (n_2+1,n_2,n_3,\dots,n_s), (n_2,1,\dots,1)\}$, we observe $(n_2+1, n_2, n_3, \dots, n_s) \in C_{n_2}$. By the $\mathcal{C}$-edge $\{(n_2,n_2,n_3,\dots,n_s), (n_2+1,1,\dots,1), (n_2,1,\dots,1)\}$ and the $\mathcal{D}$-edge $\{(n_2,n_2,n_3,\dots,n_s), (n_2+1,1,\dots,1)\}$, we observe $(n_2+1,1,\dots,1) \in C_1$. Similarly, $(n_2+j,1,\dots,1) \in C_1$, $(n_2+j, n_2, n_3, \dots, n_s) \in C_{n_2}$ for any $j \in [n_1-n_2-1]$ and $(n_1, n_2, \dots, n_s) \in C_{n_2}$. Therefore, $c = c_2^{s*}$.

Case 1.2. $(n_2,1,\dots,1) \in C_{n_2}$. Note that $(n_2+j,1,\dots,1) \notin C_k$ for any $j \in [n_1-n_2-1]$ and $k \in [n_2] \setminus \{1\}$. If $(n_2+1,1,\dots,1) \in C_1$, from the $\mathcal{C}$-edge $\{(n_2+1,1,\dots,1), (n_2,n_2,n_3,\dots,n_s), (n_2+1,n_2,\dots,n_s)\}$, we observe $(n_2+1,n_2,\dots,n_s) \in C_1 \cup C_{n_2}$, contrary to the fact that both $\{(1,1,\dots,1), (n_2+1,n_2,\dots,n_s)\}$ and $\{(n_2,1,\dots,1), (n_2+1,n_2,\dots,n_s)\}$ are $\mathcal{D}$-edges. Then, $(n_2+1,1,\dots,1) \notin C_1$. Suppose $(n_2+1,1,\dots,1) \in C_{n_2+1}$. The $\mathcal{C}$-edge $\{(n_2+1,1,\dots,1), (n_2+1,n_2,n_3,\dots,n_s), (n_2,1,\dots,1)\}$ implies $(n_2+1, n_2, \dots, n_s) \in C_{n_2+1}$. Similarly, $(n_2+j,1,\dots,1), (n_2+j,n_2,\dots,n_s) \in C_{n_2+j}$ for any $j \in [n_1-n_2-1]$ and $(n_1,n_2,\dots,n_s) \in C_{n_1}$. Therefore, $c = c_1^{s*}$.

Case 2. There exists a $k \in [s] \setminus \{1,2\}$ such that $c|_{X'} = c'_k$. In this case, we have $(n_2,n_2,n_3,\dots,n_k,\dots,n_s) \in C_{n_k}$. For each $j \in [n_1-n_2-1]$, the $\mathcal{D}$-edge $\{(n_2+j,1,\dots,1), (n_2,n_2,n_3,\dots,n_k,\dots,n_s)\}$ implies that $(n_2+j,1,\dots,1) \notin C_{n_k}$. From the $\mathcal{C}$-edge $\{(1,1,\dots,1), (n_2,n_2,n_3,\dots,n_k,\dots,n_s), (n_2,1,\dots,1)\}$ and the $\mathcal{D}$-edge $\{(n_k,\dots,n_k, n_{k+1},\dots,n_s), (n_2,1,\dots,1)\}$, we get $(n_2,1,\dots,1) \in C_1$. For $j \in [n_1-n_2-1]$, the $\mathcal{C}$-edge $\{(n_2+j,1,\dots,1), (n_2,n_2,n_3,\dots,n_s), (n_2,1,\dots,1)\}$ implies that $(n_2+j,1,\dots,1) \in C_1$.

For any $j \in [n_1-n_2]$, from the $\mathcal{D}$-edge $\{(1,1,\dots,1), (n_2+j,n_2,\dots,n_s)\}$, we have $(n_2+j,n_2,\dots,n_s) \notin C_1$. Moreover, the $\mathcal{C}$-edge $\{(n_2,n_2,n_3,\dots,n_s), (n_2+j,1,\dots,1), (n_2+j,n_2,n_3,\dots,n_s)\}$ implies that $(n_2+j,n_2,n_3,\dots,n_s) \in C_{n_k}$ for any $j \in [n_1-n_2-1]$. The fact that $\{(n_1,\dots,n_s), (n_2,n_2,n_3,\dots,n_s), (n_2,1,\dots,1)\}$ is a $\mathcal{C}$-edge follows that $(n_1,n_2,n_3,\dots,n_s) \in C_{n_k}$. Hence, $c = c_k^{s*}$.

By the above discussion, the desired result follows. □

Next, we shall construct another family of mixed hypergraphs.

Construction II. Let $X'' = X^*_{n_1,\dots,n_s} \setminus \{(n_2, 1, \dots, 1)\}$ and $\mathcal{H}'' = \mathcal{H}^*_{n_1,\dots,n_s}[X'']$. Then, for any $i \in [s]$,

$$c''_i = \{X''_{i1}, X''_{i2}, \dots, X''_{in_i}\}$$

is a strict n_i-coloring of $\mathcal{H}''$, where $X''_{ij} = X'' \cap X^s_{ij}, j \in [n_i]$.

Theorem 2.5 *If $n_1 = n_2 + 1$, the $\mathcal{H}''$ is a one-realization of S.*

Proof. Referring to the proof of Theorem 2.4, all the strict colorings of $\mathcal{H}^*_{n_2,n_2,n_3,\dots,n_s}$ are

$$c'_i = \{X'_{i1}, X'_{i2}, \dots, X'_{in_i}\}, \quad i \in [s] \setminus \{1\},$$

where $X' = \{(x_2, x_2, x_3, \dots, x_s) | x_j \in [n_j], j \in [s] \setminus \{1\}\}$ and $X'_{ij} = X' \cap X^*_{ij}, j \in [n_i]$.

For any strict coloring $c = \{C_1, C_2, \dots, C_m\}$ of $\mathcal{H}''$, there are the following two possible cases.

Case 1. $c|_{X'} = c'_2$. That is to say, $(i, i, x_3, \dots, x_s) \in C_i$ under the coloring c for any $(i, i, x_3, \dots, x_s) \in X''$. By the proof of Theorem 2.4, $(n_1, n_2, n_3, \dots, n_s) \notin C_j$ for any $j \in [n_2 - 1]$. Then, there are the following two possible subcases.

Case 1.1. $(n_1, n_2, n_3, \dots, n_s) \in C_{n_2}$. It is immediate that $c = c''_2$.

Case 1.2 $(n_1, n_2, n_3, \dots, n_s) \notin C_{n_2}$. Then $(n_1, n_2, n_3, \dots, n_s) \in C_{n_1}$. It is immediate that $c = c''_1$.

Case 2. There exists a $k \in [s] \setminus \{1, 2\}$ such that $c|_{X'} = c'_k$. It is immediate that $(n_k, \dots, n_k, n_{k+1}, \dots, n_s) \in C_{n_k}$ and $(\underbrace{n_k, \dots, n_k}_{k-1}, 1, \dots, 1) \in C_1$. From the $\mathcal{C}$-edge $\{(n_1, n_2, \dots, n_s), (n_k, \dots, n_k, n_{k+1}, \dots, n_s), (n_k, \dots, n_k, 1, \dots, 1)\}$ and the $\mathcal{D}$-edge $\{(n_1, n_2, \dots, n_s), (1, 1, \dots, 1)\}$, we observe $(n_1, n_2, \dots, n_s) \in C_{n_k}$. Therefore, $c = c''_k$.

Hence, the desired result follows. □

Combining Lemma 2.2, Theorems 2.4 and 2.5, the proof of Theorem 1.1 is completed.

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