Preserving log-convexity for generalized Pascal triangles

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Abstract

We establish the preserving log-convexity property for the generalized Pascal triangles. It is an extension of a result of H. Davenport and G. Pólya *"On the product of two power series"*, who proved that the binomial convolution of two log-convex sequences is log-convex.

1 Introduction

A sequence of nonnegative numbers $\{x_k\}_k$ is *log-convex* (LX for short) if $x_{i-1}x_{i+1} \geqslant x_i^2$ for all $i > 0$, which is equivalent to (see for instance [4])

$$x_{i-1}x_{j+1} \geqslant x_i x_j \quad (j \geqslant i \geqslant 1) . \tag{1}$$

It is well known that the convolution of sequences plays an important role in mathematics, especially in combinatorics. For the situation of log-convex sequences there is a relevant and interesting result due to Davenport and Pólya, on the product of two power series, it concerns the binomial convolution:

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Theorem 1 (Davenport and Pólya)**.** *If the sequences of nonnegative numbers* $\{u_n\}_n$ *and* $\{v_n\}_n$ *are log-convex sequences, then so is the binomial convolution*

$$w_n = \sum_{k=0}^{n} \binom{n}{k} u_k v_{n-k}, \quad (n \geqslant 0)\,.$$

Our aim is to extend the result of H. Davenport and G. Pólya to "*bi*^{s}*nomial convolution*".

2 The s-Pascal triangle

The s-Pascal triangle is the triangle given by the ordinary multinomials (see for instance [2, 3]): let $s \geqslant 1$ and $n \geqslant 0$ be two integers, and $k = 0, 1, ..., sn$, the ordinary multinomial number $\binom{n}{k}_s$ is defined as the k^{th} coefficient in the development

$$(1 + x + x^2 + \cdots + x^s)^n = \sum_{k\in\mathbb{Z}} \binom{n}{k}_s x^k, \tag{2}$$

with $\binom{n}{k}_s = 0$ for $k > sn$ or $k < 0$.

Using the classical binomial coefficient, one has

$$\binom{n}{k}_s = \sum_{j_1+j_2+\cdots+j_s=k} \binom{n}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-1}}{j_s}. \tag{3}$$

Some readily well known established properties are

- the symmetry relation

$$\binom{n}{k}_s = \binom{n}{sn-k}_s, \tag{4}$$

- the longitudinal recurrence relation

$$\binom{n}{k}_s = \sum_{j=0}^{s} \binom{n-1}{k-j}_s, \tag{5}$$

- the diagonal recurrence relation

$$\binom{n}{k}_s = \sum_{j=0}^{n} \binom{n}{j}\binom{j}{k-j}_{s-1}. \tag{6}$$

These coefficients, as for usual binomial coefficients, are built as for the Pascal triangle, known as "s-Pascal triangle". One can find the first values of the s-Pascal triangle in SLOANE [8] as A027907 for $s = 2$, as A008287 for $s = 3$ and as A035343 for $s = 4$.

$n \backslash k$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	9	*10*	*11*	*12*	*13*
0	1													
1	1	1	1	1	1									
2	1	2	3	4	5	4	3	2	1					
3	1	3	6	10	15	18	19	18	15	10	6	3	1	
4	1	4	10	20	35	52	68	80	85	80	68	52	35	...
5	1	5	15	35	70	121	185	255	320	365	379	365	320	...

Triangle of quintinomial coefficients: $s = 4$

3 Preserving log-convexity for the s-Pascal triangle

Given two sequences $\{x_n\}_n$ and $\{y_n\}_n$, let us consider the following two linear transformations of sequences

$$z_n := \sum_{k=0}^{ns} \binom{n}{k}_s x_k, \quad (n \geqslant 0), \tag{7}$$

and

$$t_n := \sum_{k=0}^{ns} \binom{n}{k}_s x_k y_{ns-k}, \quad (n \geqslant 0), \tag{8}$$

respectively.

Definition 1. *Preserving the log-convexity property.*

1. *We say that the linear transformation (7) has the PLX property if it preserves the log-convexity of sequences, i.e. the log-convexity of $\{x_n\}$ implies that of $\{z_n\}$.*

2. *We say that the linear transformation (8) has the double PLX property if it preserves the log-convexity of sequences, i.e. the log-convexity of $\{x_n\}$ and $\{y_n\}$ implies that of $\{t_n\}$.*

Now, we establish the log-convexity of the s-Pascal triangle. We start by the following proposition (see for instance [7]).

Proposition 1. *If both $\{x_n\}_n$ and $\{y_n\}_n$ are log-convex, then so is the sequence $\{x_n+y_n\}_n$.*

We can extend this result as follows.

Proposition 2. *If l sequences $\{x_n^1\}_n, \{x_n^2\}_n, ..., \{x_n^l\}_n$ are log-convex, then so is the sequence $\{x_n^1 + x_n^2 + \cdots + x_n^l\}_n$.*

Proof. It suffices to proceed by induction over n. $\square$

Now, we give our main result.

Theorem 2. *If the sequences of nonnegative numbers* $\{x_n\}_n$ *and* $\{y_n\}_n$ *are log-convex, then so is the "bi^{s}nomial convolution"*

$$t_n = \sum_{k=0}^{ns} \binom{n}{k}_s x_k y_{ns-k}, \quad (n \geqslant 0)\,.$$

Proof. Taking $n = 2$ and $s = 2$, we have

$$\begin{aligned}
t_0 t_2 &= x_0 y_0 \sum_{k=0}^{4} \binom{2}{k}_2 x_k y_{4-k} \\
&= x_0 y_0 (x_0 y_4 + 2x_1 y_3 + 3x_2 y_2 + 2x_3 y_1 + x_4 y_0) \\
&= x_0^2 y_0 y_4 + 2x_0 x_1 y_0 y_3 + 3x_0 x_2 y_0 y_2 + 2x_0 x_3 y_0 y_1 + x_0 x_4 y_0^2 \\
&= x_0^2 y_0 y_4 + 2x_0 x_1 y_0 y_3 + 2x_0 x_2 y_0 y_2 + 2x_0 x_3 y_0 y_1 + x_0 x_4 y_0^2 + x_0 x_2 y_0 y_2 \\
&\geqslant x_0^2 y_2^2 + x_1^2 y_1^2 + x_2^2 y_0^2 + 2x_0 x_1 y_1 y_2 + 2x_0 x_2 y_0 y_2 + 2x_1 x_2 y_0 y_1 \\
&\qquad \text{(by the log-convexity of } \{x_k\}_k \text{ and } \{y_k\}_k) \\
&= (x_0 y_2 + x_1 y_1 + x_2 y_0)^2 = t_1^2.
\end{aligned}$$

Suppose that this hypothesis is true for $s = q$. We show that it remains true for $s = q+1$. For $s = q$, we have

$$t_0 t_2 = x_0 y_0 \sum_{k=0}^{2q} \binom{2}{k}_q x_k y_{2q-k} \geqslant \left\{ \sum_{k=0}^{q} x_k y_{q-k} \right\}^2 = t_1^2. \tag{9}$$

So, for $s = q+1$, it follows that

$$\begin{aligned}
t_0 t_2 &= x_0 y_0 \sum_{k=0}^{2(q+1)} \binom{2}{k}_{q+1} x_k y_{2(q+1)-k} \\
&= \sum_{k=0}^{2(q+1)} \sum_{m=0}^{2} \binom{2}{m} \binom{m}{k-m}_q x_0 x_k y_0 y_{2(q+1)-k} \quad \text{by relation (6)} \\
&= \sum_{k=0}^{2(q+1)} \left[\binom{0}{k}_q + 2\binom{1}{k-1}_q + \binom{2}{k-2}_q \right] x_0 x_k y_0 y_{2(q+1)-k} \\
&= x_0^2 y_0 y_{2(q+1)} + 2\sum_{k=1}^{q+1} x_0 x_k y_0 y_{2(q+1)-k} + \sum_{k=2}^{2(q+1)} \binom{2}{k-2}_q x_0 x_k y_0 y_{2(q+1)-k}
\end{aligned}$$

$$
\begin{aligned}
&= x_0^2 y_0 y_{2(q+1)} + 2\sum_{k=1}^{q+1} x_0 x_k y_0 y_{2(q+1)-k} + \sum_{k=0}^{2q} \binom{2}{k}_q x_0 x_{k+2} y_0 y_{2q-k} \\
&\geqslant x_0^2 y_0 y_{2(q+1)} + 2\sum_{k=1}^{q+1} x_0 x_k y_0 y_{2(q+1)-k} + \sum_{k=0}^{2q} \binom{2}{k}_q x_1 x_{k+1} y_0 y_{2q-k} \quad (\{x_k\} \text{ is LX}) \\
&\geqslant x_0^2 y_0 y_{2(q+1)} + 2\sum_{k=1}^{q+1} x_0 x_k y_0 y_{2(q+1)-k} + \left\{\sum_{k=0}^{q} x_{k+1} y_{q-k}\right\}^2 \quad \text{(by relation (9))} \\
&\geqslant x_0^2 y_{q+1}^2 + 2\sum_{k=1}^{q+1} x_0 x_k y_{q+1} y_{(q+1)-k} + \left\{\sum_{k=0}^{q} x_{k+1} y_{q-k}\right\}^2 \quad (\{x_k\} \text{ and } \{y_k\} \text{ are LX}) \\
&= \left\{\sum_{k=0}^{q+1} x_k y_{(q+1)-k}\right\}^2 = t_1^2.
\end{aligned}
$$

We proceed by induction over n. Notice that

$$
\begin{aligned}
t_n &= \sum_{k=0}^{ns} \binom{n}{k}_s x_k y_{ns-k} \\
&= \sum_{k=0}^{ns} \sum_{j=0}^{s} \binom{n-1}{k-j}_s x_k y_{ns-k} \quad \text{using the longitudinal recurrence relation (5)} \\
&= \sum_{k=0}^{ns} \binom{n-1}{k}_s x_k y_{ns-k} + \sum_{k=0}^{ns} \binom{n-1}{k-1}_s x_k y_{ns-k} + \cdots + \sum_{k=0}^{ns} \binom{n-1}{k-s}_s x_k y_{ns-k} \\
&= \sum_{k=0}^{(n-1)s} \binom{n-1}{k}_s x_k y_{ns-k} + \sum_{k=0}^{(n-1)s} \binom{n-1}{k}_s x_{k+1} y_{ns-k-1} + \cdots \\
&\quad \cdots + \sum_{k=0}^{(n-1)s} \binom{n-1}{k}_s x_{k+s} y_{(n-1)s-k}. \qquad (10)
\end{aligned}
$$

Hence, by the induction hypothesis, the n sums in the right hand side of relation (10) are log-convex. Thus, by Proposition 2, the sequence $\{t_n\}_n$ is log-convex. □

Taking $y_k = 1$ for $0 \leqslant k \leqslant ns$, we have the following corollary.

Corollary 1. *If the sequence of nonnegative numbers $\{x_n\}_n$ is log-convex, then so is the sequence $\{z_n\}_n$,*

$$
z_n = \sum_{k=0}^{ns} \binom{n}{k}_s x_k, \quad (n \geqslant 0).
$$

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